Short sales allowed, risk free asset exists
Single index model - Ranking stocks

The calculation of optimal portfolios is simplified by using the single index model to rank securities based on the excess return to beta ratio defined as follows:

\[
\text{Excess return to beta} = \frac{\bar{R}_i - R_f}{\beta_i}.
\]

After stocks are ranked using the above ratio the optimum portfolio (point of tangency) consists of investing in all stocks: Those for which the excess return to beta is greater than the cut-off point \( C^* \) will be held long. Those for which the excess return to beta is smaller than the cut-off point \( C^* \) will be held short. This cut-off rate is computed as follows:

\[
C^* = \frac{\sigma^2_m \sum_{j=1}^{N} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma^2_{\epsilon j}}}{1 + \sigma^2_m \sum_{j=1}^{N} \frac{\beta^2_j}{\sigma^2_{\epsilon j}}}.
\]

where

- \( \bar{R}_j \): Expected return on stock \( j \).
- \( R_f \): Return on a riskless asset.
- \( \beta_j \): Change in the rate of return of stock \( j \) associated with a 1% change in the market return.
- \( \sigma^2_m \): Variance in the market index.
- \( \sigma^2_{\epsilon j} \): Variance of the error term. Also known as unsystematic risk.

The cut-off point is computed using all the stocks because short sales are allowed (some will be held long, some will be held short)

If \( \frac{\bar{R}_i - R_f}{\beta_i} > C^* \) then \( z_i > 0 \Rightarrow x_i > 0 \).

To find the proportion of funds invested in each of these stocks we use:

\[
z_i = \frac{\beta_i}{\sigma^2_{\epsilon i}} \left( \frac{\bar{R}_i - R_f}{\beta_i} - C^* \right)
\]

Therefore \( x_i = \sum_{i=1}^{N} \frac{z_i}{z_i} \), where \( N \) is equal to the number of stocks consisting the optimum portfolio (all stocks). Below an example with 16 stocks is presented. Monthly returns on the stocks listed below were selected from January 1980 to February 2001. Using the single index model we estimate the mean return of each stock. The results of the simple regressions for each stock against the market index (DJIA) are summarized in the next table. Also for these data the expected return of the market index is \( \bar{R}_m = 0.01082 \) and its variance is \( \sigma^2_m = 0.00192 \). Assume \( R_f = 0.005 \).
Therefore, the sum of the z's:

\[ \sum_{i=1}^{N} z_i = 0 \]

Similarly, the z's for each stock can be calculated as follows:

\[ z_i = \frac{\hat{\beta}_i}{\sigma^2_{\hat{\beta}_i}} \left( R_{P} - R_f - C^* \right) \]

We find from the previous table that C* = 0.00981. Therefore, the first 10 ranked stocks will be held long and the last 6 ranked stocks will be held short. First we need to find the \( z_i \)'s as follows: We first find the values of \( z_i \):

\[ z_1 = \frac{\hat{\beta}_1}{\sigma^2_{\hat{\beta}_1}} \left( R_{P} - R_f - C^* \right) = \frac{0.22198}{0.00297} \left( 0.04352 - 0.00981 \right) = 2.520. \]

Similarly, \( z_2 = 1.766, z_3 = 1.850, z_4 = 1.329, z_5 = 1.000, z_6 = 0.493, z_7 = 1.990, \) and \( z_8 = 0.650, z_9 = 0.314, z_{10} = 0.247, z_{11} = 0.154, z_{12} = -0.302, z_{13} = -0.398, z_{14} = -0.579, z_{15} = -1.006, z_{16} = -1.421. \) The sum of the \( z_i \)'s is \( \sum_{i=1}^{16} z_i = 8.301. \)

Therefore, \( x_1 = \frac{2.520}{8.301} = 0.30, x_2 = \frac{1.766}{8.301} = 0.21, \) and similarly, \( x_3 = 0.22, x_4 = 0.16, x_5 = 0.12, x_6 = 0.06, x_7 = 0.24, x_8 = 0.08, x_9 = 0.04, x_{10} = 0.03, x_{11} = -0.02, x_{12} = -0.04, x_{13} = -0.05, x_{14} = -0.07, x_{15} = -0.12 \) and \( x_{16} = -0.17. \)

The composition of the optimum portfolio (point of tangency) consists of

- 30% Consolidated Edison,
- 21% Merck,
- 22% Coca Cola,
- 16% Johnson & Johnson,
- 12% Pepsi,
- 6% Texaco,
- 24% General Electric,
- 8% Ford Motor,
- 4% Procter & Gamble,
- 3% Citigroup,
- 2% MN Mining & Man. Co.,
- 4% Exxon Mobil,
- 5% Alcoa Inc.,
- 7% Boeing Co.,
- 12% Ibm,
- 17% Xerox Corp.