Single index model - short sales not allowed
Risk free asset exists
Kuhn-Tucker conditions

If we assume short sales then we can simply maximize the slope and find the tangent to the efficient frontier subject to the constraint $\sum_{i=1}^{N} x_1 = 1$

$$\max \theta = \frac{\bar{R}_G - R_f}{\sigma_G}$$

To find the $x_i's$ we take derivatives w.r.t. each $x_i$ set them equal to zero and solve $\cdot \cdot \cdot$

$$\frac{d\theta}{dx_i} = z_i \sigma_i^2 + \sum_{j \neq i}^{N} z_j \sigma_{ij} = 0, \ i = 1, \cdots N$$

or

$$\bar{R}_i - R_f = z_i \sigma_i^2 + \sum_{j \neq i}^{N} z_j \sigma_{ij}, \ i = 1, \cdots N$$

If short sales are not allowed we have an extra set of constraints $x_i \geq 0$. We still take the derivative w.r.t. each $x_i$ but now if the maximum occurs at $x_i < 0$ then it is not feasible for our problem. Then $\frac{d\theta}{dx_i} < 0$. But if the maximum occurs at a positive $x_i$ then $\frac{d\theta}{dx_i} = 0$ (see figure below). To summarize

$$\frac{d\theta}{dx_i} \leq 0$$

which can me written as equality

$$\frac{d\theta}{dx_i} + u_i = 0, \text{ this is the first Kuhn-Tucker condition.}$$

About $u_i$: If the maximum occurs at a positive $x_i$ then $u_i = 0$. If the maximum occurs at $x_i = 0$ then $\frac{d\theta}{dx_i} < 0$ and therefore $u_i > 0$. This is the second Kuhn-Tucker condition and can be written as

$$x_i u_i = 0$$
$$x_i \geq 0$$
$$u_i \geq 0$$

Now the system of equations with the Kuhn-Tucker conditions will be:

$$\bar{R}_i - R_f = z_i \sigma_i^2 + \sum_{j \neq i}^{N} z_j \sigma_{ij} - u_i, \ i = 1, \cdots N.$$  
$$z_i u_i = 0, \ i = 1, \cdots N.$$  
$$z_i \geq 0, \ i \cdots N.$$  
$$u_i \geq 0, \ i \cdots N.$$  

If the single index model is assumed then

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2, \text{ and } \sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_{e_i}^2$$
If we substitute this into the first Kuhn-Tucker condition we get

\[ \bar{R}_i - R_f = z_i (\beta_i^2 \sigma_m^2 + \sigma^2_{e_i}) + \sum_{j \neq i}^{N} z_j \beta_i \beta_j \sigma_m^2 - u_i, \quad i = 1, \ldots, N \]

If we combine the terms on the right side we get

\[ \bar{R}_i - R_f = z_i \sigma_e^2 + \sum_{j = 1}^{N} z_j \beta_i \beta_j \sigma_m^2 - u_i, \quad i = 1, \ldots, N \]

Suppose now that \( k \) out of \( N \) securities will be included in the optimum portfolio. For those that are not included \( z_i = 0 \) and the summation in the previous expression will concern only the securities in the set of \( k \) securities

\[ \bar{R}_i - R_f = z_i \sigma_e^2 + \sum_{j \in k} z_j \beta_i \beta_j \sigma_m^2 - u_i, \quad i = 1, \ldots, N \]

But then for those securities that have positive \( z_i \) the corresponding \( u_i = 0 \). We can write now

\[ \bar{R}_i - R_f = z_i \sigma_e^2 + \sum_{j \in k} z_j \beta_i \beta_j \sigma_m^2, \quad \text{for } j \in k \]

Solve for \( z_i \)

\[ z_i = \frac{\beta_i}{\sigma_e^2} \left[ \frac{\bar{R}_i - R_f}{\beta_i} - \sigma_m^2 \sum_{j \in k} z_j \beta_j \right], \quad \text{for } i \in k \quad (1) \]

We need an expression of \( \sum z_j \beta_j \). If we multiply (1) by \( \beta_i \) and some over the set of \( k \) securities we get

\[ \sum_{i \in k} z_i \beta_i = \sum_{i \in k} \frac{(\bar{R}_i - R_f) \beta_i}{\sigma_e^2} - \sigma_m^2 \sum_{i \in k} \frac{\beta_i^2}{\sigma_e^2} \sum_{j \in k} z_j \beta_j \]

Solve for \( \sum z_j \beta_j \) to get:

\[ \sum_{j \in k} z_j \beta_j = \frac{\sum_{j \in k} \frac{(\bar{R}_i - R_f) \beta_j}{\sigma_e^2}}{1 + \sigma_m^2 \sum_{j \in k} \frac{\beta_j^2}{\sigma_e^2}} \]

Expression (1) can be written as:

\[ z_i = \frac{\beta_i}{\sigma_e^2} \left[ \frac{\bar{R}_i - R_f}{\beta_i} - C^* \right] \]

where

\[ C^* = \sigma_m^2 \sum_{j \in k} z_j \beta_j = \sigma_m^2 \frac{\sum_{j \in k} \frac{(\bar{R}_i - R_f) \beta_j \sigma_e^2}{\beta_j^2}}{1 + \sigma_m^2 \sum_{j \in k} \frac{\beta_j^2}{\sigma_e^2}} \]