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Statistics C183/C283 **Instructor: Nicolas Christou**

A model for stock prices

From *Options Futures and Other Derivatives* by John Hull, Prentice Hall 6th Edition, 2006.

- **Stochastic process:** Any variable that changes over time in an uncertain way it follows a stochastic process.
  Discrete time
  Continuous time

- **Markov process:** Special case of stochastic process. Only the current value of a random variable is relevant for future prediction.

- **Wiener process:** A particular type of a Markov process.
  The random variable Z follows the Wiener process if:

  a. $\Delta Z = \epsilon\sqrt{\Delta t}$, where $\epsilon \sim N(0, 1)$.
     Therefore $\Delta Z \sim N(0, \sqrt{\Delta t})$.

  b. The values of ΔZ for two different short intervals Δt are independent.

Consider the change in Z over a long period of time (from 0 to T). Let $Z(T)$ be the value of Z at the end of period T, and $Z(0)$ be the value of Z now (time zero).

1. Change in value of Z from now until T:
   $Z(T) - Z(0) = \Delta Z$.
2. This change can be be viewed as the sum of changes in n small intervals each one of length Δt as follows:
3. Therefore
   $\Delta Z = \Delta Z_1 + \Delta Z_2 + \ldots + \Delta Z_n$
4. Find the distribution of the change in Z.
   Write the previous expression as:
   $\Delta Z = \epsilon_1 \sqrt{\Delta t} + \epsilon_2 \sqrt{\Delta t} + \ldots + \epsilon_n \sqrt{\Delta t}$
   But, $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are independent $N(0,1)$. Therefore, since ΔZ is a linear combination of i.i.d. $N(0,1)$ random variables, it follows that $\Delta Z \sim N(0, \sqrt{T})$.

Example: Let Z be a random variable that follows the Wiener process and time is measured in years. Initially its value is \$20. Find the distribution of its value at the end of the

(i.) First year:
$S_1 = S_0 + \Delta S_1 = S_0 + \epsilon_1 \sqrt{1}$.
$E(S_1) = 20 + 0 = 20$.
$var(S_1) = var(S_0 + \Delta S_1) = var(\Delta S_1) = var(\epsilon_1 \sqrt{1}) = 1$
$S_1 \sim N(20, 1)$.

(ii.) Second year: $S_2 = S_0 + \Delta S_1 + \Delta S_2$.
$S_2 \sim N(20, \sqrt{2})$.

(iii.) Fifth year:
$S_5 = S_0 + \Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4 + \Delta S_5$.
$S_5 \sim N(20, \sqrt{5})$.

- **Generalized Wiener Process:** So far the mean of the change in Z was assumed to be zero. If indeed it is zero, then the expected value of Z in the future is equal to the current value!

  Definition: Let X follow the generalized Wiener process. Then $\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t}$.
  Therefore $\Delta x \sim N(a\Delta t, b\sqrt{\Delta t})$. This Wiener process has expected drift rate of a per Δt and variance of b^2 per Δt.

  Example: The current price of a stock is \$50 and has expected drift rate of 20 per year, and variance 900 per year. Find the distribution of the price of the stock at the end of the

  (i.) First year: $S_1 = S_0 + \Delta S_1 = S_0 + 20\Delta_t + 30\epsilon_t\sqrt{\Delta_t}$.
  $E(S_1) = 50 + 20 = 70$.
  $var(S_1) = var(S_0 + \Delta S_1) = var(30\epsilon_t\sqrt{\Delta_t}) = 900$.
  $S_1 \sim N(70, 30)$.

  (ii.) Second year:
  $S_2 = S_0 + \Delta S_1 + \Delta S_2$.
  $E(S_2) = 50 + 20 + 20 = 90$.
  $var(S_2) = var(\Delta S_1 + \Delta S_2) = 900 + 900$.
  $S_2 \sim N(90, 30\sqrt{2})$.

  (iii.) Sixth month:
  $S_{0.5} = S_0 + \Delta S_{0.5}$, here $\Delta t = 0.5$.
  $E(S_{0.5}) = 50 + 10 = 60$.
  $var(S_{0.5}) = 900 \times \frac{1}{2}$.
  $S_{0.5} \sim N(60, 30\sqrt{\frac{1}{2}})$.

- **Process for Stock Prices:** The generalized Wiener process could have been the correct model for stock prices, however the drift rate and variance do not include the current price of the stock.

  The model now is:

  $$\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}.$$

  or

  $$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

  This model assumes a drift rate equal to μS where μ is the expected return of the stock, and variance $\sigma^2 S^2$ where σ^2 is the variance of the return of the stock. Therefore

  $$\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma \sqrt{\Delta t}).$$

S	Price of the stock.
ΔS	Change in the stock price.
Δt	Small interval of time.
ϵ	Follows $N(0, 1)$.

  Example: The current price of a stock is $S_0 = \$100$. The expected return is $\mu = 0.10$ per year, and the standard deviation of the return is $\sigma = 0.20$ (also per year).

  1. Find an expression for the process of the stock:
     $\frac{\Delta S}{S} = 0.10 \Delta t + 0.20 \epsilon \sqrt{\Delta t}$.

2. Find the distribution of the change in S divided by S (distribution of $\frac{\Delta S}{S}$). $\frac{\Delta S}{S} \sim N(0.10\Delta t, 0.20\sqrt{\Delta t})$.
   Therefore, at the end of the first year the distribution is $\frac{\Delta S}{S} \sim N(0.10, 0.20)$.

3. Divide the year in weekly intervals and find the distribution of $\frac{\Delta S}{S}$ at the end of each weekly interval.
   $\frac{\Delta S}{S} = 0.10\frac{1}{52} + 0.20\epsilon\sqrt{\frac{1}{52}}$.
   $\frac{\Delta S}{S} \sim N(0.10\frac{1}{52}, 0.20\sqrt{\frac{1}{52}})$.

4. Repeat (3) by assuming daily intervals.
   $\frac{\Delta S}{S} = 0.10\frac{1}{365} + 0.20\epsilon\sqrt{\frac{1}{365}}$.
   $\frac{\Delta S}{S} \sim N(0.10\frac{1}{365}, 0.20\sqrt{\frac{1}{365}})$.

Monte Carlo Simulation of a stock's path

$S_0 = \$20$, annual mean and standard deviation: $\mu = 0.14, \sigma = 0.20$.
Consider time intervals of 3.65 days or $\Delta t = 0.01$ years.

$$
\begin{aligned}
\Delta S &= \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t} \\
\Delta S &= (0.14)(0.01) S + 0.20\sqrt{0.01}\epsilon S \\
\Delta S &= 0.0014 S + 0.02 \epsilon S
\end{aligned}
$$

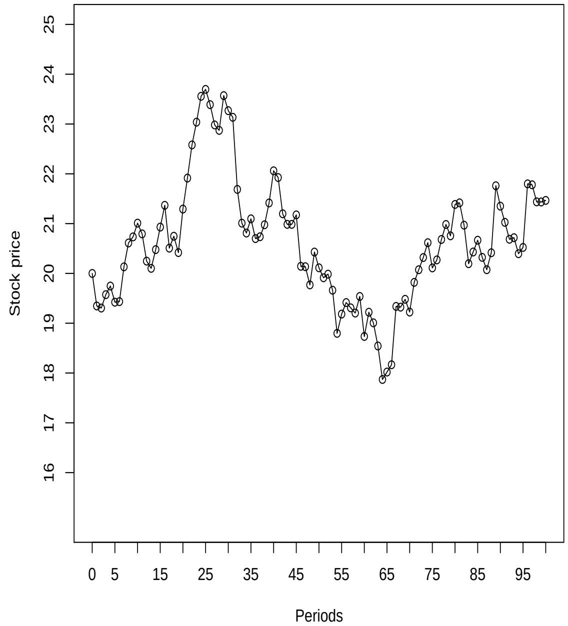
$$
\begin{aligned}
\Delta S_1 &= 0.0014 S_0 + 0.02 \epsilon_1 S_0 \\
\Delta S_1 &= 0.0014(20) + 0.02(-1.70)(20) = -0.653 \\
S_1 &= S_0 + \Delta S_1 = 20 - 0.653 = 19.347
\end{aligned}
$$

$$
\begin{aligned}
\Delta S_2 &= 0.0014 S_1 + 0.02 \epsilon_2 S_1 \\
\Delta S_2 &= 0.0014(19.347) + 0.02(-0.18)(19.347) = -0.042 \\
S_2 &= S_1 + \Delta S_2 = 19.347 - 0.042 = 19.305 \\
&\vdots \qquad \vdots \\
&\vdots \qquad \vdots \\
\Delta S_n &= 0.0014 S_{n-1} + 0.02 \epsilon_n S_{n-1} \\
S_n &= S_{n-1} + \Delta S_n
\end{aligned}
$$

i	epsilon	DS	S
			20.00000
1	-1.703034618	-0.6532138473	19.34679
2	-0.178328033	-0.0419159860	19.30487
3	0.626012018	0.2687284327	19.57360
4	0.372285880	0.1731425256	19.74674
5	-0.891222913	-0.3243295254	19.42241
6	-0.040610484	0.0114163056	19.43383
7	1.726364596	0.6982048082	20.13203
8	1.120874720	0.4794945764	20.61153
9	0.226489910	0.1222221973	20.73375
10	0.595893749	0.2761294837	21.00988
11	-0.583445574	-0.2157485874	20.79413
12	-1.385848921	-0.5472386807	20.24689
13	-0.428217124	-0.1450556662	20.10184
14	0.865628958	0.3761571979	20.47799
15	1.035171785	0.4526340065	20.93063
16	0.973033321	0.4366268329	21.36725
17	-2.081263166	-0.8595034214	20.50775
18	0.512261722	0.2388175642	20.74657
19	-0.859783037	-0.3277057531	20.41886
20	2.069083428	0.8735530064	21.29242
21	1.390716075	0.6220434713	21.91446
22	1.450279390	0.6663220053	22.58078

23	0.941865554	0.4569742885	23.03776
24	1.055756037	0.5186978406	23.55645
25	0.224687801	0.1388359872	23.69529
26	−0.715660734	−0.3059823545	23.38931
27	−0.940489092	−0.4072027270	22.98210
28	−0.308339601	−0.1095509099	22.87255
29	1.455468561	0.6978272123	23.57038
30	−0.710234853	−0.3018115790	23.26857
31	−0.356732662	−0.1334371730	23.13513
32	−3.199841196	−1.4481857549	21.68695
33	−1.631378973	−0.6772308227	21.00971
34	−0.542553852	−0.1985644345	20.81115
35	0.615265435	0.2852232422	21.09637
36	−1.010607048	−0.3968679572	20.69951
37	0.019320693	0.0369778841	20.73648
38	0.509565404	0.2403629708	20.97685
39	0.975317430	0.4385492689	21.41540
40	1.437012949	0.6454655794	22.06086
41	−0.377870717	−0.1358378649	21.92502
42	−1.729262423	−0.7275873570	21.19744
43	−0.569424217	−0.2117302605	20.98571
44	−0.071416430	−0.0005944957	20.98511
45	0.383453641	0.1903155045	21.17543
46	−2.507501365	−1.0323026462	20.14312
47	−0.089458653	−0.0078391612	20.13529
48	−0.978120617	−0.3657053529	19.76958
49	1.597830335	0.6594461006	20.42903
50	−0.844416911	−0.3164116639	20.11261
51	−0.566026512	−0.1995277986	19.91309
52	0.111094058	0.0721228328	19.98521
53	−0.876851894	−0.3225020802	19.66271
54	−2.271871464	−0.8658950808	18.79681
55	0.959173234	0.3869035196	19.18372
56	0.533664170	0.2316104363	19.41533
57	−0.345212257	−0.1068667147	19.30846
58	−0.340939728	−0.1046285748	19.20383
59	0.798637338	0.3336232902	19.53745
60	−2.124504725	−0.8027958370	18.73466
61	1.227559445	0.4861866567	19.22084
62	−0.624467688	−0.2131467492	19.00770
63	−1.292942709	−0.4649065190	18.54279
64	−1.882423867	−0.6721479645	17.87064
65	0.346697760	0.1489331441	18.01958
66	0.336584496	0.1465296117	18.16611
67	3.154811753	1.1716454740	19.33775
68	−0.107884452	−0.0146520027	19.32310
69	0.336946400	0.1572693193	19.48037
70	−0.728225530	−0.2564495276	19.22392
71	1.480563041	0.5961579893	19.82008
72	0.570817040	0.2540208709	20.07410
73	0.540706838	0.2451877856	20.31929
74	0.663504794	0.2980858797	20.61737
75	−1.297452022	−0.5061367055	20.11124
76	0.328802214	0.1604081056	20.27164
77	0.941722076	0.4101853885	20.68183
78	0.658260601	0.3012352253	20.98306
79	−0.613416879	−0.2280510263	20.75501
80	1.447850461	0.6300601289	21.38507
81	0.005486955	0.0322858814	21.41736
82	−1.117569817	−0.4487235828	20.96864
83	−1.913593757	−0.7731529170	20.19548
84	0.511726943	0.2349651292	20.43045
85	0.508464333	0.2363657082	20.66681
86	−0.903868221	−0.3446679818	20.32215
87	−0.678098181	−0.2471571957	20.07499
88	0.776638495	0.3399251599	20.41491
89	3.225883282	1.3457034482	21.76062
90	−1.012105629	−0.4100159962	21.35060
91	−0.830983810	−0.3249492345	21.02565
92	−0.876913599	−0.3393176872	20.68633
93	0.006448467	0.0316287707	20.71796
94	−0.848032954	−0.3223851576	20.39558
95	0.240807586	0.1267820054	20.52236
96	3.039850403	1.2764293740	21.79879
97	−0.112546625	−0.0185492981	21.78024
98	−0.857173384	−0.3428965014	21.43734
99	−0.075933105	−0.0025438002	21.43480
100	0.003048477	0.0313155894	21.46612

Simulation of the stock's path:



Stock simulation - R commands:

```
epsilon <- c(0,rnorm(100))
S <- c(20,rep(0,100))
DS <- rep(0,101)

for(i in(1:100)) {

         DS[i+1] <- 0.0014*S[i] + 0.02*S[i]*epsilon[i+1]

         S[i+1] = S[i] + DS[i+1]
              }

x <- seq(0,100)
xx <- as.data.frame(cbind(x, epsilon, DS, S))

plot(x, S, type="l", xlab="Periods", ylab="Stock price")

points(x,S)
```