A model for stock prices

- **Stochastic process**: Any variable that changes over time in an uncertain way it follows a stochastic process.
  - Discrete time
  - Continuous time

- **Markov process**: Special case of stochastic process. Only the current value of a random variable is relevant for future prediction.

- **Wiener process**: A particular type of a Markov process.
  - The random variable $Z$ follows the Wiener process if:
    a. $\Delta Z = \epsilon \sqrt{\Delta t}$, where $\epsilon \sim N(0, 1)$.
      Therefore $\Delta Z \sim N(0, \sqrt{\Delta t})$.
    b. The values of $\Delta Z$ for two different short intervals $\Delta t$ are independent.

Consider the change in $Z$ over a long period of time (from 0 to $T$). Let $Z(T)$ be the value of $Z$ at the end of period $T$, and $Z(0)$ be the value of $Z$ now (time zero).
1. Change in value of \( Z \) from now until \( T \):
\[
Z(T) - Z(0) = \Delta Z.
\]
2. This change can be viewed as the sum of changes in \( n \) small intervals each one of length \( \Delta t \) as follows:
3. Therefore
\[
\Delta Z = \Delta Z_1 + \Delta Z_2 + \ldots + \Delta Z_n
\]
4. Find the distribution of the change in \( Z \).
Write the previous expression as:
\[
\Delta Z = \epsilon_1 \sqrt{\Delta t} + \epsilon_2 \sqrt{\Delta t} + \ldots + \epsilon_n \sqrt{\Delta t}
\]
But, \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \) are independent \( N(0, 1) \). Therefore, since \( \Delta Z \) is a linear combination of i.i.d. \( N(0, 1) \) random variables, it follows that \( \Delta Z \sim N(0, \sqrt{T}) \).

Example: Let \( Z \) be a random variable that follows the Wiener process and time is measured in years. Initially its value is $20. Find the distribution of its value at the end of the

(i.) First year:
\[
S_1 = S_0 + \Delta S_1 = S_0 + \epsilon_1 \sqrt{1}.
\]
\[
E(S_1) = 20 + 0 = 20.
\]
\[
var(S_1) = var(S_0 + \Delta S_1) = var(\Delta S_1) = var(\epsilon_1 \sqrt{1}) = 1
\]
\[
S_1 \sim N(20, 1).
\]

(ii.) Second year: \( S_2 = S_0 + \Delta S_1 + \Delta S_2 \).
\[
S_2 \sim N(20, \sqrt{2}).
\]

(iii.) Fifth year:
\[
S_5 = S_0 + \Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4 + \Delta S_5.
\]
\[
S_5 \sim N(20, \sqrt{5}).
\]
• Generalized Wiener Process: So far the mean of the change in $Z$ was assumed to be zero. If indeed it is zero, then the expected value of $Z$ in the future is equal to the current value!

Definition: Let $X$ follow the generalized Wiener process. Then $\Delta x = a\Delta t + b\epsilon\sqrt{\Delta t}$.
Therefore $\Delta x \sim N(a\Delta t, b\sqrt{\Delta t})$. This Wiener process has expected drift rate of $a$ per $\Delta t$ and variance of $b^2$ per $\Delta t$.

Example: The current price of a stock is $50 and has expected drift rate of 20 per year, and variance 900 per year. Find the distribution of the price of the stock at the end of the

(i.) First year: $S_1 = S_0 + \Delta S_1 = S_0 + 20\Delta t + 30\epsilon_t\sqrt{\Delta t}$.
$E(S_1) = 50 + 20 = 70$.
$\text{var}(S_1) = \text{var}(S_0 + \Delta S_1) = \text{var}(30\epsilon_t\sqrt{\Delta t}) = 900$.
$S_1 \sim N(70, 30)$.

(ii.) Second year:
$S_2 = S_0 + \Delta S_1 + \Delta S_2$.
$E(S_2) = 50 + 20 + 20 = 90$.
$\text{var}(S_2) = \text{var}(\Delta S_1 + \Delta S_2) = 900 + 900$.
$S_2 \sim N(90, 30\sqrt{2})$.

(iii.) Sixth month:
$S_{0.5} = S_0 + \Delta S_{0.5}$, here $\Delta t = 0.5$.
$E(S_{0.5}) = 50 + 10 = 60$.
$\text{var}(S_{0.5}) = 900 \times \frac{1}{2}$.
$S_{0.5} \sim N(60, 30\sqrt{\frac{1}{2}})$.
• **Process for Stock Prices:** The generalized Wiener process could have been the correct model for stock prices, however the drift rate and variance do not include the current price of the stock.

The model now is:

\[
\frac{\Delta S}{S} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}.
\]

or

\[
\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}
\]

This model assumes a drift rate equal to \( \mu S \) where \( \mu \) is the expected return of the stock, and variance \( \sigma^2 S^2 \) where \( \sigma^2 \) is the variance of the return of the stock. Therefore

\[
\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma \sqrt{\Delta t}).
\]

\( S \) Price of the stock.
\( \Delta S \) Change in the stock price.
\( \Delta t \) Small interval of time.
\( \epsilon \) Follows \( N(0, 1) \).

Example: The current price of a stock is \( S_0 = $100 \). The expected return is \( \mu = 0.10 \) per year, and the standard deviation of the return is \( \sigma = 0.20 \) (also per year).

1. Find an expression for the process of the stock:

\[
\frac{\Delta S}{S} = 0.10 \Delta t + 0.20 \epsilon \sqrt{\Delta t}.
\]
2. Find the distribution of the change in $S$ divided by $S$ (distribution of $\frac{\Delta S}{S}$). $\frac{\Delta S}{S} \sim N(0.10\Delta t, 0.20\sqrt{\Delta t})$.

Therefore, at the end of the first year the distribution is $\frac{\Delta S}{S} \sim N(0.10, 0.20)$.

3. Divide the year in weekly intervals and find the distribution of $\frac{\Delta S}{S}$ at the end of each weekly interval.

$\frac{\Delta S}{S} = 0.10\frac{1}{52} + 0.20\epsilon\sqrt{\frac{1}{52}}$.

$\frac{\Delta S}{S} \sim N(0.10\frac{1}{52}, 0.20\sqrt{\frac{1}{52}})$.

4. Repeat (3) by assuming daily intervals.

$\frac{\Delta S}{S} = 0.10\frac{1}{365} + 0.20\epsilon\sqrt{\frac{1}{365}}$.

$\frac{\Delta S}{S} \sim N(0.10\frac{1}{365}, 0.20\sqrt{\frac{1}{365}})$. 
Monte Carlo Simulation of a stock’s path

$S_0 = $20, annual mean and standard deviation: $\mu = 0.14, \sigma = 0.20$. Consider time intervals of 3.65 days or $\Delta t = 0.01$ years.

$$\Delta S = \mu S \Delta t + \sigma S \epsilon \sqrt{\Delta t}$$

$$\Delta S = (0.14)(0.01)S + 0.20\sqrt{0.01}S$$

$$\Delta S = 0.0014S + 0.02\epsilon S$$

$$\Delta S_1 = 0.0014S_0 + 0.02\epsilon_1S_0$$

$$\Delta S_1 = 0.0014(20) + 0.02(-1.70)(20) = -0.653$$

$$S_1 = S_0 + \Delta S_1 = 20 - 0.653 = 19.347$$

$$\Delta S_2 = 0.0014S_1 + 0.02\epsilon_2S_1$$

$$\Delta S_2 = 0.0014(19.347) + 0.02(-0.18)(19.347) = -0.042$$

$$S_2 = S_1 + \Delta S_2 = 19.347 - 0.042 = 19.305$$

$$\vdots$$

$$\Delta S_n = 0.0014S_{n-1} + 0.02\epsilon_nS_{n-1}$$

$$S_n = S_{n-1} + \Delta S_n$$
Simulation of the stock’s path:

Stock simulation - R commands:

```r
epsilon <- c(0, rnorm(100))
S <- c(20, rep(0, 100))
DS <- rep(0, 101)

for(i in 1:100) {
  DS[i+1] <- 0.0014 * S[i] + 0.02 * S[i] * epsilon[i+1]
  S[i+1] = S[i] + DS[i+1]
}

x <- seq(0, 100)
xx <- as.data.frame(cbind(x, epsilon, DS, S))

plot(x, S, type="l", xlab="Periods", ylab="Stock price")
points(x, S)
```

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