University of California, Los Angeles
Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

Multi-index model
Short sales allowed
From: "Simple Rules for Optimal Portfolio Selection: The Multi Group Case"
Elton, E., Gruber, M., Padberg, M. (1977), Journal of Financial and Quantitative Analysis

Stocks are grouped by industry. The multi-index model used here gives a diagonal form of the covariance matrix between stocks. The assumptions for this model is that stocks are linearly related to the group index (industry) and the industry is linearly related to the market index. Here is the model:

$$\begin{aligned} R_i &= \alpha_i + \beta_i I_j + \epsilon_i \\ I_j &= \gamma_j + b_j R_m + c_j \end{aligned}$$

with,

$$\begin{aligned} &E(\epsilon_i \epsilon_k) = 0 \qquad i = 1, \ldots, n, \; k = 1, \ldots, n, \; i \neq k \\ &E(c_j c_l) = 0 \qquad j = 1, \ldots, p, \; l = 1, \ldots, p, \; j \neq l \\ &E(\epsilon_i c_j) = 0 \qquad i = 1, \ldots, n, \; j = 1, \ldots, p \end{aligned}$$

where,

R_i	Return of stock i
I_j	Return of index j
R_m	Return of the market
α_i, β_i	Parameters associated with stock i
γ_j, b_j	Parameters associated with index j
ϵ_i	Error term with mean zero and variance $\sigma^2_{\epsilon_i}$
c_j	Error term with mean zero and variance $\sigma^2_{c_i}$

Using these assumptions we get the following.
Variance of the return of stock i:

$$\begin{aligned} \sigma_i^2 &= \beta_i^2 \sigma_j^2 + \sigma^2_{\epsilon_i} \\ \text{But } \sigma_j^2 &= b_j^2 \sigma_m^2 + \sigma^2_{c_j} \\ \text{Therefore, } \sigma_i^2 &= \beta_i^2 (b_j^2 \sigma_m^2 + \sigma^2_{c_j}) + \sigma^2_{\epsilon_i} \end{aligned} \tag{1}$$

Covariance between stocks i and k in the same group (industry):

$$\sigma_{ik} = \beta_i \beta_k (b_j^2 \sigma_m^2 + \sigma^2_{cj}) \tag{2}$$

Covariance between stocks i and k in different industries:

$$\sigma_{ik} = \beta_i \beta_k b_j b_l \sigma_m^2 \tag{3}$$

Assume two stocks and two industries (two per industry). The solution as always is given by the following system of equations:

$$\bar{R}_1 - R_f = z_1\sigma_1^2 + z_2\sigma_{12} + z_3\sigma_{13} + z_4\sigma_{14} \tag{4}$$

$$\bar{R}_2 - R_f = z_1\sigma_{21} + z_2\sigma_2^2 + z_3\sigma_{23} + z_4\sigma_{24} \tag{5}$$

$$\bar{R}_3 - R_f = z_1\sigma_{31} + z_2\sigma_{32} + z_3\sigma_3^2 + z_4\sigma_{34} \tag{6}$$

$$\bar{R}_4 - R_f = z_1\sigma_{41} + z_2\sigma_{42} + z_3\sigma_{43} + z_4\sigma_4^2 \tag{7}$$

Let's examine equation (4) and see how it can be written using equations (1), (2), and (3).

$$\begin{aligned} \bar{R}_1 - R_f &= z_1\left(\beta_1^2[b_1^2\sigma_m^2 + \sigma_{c_1}^2] + \sigma_{\epsilon_1}^2\right) + z_2\left(\beta_1\beta_2[b_1^2\sigma_m^2 + \sigma_{c_1}^2]\right) \\ &+ z_3\left(\beta_1\beta_3 b_1 b_2\sigma_m^2\right) + z_4\left(\beta_1\beta_4 b_1 b_2\sigma_m^2\right) \end{aligned}$$

Rearrange

$$\begin{aligned} \bar{R}_1 - R_f &= z_1\sigma_{\epsilon_1}^2 + \beta_1\left[z_1\beta_1 b_1^2\sigma_m^2 + z_2\beta_2 b_1^2\sigma_m^2 + z_1\beta_1\sigma_{c_1}^2 + z_2\beta_2\sigma_{c_1}^2\right] \\ &+ \beta_1\left[z_3\beta_3 b_1 b_2\sigma_m^2 + z_4\beta_4 b_1 b_2\sigma_m^2\right] \end{aligned}$$

Or

$$\bar{R}_1 - R_f = z_1\sigma_{\epsilon_1}^2 + \beta_1\left[b_1^2\sigma_m^2(z_1\beta_1 + z_2\beta_2) + \sigma_{c_1}^2(z_1\beta_1 + z_2\beta_2)\right] + \beta_1\left[b_1 b_2\sigma_m^2(z_3\beta_3 + z_4\beta_4)\right]$$

If we let $\Phi_1 = z_1\beta_1 + z_2\beta_2$ and $\Phi_2 = z_3\beta_3 + z_4\beta_4$ we get:

$$\bar{R}_1 - R_f = z_1\sigma_{\epsilon_1}^2 + \beta_1\left[(\sigma_{c_1}^2 + b_1\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]$$

Now solve for z_1:

$$z_1 = \frac{\beta_1}{\sigma_{\epsilon_1}^2}\left[\frac{\bar{R}_1 - R_f}{\beta_1} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]\right] \tag{8}$$

Similarly, stock 2 will give the following expression:

$$z_2 = \frac{\beta_2}{\sigma_{\epsilon_2}^2}\left[\frac{\bar{R}_2 - R_f}{\beta_2} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]\right] \tag{9}$$

These expressions look similar to the single index model solution (see class notes). But now the C^* cut-off point is a more complicated expression and of course it is the same for stocks in the same industry. In our example the C^* for industry 1 is equal to:

$$C_1^* = (\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2$$

If we know C_1^* it will be easy to compute the $z_i's$ and from there the $x_i's$. In order to find C_1^* we need to find Φ_1 and Φ_2.

Multiply (8) by β_1 and (9) by β_2:

$$z_1\beta_1 = \frac{\beta_1^2}{\sigma_{\epsilon_1}^2}\left[\frac{\bar{R}_1 - R_f}{\beta_1} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]\right] \tag{10}$$

and

$$z_2\beta_2 = \frac{\beta_2^2}{\sigma_{\epsilon_2}^2}\left[\frac{\bar{R}_2 - R_f}{\beta_2} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2\sigma_m^2\Phi_2\right]\right] \tag{11}$$

To produce Φ_1 on the left hand side add (10) and (11):

$$\sum_{i=1}^{2} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma^2_{\epsilon_i}} = \Phi_1 + \frac{\beta_1^2}{\sigma^2_{\epsilon_1}} \left[\sigma^2_{c_1} + b_1^2\sigma^2_m\right] \Phi_1 + \frac{\beta_1^2}{\sigma^2_{\epsilon_1}} b_1 b_2 \sigma^2_m \Phi_2$$
$$+ \frac{\beta_2^2}{\sigma^2_{\epsilon_2}} \left[\sigma^2_{c_1} + b_1^2\sigma^2_m\right] \Phi_1 + \frac{\beta_2^2}{\sigma^2_{\epsilon_2}} b_1 b_2 \sigma^2_m \Phi_2$$

Or

$$\sum_{i=1}^{2} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma^2_{\epsilon_i}} = \Phi_1 \left[1 + \frac{\beta_1^2}{\sigma^2_{\epsilon_1}} \left[\sigma^2_{c_1} + b_1^2\sigma^2_m\right] + \frac{\beta_2^2}{\sigma^2_{\epsilon_2}} \left[\sigma^2_{c_1} + b_1^2\sigma^2_m\right]\right]$$
$$+ \Phi_2 \left[\frac{\beta_1^2}{\sigma^2_{\epsilon_1}} b_1 b_2 \sigma^2_m + \frac{\beta_2^2}{\sigma^2_{\epsilon_2}} b_1 b_2 \sigma^2_m\right] \tag{12}$$

Similarly, for stocks 3 and 4 that belong in industry 2 we get:

$$\sum_{i=3}^{4} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma^2_{\epsilon_i}} = \Phi_1 \left[\frac{\beta_3^2}{\sigma^2_{\epsilon_3}} b_1 b_2 \sigma^2_m + \frac{\beta_4^2}{\sigma^2_{\epsilon_4}} b_1 b_2 \sigma^2_m\right]$$
$$+ \Phi_2 \left[1 + \frac{\beta_3^2}{\sigma^2_{\epsilon_3}} \left[\sigma^2_{c_2} + b_2^2\sigma^2_m\right] + \frac{\beta_4^2}{\sigma^2_{\epsilon_4}} \left[\sigma^2_{c_2} + b_2^2\sigma^2_m\right]\right] \tag{13}$$

The system above can be written in vector and matrix form as $\boldsymbol{M\Phi = R}$ and therefore: $\boldsymbol{\Phi = M^{-1}R}$. The dimensions of the matrix $\boldsymbol{M}$ in our example are 2×2 (two industries).

Multi-index system

$$\mathbf{\Phi} = \begin{pmatrix} \Phi_1 \\ \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} 1 + \frac{\beta_1^2}{\sigma_{\epsilon_1}^2}\left[\sigma_{c_1}^2 + b_1^2\sigma_m^2\right] + \frac{\beta_2^2}{\sigma_{\epsilon_2}^2}\left[\sigma_{c_1}^2 + b_1^2\sigma_m^2\right] & \left[\frac{\beta_1^2 b_1 b_2}{\sigma_{\epsilon_1}^2} + \frac{\beta_2^2 b_1 b_2}{\sigma_{\epsilon_2}^2}\right]\sigma_m^2 \\ \\ \left[\frac{\beta_3^2 b_1 b_2}{\sigma_{\epsilon_3}^2} + \frac{\beta_4^2 b_1 b_2}{\sigma_{\epsilon_4}^2}\right]\sigma_m^2 & 1 + \frac{\beta_3^2}{\sigma_{\epsilon_3}^2}\left[\sigma_{c_2}^2 + b_2^2\sigma_m^2\right] + \frac{\beta_4^2}{\sigma_{\epsilon_4}^2}\left[\sigma_{c_2}^2 + b_2^2\sigma_m^2\right] \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^{2} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon_i}^2} \\ \\ \sum_{i=3}^{4} \frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon_i}^2} \end{pmatrix}$$

Once Φ_1 and Φ_2 are obtained we can compute the $z_i's$ (see equations (8) and (9)).

$$z_1 = \frac{\beta_1}{\sigma_{\epsilon_1}^2}\left[\frac{\bar{R}_1 - R_f}{\beta_1} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2\right]\right]$$

$$z_2 = \frac{\beta_2}{\sigma_{\epsilon_2}^2}\left[\frac{\bar{R}_2 - R_f}{\beta_2} - \left[(\sigma_{c_1}^2 + b_1^2\sigma_m^2)\Phi_1 + b_1 b_2 \sigma_m^2 \Phi_2\right]\right]$$

$$z_3 = \frac{\beta_3}{\sigma_{\epsilon_3}^2}\left[\frac{\bar{R}_3 - R_f}{\beta_3} - \left[(\sigma_{c_2}^2 + b_2^2\sigma_m^2)\Phi_2 + b_1 b_2 \sigma_m^2 \Phi_1\right]\right]$$

$$z_4 = \frac{\beta_4}{\sigma_{\epsilon_4}^2}\left[\frac{\bar{R}_4 - R_f}{\beta_4} - \left[(\sigma_{c_2}^2 + b_2^2\sigma_m^2)\Phi_2 + b_1 b_2 \sigma_m^2 \Phi_1\right]\right]$$

Write the system in vector and matrix form when there are three industries with three stocks in each industry.