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## Multigroup model
## Short sales allowed with riskless lending and borrowing
## From: "Simple Rules for Optimal Portfolio Selection: The Multi Group Case"
## Elton, E., Gruber, M., Padberg, M. (1977), Journal of Financial and Quantitative Analysis

The stocks are grouped by industry. Let's examine the simple case of two industries. The assumption here is that the correlations within the first group are the same for all the pairs in group 1 (called it $\rho_{11}$), similarly within the second group correlations are the same for all pairs (called it $\rho_{22}$), and the correlations for all pairs of stocks between the first group and second group is the same (called it $\rho_{12}$). For example, suppose stocks 1, 2, 3 belong to industry 1 (automobile stocks), and stocks 4, 5, 6 belong to industry 2 (chemical stocks). Then the correlations are as follows:
Group 1: $\rho_{12} = \rho_{13} = \rho_{23} = \rho_{11}$,
Group 2: $\rho_{45} = \rho_{46} = \rho_{56} = \rho_{22}$,
Between stocks of group 1 and group 2: $\rho_{14} = \rho_{15} = \rho_{16} = \rho_{24} = \rho_{25} = \rho_{26} = \rho_{34} = \rho_{35} = \rho_{36} = \rho_{12}$.

$$\boldsymbol{\rho} = \left(\begin{array}{ccc|ccc} 1 & \rho_{11} & \rho_{11} & \rho_{12} & \rho_{12} & \rho_{12} \\ \rho_{11} & 1 & \rho_{11} & \rho_{12} & \rho_{12} & \rho_{12} \\ \rho_{11} & \rho_{11} & 1 & \rho_{12} & \rho_{12} & \rho_{12} \\ \hline \rho_{21} & \rho_{21} & \rho_{21} & 1 & \rho_{22} & \rho_{22} \\ \rho_{21} & \rho_{21} & \rho_{21} & \rho_{22} & 1 & \rho_{22} \\ \rho_{21} & \rho_{21} & \rho_{21} & \rho_{22} & \rho_{22} & 1 \end{array}\right)$$

The solution was found and it was the point of tangency to the efficient frontier. By grouping the stocks into industries we can solve the problem using the ranking method within each industry. Assume two stocks and two industries (two per industry). The solution as always is given by the following system of equations:

$$\bar{R}_1 - R_f = z_1\sigma_1^2 + z_2\sigma_{12} + z_3\sigma_{13} + z_4\sigma_{14} \tag{1}$$
$$\bar{R}_2 - R_f = z_1\sigma_{21} + z_2\sigma_2^2 + z_3\sigma_{23} + z_4\sigma_{24} \tag{2}$$
$$\bar{R}_3 - R_f = z_1\sigma_{31} + z_2\sigma_{32} + z_3\sigma_3^2 + z_4\sigma_{34} \tag{3}$$
$$\bar{R}_4 - R_f = z_1\sigma_{41} + z_2\sigma_{42} + z_3\sigma_{43} + z_4\sigma_4^2 \tag{4}$$

The equations (1), (2), (3), and (4) using $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$, and the argument at the beginning of the page can be written as follows:

$$\begin{aligned} \bar{R}_1 - R_f &= z_1\sigma_1^2 + z_2\rho_{11}\sigma_1\sigma_2 + z_3\rho_{12}\sigma_1\sigma_3 + z_4\rho_{12}\sigma_1\sigma_4 \\ \bar{R}_2 - R_f &= z_1\rho_{11}\sigma_1\sigma_2 + z_2\sigma_2^2 + z_3\rho_{12}\sigma_2\sigma_3 + z_4\rho_{12}\sigma_2\sigma_4 \\ \bar{R}_3 - R_f &= z_1\rho_{12}\sigma_1\sigma_3 + z_2\rho_{12}\sigma_2\sigma_3 + z_3\sigma_3^2 + z_4\rho_{22}\sigma_3\sigma_4 \\ \bar{R}_4 - R_f &= z_1\rho_{12}\sigma_1\sigma_4 + z_2\rho_{12}\sigma_2\sigma_4 + z_3\rho_{22}\sigma_3\sigma_4 + z_4\sigma_4^2 \end{aligned}$$

To solve for $\mathbf{Z}$:

$$\mathbf{Z} = \boldsymbol{\Sigma}^{-1}\mathbf{R}$$

The elements of $\boldsymbol{\Sigma}$ are computed based on the assumption that stocks in the same industry have equal correlations, etc.

But there is another approach, similar to the $C^*$ cut-off point...

Consider equation (1): Add and subtract $z_1\rho_{11}\sigma_1^2$ to get:

$$\begin{aligned}
\bar{R}_1 - R_f &= z_1\sigma_1^2 - z_1\rho_{11}\sigma_1^2 + z_1\rho_{11}\sigma_1^2 + z_2\rho_{11}\sigma_1\sigma_2 + z_3\rho_{12}\sigma_1\sigma_3 + z_4\rho_{12}\sigma_1\sigma_4 \\
\bar{R}_1 - R_f &= z_1\sigma_1^2(1-\rho_{11}) + \sigma_1\left[\rho_{11}(z_1\sigma_1 + z_2\sigma_2) + \rho_{12}(z_3\sigma_3 + z_4\sigma_4)\right] \\
\bar{R}_1 - R_f &= z_1\sigma_1^2(1-\rho_{11}) + \sigma_1\left[\rho_{11}\Phi_1 + \rho_{12}\Phi_2\right] \\
\bar{R}_1 - R_f &= z_1\sigma_1^2(1-\rho_{11}) + \sigma_1\sum_{g=1}^{2}\rho_{1g}\Phi_g
\end{aligned}$$

Solve for $Z_1$:

$$z_1 = \frac{1}{\sigma_1(1-\rho_{11})}\left[\frac{\bar{R}_1 - R_f}{\sigma_1} - \sum_{g=1}^{2}\rho_{1g}\Phi_g\right] \tag{5}$$

Similarly, because stock 2 belongs in the same group with stock 1:

$$z_2 = \frac{1}{\sigma_2(1-\rho_{11})}\left[\frac{\bar{R}_2 - R_f}{\sigma_2} - \sum_{g=1}^{2}\rho_{1g}\Phi_g\right] \tag{6}$$

Now multiply (5) by $\sigma_1$ and (6) by $\sigma_2$ and add them together:

$$\begin{aligned}
z_1\sigma_1 &= \frac{1}{1-\rho_{11}}\left[\frac{\bar{R}_1 - R_f}{\sigma_2} - \sum_{g=1}^{2}\rho_{1g}\Phi_g\right] \\
z_2\sigma_2 &= \frac{1}{1-\rho_{11}}\left[\frac{\bar{R}_2 - R_f}{\sigma_2} - \sum_{g=1}^{2}\rho_{1g}\Phi_g\right]
\end{aligned}$$

Sum the two equations:

$$\begin{aligned}
z_1\sigma_1 + z_2\sigma_2 &= \frac{1}{1-\rho_{11}}\left[\frac{\bar{R}_1 - R_f}{\sigma_1} + \frac{\bar{R}_2 - R_f}{\sigma_2} - 2\sum_{g=1}^{2}\rho_{1g}\Phi_g\right] \\
(1-\rho_{11})\Phi_1 &= \sum_{i=1}^{2}\frac{\bar{R}_i - R_f}{\sigma_i} - 2\sum_{g=1}^{2}\rho_{1g}\Phi_g
\end{aligned}$$

Or

$$(1-\rho_{11})\Phi_1 + 2\sum_{g=1}^{2}\rho_{1g}\Phi_g = \sum_{i=1}^{2}\frac{\bar{R}_i - R_f}{\sigma_i} \tag{7}$$

Consider now stocks 3 and 4 that belong in group 2. A similar equation with equation (7) for stocks 3 and 4 is below:

$$(1-\rho_{22})\Phi_2 + 2\sum_{g=1}^{2}\rho_{2g}\Phi_g = \sum_{i=3}^{4}\frac{\bar{R}_i - R_f}{\sigma_i} \tag{8}$$

Our goal is to find $\Phi_1$ and $\Phi_2$. We divide equation (7) by $1-\rho_{11}$ and equation (8) by $1-\rho_{22}$ to get:

$$\Phi_1 + \frac{2\rho_{11}\Phi_1}{1-\rho_{11}} + \frac{2\rho_{12}\Phi_2}{1-\rho_{11}} = \sum_{i=1}^{2}\frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{11})} \tag{9}$$

and

$$\Phi_2 + \frac{2\rho_{21}\Phi_1}{1-\rho_{22}} + \frac{2\rho_{22}\Phi_2}{1-\rho_{22}} = \sum_{i=3}^{4}\frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{22})} \tag{10}$$

We can write equations (9) and (10) in matrix/vector form as follows:

$$\begin{pmatrix} 1 + \frac{2\rho_{11}}{1-\rho_{11}} & \frac{2\rho_{12}}{1-\rho_{11}} \\ \frac{2\rho_{21}}{1-\rho_{22}} & 1 + \frac{2\rho_{22}}{1-\rho_{22}} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2}\frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{11})} \\ \sum_{i=3}^{4}\frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{22})} \end{pmatrix}$$

Or $\mathbf{A\Phi} = \mathbf{C}$. Therefore $\boldsymbol{\Phi} = \boldsymbol{A}^{-1}\boldsymbol{C}$. Once we find $\Phi_1$ and $\Phi_2$ we can go back to equations (5) and (6) to find $z_1$ and $z_2$. Similarly we can find $z_3$ and $z_4$.

## Multigroup model
## Short sales allowed
## Example

The stocks are grouped into industries. Here we have nine stocks as follows:

| Auto manufacturers | Drug | Beverage |
|---|---|---|
| Ford | Johnston&Johnston | Coca Cola |
| GM | Merck | Hansen |
| Toyota | Pfizer | Pepsico |

The correlation matrix is given below.

$$\boldsymbol{\rho} = \begin{pmatrix} 1.0000 & 0.4980 & 0.3029 & -0.0283 & 0.1803 & 0.1915 & 0.2285 & 0.0612 & 0.1147 \\ 0.4980 & 1.0000 & 0.4212 & -0.1537 & -0.0083 & 0.0906 & 0.1779 & 0.0643 & -0.1178 \\ 0.3029 & 0.4212 & 1.0000 & -0.1157 & 0.1016 & 0.1045 & 0.1231 & 0.0035 & -0.0059 \\ -0.0283 & -0.1537 & -0.1157 & 1.0000 & 0.4387 & 0.4164 & -0.0211 & -0.0403 & 0.3426 \\ 0.1803 & -0.0083 & 0.1016 & 0.4387 & 1.0000 & 0.5349 & 0.0517 & -0.0382 & 0.4162 \\ 0.1915 & 0.0906 & 0.1045 & 0.4164 & 0.5349 & 1.0000 & 0.0526 & -0.0035 & 0.3876 \\ 0.2285 & 0.1779 & 0.1231 & -0.0211 & 0.0517 & 0.0526 & 1.0000 & 0.1524 & 0.0757 \\ 0.0612 & 0.0643 & 0.0035 & -0.0403 & -0.0382 & -0.0035 & 0.1524 & 1.0000 & 0.0597 \\ 0.1147 & -0.1178 & -0.0059 & 0.3426 & 0.4162 & 0.3876 & 0.0757 & 0.0597 & 1.0000 \end{pmatrix}$$

The multigroup model requires the following inputs: the average return and standard deviation of the return for each stock, the average correlation within each industry and between industries. These are summarized below:

| | ford | gm | toyota | jnj | merck | pfizer | coke | hansen | pepsico |
|---|---|---|---|---|---|---|---|---|---|
| mean | 0.0169 | 0.0142 | 0.0051 | 0.0009 | 0.0089 | 0.0043 | 0.0002 | 0.0545 | 0.0091 |
| sd | 0.1314 | 0.1065 | 0.0726 | 0.0823 | 0.0857 | 0.0607 | 0.0649 | 0.1745 | 0.0525 |

Within and between group correlations:

$$\bar{\boldsymbol{\rho}} = \begin{pmatrix} 0.4074 & 0.0403 & 0.0722 \\ 0.0403 & 0.4633 & 0.1275 \\ 0.0722 & 0.1275 & 0.0960 \end{pmatrix}$$

For example, $\rho_{11} = 0.4073, \rho_{12} = 0.0403$, etc.

To find the optimum portfolio (point of tangency) we will use the following equations for the $z_i$'s:

$$z_i = \frac{1}{\sigma_i(1-\rho_{kk})}\left[\frac{\bar{R}_i - R_f}{\sigma_i} - \sum_{g=1}^{p} \rho_{kg}\Phi_g\right], \text{ for } k = 1, \cdots, p, \tag{11}$$

where, $p$ is the number of industries (in our example 3). The unknowns are the $\Phi_i$'s. These can be found from $\boldsymbol{\Phi} = \boldsymbol{A}^{-1}\boldsymbol{C}$. Here $\boldsymbol{A} = \begin{pmatrix} 1 + \frac{N_1\rho_{11}}{1-\rho_{11}} & \frac{N_1\rho_{12}}{1-\rho_{11}} & \frac{N_1\rho_{13}}{1-\rho_{11}} \\ \frac{N_2\rho_{21}}{1-\rho_{22}} & 1 + \frac{N_2\rho_{22}}{1-\rho_{22}} & \frac{N_2\rho_{23}}{1-\rho_{22}} \\ \frac{N_3\rho_{31}}{1-\rho_{33}} & \frac{N_3\rho_{32}}{1-\rho_{33}} & 1 + \frac{N_3\rho_{33}}{1-\rho_{33}} \end{pmatrix}$ and $\boldsymbol{C} = \begin{pmatrix} \sum_{i=1}^{N_1} \frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{11})} \\ \sum_{i=1}^{N_2} \frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{22})} \\ \sum_{i=1}^{N_3} \frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{33})} \end{pmatrix}$. In our example, $N_1 = N_2 = N_3 = 3$, and $R_f = 0.002$, and the solution is:

$$\boldsymbol{\Phi} = \begin{pmatrix} 3.0616 & 0.2040 & 0.3654 \\ 0.2253 & 3.5897 & 0.7127 \\ 0.2396 & 0.4231 & 1.3186 \end{pmatrix}^{-1} \begin{pmatrix} 0.4566 \\ 0.1957 \\ 0.4517 \end{pmatrix} = \begin{pmatrix} 0.1112 \\ -0.0176 \\ 0.3280 \end{pmatrix}.$$

Therefore, $\Phi_1 = 0.1112, \Phi_2 = -0.0176, \Phi_3 = 0.3280$. The cut-off points for the three groups (this is the summation in the square bracket of equation (1)) are: 0.0683, 0.0381, and 0.0373 respectively. For example, the cut-off point for the first group is:

$$C_1^* = \sum_{g=1}^{p} \rho_{1g}\Phi_g = \rho_{11}\Phi_1 + \rho_{12}\Phi_2 + \rho_{13}\Phi_3 = 0.4073(0.1112) + 0.0403(-0.0176) + 0.0722(0.3280) = 0.0683.$$

Using equation (11) we can find the $z_i$'s:

$$\begin{aligned}
z_1 &= \frac{1}{0.1314(1-0.4073)}\left[\frac{0.0169-0.002}{0.1314}-0.0683\right] = 0.5782 \\
z_2 &= \frac{1}{0.1065(1-0.4073)}\left[\frac{0.0142-0.002}{0.1065}-0.0683\right] = 0.7294 \\
z_3 &= \frac{1}{0.0726(1-0.4073)}\left[\frac{0.0051-0.002}{0.0726}-0.0683\right] = -0.6018 \\
z_4 &= \frac{1}{0.0823(1-0.4633)}\left[\frac{0.0009-0.002}{0.0823}-0.0381\right] = -1.1757 \\
z_5 &= \frac{1}{0.0857(1-0.4633)}\left[\frac{0.0089-0.002}{0.0857}-0.0381\right] = 0.9311 \\
z_6 &= \frac{1}{0.0607(1-0.4633)}\left[\frac{0.0043-0.002}{0.0607}-0.0381\right] = 0.0175 \\
z_7 &= \frac{1}{0.0649(1-0.0960)}\left[\frac{0.0002-0.002}{0.0649}-0.0373\right] = -1.182 \\
z_8 &= \frac{1}{0.1745(1-0.0960)}\left[\frac{0.0545-0.002}{0.1745}-0.0373\right] = 1.6693 \\
z_9 &= \frac{1}{0.0525(1-0.0960)}\left[\frac{0.0091-0.002}{0.0525}-0.0373\right] = 2.0622
\end{aligned}$$

The sum of the $z_i$'s is: $\sum_{i=1}^{9} z_i = 3.092$, and the fractions invested in each stock will be $\frac{z_i}{\sum_{i=1}^{9} z_i}$. We have:
$x_1 = 0.187, x_2 = 0.236, x_3 = -0.195,$
$x_4 = -0.380, x_5 = 0.301, x_6 = 0.006,$
$x_7 = -0.362, x_8 = 0.540, x_9 = 0.667.$