Combinations of two risky assets: Short sales not allowed

Define:
- $x_A$ is the fraction of available funds invested in asset $A$.
- $x_B$ is the fraction of available funds invested in asset $B$.
- $\bar{R}_A$ is the expected return on the asset $A$.
- $\bar{R}_B$ is the expected return on the asset $B$.
- $\bar{R}_p$ is the expected return on the portfolio.
- $\sigma_A^2$ is the variance of the return on asset $A$.
- $\sigma_B^2$ is the variance of the return on asset $B$.
- $\sigma_{AB}$ ($\text{cov}(R_A, R_B)$) is the covariance between the returns on asset $A$ and asset $B$.
- $\rho_{AB}$ is the correlation coefficient between the returns on asset $A$ and asset $B$.
- $\sigma_p$ is the standard deviation of the return on the portfolio.

**Correlation coefficient and the efficient frontier**

The inputs of portfolio are:
- Expected return for each stock.
- Standard deviation of the return of each stock.
- Covariance between two stocks.

The correlation coefficient ($\rho$) between stocks $A, B$ is always between -1, 1 and it is equal to:

$$\rho = \frac{\text{cov}(R_A, R_B)}{\sigma_A \sigma_B} \Rightarrow \text{cov}(R_A, R_B) = \rho \sigma_A \sigma_B$$

Expected return of the portfolio:

$$E(X_A R_A + X_B R_B) = X_A \bar{R}_A + X_B \bar{R}_B$$

Variance of the portfolio:

$$\text{var}(X_A R_A + X_B R_B) = X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2X_A X_B \rho \sigma_A \sigma_B$$

Or

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A) \bar{R}_B$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A (1 - X_A) \rho \sigma_A \sigma_B$$

The standard deviation of the portfolio:

$$\sigma_p = \sqrt{X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A (1 - X_A) \rho \sigma_A \sigma_B}$$

Next pages we explore the shape of the efficient frontier for different values of the correlation coefficient.

What do you observe when $\rho = 1$, $\rho = -1$?
Suppose $\rho = +1$:

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A) \bar{R}_B$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\sigma_A \sigma_B$$

or

$$\sigma_p^2 = [X_A \sigma_A + (1 - X_A) \sigma_B]^2$$

$$\sigma_p = X_A \sigma_A + (1 - X_A) \sigma_B \Rightarrow X_A = \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B}$$

By combining (1) and (2) we get:

$$\bar{R}_p = \sigma_B \bar{R}_A + \sigma_A \bar{R}_B$$

Finally:

$$\bar{R}_p = \frac{\sigma_A \bar{R}_B - \sigma_B \bar{R}_A}{\sigma_A - \sigma_B} + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B} \sigma_p$$

Conclusion: All the portfolios (combinations of stocks $A$ and $B$) will be found on this line.

Suppose $\rho = -1$:

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A) \bar{R}_B$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 - 2X_A(1 - X_A)\sigma_A \sigma_B$$

Therefore,

$$\sigma_p^2 = [X_A \sigma_A - (1 - X_A) \sigma_B]^2$$

$$\sigma_p = X_A \sigma_A - (1 - X_A) \sigma_B \Rightarrow X_A = \frac{\sigma_p + \sigma_B}{\sigma_A + \sigma_B}$$

or

$$\sigma_p^2 = [-X_A \sigma_A + (1 - X_A) \sigma_B]^2$$

$$\sigma_p = -X_A \sigma_A + (1 - X_A) \sigma_B \Rightarrow X_A = \frac{\sigma_p - \sigma_B}{-\sigma_A - \sigma_B}$$

By combining (3), (4), and (5) we get two equations:

$$\bar{R}_p = \frac{\sigma_B \bar{R}_A + \sigma_A \bar{R}_B}{\sigma_A + \sigma_B} + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A + \sigma_B} \sigma_p$$

and

$$\bar{R}_p = \frac{\sigma_B \bar{R}_A + \sigma_A \bar{R}_B}{\sigma_A + \sigma_B} + \frac{\bar{R}_B - \bar{R}_A}{\sigma_A + \sigma_B} \sigma_p$$

Conclusion: All the portfolios (combinations of stocks $A$ and $B$) will be found on these two lines.