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Statistics C183/C283 **Instructor: Nicolas Christou**

Combinations of two risky assets: Short sales not allowed

Define:

x_A is the fraction of available funds invested in asset A.

x_B is the fraction of available funds invested in asset B.

$\bar{R}_A$ is the expected return on the asset A.

$\bar{R}_B$ is the expected return on the asset B.

$\bar{R}_p$ is the expected return on the portfolio.

σ_A^2 is the variance of the return on asset A.

σ_B^2 is the variance of the return on asset B.

σ_{AB} $(cov(R_A, R_B))$ is the covariance between the returns on asset A and asset B.

ρ_{AB} is the correlation coefficient between the returns on asset A and asset B.

σ_p is the standard deviation of the return on the portfolio.

Correlation coefficient and the efficient frontier

The inputs of portfolio are:

- Expected return for each stock.
- Standard deviation of the return of each stock.
- Covariance between two stocks.

The correlation coefficient (ρ) between stocks A, B is always between -1, 1 and it is equal to:

$$\rho = \frac{cov(R_A, R_B)}{\sigma_A \sigma_B} \;\Rightarrow\; cov(R_A, R_B) = \rho \sigma_A \sigma_B$$

Expected return of the portfolio:

$$E(X_A R_A + X_B R_B) = X_A \bar{R}_A + X_B \bar{R}_B$$

Variance of the portfolio:

$$var(X_A R_A + X_B R_B) = X_A^2 \sigma_A^2 + X_B^2 \sigma_B^2 + 2 X_A X_B \rho \sigma_A \sigma_B$$

Or

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A) \bar{R}_B$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2 X_A (1 - X_A) \rho \sigma_A \sigma_B$$

The standard deviation of the portfolio:

$$\sigma_p = \left[X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2 X_A (1 - X_A) \rho \sigma_A \sigma_B \right]^{\frac{1}{2}}$$

Next pages we explore the shape of the efficient frontier for different values of the correlation coefficient.

What do you observe when $\rho = 1$, $\rho = -1$?

Suppose $\rho = +1$:

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A)\bar{R}_B \tag{1}$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 + 2X_A(1 - X_A)\sigma_A \sigma_B$$

or

$$\sigma_p^2 = [X_A \sigma_A + (1 - X_A)\sigma_B]^2$$

$$\sigma_p = X_A \sigma_A + (1 - X_A)\sigma_B \Rightarrow X_A = \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B} \tag{2}$$

By combining (1) and (2) we get:

$$\bar{R}_p = \frac{\sigma_p - \sigma_B}{\sigma_A - \sigma_B}\bar{R}_A + \frac{\sigma_A - \sigma_p}{\sigma_A - \sigma_B}\bar{R}_B$$

Finally:

$$\bar{R}_p = \frac{\sigma_A \bar{R}_B - \sigma_B \bar{R}_A}{\sigma_A - \sigma_B} + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A - \sigma_B}\sigma_p$$

Conclusion: All the portfolios (combinations of stocks A and B) will be found on this line.

Suppose $\rho = -1$:

$$\bar{R}_p = X_A \bar{R}_A + (1 - X_A)\bar{R}_B \tag{3}$$

and

$$\sigma_p^2 = X_A^2 \sigma_A^2 + (1 - X_A)^2 \sigma_B^2 - 2X_A(1 - X_A)\sigma_A \sigma_B$$

Therefore,

$$\sigma_p^2 = [X_A \sigma_A - (1 - X_A)\sigma_B]^2$$

$$\sigma_p = X_A \sigma_A - (1 - X_A)\sigma_B \Rightarrow X_A = \frac{\sigma_p + \sigma_B}{\sigma_A + \sigma_B} \tag{4}$$

or

$$\sigma_p^2 = [-X_A \sigma_A + (1 - X_A)\sigma_B]^2$$

$$\sigma_p = -X_A \sigma_A + (1 - X_A)\sigma_B \Rightarrow X_A = \frac{\sigma_p - \sigma_B}{-\sigma_A - \sigma_B} \tag{5}$$

By combining (3), (4), and (5) we get two equations:

$$\bar{R}_p = \frac{\sigma_B \bar{R}_A + \sigma_A \bar{R}_B}{\sigma_A + \sigma_B} + \frac{\bar{R}_A - \bar{R}_B}{\sigma_A + \sigma_B}\sigma_p$$

and

$$\bar{R}_p = \frac{\sigma_B \bar{R}_A + \sigma_A \bar{R}_B}{\sigma_A + \sigma_B} + \frac{\bar{R}_B - \bar{R}_A}{\sigma_A + \sigma_B}\sigma_p$$

Conclusion: All the portfolios (combinations of stocks A and B) will be found on these two lines.