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Statistics C183/C283 **Instructor: Nicolas Christou**

Constructing the optimal portfolios - Constant correlation model
Short sales not allowed

The calculation of optimal portfolios is simplified by using the constant correlatation model to rank securities based on the excess return to standard deviation ratio.

$$\text{Excess return to standard deviation} = \frac{\bar{R}_i - R_f}{\sigma_i}.$$

After stocks are ranked using the above ratio the optimum portfolio consists of investing in all stocks for which the excess return to beta is greater than the cut-off point C^*. This cut-off point is computed as follows:

$$C^* = \frac{\rho}{1-\rho+i\rho}\sum_{j=1}^{i}\frac{\bar{R}_j - R_f}{\sigma_j}$$

where

$\bar{R}_j$ Expected return on stock j.
R_f Return on a riskless asset.
σ_j Standard deviation of the returns of stock j.
ρ The correlation coefficient - it is constant for all pairs of stocks.

To find C^* we compute all C_i's using portfolios that consist with the first ranked stock, the first and second ranked stocks, the first, second, and third ranked stock etc. If short sales are not allowed we know we have found the cut-off poit C^* when all stocks used in calculating C_i satisfy:

$$\frac{\bar{R}_i - R_f}{\sigma_i} > C^*.$$

Once C^* is found, we know that the optimum portfolio consists of the first i stocks which satisfy the above inequality. To find the proportion of funds invested in each of these stocks we use:

$$z_i = \frac{1}{(1-\rho)\sigma_i}\left(\frac{\bar{R}_i - R_f}{\sigma_i} - C^*\right)$$

Therefore

$$x_i = \frac{z_i}{\sum_{i=1}^{n} z_i}.$$

where n is equal to the number of stocks consisting the optimum portfolio.
Below an example with 10 stocks is presented. The first table shows the expected return, standard deviation, and the excess return to standard deviation ratio assuming $R_f = 0.02$. The second tables shows the prcedure for the calculation of the cut-off point C^* and the optimum portfolio by assuming $\rho = 0.2$ for all pairs of stocks.

Stock i	$\bar{R}_i$	$\bar{R}_i - R_f$	σ_i	$\frac{\bar{R}_i - R_f}{\sigma_i}$
1	0.09	0.07	0.015	4.667
2	0.13	0.11	0.025	4.400
3	0.08	0.06	0.02	3.000
4	0.12	0.10	0.04	2.500
5	0.08	0.06	0.03	2.000
6	0.15	0.13	0.10	1.300
7	0.17	0.15	0.13	1.154
8	0.10	0.08	0.08	1.000
9	0.05	0.03	0.035	0.857
10	0.06	0.04	0.06	0.667

Using the above table we compute the entries in the next table. The last column contains the C_i's.

Stock i	$\frac{\bar{R}_i - R_f}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	$\sum_{j=1}^{i} \frac{\bar{R}_j - R_f}{\sigma_j}$	C_i
1	4.667	0.2	4.667	0.933
2	4.400	0.167	9.067	1.511
3	3.000	0.143	12.067	1.724
4	2.500	0.125	14.567	1.821
5	2.000	0.111	16.567	1.841
6	1.300	0.100	17.867	1.787
7	1.154	0.091	19.021	1.729
8	1.000	0.083	20.021	1.668
9	0.857	0.077	20.878	1.606
10	0.667	0.071	21.544	1.539

We find from the previous table that $C^* = 1.841$. Therefore the optimum portfolio consists of the first 5 ranked stocks. In solving this problem there is no need to fill in all the enries of the previous table. The reason is simple: Once we find the cut-off point C^* we can ignore all stocks that are ranked below the last stock included in the optimum portfolio.

We first find the values of z_i's:

$$z_1 = \frac{1}{(1-\rho)\sigma_1}\left(\frac{\bar{R}_1 - R_f}{\sigma_1} - C^*\right) = \frac{1}{(1-0.2)0.015}(4.67 - 1.841) = 235.494.$$

Similarly $z_2 = 127.963$, $z_3 = 72.454$, $z_4 = 20.602$, $z_5 = 6.636$. The sum of the z_i's is $\sum_{i=1}^{5} z_i = 463.148$. Therefore $x_1 = \frac{235.494}{463.148} = 0.508$, $x_2 = \frac{127.963}{463.148} = 0.276$, $x_3 = \frac{72.454}{463.148} = 0.156$, $x_4 = \frac{20.602}{463.148} = 0.044$, $x_5 = \frac{6.636}{463.148} = 0.014$. We conclude that the optimum portfolio consists of 50.8% of stock 1, 27.6% of stock 2, 15.6% of stock 3, 4.4% of stock 4, 1.4% of stock 5.

If short sales are allowed then $C^* = 1.539$ (last C_i). This is true since all the stocks will be included in the optimum portfolio, but some of them will be held short. The precedure for finding the z_i's and the x_i's is the same as above.