Short sales allowed, risk-free lending and borrowing
Point of tangency - solution


When the investor faces the efficient frontier and riskless lending and borrowing the combinations of the risk-free asset with the risky portfolio lie on the line:

\[
\bar{R}_p = R_f + \left( \frac{\bar{R}_A - R_f}{\sigma_A} \right) \sigma_p
\]

(1)

The solution is to find the point of tangency of this line to the efficient frontier. Let’s call this point \( G \). To find this point we want to maximize the slope of the line in (1) as follows:

\[
\max \theta = \frac{\bar{R}_p - R_f}{\sigma_p}
\]

Subject to

\[
\sum_{i=1}^{n} x_i = 1
\]

Since,

\[
R_f = (\sum_{i=1}^{n} x_i) R_f = \sum_{i=1}^{n} x_i R_f
\]

we can write the maximization problem as

\[
\max \theta = \frac{\sum_{i=1}^{n} x_i (\bar{R}_i - R_f)}{\left( \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n}, j \neq i x_i x_j \sigma_{ij} \right)^{\frac{1}{2}}}
\]

or

\[
\max \theta = \left[ \sum_{i=1}^{n} x_i (\bar{R}_i - R_f) \right] \left[ \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1}^{n}, j \neq i x_i x_j \sigma_{ij} \right]^{-\frac{1}{2}}
\]

Take now the partial derivative with respect to each \( x_i, i = 1, \cdots, n \), set them equal to zero and solve. Let’s find the partial derivative w.r.t. \( x_k \):

\[
\frac{\partial \theta}{\partial x_k} = (\bar{R}_k - R_f) \left[ \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} x_i x_j \sigma_{ij} \right]^{-\frac{1}{2}}
\]

\[
+ \left[ \sum_{i=1}^{n} x_i (\bar{R}_i - R_f) \right] \left[ 2x_k \sigma_k^2 + 2 \sum_{j=1, j \neq k}^{n} x_j \sigma_{kj} \right]
\]

\[
\times \left[ \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} x_i x_j \sigma_{ij} \right]^{-\frac{3}{2}} \times (-\frac{1}{2}) = 0
\]
Multiply both sides by

\[ \left[ \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i} x_i x_j \sigma_{ij} \right]^{\frac{1}{2}} \]

to get

\[
(\bar{R}_k - R_f) - \frac{\sum_{i=1}^{n} x_i (\bar{R}_i - R_f)}{\sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i} x_i x_j \sigma_{ij}} (x_k \sigma_k^2 + \sum_{j=1, j \neq k}^{n} x_j \sigma_{kj}) = 0
\]

Now, if we let

\[
\lambda = \frac{\sum_{i=1}^{n} x_i (\bar{R}_i - R_f)}{\sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i} x_i x_j \sigma_{ij}}
\]

the previous expression will be

\[
(\bar{R}_k - R_f) - \lambda x_k \sigma_k^2 - \sum_{j=1, j \neq k}^{n} \lambda x_j \sigma_{kj} = 0
\]
or

\[
\bar{R}_k - R_f = \lambda x_k \sigma_k^2 + \sum_{j=1, j \neq k}^{n} \lambda x_j \sigma_{kj}
\]

Let’s define now a new variable,

\[ z_k = \lambda x_k \]

and finally

\[
\bar{R}_k - R_f = z_k \sigma_k^2 + \sum_{j=1, j \neq k}^{n} z_j \sigma_{kj}
\] (2)

We have one equation like (2) for each \( k = 1, \cdots, n \). Here they are:

\[
\begin{align*}
\bar{R}_1 - R_f &= z_1 \sigma_1^2 + z_2 \sigma_{12} + z_3 \sigma_{13} + \cdots + z_n \sigma_{1n} \\
\bar{R}_2 - R_f &= z_1 \sigma_{21} + z_2 \sigma_2^2 + z_3 \sigma_{23} + \cdots + z_n \sigma_{2n} \\
\vdots &= \vdots \\
\bar{R}_n - R_f &= z_1 \sigma_{n1} + z_2 \sigma_{n2} + z_3 \sigma_{n3} + \cdots + z_n \sigma_n^2
\end{align*}
\]

The solution involves solving the system of these simultaneous equations, which can be written in matrix form as:

\[ \mathbf{R} = \mathbf{\Sigma} \mathbf{Z}, \] where \( \mathbf{\Sigma} \) is the variance-covariance matrix of the returns of the \( n \) stocks.
To solve for $Z$:

$$Z = \Sigma^{-1}R$$

Once we find the $z_i'$s it is easy to find the $x_i'$s (the fraction of funds to be invested in each security). Earlier we defined

$$z_k = \lambda x_k \Rightarrow x_k = \frac{z_k}{\lambda}$$

We need to find $\lambda$ as follows:

$$z_1 + z_2 + \cdots + z_n = \sum_{i=1}^{n} z_i$$

$$\lambda(x_1 + x_2 + \cdots + x_n) = \sum_{i=1}^{n} z_i$$

$$\lambda = \sum_{i=1}^{n} z_i$$

Therefore the composition $G$ (the point of tangency) is given by:

$$x_1 = \frac{z_1}{\lambda}$$

$$x_2 = \frac{z_2}{\lambda}$$

$$x_3 = \frac{z_3}{\lambda}$$

$$\vdots$$

$$x_n = \frac{z_n}{\lambda}$$