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Statistics C183/C283 **Instructor: Nicolas Christou**

Short sales allowed, risk-free lending and borrowing
Point of tangency - solution

See also, *Modern Portfolio Theory and Investments Analysis* by Elton, Gruber, Brown, Goetzmann, Wiley 6th Edition, 2003.

When the investor faces the efficient frontier and riskless lending and borrowing the combinations of the risk-free asset with the risky portfolio lie on the line:

$$\bar{R}_p = R_f + \left(\frac{\bar{R}_A - R_f}{\sigma_A}\right)\sigma_p \tag{1}$$

The solution is to find the point of tangency of this line to the efficient frontier. Let's call this point G. To find this point we want to maximize the slope of the line in (1) as follows:

$$\max \quad \theta = \frac{\bar{R}_p - R_f}{\sigma_p}$$

Subject to

$$\sum_{i=1}^{n} x_i = 1$$

Since,

$$R_f = (\sum_{i=1}^{n} x_i)R_f = \sum_{i=1}^{n} x_i R_f$$

we can write the maximization problem as

$$\max \quad \theta = \frac{\sum_{i=1}^{n} x_i(\bar{R}_i - R_f)}{\left(\sum_{i=1}^{n} x_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} x_i x_j \sigma_{ij}\right)^{\frac{1}{2}}}$$

or

$$\max \quad \theta = \left[\sum_{i=1}^{n} x_i(\bar{R}_i - R_f)\right]\left[\sum_{i=1}^{n} x_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} x_i x_j \sigma_{ij}\right]^{-\frac{1}{2}}$$

Take now the partial derivative with respect to each $x_i, i = 1, \cdots, n$, set them equal to zero and solve. Let's find the partial derivative w.r.t. x_k:

$$\begin{aligned}
\frac{\partial\theta}{\partial x_k} &= (\bar{R}_k - R_f)\left[\sum_{i=1}^{n} x_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} x_i x_j \sigma_{ij}\right]^{-\frac{1}{2}} \\
&+ \left[\sum_{i=1}^{n} x_i(\bar{R}_i - R_f)\right]\left[2x_k\sigma_k^2 + 2\sum_{j=1,j\neq k}^{n} x_j\sigma_{kj}\right] \\
&\times \left[\sum_{i=1}^{n} x_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} x_i x_j \sigma_{ij}\right]^{-\frac{3}{2}} \times (-\frac{1}{2}) = 0
\end{aligned}$$

Multiply both sides by

$$\left[\sum_{i=1}^{n} x_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} x_i x_j \sigma_{ij}\right]^{\frac{1}{2}} \quad \text{to get}$$

$$(\bar{R}_k - R_f) - \frac{\sum_{i=1}^{n} x_i(\bar{R}_i - R_f)}{\sum_{i=1}^{n} x_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} x_i x_j \sigma_{ij}}(x_k\sigma_k^2 + \sum_{j=1, j\neq k}^{n} x_j\sigma_{kj}) = 0$$

Now, if we let

$$\lambda = \frac{\sum_{i=1}^{n} x_i(\bar{R}_i - R_f)}{\sum_{i=1}^{n} x_i^2\sigma_i^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} x_i x_j \sigma_{ij}}$$

the previous expression will be

$$(\bar{R}_k - R_f) - \lambda x_k\sigma_k^2 - \sum_{j=1, j\neq k}^{n} \lambda x_j\sigma_{kj} = 0$$

or

$$\bar{R}_k - R_f = \lambda x_k\sigma_k^2 + \sum_{j=1, j\neq k}^{n} \lambda x_j\sigma_{kj}$$

Let's define now a new variable,

$$z_k = \lambda x_k$$

and finally

$$\bar{R}_k - R_f = z_k\sigma_k^2 + \sum_{j=1, j\neq k}^{n} z_j\sigma_{kj} \tag{2}$$

We have one equation like (2) for each $k = 1, \cdots, n$. Here they are:

$$\begin{aligned}
\bar{R}_1 - R_f &= z_1\sigma_1^2 + z_2\sigma_{12} + z_3\sigma_{13} + \cdots + z_n\sigma_{1n} \\
\bar{R}_2 - R_f &= z_1\sigma_{21} + z_2\sigma_2^2 + z_3\sigma_{23} + \cdots + z_n\sigma_{2n} \\
\cdots &= \cdots \\
\cdots &= \cdots \\
\cdots &= \cdots \\
\bar{R}_n - R_f &= z_1\sigma_{n1} + z_2\sigma_{n2} + z_3\sigma_{n3} + \cdots + z_n\sigma_n^2
\end{aligned}$$

The solution involves solving the system of these simultaneous equations, which can be written in matrix form as:

$\mathbf{R} = \mathbf{\Sigma Z}$, where $\mathbf{\Sigma}$ is the variance-covariance matrix of the returns of the n stocks.

To solve for **Z**:

$$\mathbf{Z} = \mathbf{\Sigma^{-1}R}$$

Once we find the z_i's it is easy to find the x_i's (the fraction of funds to be invested in each security). Earlier we defined

$$z_k = \lambda x_k \Rightarrow x_k = \frac{z_k}{\lambda}$$

We need to find λ as follows:

$$\begin{aligned} z_1 + z_2 + \cdots + z_n &= \sum_{i=1}^{n} z_i \\ \lambda(x_1 + x_2 + \cdots + x_n) &= \sum_{i=1}^{n} z_i \\ \lambda &= \sum_{i=1}^{n} z_i \end{aligned}$$

Therefore the composition G (the point of tangency) is given by:

$$\begin{aligned} x_1 &= \frac{z_1}{\lambda} \\ x_2 &= \frac{z_2}{\lambda} \\ x_3 &= \frac{z_3}{\lambda} \\ &\vdots \\ &\vdots \\ x_n &= \frac{z_n}{\lambda} \end{aligned}$$