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Statistics C183/C283 **Instructor: Nicolas Christou**

Portfolio expected return and risk

Suppose a portfolio consists of n stocks. Let $\bar{R}_i$ and σ_i^2 the expected return and variance of stock i, $i = 1, 2, \ldots, n$. Also, let σ_{ij} the covariance between stocks i and j. Let $x_1, x_2, \ldots, x_n$ the fractions of the investors wealth invested in each one of the n stocks ($\sum_{i=1}^n x_i = 1$). The resulting portfolio is $x_1 R_1 + x_2 R_2 + \cdots + x_n R_n$ and at time t it has return:

$$R_{pt} = x_1 R_{1t} + x_2 R_{2t} + \cdots + x_n R_{nt}$$

The expected return of this portfolio is given by:

$$\bar{R}_p = x_1 \bar{R}_1 + x_2 \bar{R}_2 + \cdots + x_n \bar{R}_n = \sum_{i=1}^n x_i \bar{R}_i = \mathbf{x}'\bar{\mathbf{R}}$$

where,

$$\mathbf{x}' = (x_1, x_2, \cdots, x_n), \quad \text{and} \quad \bar{\mathbf{R}}' = (\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_n)$$

And its risk (variance) by:

$$\sigma_p^2 = var(x_1 R_1 + x_2 R_2 + \cdots + x_n R_n) = \sum_{i=1}^n x_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j \neq i}^n x_i x_j \sigma_{ij} = \sum_{i=1}^n \sum_{j=1}^n x_i x_j \sigma_{ij}$$

Or in matrix form:

$$\sigma_p^2 = \mathbf{x}'\mathbf{\Sigma}\mathbf{x}$$

where, $\mathbf{\Sigma}$ is the symmetric, positive definite $n \times n$ variance covariance matrix of the returns of the n stocks as shown below:

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \sigma_{24} & \cdots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \sigma_{34} & \cdots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \sigma_{n4} & \cdots & \sigma_n^2 \end{pmatrix}$$

There are n variances and $\frac{n(n-1)}{2}$ covariances.

Why does diversification work?

We will explain here very briefly (Elton et al, 2003) why investing in more than one securities reduces the risk. Suppose in a portfolio there are n securities. Then, the variance of the return on the portfolio (risk) is:

$$\sigma_p^2 = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j \neq i}^{n} x_i x_j \sigma_{ij}$$

Let us consider equal allocation into the n securities. This means that $\frac{1}{n}$ of our wealth will be invested in each security. So, $x_i = \frac{1}{n}$ and the above expression becomes:

$$\sigma_p^2 = \sum_{i=1}^{n} (\frac{1}{n})^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j \neq i}^{n} (\frac{1}{n})(\frac{1}{n}) \sigma_{ij}$$

We can factor out from the first summation $\frac{1}{n}$ and from the second summation $\frac{n-1}{n}$ to get

$$\sigma_p^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_i^2}{n} + \frac{n-1}{n} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{\sigma_{ij}}{n(n-1)}$$

and since there are all together $n(n-1)$ covariances we have:

$$\sigma_p^2 = \frac{1}{n} \bar{\sigma}_i^2 + \frac{n-1}{n} \bar{\sigma}_{ij}$$

where $\bar{\sigma}_i^2 = \sum_{i=1}^{n} \frac{\sigma_i^2}{n}$, and $\bar{\sigma}_{ij} = \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{\sigma_{ij}}{n(n-1)}$. We see that when n is large the risk of the portfolio is approximately equal the average covariance. The individual risk of securities can be diversified away. Even though equal allocation is not the optimum solution the above can explain the reduction of risk by holding many securities.