Kalman Filters & Particle Filters.

Another way to deal with one-dimensional graphs.

\[ x_1, x_2, x_3, x_4, x_5, x_6, x_7, \ldots \]
\[ y_1, y_2, y_3, y_4, y_5, y_6, y_7, \ldots \]

update model \[ P(X_{t+1} | X_t) \] observation model \[ P(Y_t | X_t) \]

\( \{ X_t : t = 1, \ldots \} \) is the state of the system.
\( \{ Y_t : t = 1, \ldots \} \) are the observations.

Let \[ Y_t = \langle y_t, y_{t-1}, y_{t-2}, \ldots, y_1 \rangle \]
all observations prior to time \( t \).

Want to know \[ P(X_t | Y_t) \]
and update to \[ P(X_{t+1} | Y_{t+1}) \].

Two stages:

(i) prediction \[ p(X_{t+1} | Y_t) \]

(ii) correction for new observation \[ p(X_{t+1} | Y_{t+1}) \].

Tracking Airplanes, Spacecraft,
Prediction

(1) \[ p(x_{t+1} | y_t) = \sum_{x_t} p(x_{t+1} | x_t) p(x_t | y_t) \]

Correction

(2) \[ p(x_{t+1} | y_{t+1}) = \frac{p(y_{t+1} | x_{t+1}) p(x_{t+1} | y_{t})}{p(y_{t+1})} \]

with \[ p(x_{t+1}) = \sum_{x_{t+1}} p(y_{t+1} | x_{t+1}) p(x_{t+1} | y_{t}) \]

normalization constant.

Problems:
It may be difficult to compute these two stages.

There is a very important special case — The Kalman Filter. Described in 1-5.

In this case:

\[ p(y_t | x_t) = \frac{1}{\sqrt{2\pi \delta_m^2}} e^{-\frac{(x_t - y_t)^2}{2\delta_m^2}} \text{ Gaussian Model} \]

\[ p(x_{t+1} | x_t) = \frac{1}{\sqrt{2\pi \delta_p^2}} e^{-\frac{(x_{t+1} - x_t - \mu)^2}{2\delta_p^2}} \text{ Gaussian} \]
Then the distributions are all Gaussian

\[ p(x_{t+1} | y_t) \sim N(\mu_t, \bar{\sigma}_t) \]

\[
p(x_{t+1} | y_t) = \int dx_t \ p(x_{t+1} | x_t) p(x_t | y_t)
= \int dx_t \ \frac{1}{\sqrt{2\pi\bar{\sigma}_t}} e^{-\frac{(x_{t+1} - x_t - \mu_t)^2}{2\bar{\sigma}_t}}  \frac{1}{\sqrt{2\pi\bar{\sigma}_t}} e^{-\frac{(x_t - \mu_t)^2}{2\bar{\sigma}_t}}
= \frac{1}{\sqrt{2\pi(\bar{\sigma}_t^2 + \bar{\sigma}_t^2)}} e^{-\frac{(x_{t+1} - \mu_t - \mu_t)^2}{2(\bar{\sigma}_t^2 + \bar{\sigma}_t^2)}}
\]

\[
p(x_{t+1} | y_{t+1}) = \frac{p(y_{t+1} | x_{t+1}) p(x_{t+1} | y_t)}{p(y_t) p(x_{t+1} | y_{t+1}) = N(\mu_{t+1}, \bar{\sigma}_{t+1})}
\]

\[
\mu_{t+1} = \mu + \bar{\mu}_t - \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_t^2} \left( (\mu + \bar{\mu}_t) - y_{t+1} \right)
\]

\[
\bar{\sigma}_{t+1}^2 = \frac{\bar{\sigma}_t^2 \left( \frac{\bar{\sigma}_t^2 + \bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_t^2} \right)}{\bar{\sigma}_t^2 + \left( \frac{\bar{\sigma}_t^2 + \bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_t^2} \right)}
\]
Kalman Filter is very efficient, but requires the distributions \( p(x_{t+1} | x_t) \) and \( p(y_{t+1} | x_t) \) to be Gaussian.

Breakdown:

1. Tracking an object, and another object appears nearly (next lecture).
2. \( x_t, y_t \) take finite set of values (e.g., Biology Applications).
Special Cases:

1. Suppose $\tilde{\sigma}_m = 0$, i.e., the measurements are perfect. Then it follows that:
   
   (a) $M_{t+1} = (\mu + \mu_t) - (\mu + \mu_t) + y_{t+1} = y_{t+1}$
   
   (b) $\tilde{\sigma}_{t+1} = 0$. Extreme case with perfect measurement.

2. Suppose $\tilde{\sigma}_p = 0$, i.e., we have perfect prediction.

   Then:
   
   (a) $\mu_{t+1} = \frac{\tilde{\sigma}_m^2}{\tilde{\sigma}_m^2 + \tilde{\sigma}_t^2} y_{t+1} + \frac{\tilde{\sigma}_m^2}{\tilde{\sigma}_m^2 + \tilde{\sigma}_t^2} (\mu + \mu_t)$, weighted average
   
   (b) $\tilde{\sigma}_{t+1} = \frac{\tilde{\sigma}_m^2 \tilde{\sigma}_t^2}{\tilde{\sigma}_m^2 + \tilde{\sigma}_t^2}$

If we also have $\mu = 0$ (so $x_t$ is constant)
then 

$\mu_{t+1} = \frac{\tilde{\sigma}_m^2}{\tilde{\sigma}_m^2 + \tilde{\sigma}_t^2} y_{t+1} + \frac{\tilde{\sigma}_m^2}{\tilde{\sigma}_m^2 + \tilde{\sigma}_t^2} \mu$

This is an incremental way to estimate the MAP for the

$P(y_1, \ldots, y_{t+1} | X) = \prod_{i=1}^{t} P(y_i | x) \overset{t}{\sim} \text{Gaussian}$

$P(x) \sim \text{Gaussian } N(0, \tilde{\sigma}_p^2)$

Here, reduces to MAP estimation for the static case.