Swendsen-Wang. Ising \( X_i \in \{-1, 1\} \) Edges \( E \).

Standard Metropolis is slow to converge when sampling from the Ising model.

\[
P(x) = \frac{1}{Z} e^{\sum_{i,j} x_i x_j / T}
\]

particularly slow at low temperature.

Swendsen-Wang (SW) is a way to speed up the sampling by “dynamically grouping” sites into clusters.

Then the state of the entire cluster can be switched.

SW is also an example of data augmentation.
(*)& Each subset $V_i$ is connected.

First, Swendsen-Wang for the Potts model.

Each node $s$, can take colors $\mathcal{S}_s \in \{1, 2, \ldots, \Omega\}$.

$$p(s) = \frac{1}{\mathcal{Z}} e^{\frac{\beta}{\mathcal{Z}} \sum_{s_t \in \mathcal{E}} I(s_t = s'_t) + \sum_{s_i \in \mathcal{S}} I(s_i = s)}$$

Swendsen-Wang model is a special case of Potts with $\Omega = 2$.

Notation: set of nodes $V$, partition into subsets $\{V_i; i = 1 \text{ to } n\}$ such that $\bigcup_{i=1}^{n} V_i = V$, $V_i \cap V_j = \emptyset$, $V_i \neq j$.

All nodes in each subset have the same color.

State $A$:

State $B$:

Partition differ only by the state of the regions.

$\Pi_A = (V_0, V_1, V_2)$  $\Pi_B = (V_0, V_1 \cup V_2)$
**SW Algorithm.**

1. Define a new variable $V_{s,t}$ (binary-valued).
   - Set: $V_{s,t} = 0$ if $s \neq c_t$.
   - If $s = c_t$, set $V_{s,t} = 1$ with prob $q_0 = 1 - e^{-\beta}$ (otherwise $V_{s,t} = 0$).

   This yields a number of connected components (each is a subset of vertices of the same color).

2. Randomly select a connected component (e.g. the region on previous page).

3. Randomly select a color for this connected component (uniform prob).

**Claim:** this algorithm is a Markov Chain Monte Carlo. It satisfies detailed balance.

It encouages large groups at small temperature ($\text{large } \beta$).
Detailed Balance of SW.

Define:
\[ C_A = C(V_0, V_1) = \{ (s, t) : s \in V_0, t \in V_1 \} \]
\[ C_B = C(V_0, V_1) = \{ (s, t) : s \in V_0, t \in V_2 \} \]

These are the Swendsen-Wang cuts at \( T_A \), \( T_B \).

Then, it follows that:
\[ q \left( \frac{T_A \rightarrow T_B} \right) = \left( 1 - q_0 \right) \frac{|C_A|}{|C_B|} \]
\[ q \left( \frac{T_B \rightarrow T_A} \right) = \left( 1 - q_0 \right) \frac{|C_B|}{|C_A|} \]

where \( |C_A|, |C_B| \) are the sizes of \( C_A \) and \( C_B \).

For the Potts model:
\[ p \left( \frac{T_A} \right) = e^{-\beta |C_B|} \]
\[ p \left( \frac{T_B} \right) = e^{-\beta |C_A|} \]

Hence:
if \( q_0 = 1 - e^{-\beta} \),
then
\[ q \left( \frac{T_A \rightarrow T_B} \right) \frac{p \left( \frac{T_A} \right)}{p \left( \frac{T_B} \right)} = 1 \]
\[ q \left( \frac{T_B \rightarrow T_A} \right) \frac{p \left( \frac{T_B} \right)}{p \left( \frac{T_A} \right)} \]
which is detailed balance.

on Potts model.

\[ T \left( \leq \right) = \bigcap_{s \in E} T_I \left( V_{s,t} = s, s^t \right) \]

with \[ T_I \left( V_{s,t} = 1, s^t \right) = 0, \quad \text{if} \quad s \neq s^t \]
\[ T_I \left( V_{s,t} = 1, s^t, s \right) = 1 - e^{-\beta}, \quad \text{if} \quad s = s^t \]

Then do Data Augmentation:

Fix \( \mathcal{I} \),

sample from \( T_I \left( \mathcal{I}, \leq \right) \) to get connected components.

Then sample from \( T \left( \leq, \mathcal{I} \right) \) to get the colour of the connected components.

It can be shown that \( T \left( \leq, \mathcal{I} \right) \) is a uniform distribution on each colour for each connected component.

(by definition each connected component must have the same colour).
Extending Swendsen-Wang (Barbu & Zhu)

Standard SW is restricted to a few special distributions — e.g. Ising model & Potts model.

The problem is that detailed balance only occurs for these models.

But, we can extend Swendsen-Wang by Metropolis-Hastings.

Use SW to make proposals

\[ q(\tau_A \rightarrow \tau_B) \] then accept proposals

with probability

\[ \alpha(\tau_A \rightarrow \tau_B) = \min \left( 1, \frac{q(\tau_B \rightarrow \tau_A) p(\tau_B)}{q(\tau_A \rightarrow \tau_B) p(\tau_A)} \right) \]

(Note: original SW is a special case

where \( \alpha(\tau_A \rightarrow \tau_B) = 1 \), the proposal is automatically accepted because of detailed balance.)