Problem 1. A prize is hidden behind one of three doors A, B, and C. The contestant picks a door, say A, but it is left closed. The host opens door C and shows that there is no prize behind it. Should the contestant change his mind and select door B?

Formulate this problem as Bayes inference. What is the prior probability for the position of the prize before door C is opened? What is the posterior after door C has been opened? What should the contestant’s decision rule be?

Problem 2. Suppose you have an image $I$ and want to decide whether it is a Sea Bass $S$ or a Flounder $F$. Formulate this as a decision problem defining distributions $P(I|f) \& P(f)$, where $f \in \{S, F\}$. Define the loss function, the risk, the Bayes rule and the Bayes risk. Write down the loss function if all errors are weighted equally.

Now suppose we have a set of labeled samples $\{(I_\mu, f_\mu) : \mu = 1, ..., N\}$ (i.e. each of the $N$ images is labeled as being Sea Bass or Flounder). Define the empirical risk and say how it relates to the risk. Describe two strategies for obtaining a decision rule to label new images as Sea Bass or Flounder. Describe the difference between generalization and memorization.

Problem 3. Suppose the feature space is a two-dimensional plane and the two likelihood functions $p(x, y|\omega_1), p(x, y|\omega_2)$ are Gaussians with means $\mu_1 = (2, 8), \mu_2 = (8, 2)$, and identical covariances $\Sigma_1 = \Sigma_2 = I$, where $I$ is the (two-dimensional) identity matrix. The prior probabilities are $P(\omega_1) = 1/5$ and $P(\omega_2) = 4/5$. Derive the Bayes rule and Bayes risk for this problem assuming that the loss function weights all errors equally. Draw the decision boundary in the two-dimensional plane.

Now repeat the problem but changing the covariances of the likelihood functions, so that $\Sigma_1 = 4I$ and $\Sigma_2 = 16I$. 

\[\text{Statistics 231. Fall 2008. Homework 1.}\]

\[\text{Due: Monday 3/Nov. 2008.}\]

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\[\text{Now repeat the problem but changing the covariances of the likelihood functions, so that \Sigma_1 = 4I and \Sigma_2 = 16I.}\]
**Problem 4.** Define the errors for the two class discrimination problem. Derive the form of the ROC curve if the data is generated by two Gaussian distributions with the same variance $\sigma^2$:

$$
P(x|\mu_1, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu_1)^2/(2\sigma^2)}, \quad P(x|\mu_2, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu_2)^2/(2\sigma^2)}. \tag{1}
$$

(Note: the form of the ROC curve is sufficient. No need to calculate it exactly.)

**Problem 5.** Let $p_1(x)$ and $p_2(x)$ be two one-dimensional Gaussian distributions $p_1(x) = N(\mu_1, \sigma^2)$ and $p_2(x) = N(\mu_2, \sigma^2)$. Compute the entropy of $p_1(x)$, the difference in the entropies of $p_1(x)$ and $p_2(x)$, and the Kullback-Leibler divergence between $p_1(x)$ and $p_2(x)$. (Hint: for any distribution $p_1(x)$ of a continuous variable $x \in \{-\infty, \infty\}$ then, by definition, $\int_{-\infty}^{\infty} p_1(x) dx = 1$, $\int_{-\infty}^{\infty} p_1(x) x dx = \mu$, and $\int_{-\infty}^{\infty} p_1(x) (x-\mu)^2 dx = \sigma^2$ where $\mu, \sigma^2$ are the mean and variance of the distribution).

**Problem 6.** Use the Maximum Entropy principle to derive a distribution $P(x|\mu)$ defined for $0 \leq x < \infty$ with constraint $\int_{0}^{\infty} x P(x) dx = \mu$.

What is the distribution for $M$ samples $x_1, ..., x_M$ independently and identically distributed from $P(x|\mu)$? What is the sufficient statistic for this distribution? Use Maximum Likelihood to estimate the $\mu$ of the distribution.