Global Segmentation: Weak Membranes

In the early 1980's three closely related, but independent, models emerged:

1. Weak Membrane: Blake & Zisserman
2. Geman & Geman
3. Mumford & Shah

These models have a close relation: Rudin-Osher-Fatemi, TV model.

These first generation models are good for segmentation and de-noising for a restricted type of images.

We will cover the second generation, which is more successful for natural images.

Region Competition / Image Parsing

- Zhu & Yuille, Ted Zhu

Normalized Cuts

- Shi & Malik
- Sharon et al.

Two Issues:

1. Global Criteria
2. Algorithms
(2) Weak-Membrane

Mumford-Shah. \( D(x,y) \) input image.

\[
\min_{M,F} E(M,F) = \int \left( \frac{1}{2} (F(x,y) - D(x,y))^2 + \alpha \int \sqrt{\nabla F(x,y)^2} \, dx \, dy \right) \, dA + \int_{\partial M} c \, ds.
\]

Minimize \( E(M,F) \)

w.r.t. \( M \) and \( F \)

to find the boundary, \( M^* \)
and the smoothed intensity values \( F^*(x,y) \).

Assumption: (1) Images are piecewise
slowly varying, or smooth,
(2) The boundaries \( M \) are as
short as possible.

From probabilistic perspective, see later, the
image is corrupted by additive Gaussian noise.

Ideal image (1D) is

How to minimize this
functional? (Discrete & Continuous Variables)

(Historically it was hard to prove that it had
a well-defined minimum).
(3) **Weak-Membrane**

Ambrosio-Tortorelli

Minimizing Mumford-Shah with $F \in \Gamma$ is equivalent to minimizing the following functional Ambrosio-Tortorelli in the limit as $\varepsilon > 0$

$$E_{\varepsilon} = \frac{1}{2} \int \left\{ F(\nabla u) + \lambda \right\} \text{d}x \text{d}y$$

$$+ \int (1 - \alpha \nabla u)^2 \left| \nabla F(\nabla u) \right|^2 \text{d}x \text{d}y$$

$$- \int \left\{ \varepsilon \left| \nabla \alpha \right|^2 + \varepsilon^{-1} \alpha^2 \right\} \text{d}x \text{d}y.$$

$\alpha(x,y)$ is a *thin process*. 

**Intuitively:** if $\alpha(x,y) = 1$, then the smoothness constraint $\left| \nabla F(\nabla u) \right|^2$ is cut, and there is a boundary (i.e., $\alpha(x,y) \approx 1$, corresponds to $\Gamma$).

**Fix edges smooth $F$.**

**Fix $F$ smooth $\alpha$.**

**Algorithm:**

1. Fix $\alpha$: minimize $E_{\varepsilon} = \left\{ F(\nabla u) + \lambda \right\}$ with $\alpha$.

2. Fix $F$: minimize $E_{\varepsilon} = \left\{ F(\nabla u) + \lambda \right\}$ with $\alpha$.

At each step solve a linear differential equation.
Week-Membrane

Hence steps (1) & (2) while decreasing $c$

Continuation Method

Both processes — for $c(x,y) = f(x,y)$ — are forms of diffusion.

Alternatively, direct steepest descent on

$$E_{A-T}[F, l, c]$$

$$\frac{df}{dt} = -S E_{A-T}, \quad \frac{dl}{dt} = -S E_{A-T}$$

Problem determining the step size for $t$

when doing the discretization of $\frac{d}{dt}$

Note: $E[F, l, c]$ and $E_{A-T}[F, l, c]$

are functionals (i.e., their arguments are functions)

Taking derivatives $S$

requires calculus of variations.
Weak Membrane

Rudin-Osher-Fatemi. TV restoration.

\[ E[F] = \int_{\Omega} \left[ |\nabla F(x,y)|^2 + \frac{1}{2} \left( \frac{F(x,y) - D(x,y)}{\sqrt{1 - \delta(x,y)}} \right)^2 \right] dx \]

\[ \delta(x,y) \text{ is a non-negative function.} \]

This is a convex function of \( F \). Hence there is a unique minimum. Any algorithm that decreases the energy is guaranteed to find it.

Alternating steepest descent:

\[ \frac{dF}{dt} = -\frac{\delta F}{\delta F} \]

Alternative Insight (Shek & Yulke?)

Equivalent to minimizing:

\[ E[F, \delta] = \frac{1}{2} \int_{\Omega} \left( (1 - \delta(x,y)) |\nabla F(x,y)|^2 + \frac{1}{2} \left( \frac{F(x,y) - D(x,y)}{\sqrt{1 - \delta(x,y)}} \right)^2 \right) dx \]

Proof: Minimize w.r.t. \( \delta \) given \( 1 - \delta(x,y) = \frac{1}{\sqrt{1 - \delta(x,y)}} \)

result follows by substitution.

Algorithm, minimize \( E[F, \delta] \) w.r.t. \( \delta \) and \( F \) alternately:

1. Fix \( \delta \), solve
   \[ -\nabla \left( (1 - \delta(x,y)) |\nabla F(x,y)|^2 \right) \]
   for \( F(x,y) \)

2. Fix \( F \), solve
   \[ 1 - \delta(x,y) = \frac{1}{\sqrt{1 - \delta(x,y)}} \]
   for \( \delta(x,y) \)

Each stage decreases the energy, guaranteed to converge to global optimum.

Algorithm known as lagged diffusion.