Part 1 : One Random Variable

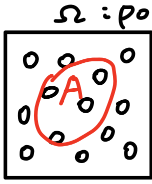
Discrete R.V.

Recall. Axiom 0 : Under equally likely sampling,

$$P(A) = \frac{|A|}{|\Omega|}$$

Picture 1:

Ω : population N people

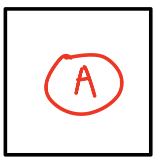


$$P(A) = \frac{|A|}{|\Omega|} = \frac{M}{N}$$

under random sampling, prob = population proportion

M people : fixed

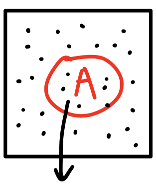
2: Ω : region, population of uncountably infinitely many points



$$P(A) = \frac{|A|}{|\Omega|} = \frac{\text{area of } A}{\text{area of } \Omega}$$

math idealization of Pic. 1

3: Ω : Repeat random sampling in Pic 2 $\underline{\underline{n}}$ times



$\Rightarrow$ n points in total

$$\frac{m}{n} \xrightarrow{n \to \infty} P(A) = \frac{|A|}{|\Omega|}$$

long run frequency

m points fall into A

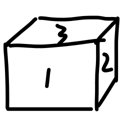
Differ from Pic. 1 : m is random ;

If repeat Pic. 3 many times, we can see many different scenarios.

$\Rightarrow$ m fluctuates around $P(A) = \frac{|A|}{|\Omega|}$

The magnitude of fluctuation is called **variance**.

Roll a **Fair Dice**



X	1	2	3	4	5	6
P(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

lower case   capital

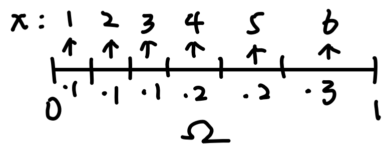
$$p(x) = P(X = x)$$

prob mass function

What about a biased dice ?

X	1	2	3	4	5	6
$p(x)$	.1	.1	.1	.2	.2	.3

$X \sim p(w)$ prob distribution

$\Rightarrow$ 

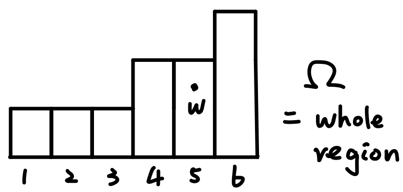
like Pic. 2

Imagine randomly throwing a point to Ω

$w \sim$ uniform (Ω)

$X(w)$

Another way to think about it (like Pic.2):

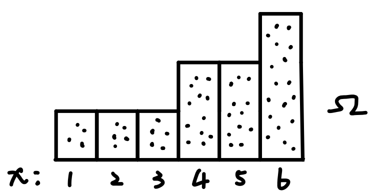


Think as regions

$w \sim$ uniform (Ω)

$X(w)$

Use this to build Pic. 3 :



n points in total

$n(x) = \#$ of points in bin x.

$\frac{n(x)}{n} \longrightarrow p(x)$ How often $X = x$

$P(X \in (a, b)) = \sum_{x \in (a,b)} p(x)$ $\quad \sum_x p(x) = 1$

X	1	2	3	4	5	6
$p(x)$	.1	.1	.1	.2	.2	.3
long rn frq	10%	10%	10%	20%	20%	30%

Prob Mass Function : $X \sim p(x)$

$\downarrow$ summarize

expectation, variance

$\equiv$ $E(X)$ $\equiv$ $Var(X)$

Expectation

$$E(X) = \sum_x x \cdot p(x)$$

$$\overset{e.g.}{=} 1 \cdot p(1) + 2 \cdot p(2) + \cdots + 6\, p(6)$$

$$= 1 \times .1 + 2 \times .1 + \cdots + 5 \times .2 + 6 \times .3$$

Interpretation : in this example

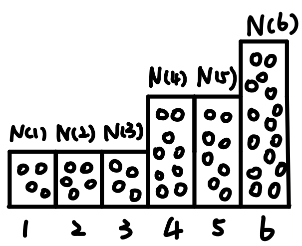
① Long run average $= \dfrac{1 \times n(1) + 2 \times n(2) + \cdots + 5 \times n(5) + 6 \times n(6)}{n}$

$$= 1 \times \frac{n(1)}{n} + 2 \times \frac{n(2)}{n} + \cdots + 5 \times \frac{n(5)}{n} + 6 \times \frac{n(6)}{n}$$

$$\xrightarrow[\infty]{n} 1 \times p(1) + 2 \times p(2) + \cdots + 6 \times p(6)$$

$$= E(X)$$

② Population average



$X(w)$ = # of states been to

population average $= \dfrac{1 \times N(1) + \cdots \; 6 \times N(6)}{N}$

$$= \sum_x x \frac{N(x)}{N}$$

$$= \sum_x x\, p(x)$$

Roll a Dice Example :

X	1	2	3	4	5	6
$p(x)$	.1	.1	.1	.2	.2	.3
$h(x)$	-\$30	-\$20	\$0	\$20	\$40	\$60

where $h(x)$ = pay-off at a Casino

$$E(h(x)) = \sum_x h(x)p(x)$$

Example :

Offer 1

Face value x	\$100
$p(x)$	1

$E(X) = \$100 \times 1 = \$100$

Offer 2

x	\$0	\$200
$p(x)$	$\frac{1}{2}$	$\frac{1}{2}$

$E(X) = \$0 \times \frac{1}{2} + \$200 \times \frac{1}{2} = \$100$

utility / perceived value :

face value x	\$0	\$100	\$200
perceived value $h(x)$	\$0	\$100	\$150

Offer 1: $E(h(x)) = \$100 \times 1 = \$100$

Offer 2: $E(h(x)) = \$0 \times \frac{1}{2} + \$150 \times \frac{1}{2} = \$75$

Another example :

face value x	\$0	\$100	\$200
perceived value $h(x)$	\$0	\$100	\$400

Offer 1: $E(h(x)) = \$100 \times 1 = \$100$

Offer 2: $E(h(x)) = \$0 \times \frac{1}{2} + \$400 \times \frac{1}{2} = \$200$

Variance :

always in squared units

$Var(X)$ is based on $E(X)$:

$$Var(X) = E\left(\boxed{(X - E(X))^2}\right)$$

special case of $h(x)$, squared deviation

Offer 1: $Var(X) = E((X - E(X))^2)$

$= E((X - 100)^2)$

$= (100 - 100)^2 \times 1 = 0$

Offer 2: $Var(X) = (0 - 100)^2 \times \frac{1}{2} + (200 - 100)^2 \times \frac{1}{2}$

$= 10{,}000$ (unit is $\$^2$)

Standard Deviation : $SD(X) = \sqrt{Var(X)} = \$100$

goes back to the original unit

Roll a Dice Example :

X	1	2	3	4	5	6
$p(x)$	.1	.1	.1	.2	.2	.3
$h(x)$	-\$30	-\$20	\$0	\$20	\$40	\$60

$$Var(h(X)) = E((h(X) - E(h(X)))^2)$$

$$SD(h(X)) = \sqrt{Var(h(X))}$$