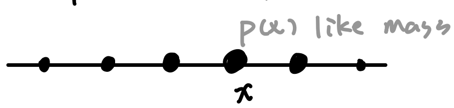
Cont. Discrete R.V.

$X \sim p(x)$ (X follows $p(x)$ / X is a sample from $p(x)$)

X: capital $p(x)$: lower case

$p(x)$: prob mass function



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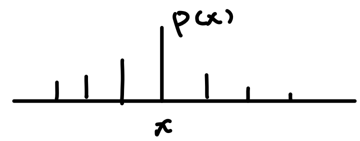


or

X	x_1	x_2	$\cdots$	x_n
$p(x)$	p_1	p_2	$\cdots$	p_n

$\Rightarrow$ $p(x) = P(X)$ — population proportion / long run frequency

$$P(X \in A) \overset{eg}{=} P(X \in (a,b)) = \sum_{x \in (a,b)} p(x)$$

$\mu = E(X) = \sum_x x \cdot p(x)$ — population average / long run average / center

weighted average

$$E(h(X)) = \sum_x h(x) \cdot p(x)$$

eg $E(h(X))$: \$; $h(x)$: dice #

$\sigma^2 = Var(X) = E((X - E(X))^2)$ (same)

long run average of squared deviation
magnitude of fluctuation
spread of distribution
stock market : volatility

$Var(h(X)) = E((h(X) - E(h(X)))^2)$ (same)

$\sigma = SD(X) = \sqrt{Var(X)}$

Properties :

1) $E(h(X)+g(X)) = \sum_x (h(x)+g(x))\cdot p(x)$

$$= \sum_x h(x)\,p(x) + \sum_x g(x)\,p(x)$$

$$= E(h(X)) + E(g(X))$$

2) $E(\underbrace{a}_{\text{constants}}X+\underbrace{b}_{}) = \sum_x (ax+b)\cdot p(x)$

$$= \sum_x ax\cdot p(x) + \sum_x b\cdot p(x)$$

$$= a\sum_x x\cdot p(x) + b\boxed{\sum_x p(x)} = 1$$

$$= aE(X) + b$$

3) $Var(aX+b) = E(((aX+b) - E(aX+b))^2)$

$$= E(((aX+b)-(aE(X)+b))^2)$$

$$= E((a(X-E(X)))^2)$$

$$= E(a^2(X-E(X))^2)$$

$$= a^2E((X-E(X))^2)$$

$$= a^2\,Var(X)$$

4) $Var(X) = E((X-\mu)^2)$ $\quad \mu \to E(X)$

$$= E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2\mu^2 + \mu^2$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - E(X) \geq 0$$

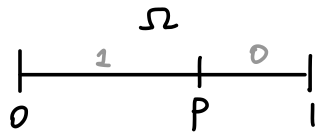
Concrete Models

Bernoulli model (coin flipping)

$Z \sim \text{Bernoulli}(p)$

z	tail	head
prob	$1-p$	p

(fair coin : $p = \frac{1}{2}$)



$$Z(w) = \begin{cases} 1 & w \leq p \\ 0 & w > p \end{cases}$$

$w \sim$ Uniform (Ω) $\qquad P(A) = \frac{|A|}{|\Omega|} = p$

$$Z = 1(w \in A) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$$

$$E(Z) = 0 \cdot (1-p) + 1 \cdot p = p$$

$$E(1(w \in A)) = P(A)$$

$$\begin{aligned} Var(Z) &= E((Z - E(Z))^2) \\ &= E((Z-p)^2) \\ &= (0-p)^2 \cdot (1-p) + (1-p)^2 \cdot p \\ &= p^2 \cdot (1-p) + (1-p)^2 \cdot p \\ &= p(1-p)(\cancel{p} + (1-\cancel{p})) \\ &= p(1-p) \end{aligned}$$

In another way:

$$Var(Z) = E(Z^2) - E(Z)^2$$

$E(Z^2) = E(Z) = p$ bc in this case $Z^2 = Z$

$$Var(Z) = p - p^2 = p(1-p)$$

$$Z \sim Ber(\tfrac{1}{2})$$

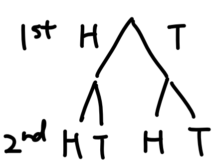
$E(Z) = \frac{1}{2}$, $Var(Z) = \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4}$, $SD(Z) = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Flip Fair Coin Independently Twice

$$\Omega = \{HH, HT, TH, TT\}$$

2nd \ 1st	H	T
H	HH	TH
T	HT	TT

or



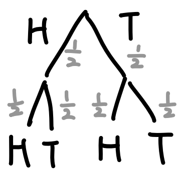
2^2 branches

$X(w) = $ # of heads in $w \in \Omega$

a sequence

w	HH	HT	TH	TT
$X(w)$	2	1	1	0

Independence : All $2^2 = 4$ outcomes are equally likely



$P(X(w)=2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$P(X(w)=1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ $\Rightarrow$

x	0	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$P(X(w)=0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$X \sim \text{Binomial}(n=2, p=\frac{1}{2})$

# of flips   prob of Head

Binomial Model

$X \sim \text{Bin}(n, p)$, where X = # of heads

flip a coin independently n times ,

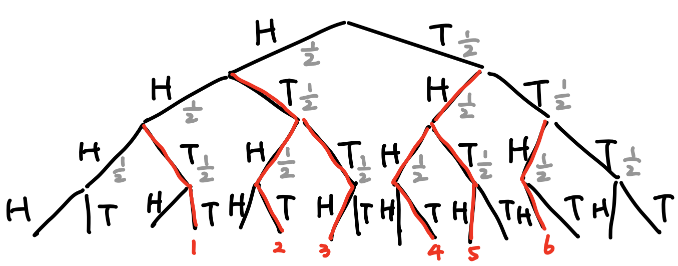
$p(\text{head}) = p$ in each flip , $\text{Ber}(p) = \text{Bin}(1, p)$

↳ Change of Perspectives :

Originally, flip a coin twice

Reimagine : randomly pick a sequence $w \in \Omega$

Ex. $X \sim \text{Bin}(n=4, p=\frac{1}{2})$



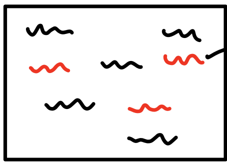
$\Omega = \{$ HHHH, HHHT, HHTH ... ... TTTT $\}$

2^4 sequences

randomly sample a $w \in \Omega$

$$P(X=2) = \frac{\#\text{ of seqs with 2H's}}{2^4} = \frac{6}{2^4} = \frac{3}{8}$$

Ω = 16 seqs



→ seq with 2 heads

imagine picking a sequence randomly, what is the probability that the seq has two heads?

$X \sim \text{Bin}(n, p = \frac{1}{2})$

x	0	1	2	$\cdots$	k	$\cdots$	n
p(x)					$P(X=k)$		

Independence: all 2^n sequences in Ω are equally likely

$$P(X=k) = P(A) \overset{\text{axiom 0}}{=\!=} \frac{|A|}{|\Omega|} \quad |A| = \binom{n}{k}, \quad |\Omega| = 2^n$$

where $A = \{w : X(w) = k\}$ $= \dfrac{\binom{n}{k}}{2^n}$, $k = 0, 1, \cdots, n$