Axiom 0: Equally likely sampling from Ω

$$P(A) = \frac{|A|}{|\Omega|}$$

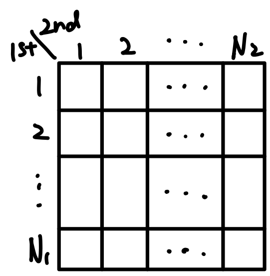
Counting techniques to calculate $|A|$ & $|\Omega|$

Multiplication Rule:

step 1: pick a ball from box 1
step 2: pick a ball from box 2 } ordered pairs

\# of pairs $= N_1 \times N_2$

$\Rightarrow$



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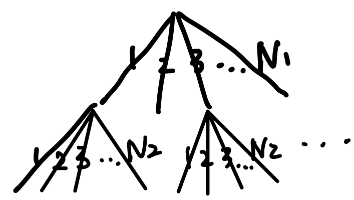
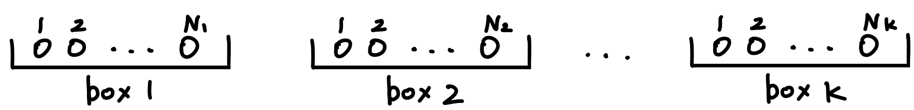


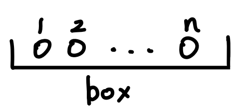


k tuples

step 1 : pick a ball from box 1
2 : 2
⋮ ⋮
k : k } ordered k tuple

\# of k-tuples $= N_1 \times N_2 \cdots \times N_k$

Permutation:



Sequentially pick k balls without replacement
(order matters)

do not put the ball back

\# of ordered sequences $= n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}$

if $k = n$, # of ordered sequences $= n(n-1)\cdots(1) = n!$

$\Rightarrow$ Each sequence is a permutation

Combination:

pick k balls from a box of n balls without replacement (order does not matter)

$\Rightarrow$ Each unordered sequence is a combination

# of unordered sequences $= \binom{n}{k} = \dfrac{P_{n,k}}{k!}$ $\rightarrow$ $k!$ permutations for each combination

$$= \frac{n!}{(n-k)!\,k!}$$

$$= \binom{n}{n-k}$$

Back to Coin Flipping

Fair coin $\rightarrow p = \frac{1}{2}$

$X \sim Bin(n, \frac{1}{2})$ $\quad \Omega = \{2^n \text{ sequences } w\}$

$X(w) =$ # of heads in seq w

$A_k = \{X = k\} = \{w : X(w) = k\}$ — Whole Sale Viewpoint

$P(A_k) = p(X = k) = \dfrac{|A|}{|\Omega|} = \dfrac{\binom{n}{k}}{2^n}$

How to create a seq with k heads?

$$\underset{1}{\square}\ \underset{2}{\square}\ \underset{3}{\square}\ \cdots\ \square\ \cdots\ \underset{n}{\square}$$

$\binom{n}{k}$ ways to write

n blanks, choose k blanks to write H;

The rest ($n-k$ blanks) write T.

Identity : $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ = all possible sequences

$E(X) = \frac{n}{2}$

Proof :

$E(X) = \sum_{x} x \cdot p(x)$

convention:
i, j, k for integers;
x, y for continuous variables

$\underline{\underline{\text{change notation}}} \sum_{k=0}^{n} k \cdot p(k)$

$= \sum_{k=0}^{n} k \cdot \frac{\binom{n}{k}}{2^n}$

$= \frac{1}{2^n} \sum_{k=0}^{n} k \cdot \frac{n!}{k!(n-k)!}$

$= \frac{n}{2^n} \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}$

$= \frac{n}{2} \frac{1}{2^{n'}} \sum_{k'=0}^{n'} \frac{n'!}{k'!(n'-k')!}$

$= \frac{n}{2} \frac{1}{2^{n'}} \boxed{\sum_{k'=0}^{n'} \binom{n'}{k'}} = 2^{n'}$ according to the identity eqn

$= \frac{n}{2}$

$Var(X) = \frac{n}{4}$

Recall: $Var(X) = E(X^2) - E(X)^2$

$E(X^2) = ?$, $E(X)^2 = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$

$\Rightarrow E(X(X-1)) = E(X^2 - X)$

$= E(X^2) - E(X)$

so, $E(X^2) = E(X(X-1)) + E(X) = \frac{n}{2}$

$E(X(X-1)) = \sum_{x} x(x-1) \cdot p(x)$

$= \sum_{k=0}^{n} k(k-1) \cdot p(k)$

$= \sum_{k=0}^{n} k(k-1) \cdot \frac{\binom{n}{k}}{2^n}$

$= \frac{1}{2^n} \sum_{k=0}^{n} k(k-1) \cdot \frac{n!}{k!(n-k)!}$

$= \frac{n(n-1)}{2^n} \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!}$

$$= \frac{n(n-1)}{2^n} \sum_{k''=0}^{n''} \frac{n''!}{k''!(n''-k'')!} \qquad n'' = n-2, \quad k'' = k-2$$

$$= \frac{n(n-1)}{2^n} 2^{n''}$$

$$= \frac{n(n-1)}{4}$$

$$E(X^2) = \frac{n(n-1)}{4} + \frac{n}{2} = \frac{n^2-n+2n}{4} = \frac{n^2+n}{4}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \frac{n^2+n}{4} - \frac{n^2}{4}$$

$$= \frac{n}{4}$$

$$\text{Frequency of heads} = \frac{X}{n}$$

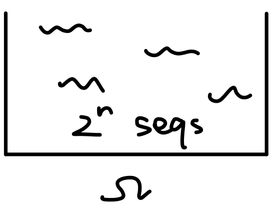
$$E\left(\frac{X}{n}\right) = \frac{E(X)}{n} = \frac{n/2}{n} = \frac{1}{2} \qquad (E(aX+b) = aE(X)+b)$$

$$Var\left(\frac{X}{n}\right) = \frac{Var(X)}{n^2} = \frac{n/4}{n^2} = \frac{1}{4n} \qquad (Var(aX+b) = a^2 Var(X))$$

$$\xrightarrow[\infty]{n} 0$$

This implies the frequency $\frac{X}{n} \xrightarrow[\infty]{n} \frac{1}{2}$ (probability)

Whole Sale Viewpoint : product space $\{H, T\}^n$



Each seq has a freq of heads $\frac{X(w)}{n}$

2^n freqs $\longrightarrow$ their average $= E\left(\frac{X}{n}\right) = \frac{1}{2}$

their variance $= Var\left(\frac{X}{n}\right) = \frac{1}{4n} \to 0$

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| > \underset{.0001}{\varepsilon}\right) = \frac{\#\text{ of "bad" seqs}}{2^n} \xrightarrow[\infty]{n} 0$$

Law of large # / Concentration of measure

Consider a General Coin

$X \sim Bin(n, p)$, where p is not necessarily $\frac{1}{2}$

$P(X=k) = \binom{n}{k} \boxed{p^k(1-p)^{n-k}}$ → prob of getting k heads

$\binom{n}{k}$ ↓ # of seq with k heads

$E(X) = np \rightarrow E\left(\frac{X}{n}\right) = p$

$Var(X) = np(1-p) \rightarrow Var\left(\frac{X}{n}\right) = \frac{p(1-p)}{n} \longrightarrow 0$

$P\left(\left|\frac{X}{n} - p\right| > \varepsilon\right) \longrightarrow 0$

Similar to Bernoulli :

$Z \sim Ber(p)$

Z	tail 0	head 1
prob	$1-p$	p

Z = # of head $\sim Bin(n=1, p)$

⇒ Bernoulli is a special case

$Z_1, Z_2, \cdots Z_n \sim Ber(p)$ independently

$X \sim Bin(n, p)$

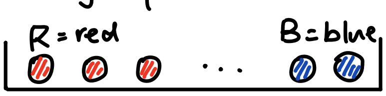
$$X = Z_1 + Z_2 + \cdots + Z_n = \sum_{i=1}^{n} Z_i$$

$$E(X) = \sum_{i=1}^{n} E(Z_i) = np$$

$$Var(X) = \sum_{i=1}^{n} Var(Z_i) = np(1-p)$$

Another Example : $X \sim Bin(n, p)$, p is not necessarily $\frac{1}{2}$

survey / poll



$N = R + B$

$\frac{X}{n}$ = frequency of red

$E\left(\frac{X}{n}\right) = p$

$Var\left(\frac{X}{n}\right) = \frac{p(1-p)}{n}$

$SD\left(\frac{X}{n}\right) = \sqrt{\frac{p(1-p)}{n}}$

randomly pick a ball :

$P(red) = \frac{R}{N} = p$ (population proportion)

randomly pick n balls :

X = # of red balls

$\sim Bin\left(n, p = \frac{R}{N}\right)$