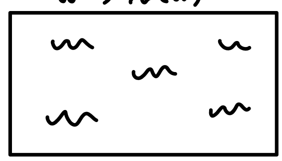
More on Binomial

$X \sim \text{Bin}(n, \frac{1}{2})$

"whole sale viewpoint"

$\Omega = \{ 2^n \text{ sequences } \underset{w}{} \}$

$w \to X(w)$



Ω

$X(w) = \#$ of heads in sequence w

$$P(X=k) = \frac{\#\text{ of seqs w/ k heads}}{2^n} = \frac{\binom{n}{k}}{2^n}$$

$$E(X) = \frac{1}{2^n} \sum_{w \in \Omega} X(w) = \frac{n}{2}$$

$$Var(X) = \frac{1}{2^n} \sum_{w \in \Omega} \left(X(w) - \frac{n}{2}\right)^2 = \frac{n}{4}$$

$$E\left(\frac{X}{n}\right) = \frac{1}{2^n} \sum_{w \in \Omega} \frac{X(w)}{n} = \frac{1}{2}$$

$$Var\left(\frac{X}{n}\right) = \frac{1}{2^n} \sum_{w \in \Omega} \left(\frac{X(w)}{n} - \frac{1}{2}\right)^2 = \frac{1}{4n} \xrightarrow{n \to \infty} 0$$

$$P\left(\left|\frac{X}{n} - \frac{1}{2}\right| < \varepsilon\right) = \frac{\#\text{ of seqs where freq} \in (0.49, 0.51)}{2^n} \xrightarrow{n \to \infty} 1$$

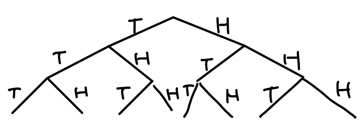
law of large #

e.g. $\frac{X}{n} \in (0.49, 0.51)$ $\quad \frac{X}{n} \to \frac{1}{2}$ in prob

long run freq $\quad$ prob

Binomial Formula

$(T+H)^n$



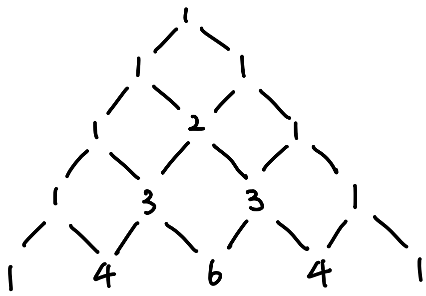
$n=0$	1
$n=1$	$T+H$
$n=2$	$T^2 + 2TH + H^2$
$n=3$	$T^3 + 3T^2H + 3TH^2 + H^3$

$$(T+H)^n = \binom{n}{0} T^n + \binom{n}{1} T^{n-1} H + \binom{n}{2} T^{n-2} H^2 + \cdots + \binom{n}{n} H^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} H^k T^{n-k}$$

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

Consider **Pascal's Triangle** :



shift and summation

$$(T+H^3) = (T^2+2TH+H^2)(T+H)$$

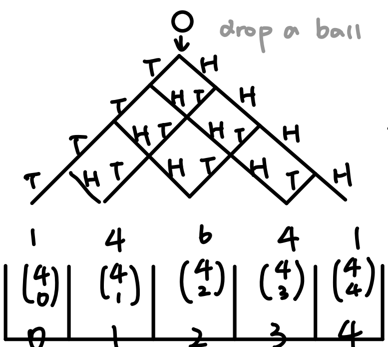
$$= T^3+2T^2H+H^2T+$$
$$T^2H+2H^2T+H^3$$

aligns like

1	2	1	
	1	2	1
1	3	3	1

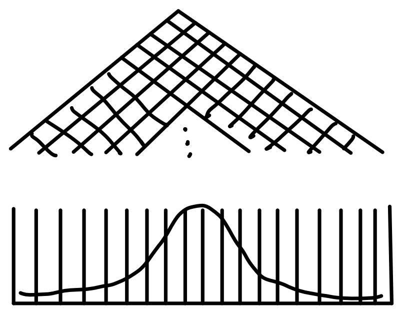
Consider **Gaston Board** :

drop a ball



make at least one right turn

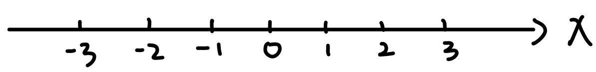
$$\xrightarrow[\infty]{n}$$



$N(\mu, \sigma^2)$

Central Lim

Random Walk :



$$X_t = 0$$

$$X_{t+1} = X_t + Z_{t+1}$$

$$Z_1, Z_2, \dots, Z_n \sim Ber\left(\tfrac{1}{2}\right)$$

General Binomial

$X \sim Bin(n, p)$

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ ($1-p \to q$)

$\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1$

$\boxed{\begin{array}{l} \frac{X}{n} \to p \\ P(|\frac{X}{n} - p| < \varepsilon) \to 1 \end{array}}$

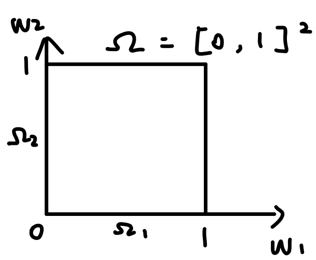
$E(X) = np$

$Var(X) = np(1-p)$

$E(\frac{X}{n}) = p$

$Var(\frac{X}{n}) = \frac{p(1-p)}{n} \to 0$

Independence



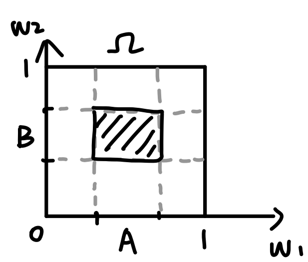
$\Omega = [0,1]^2 = \Omega_1 \times \Omega_2$

$\Rightarrow w \in (w_1, w_2) \sim \text{Uniform}(\Omega)$

$A: w_1 \in A$

$B: w_2 \in B$

$(w_1, w_2) \sim$ Uniform $[0,1]$ independently



$P(A) = \frac{|A|}{|\Omega_1|} = |A|$ bc $|\Omega_1| = 1$

$P(B) = \frac{|B|}{|\Omega_2|} = |B|$

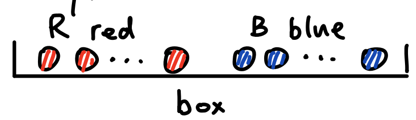
$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} \to |A \cap B| = |A| \times |B|$

$= P(A)P(B)$

independence : $A \perp B$

$P(A \cap B) = P(A)P(B)$

Example:



$N = R + B$

Sample one ball, put it back

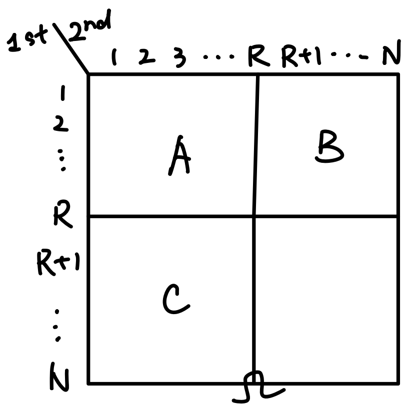
... one ball again independently

$P(\text{red}, \text{red}) = P(\text{red})\,P(\text{red}) = \left(\frac{R}{N}\right)^2$

$P(\text{red}, \text{blue}) = P(\text{red})\,P(\text{blue}) = \frac{R}{N} \cdot \frac{B}{N}$

$\Omega = \{N^2 \text{ pairs}\}$ all equally likely

A = red, red



$|A| = R^2$

$P(A) = \frac{|A|}{|\Omega|} = \frac{R^2}{N^2}$

B: red, blue

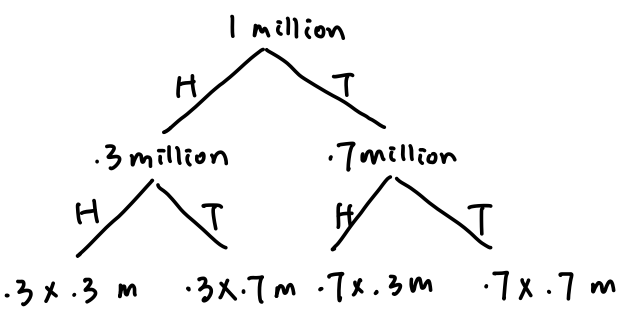
$P(B) = \frac{|B|}{|\Omega|} = \frac{RB}{N^2}$

C: blue, red

$P(C) = \frac{|C|}{|\Omega|} = \frac{BR}{N^2}$

Another Example:

A million people during experiment, flip a biased coin, prob of head = 0.3 independently



Think:

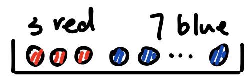
$X \sim Bin(n, p)$

$P(\text{any seq with } k \text{ heads}) = p^k (1-p)^{n-k}$

$\underbrace{H\,H \cdots H}_{k}\ \underbrace{T\,T \cdots T}_{n-k}$

$\#$ of seqs with k heads $= \binom{n}{k}$

$\Rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

Example: 

Sample the box independently n times.

$X = \#$ of red balls

$X \sim Bin(n, p=.3)$

$\Omega = \{ 10^n \underset{w}{\text{seqs}} \}$

Finished Binomial Model

$T \sim Geometric(p)$

waiting time

$\#$ of flips until 1^{st} head

$P(T=k) = P(\underbrace{T\,T \cdots T}_{k-1} H \cdots)$

$= (1-p)^{k-1} p$

k	1	2	3	$\cdots$	k	$\cdots$
prob						

$P(T=1) = P(H) = p$

$P(T=2) = P(TH) = (1-p)p$

$P(T=3) = P(TTH) = (1-p)^2 p$

$E(T) = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p$

$\sum_{x} x \cdot p(x)$

$= \frac{1}{p}$