Geometric Sequence

$$S = 1 + a + a^2 + \cdots + a^m$$

$$(1-a)\cdot S = S - aS = 1 + a + a^2 + \cdots + a^m$$
$$- (a + a^2 + \cdots + a^m + a^{m+1})$$
$$= 1 - a^{m+1}$$

$$\Rightarrow S = \frac{1-a^{m+1}}{1-a}$$

If $|a|<1$ & $m \to \infty$ $\Rightarrow a^{m+1} \xrightarrow[\infty]{m} 0$

$$S = 1 + a + a^2 + \cdots = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$$

Recall. $T \sim Geo(p)$

$$P(T=k) = P(\underbrace{\text{Tail Tail} \cdots \text{Tail}}_{k-1}\ \text{Head}) = (1-p)^{k-1} p$$

$$E(T) = \sum_{k=1}^{\infty} k \cdot p(T=k) = \frac{1}{p}$$

Proof:

$$E(T) = \sum_{k=1}^{\infty} k \cdot (\underbrace{1-p}_{q})^{k-1} p$$

$$= \sum_{k=1}^{\infty} \boxed{k \cdot q^{k-1}} \cdot p \qquad k \cdot q^{k-1} = \frac{d}{dq} q^k$$

$$= p \frac{d}{dq} \sum_{k=1}^{\infty} q^k$$

Recall $S = 1 + a + a^2 + \cdots = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a}$

$$= p \frac{d}{dq}\left(\frac{1}{1-q} - 1\right)$$

$$= p\left(-\frac{-1}{(1-q)^2}\right)$$

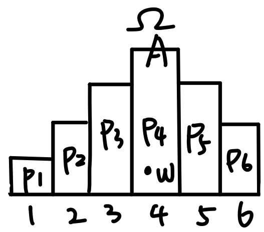
$$= p \frac{1}{(1-q)^2}$$

$$= p \frac{1}{p^2}$$

$$= \frac{1}{p}$$

<u>Continous Random Variable</u>

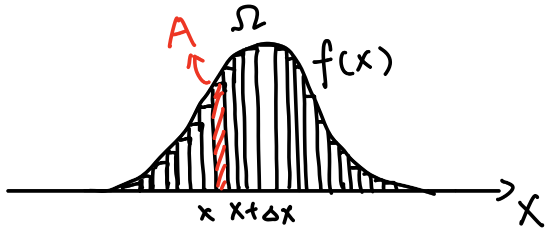
<u>Recall</u> Biased dice



$$P(X=4) = P(A) = \frac{|A|}{|\Omega|} = p_4$$

$$A = \{w : X(w) = 4\}$$

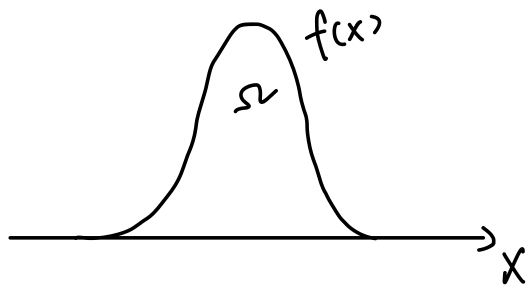
Continuous



$$P(X(w) \in (x, x+\Delta x)) = P(w \in A)$$

$$= P(A) = \frac{|A|}{|\Omega|} = \frac{f(x)\Delta x}{|\Omega|} = f(x)\Delta x$$

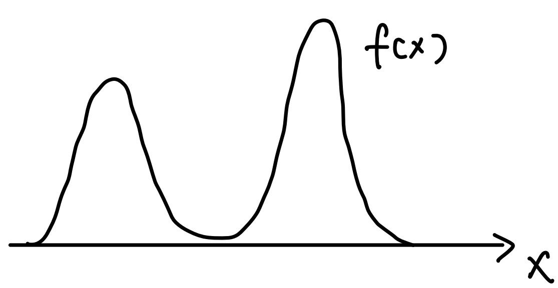
$$|\Omega| = \sum_{x\,\text{bins}} f(x)\cdot\Delta x \xrightarrow[0]{\Delta x} \int_{-\infty}^{\infty} f(x)dx = 1$$



randomly throw a point into Ω

get $w \sim \text{Uniform}(\Omega)$

$X(w) \sim f(x)$



$$f(x) = \lim_{x \to 0} \frac{P(X \in (x, x+\Delta x))}{\Delta x}$$

prob mass

prob density function

Discrete: prob mass function $P(X = x) = p(x)$

$$p(X \in (a,b)) = \sum_{x \in (a,b)} p(x)$$

Continuous: prob density function $P(X \in (x, x+\Delta x)) = f(x)\Delta x$

$$p(X \in (a,b)) = \int_a^b f(x)\,dx$$

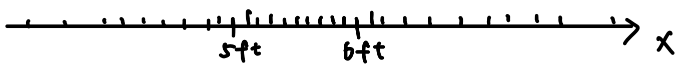
, $P(X = x) = 0$ in infinite precision

Another Intuition

Consider U.S. population of 300 million people

Ω = population $\qquad X(w)$ = height of w

w = person $\in \Omega$ $\qquad w \sim \text{Uniform}(\Omega)$



N = 300 million points

Density at X (e.g. 6ft) $= \dfrac{\#\text{ of points} \in (x, x+\Delta x)}{\Delta x}$

(for example : Density (LA) $= \dfrac{\#\text{ of people in LA}}{\text{area of LA}}$)

prob density : $f(x) = \dfrac{\#\text{ of points} \in (x, x+\Delta x) / N}{\Delta x}$

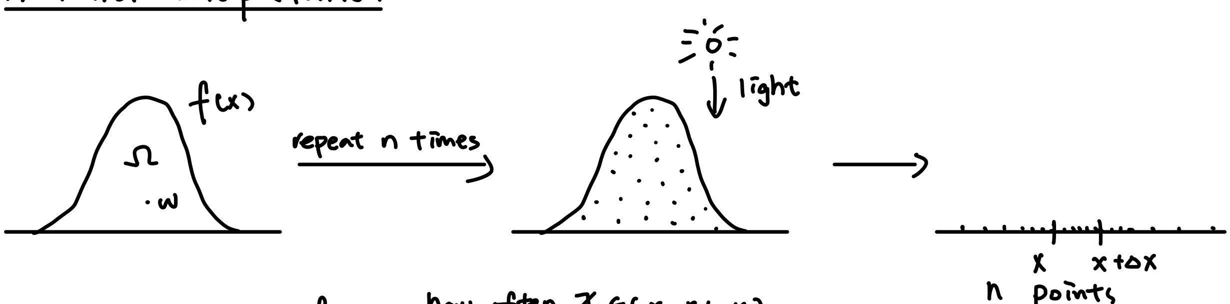
$= \dfrac{\text{population proportion of points} \in (x, x+\Delta x)}{\Delta x}$ $\quad |\Omega|$

($f(x)$ describes the deterministic population)

under random sampling :

$w \sim \Omega \qquad \dfrac{\text{Prob}(X(w) \in (x, x+\Delta x))}{\Delta x}$

A Third Interpretation



$f(x) = \dfrac{\text{how often } X \in (x, x+\Delta x)}{\Delta x}$

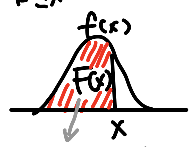
Discrete : $P(X = x) = p(x)$ how often $X = x$

Continuous : $P(X \in (x, x+\Delta x)) = f(x)\Delta x$ how often $X \in (x, x+\Delta x)$

$$f(x) = \frac{P(X \in (x, x+\Delta x))}{\Delta x}$$ (smear prob mass evenly over $(x, x+\Delta x)$)

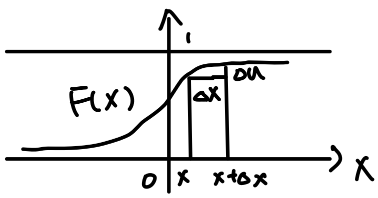
Cumulative Prob Function

$$F(x) = P(X \le x) = \begin{cases} \text{discrete} & \sum_{k \le x} p(x) \\ \text{continuous} & \end{cases}$$



$= \int_{-\infty}^{x} f(x)\,dx$

F: cumulative density function (cdf)



$$F(X) = \int_{-\infty}^{x} f(x)\,dx$$

notation :

$$F'(x) = \frac{dF(x)}{dx} = \frac{d}{dx} F(x)$$

$$dF = dF(x) = F'(x)dx$$

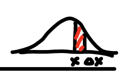
$$f(x) = F'(x)$$

$$= \frac{\boxed{F(x+\Delta x) - F(x)}}{\Delta X} = \Delta F$$

$=$

$/ \ \Delta X$

$=$



$/ \ \Delta X$

geometric meaning : slope of $F(x)$

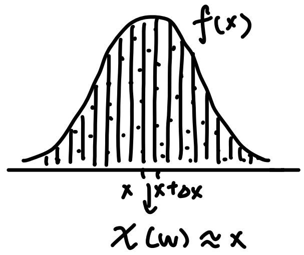
$$E(X) = \begin{cases} \text{discrete} & \sum_{x} x \cdot p(x) \\ \text{continuous} & \int_{-\infty}^{\infty} x f(x)\,dx \end{cases}$$

$$E(h(x)) = \begin{cases} \text{discrete} & \sum_{x} h(x) \cdot p(x) \\ \text{continuous} & \int_{-\infty}^{\infty} h(x) f(x)\,dx \end{cases}$$

$$Var(X) = E((X - E(X))^2)$$

same

Intepretation of $E(X)$



repeat n times $(n \to \infty)$

$$\text{average of } X(w) = \frac{1}{n} \sum_{x \text{ bins}} x \cdot \#\text{ of points} \in (x, x+\Delta x)$$

$$= \sum_{x \text{ bins}} x \cdot \boxed{\frac{\#\text{ of points} \in (x, x+\Delta x)}{n}}$$

$$= p(X \in (x, x+\Delta x))$$

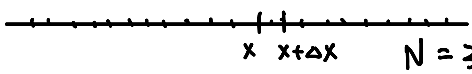
$$E(X) = \begin{cases} \text{discrete} & \sum_x x \cdot p(X = x) \\ \text{continuous} & \sum_x x \cdot \boxed{p(X \in (x, x+\Delta x))} \end{cases}$$

$$p(X \in (x, x+\Delta x)) = f(x)\Delta x$$

$$\Downarrow \Delta x \to 0$$

$$\int x \cdot f(x)\, dx$$

The same logic applies to large population



$N = 300$ million points

$$\text{average} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$= \frac{1}{N} \sum_{x \text{ bins}} x \cdot \#\text{ of points in } (x, x+\Delta x)$$

$E(X)$ = population average or long run average