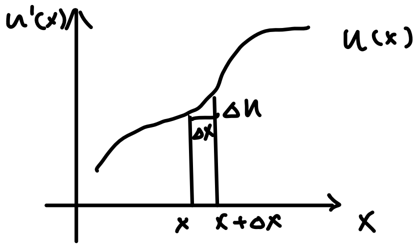
# Calculus (review)




$\Delta x$ is infinitesimal $\Delta x \to 0$

$$u'(x) = \frac{\Delta u}{\Delta x}$$ slope

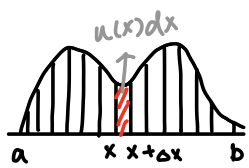
$$= \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$= \frac{du(x)}{dx}$$

$$= \frac{d}{dx} u(x)$$

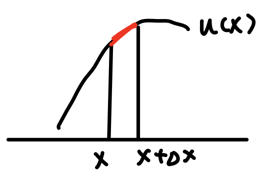
$$du = du(x) = u'(x)\,dx$$

notation: $\lim\limits_{\Delta x \to 0}$

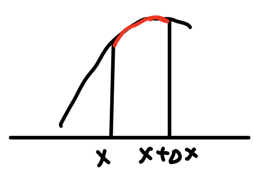
$\int_a^b u(x)\,dx =$  $= \sum_a^b u(x)\Delta x$


## Taylor Expasion :

$$u(x+\Delta x) = \underbrace{u(x) + u'(x)\Delta x}_{\text{linear}} + \frac{1}{2}u''(x)\Delta x^2 + \cdots$$




$1^{st}$ order




$2^{nd}$ order

$u(x) = e^x$ $\qquad e^{x+\Delta x} = e^x + e^x\Delta x + \frac{1}{2}e^x\Delta x^2 + \cdots$

$x = 0 \Rightarrow e^{\Delta x} = 1 + \Delta x + \frac{1}{2}\Delta x^2 + \cdots$

## Small o notation :

$$e^{\Delta x} = 1 + \Delta x + o(\Delta x) \qquad \frac{o(\Delta x)}{\Delta x} \xrightarrow[0]{\Delta x} 0$$

$$\boxed{1 + \Delta x = e^{\Delta x}} + o(\Delta x)$$

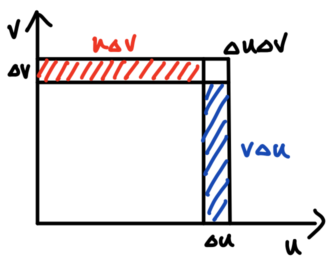
# Integral by parts

$$\frac{d}{dx}(u(x)v(x)) = u'(x)v(x) + u(x)v'(x)$$

$$d(u(x)v(x)) = (u'(x)v(x) + u(x)v'(x))\,dx$$

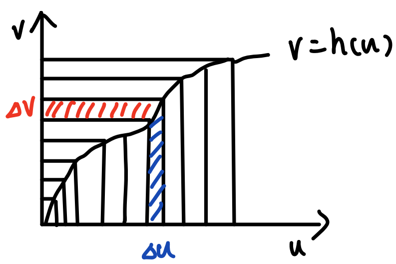
$$= (du(x))v(x) + u(x)(dv(x))$$

$$\xrightarrow{\text{simplify}} \quad duv = udv + vdu$$




$$\int [u'(x)v(x) + u(x)v'(x)]\,d(x) = u(x)v(x)$$

$$\int udv + vdu = uv$$




$$\int udv + vdu = uv$$

$\downarrow$


# Continuous R.V.

Intuitions:

(1) Large population $\Omega$

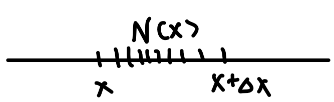
$w \sim \text{Uniform}(\Omega)$ $\quad X(w)$ e.g. height of $w$


N points ($N \approx \infty$)

$X \sim f(x)$

$\downarrow$

population

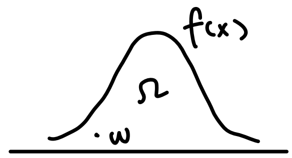


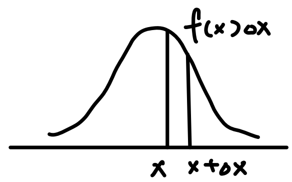

\# of points in $(x, x+\Delta x) = N(x)$

$$f(x) = \frac{\boxed{N(x)/N}}{\Delta x}$$ — population proportion = prob

under $w \sim \text{Uniform}(\Omega)$ $\quad f(x) = \dfrac{P(X \in (x, x+\Delta x))}{\Delta x}$

(2)

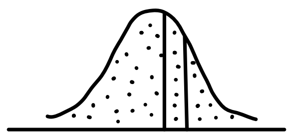



$w \sim \text{Unif}(\Omega)$

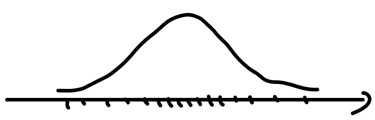
$X(w) \sim f(x)$ $\qquad\qquad$ $P(X \in (x, x+\Delta x)) = f(x)\Delta x$

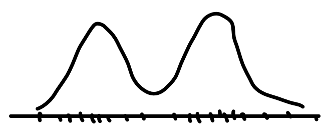
Repeat n times :



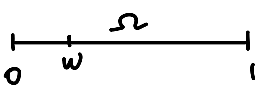
(3) Repeat n times (e.g. 1 million people doing the same experiment)

$$f(x)\ \frac{\text{how often } X \in (x, x+\Delta x)}{\Delta x} = \frac{\text{sample proportion} \in (x, x+\Delta x)}{\Delta x}$$





$\boxed{\text{Uniform } [0, 1]}$




computer simulation

Seed $x_0$

Iterate $x_{t+1} = \overset{7^5}{a} x_t + \overset{0}{b} \quad \text{mod } \overset{2^{31}-1}{M}$

carefully chosen integers

$u_t = \dfrac{x_t}{M} \sim \text{Unif}[0, 1]$ indep

(pseudo random)

divide & remainder

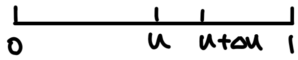
ex $7 = \boxed{2} \text{ mod } 5$

$\boxed{\text{Linear Congruential Method}}$

$x_0 = 1$

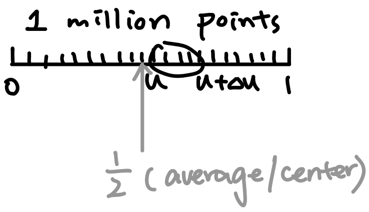
$x_1 = 7x_t + 0 \text{ mod } 5$

$1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow 1$




$$f(u) = \frac{P(U \in (u, u+\Delta u))}{\Delta u} = \frac{\Delta u}{\Delta u} = 1$$

$$\Rightarrow f(u) = \begin{cases} 1 & u \in [0,1] \\ 0 & u \notin [0,1] \end{cases}$$




$1 \cdot \Delta u$ million in $(u, u+\Delta u)$

$$E(u) = \int_0^1 u\, f(x)\, dx$$

$$E(X) = \int x f(x)\, dx = \sum_x x \cdot P(X \in (x, x+\Delta x))$$

$$= \int_0^1 u\, du$$

$$= \frac{u^2}{2}\Big|_0^1$$

$$= \frac{1}{2}$$

$$E(u^2) = \int_0^1 u^2 f(x)\, dx$$

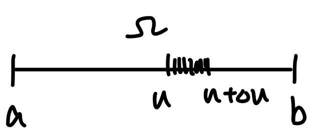
$$= \int_0^1 u^2\, du$$

$$= \frac{u^3}{3}\Big|_0^1$$

$$= \frac{1}{3}$$

$$Var(u) = E(u^2) - E(u)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

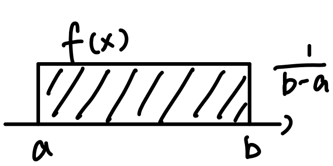
$$\boxed{u \sim Unif[a,b]}$$




$$f(u) = \frac{P(U \in (u, u+\Delta u))}{\Delta u} = \frac{\frac{|A|}{|\Omega|}}{\Delta u}$$

$$= \frac{\frac{\Delta u}{b-a}}{\Delta u}$$

$$= \begin{cases} \frac{1}{b-a} & u \in [a,b] \\ 0 & u \notin [a,b] \end{cases}$$

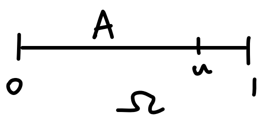



$$E(u) = \frac{a+b}{2}$$

$$Var(u) = \frac{(b-a)^2}{12}$$

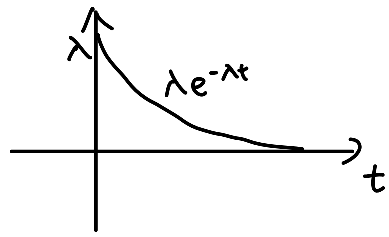
## Back to $U \sim \text{Unif}[0,1]$

$$F(u) = P(U \le u) = \frac{|A|}{|\Omega|} = u, \quad u \in [0,1]$$



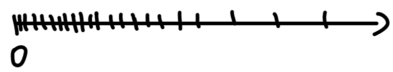

## $\boxed{T \sim \text{Exponential}(\lambda)}$

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

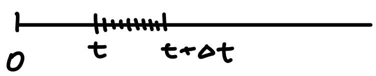



waiting time, e.g. time until a particle decays

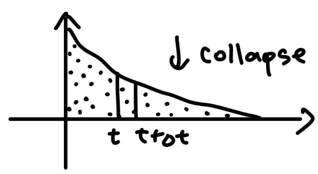
1 million particles $\Rightarrow$



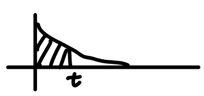

decay time




$\Rightarrow$



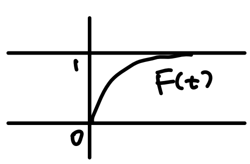

\# of particles decaying in $(t, t+\Delta t)$ $= P(T \in (t, t+\Delta t)) = f(t)\Delta t$
(or proportion)

$F(t) = P(T \le t) =$  $= \int_0^t f(t)\,dt = \int_0^t \lambda e^{-\lambda t}\,dt$

$$= -e^{-\lambda t}\Big|_0^t$$

$$= -e^{-\lambda t} - (-1)$$

$$= 1 - e^{-\lambda t}$$




Interpretation: how many particles decayed at $t$?

survived $= 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$

$\Downarrow$

$$\overline{F}(t) = P(T > t) = e^{-\lambda t}$$

$$E(T) = \int_0^\infty t f(t)\, d(t)$$

$$= \int_0^\infty \underbrace{t}_{u(t)} \cdot \underbrace{\boxed{\lambda e^{-\lambda t}}}_{v'(t)} dt$$

$$= -\int_0^\infty t \,\boxed{d\, e^{-\lambda t}} \rightarrow dv(t) = v'(t)dt = -\lambda e^{-\lambda t} dt$$

$$= -\left[ \underbrace{t \cdot e^{-\lambda t} \Big|_0^\infty}_{=0} - \int_0^\infty e^{-\lambda t} dt \right] \quad \left( \int u\,dv = uv - \int v\,du \right)$$

$$= \int_0^\infty e^{-\lambda t} dt$$

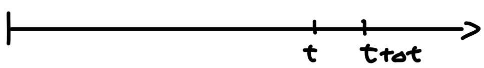
$$= -\frac{e^{-\lambda t}}{\lambda} \Big|_0^\infty$$

$$= 0 - \left(-\frac{1}{\lambda}\right)$$

$$= \frac{1}{\lambda}$$

$T \sim \text{Exponential}(\lambda)$

Continuous time → discrete




each particle flips a coin in $(t, t+\Delta t)$

unit : $\frac{1}{\text{unit of time}}$

$P(\text{head}) = \lambda \Delta t$

↓ decay

1 million particles

$P(\text{head})$ = how many decay in $(t, t+\Delta t)$ (proportion)

e.g. 1 million

$$P(T \in (t, t+\Delta t)) = P(\; \overset{T\ \ T\ \ \cdots\ \ T\ \ H}{\longrightarrow} \text{ at } t,\ t+\Delta t\;)$$

$$= (1-\lambda\Delta t)^{\frac{t}{\Delta t}} \lambda \Delta t$$

$$= (e^{-\lambda \Delta t})^{\frac{t}{\Delta t}} \lambda \Delta t = \lambda e^{-\lambda t} \Delta t$$