Language :

e.g. 1 million people flip fair coins independently

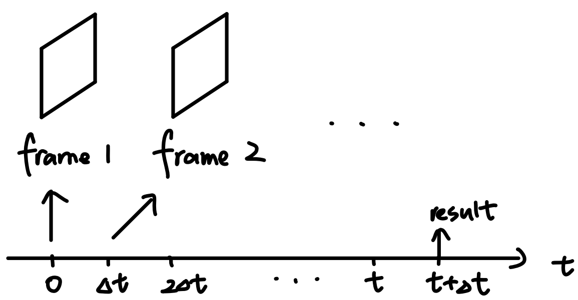
$\cdot 5$ million get Head

(not exactly $\cdot 5$ million, but the proportion $\to .5$)

Repeat $n = 1$ million times, $X = \#$ of people who get Head

$\frac{X}{n} \to .5$ in probability (frequency / sample proportion)

## Continuous Process $\longrightarrow$ Make a movie




Continuous Process

discrete time

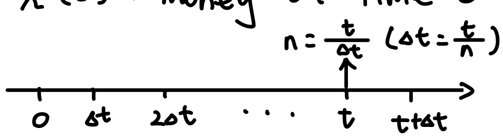
study what happens in each period

$(t, t+\Delta t)$

$\boxed{e^{\Delta X} = 1 + \Delta X} + \frac{1}{2}\Delta X^2 + \cdots$

Bank Account :

$X(0)$ : money at time 0




interest rate $= r$

$$X(t+\Delta t) = X(t)\left[1 + r\cdot\Delta t\right]$$

$$\frac{X(t+\Delta t) - X(t)}{\Delta t} = r \cdot X(t)$$

$$\frac{dX(t)}{dt} = r \cdot X(t) \quad \text{differential eqn}$$

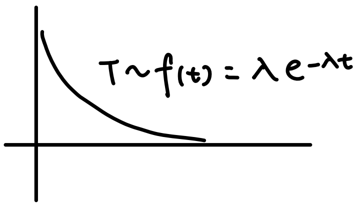
$$X(t) = X(0)\,\underline{(1 + r\cdot\Delta t)}^{\frac{t}{\Delta t}} \qquad e^{\Delta X} = 1 + \Delta X$$

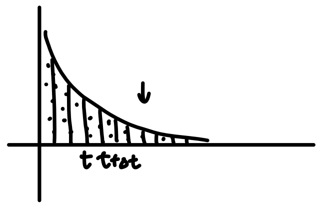
$$= X(0)\left(e^{r\cdot\Delta t}\right)^{\frac{t}{\Delta t}}$$

$$= X(0)\cdot e^{rt}$$

## Exponential ($\lambda$)

rate

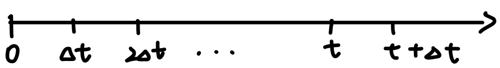



1 million particles

Sample proportion of particles decay in $(t, t+\Delta t) = f(t)\Delta t$

## A Way to Interpret This :




Each surviving particle flips a coin at period $(t, t+\Delta t)$ indep

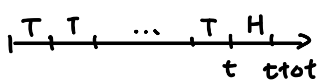
Head → decay

Tail → continue

$P(\text{head}) = \lambda \cdot \Delta t$

(proportion of surviving particles that decay in $(t, t+\Delta t)$ )

(e.g. half million)

$P(T \in (t, t+\Delta t)) = P($  $)$


$$= (1-\lambda\Delta t)^{\frac{t}{\Delta t}} \lambda\Delta t$$

$$= (e^{-\lambda\Delta t})^{\frac{t}{\Delta t}} \lambda\Delta t = \lambda e^{-\lambda t}\Delta t = f(t)\Delta t$$

$$P(T > t) = \int_t^{\infty} f(t)\, dt$$

$$= \int_t^{\infty} \lambda \cdot e^{-\lambda t}\, dt$$

$$= -e^{-\lambda t} \Big|_t^{\infty} = e^{-\lambda t}$$

**Interpretation**:

Among the 1 million particles, how many survive at time $t$?

$$P(T>t) = P(\ T\ T\ T\ \cdots\ T\ \text{up to } t\ )$$

$$= (1-\underbrace{\lambda\Delta t}_{\text{interest rate in the bank account example}})^{\frac{t}{\Delta t}} = e^{-\lambda t}$$

Suppose $\tilde{T} \sim \text{Geometric}(p=\lambda\Delta t)$ → only counts the number

$$T = \tilde{T}\cdot\Delta t \quad \Rightarrow \quad E(T) = E(\tilde{T})\cdot\Delta t$$

$$= \frac{1}{p}\cdot\Delta t$$

$$= \frac{1}{\lambda\Delta t}\cdot\Delta t$$

$$= \frac{1}{\lambda}$$

## Poisson Model.

$\Delta t = \frac{t}{n}\ (n\to\infty,\ \Delta t\to 0)$

$0 \quad \Delta t \quad 2\Delta t \quad \cdots \quad t$ → fixed (e.g. 10 yrs)

(A person keeps flipping until time $t$)

Head → earthquake

Tail → nothing

$P(\text{head}) = \lambda\Delta t$

$X = \#$ of heads in $(t, t+\Delta t)$

$X \sim \text{Binomial}(n=\frac{t}{\Delta t},\ p=\lambda\Delta t) \Rightarrow E(X) = np = \lambda t$

$\big\downarrow \Delta t\to 0$

$\text{Poisson}(\lambda t)$

$$\Rightarrow \lambda = \frac{E(X)}{t}$$

$E(X)$ → expected # of earthquake

$t$ → total time

So, $\lambda$ is the frequency of something happening

$\Delta t = \frac{t}{n}\ (n\to\infty,\ \Delta t\to 0) \Rightarrow n\Delta t = t$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-(k-1))}{k!} \boxed{p^k (1-p)^{n-k}} = (\lambda\Delta t)^k \overset{e^{-\lambda\Delta t}}{\overset{\|}{(-\lambda\Delta t)}}{}^{n-k}$$

$$= \frac{\overset{t}{(n\Delta t)}(\overset{t}{n\Delta t}-\overset{0}{\Delta t})\cdots(n\Delta t-(k-1)\Delta t)}{k!} \lambda^k \left(e^{-\lambda \overset{t}{\Delta t}}\right)^{\overset{0}{n-k}} \quad \Delta t \to 0$$

$$= \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

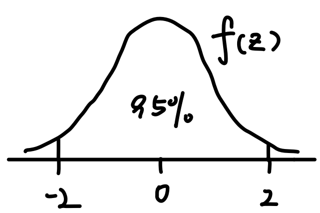
## Normal (Guassian)

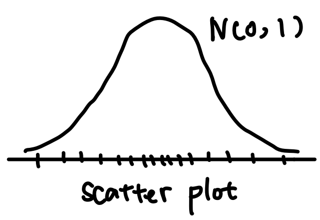
Standard Normal $\quad Z \sim N(\underset{\substack{\mu \\ E(\,)}}{0}, \underset{\substack{\sigma^2 \\ Var(\,)}}{1})$

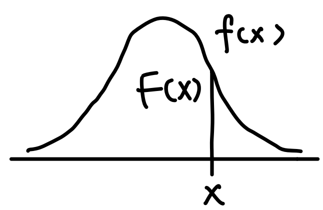
$\mu = 0,\ \sigma^2 = 1$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$P(Z \in [-2, 2]) = 95\%$$

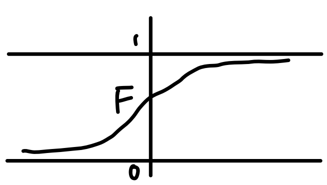



$F(-2) = 2.5\%$

$F(2) = 97.5\%$




$$\int f(z)\,dz = 1$$

$$E(Z) = \int_{-\infty}^{\infty} z f(z)\,dz = \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big|_{-\infty}^{\infty} = 0$$

$$E(Z^2) = \int_{-\infty}^{\infty} z^2 f(z)\,dz = \int_{-\infty}^{\infty} \underbrace{(-z)}_{u}\ \boxed{d\,\underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}_{dv}} = uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\,du$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= 1$$

$$Var(Z) = E(Z^2) - E(Z)^2 = 1$$

## General Random Variable

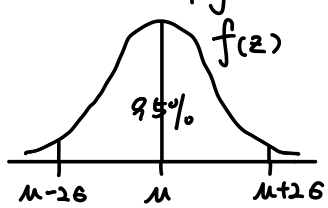
$X = \mu + \sigma Z \Rightarrow \underbrace{Z}_{\text{unit-less}} = \frac{X-\mu}{\sigma}$ (standardization) $E(Z)=0,\ Var(Z)=1$

$$E(X) = \mu + \sigma E(Z) = \mu$$

$$Var(X) = \sigma^2 Var(Z) = \sigma^2$$

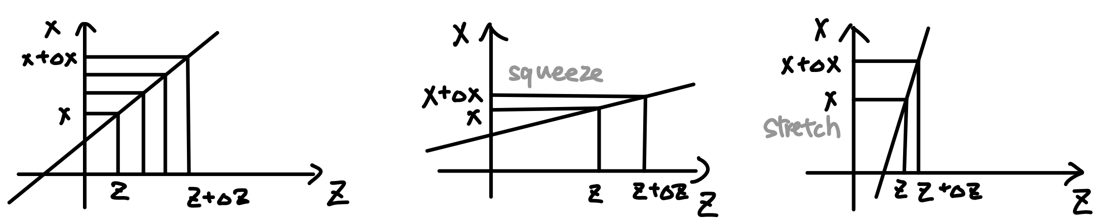
$X \sim N(\mu, \sigma^2)$ general normal / guassian distribution

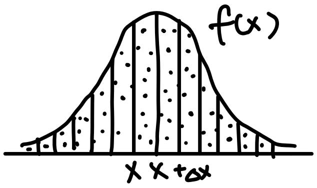
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$




$$P(\underbrace{\boxed{X \in (\mu-2\sigma,\ \mu+2\sigma)}) = P(\boxed{Z = \frac{X-\mu}{\sigma} \in (-2, 2)}}_{\text{equivalent}}) = 95\%$$

**Change of Variable** : $x = \mu + \sigma z \iff z = \frac{x-\mu}{\sigma}$







$$P(Z \in (z, z+\Delta z)) = P(X \in (x, x+\Delta x))$$

$$\boxed{f_Z(z)\Delta z = f_X(x)\Delta x}$$

$$f_X(x) = f_Z(z)\boxed{\frac{\Delta z}{\Delta x}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \cdot \frac{1}{\sigma} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2}}$$

capital lower slope