STATS 100A: RANDOM VARIABLES

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Random variables

100A

Ying Nian Wu

Discrete

Continuous

Connection to events:

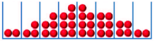
Randomly sample a person ω from a population Ω.

$X(\omega)$: gender of ω, $\Omega \to \{0, 1\}$.
$Y(\omega)$: height of ω, $\Omega \to \mathbb{R}^+$.
$A = \{\omega : X(\omega) = 1\}$. $P(A) = P(X = 1)$. Discrete.
$B = \{\omega : Y(\omega) > 6\}$. $P(B) = P(Y > 6)$. Continuous.
We shall study random variables more systematically.
$\omega \in \Omega$ equally likely, but $X(\omega)$ and $Y(\omega)$ are not necessarily equally likely.

Discrete random variable

Randomly sample a person ω from a population Ω of N people.





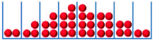
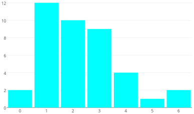
$X(\omega)$: eye color of person ω, $\Omega \to \{1(blue), 2(brown), 3, 4\}$.
$N(x)$ = number of people with eye color x.
Probability mass function, probability distribution, law:

$$X \sim p(x) = P(X = x) = \frac{N(x)}{N}.$$

x	1	2	3	4
$p(x)$	$p(1)$	$p(2)$	$p(3)$	$p(4)$
number	$N(1)$	$N(2)$	$N(3)$	$N(4)$

Discrete random variable

Randomly sample a person ω from a population Ω of N people.

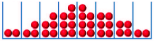
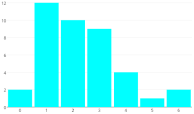




$X(\omega)$: number of siblings of person ω, $\Omega \to \{0, 1, 2, ...\}$.
$N(x)$ = number of people with x siblings.
Probability mass function, probability distribution, law:

$$X \sim p(x) = P(X = x) = \frac{N(x)}{N}.$$

x	0	1	2	3	...
$p(x)$	$p(0)$	$p(1)$	$p(2)$	$p(3)$	...
number	$N(0)$	$N(1)$	$N(2)$	$N(3)$	...

Population average

$N(x)$ = number of people with x siblings.

$$X \sim p(x) = P(X = x) = \frac{N(x)}{N}.$$

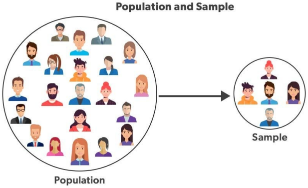
x	0	1	2	3	...
$p(x)$	$p(0)$	$p(1)$	$p(2)$	$p(3)$	...
number	$N(0)$	$N(1)$	$N(2)$	$N(3)$	...

Population average:

$$\mathbb{E}(X) = \frac{1}{N}\sum_{\omega\in\Omega} X(\omega) = \frac{1}{N}\sum_{x} xN(x) = \sum_{x} x\frac{N(x)}{N} = \sum_{x} xp(x).$$

Long run average: N^n reasoning

Randomly sample a person ω from a population Ω of N people. Each person ω carries a number $X(\omega)$.

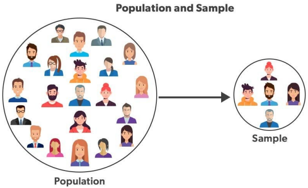


Population average:

$$\mu = \mathbb{E}(X) = \sum_x x p(x).$$

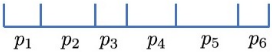
Repeat random sampling n times independently $\rightarrow$
N^n equally likely sequences: Ω_n.
$\bar{X}(\text{sequence})$ = average of the sequence.
$\bar{X} \rightarrow \mu$ in probability as $n \rightarrow \infty$.

Law of large number: N^n reasoning



Repeat random sampling n times independently $\rightarrow$
N^n equally likely sequences: Ω_n.
$\bar{X}(\text{sequence})$ = average of the sequence.
$A = \{\text{sequence} : |\bar{X}(\text{sequence}) - \mu| \leq .01\}$: representative sequences.
$P(A) = \frac{|A|}{N^n} \rightarrow 1$ as $n \rightarrow \infty$.

Die rolling



Randomly throw a point into [0, 1], which bin (1, 2, ..., 6) it falls into?

$\omega \in \Omega = [0, 1]$, population of points (tiny balls), equally likely.

$X(\omega)$ is the bin that ω belongs to, not necessarily equally likely.

Probability mass function, probability distribution, law:

$$X \sim p(x) = P(X = x) = \text{length of bin } x.$$

Population average (each point or tiny ball ω carries a number $X(\omega)$).

$$\mathbb{E}(X) = \sum_x x p(x).$$

Long run average

Independent repetitions:
$p(x)$: how often $X = x$ in the long run (e.g., throw 1 million points into [0, 1]).

x	1	2	3	4	5	6
p(x)	0.1	0.1	0.2	0.2	0.1	0.3

x	1	2	3	4	5	6
#	0.1m	0.1m	0.2m	0.2m	0.1m	0.3m
%	*10%*	*10%*	*20%*	*20%*	*10%*	*30%*

$$average = \frac{(1\times0.1m + 2\times0.1m + 3\times0.2m + 4\times0.2m + 5\times0.1m + 6\times0.3m)}{1m}$$

$$\mathbb{E}(X) = \sum_x xp(x).$$

Expectation of function

Function of random variable

x	1	2	3	4	5	6
p(x)	0.1	0.1	0.2	0.2	0.1	0.3

x	1	2	3	4	5	6
#	0.1m	0.1m	0.2m	0.2m	0.1m	0.3m
%	*10%*	*10%*	*20%*	*20%*	*10%*	*30%*

$$average = \frac{(1\times 0.1m + 2\times 0.1m + 3\times 0.2m + 4\times 0.2m + 5\times 0.1m + 6\times 0.3m)}{1m}$$

x	1	2	3	4	5	6	x
payoff	-$30	-$20	$0	$20	$30	$100	h(x)
	h(1)	*h(2)*	*h(3)*	*h(4)*	*h(5)*	*h(6)*	

longrun average = (-$30)×0.1 + (-$20) ×0.1 + ($0)×0.2 + ($20)×0.2 + ($30)×0.1 + ($100)×0.3

$$\mathbb{E}(h(X)) = \sum_x h(x)p(x).$$

Utility

Utility, reward, value

$$\mathbb{E}(h(X)) = \sum_x h(x)p(x).$$

Offer 1	
x	\$100
p(x)	1

$E(X) = (\$100)\times 1 = \$100$

Offer 2		
x	\$0	\$200
p(x)	1/2	1/2

$E(X) = (\$0)\times\frac{1}{2} + (\$200)\times\frac{1}{2} = \$100$

x: face value	\$0	\$100	\$200
h(x): perceived value	\$0	\$100	\$150

Offer 1: $\mathbb{E}[h(X)] = \$100 \times 1 = \$100.$

Offer 2: $\mathbb{E}[h(X)] = \$0 \times \frac{1}{2} + \$150 \times \frac{1}{2} = \$75.$

Variance

$$\mathbb{E}(X) = \sum_x x p(x) = \mu (= \$0 \times 1/2 + \$200 \times 1/2 = \$100)$$

$$\begin{aligned}\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = \sum_x (x-\mu)^2 p(x) = \sigma^2 \\ = (\$0 - \$100)^2 \times 1/2 + (\$200 - \$100)^2 \times 1/2 \\ = \$^2 10,000.\end{aligned}$$

Long run average of squared deviation from the mean.

$$SD(X) = \sqrt{\mathrm{Var}(X)} = \sigma (= \$100).$$

Extent of variation from the mean.

Variance

x	1	2	3	4	5	6	x
payoff	-$30	-$20	$0	$20	$30	$100	h(x)
	h(1)	*h(2)*	*h(3)*	*h(4)*	*h(5)*	*h(6)*	

longrun average = (-$30)×0.1 + (-$20) ×0.1 + ($0)×0.2 + ($20)×0.2 + ($30)×0.1 + ($100)×0.3

$$\mathbb{E}(h(X)) = \sum_x h(x)p(x).$$

$$\text{Var}[h(X)] = \mathbb{E}[(h(X) - \mathbb{E}(h(X)))^2].$$

Data

$$\mathbb{E}(X) = \sum_x x p(x) = \mu.$$

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = \sum_x (x-\mu)^2 p(x) = \sigma^2.$$

Long run average of squared deviation from the mean.
Sampling $p(x) \to x_1, ..., x_i, ..., x_n$
(e.g., rolling a die → 2, 1, 6, 5, 3, 2, 5, 4, 3, ...)

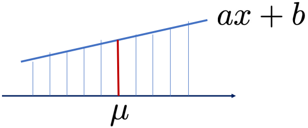
$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \to \mathbb{E}(X) = \mu.$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2 \to \mathrm{Var}(X) = \sigma^2$$

Linear transformation

$$\mathbb{E}(h(X)) = \sum_x h(x)p(x).$$

$$Y = aX + b.$$



$$\mathbb{E}(Y) = \mathbb{E}(aX + b) = \sum_x (ax + b)p(x)$$
$$= \sum_x axp(x) + \sum_x bp(x)$$
$$= a\sum_x xp(x) + b\sum_x p(x) = a\mathbb{E}(X) + b.$$

Data

Sampling $p(x) \to x_1, ..., x_i, ..., x_n$
(e.g., rolling a die $\to$ 2, 1, 6, 5, 3, 2, 5, 4, 3, ...)

$$y_i = ax_i + b.$$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \to \mathbb{E}(X) = \mu.$$

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{1}{n} \sum_{i=1}^{n} (ax_i + b) = a \frac{1}{n} \sum_{i=1}^{n} x_i + b = a\bar{x} + b.$$

Linear transformation

$$\mathrm{Var}(h(X)) = \mathbb{E}[(h(X) - \mathbb{E}(h(X)))^2].$$

$$Y = aX + b.$$

$$\mathbb{E}(Y) = a\mathbb{E}(X) + b.$$

$$\mathrm{Var}(Y) = \mathbb{E}[(Y - \mathbb{E}(Y))^2].$$

$$\begin{aligned}
\mathrm{Var}(aX+b) &= \mathbb{E}[((aX+b) - \mathbb{E}(aX+b))^2] \\
&= \mathbb{E}[(aX + b - (a\mathbb{E}(X) + b))^2] \\
&= \mathbb{E}[(a(X - \mathbb{E}(X)))^2] \\
&= a^2\mathbb{E}[(X - \mathbb{E}(X))^2] = a^2\mathrm{Var}(X).
\end{aligned}$$

Data

Sampling $p(x) \to x_1, ..., x_i, ..., x_n$
(e.g., rolling a die $\to$ 2, 1, 6, 5, 3, 2, 5, 4, 3, ...)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \to \mathbb{E}(X) = \mu.$$

$$y_i = a x_i + b.$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n}(a x_i + b) = a\frac{1}{n}\sum_{i=1}^{n} x_i + b = a\bar{x} + b.$$

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n}(a x_i + b - (a\bar{x} + b))^2 = \frac{1}{n}\sum_{i=1}^{n} a^2 (x_i - \bar{x})^2.$$

Short-cut for variance

$$\mu = \mathbb{E}(X).$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[(X-\mu)^2] \\
&= \mathbb{E}[X^2 - 2\mu X + \mu^2] \\
&= \mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mu^2 \\
&= \mathbb{E}(X^2) - \mu^2 = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2.
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[h(X) + g(X)] &= \sum_x [h(x) + g(x)]p(x) \\
&= \sum_x h(x)p(x) + \sum_x g(x)p(x) \\
&= \mathbb{E}[h(X)] + \mathbb{E}[g(X)].
\end{aligned}$$

Bernoulli

Flip a coin (probability of head is p)
$Z \sim \text{Bernoulli}(p)$
$Z \in \{0, 1\}$, $P(Z = 1) = p$ and $P(Z = 0) = 1 - p$.

$$\mathbb{E}(Z) = 0 \times (1 - p) + 1 \times p = p.$$

$$\begin{aligned} \text{Var}(Z) &= (0 - p)^2 \times (1 - p) + (1 - p)^2 \times p \\ &= p(1 - p)[p + (1 - p)] = p(1 - p). \end{aligned}$$

$$\mathbb{E}(Z^2) = p.$$

$$\text{Var}(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 = p - p^2 = p(1 - p).$$

Binomial

Flip a coin (probability of head is p) n times independently.
X = number of heads.
$X \sim \text{Binomial}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

$\binom{n}{k}$ is the number of sequences with exactly k heads.
$p^k(1-p)^{n-k}$ is the probability of each sequence with k heads.
e.g., $n = 3$,
$P(X = 2) = P(HHT) + P(HTH) + P(THH) = 3p^2(1-p)$.
$p = 1/2$, we have $P(X = k) = \binom{n}{k}/2^n$.

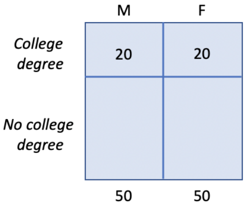
Recall independence

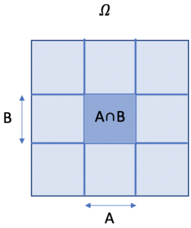
Definition 1:

$$P(A|B) = P(A).$$

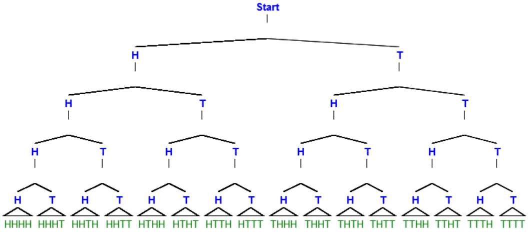
Definition 2:

$$P(A \cap B) = P(A)P(B).$$





Binomial formula



$$(H+T)^n = \sum_{k=0}^{n} \binom{n}{k} H^k T^{n-k}.$$

$n = 1$, $H + T$.
$n = 2$, $(H+T)(H+T) = HH + HT + TH + TT$.
$n = 3$, above $\times(H+T) =$
$HHH + HHT + HTH + HTT + THH + THT + TTH + TTT$.

Binomial and Bernoulli

$$X = Z_1 + Z_2 + ... + Z_n,$$

where $Z_i \sim \text{Bernoulli}(p)$ independently.

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(Z_i) = np.$$

Due to independence of Z_i, $i = 1, ..., n$,

$$\text{Var}(X) = \sum_{i=1}^{n} \text{Var}(Z_i) = np(1-p).$$

Frequency

X/n is the frequency of heads.

$$\mathbb{E}(X/n) = \mathbb{E}(X)/n = p.$$

$$\text{Var}(X/n) = \text{Var}(X)/n^2 = p(1-p)/n.$$

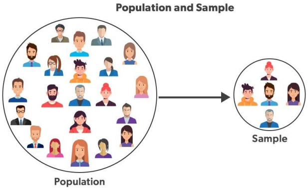
$\text{Var}(X/n) \to 0$ as $n \to \infty$.

$$X/n \to p, \text{ in probability}$$

Law of large number
Probability = long run frequency

Recall survey sampling

Population of N people, M males.
Repeat random sampling n times independently



$\rightarrow N^n$ equally likely sequences.
For a sequence ω, $X(\omega)$ = number of males in ω.
$A_m = \{\omega : X(\omega) = m\}$: sequences with m males.
$|A_m| = \binom{n}{m} M^m (N-M)^{n-m}$. n blanks. Choose m blanks for males, the rest $n-m$ blanks for females. Each male blank has M choices. Each female blank has $N-M$ choices.

Survey sampling

Population of N people. M males.
Sample a person, $p = M/N =$ Prob(male).

$$
\begin{aligned}
P(A_m) = P(X = m) &= \frac{|A_m|}{|\Omega_n|} \\
&= \frac{\binom{n}{m} M^m (N-M)^{n-m}}{N^n} \\
&= \binom{n}{m} p^m (1-p)^{n-m}.
\end{aligned}
$$

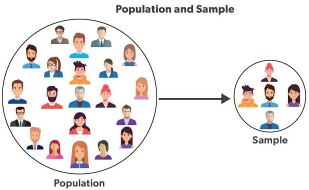
Most sequences are representative, $X/n \approx M/N = p$.

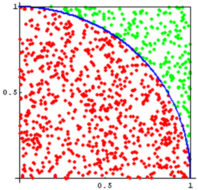
Binomial distribution

Flip a coin n times independently, $p =$ probability of head.

$$p(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$
$$x = 0, 1, ..., n.$$

$p(x)$: probability mass function, probability distribution.

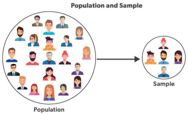




Survey sampling, poll before election, $p = M/N$.
Monte Carlo, $p = \pi/4$.

Law of large number: N^n reasoning

Survey sampling, poll before election, $p = M/N$.



Among all N^n sequences in the hyper-population of sequences Ω_n, let

$$A = \left\{\omega : \left|\frac{X(\omega)}{n} - p\right| \leq .01\right\}.$$

consist of representative sequences.

$$P(A) = \frac{|A|}{|\Omega_n|} = \sum_{x \in [n(p-.01), n(p+.01)]} p(x) \to 1,$$

$X/n \to p$ in probability.
$\mathbb{E}(X/n) = p$: average of $X(\omega)/n$ in Ω_n.
$\mathrm{Var}(X/n) = p(1-p)/n \to 0$: variance of $X(\omega)/n$ in Ω_n.

Binomial expectation

$$
\begin{aligned}
\mathbb{E}(X) &= \sum_{x=0}^{n} x p(x) = \sum_{k=0}^{n} k P(X = k) \\
&= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} np \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\
&= \sum_{k'=0}^{n'} np \binom{n'}{k'} p^{k'} (1-p)^{n'-k'} = np.
\end{aligned}
$$

$$k' = k - 1;\ n' = n - 1.$$

Binomial variance

$$\mathbb{E}(X(X-1)) = \sum_{x=0}^{n} x(x-1)p(x) = \sum_{k=0}^{n} k(k-1)P(X=k)$$
$$= \sum_{k=0}^{n} k(k-1)\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$$
$$= \sum_{k=2}^{n} n(n-1)p^2\frac{(n-2)!}{(k-2)!(n-k)!}p^{k-2}(1-p)^{n-k}$$
$$= \sum_{k'=0}^{n'} n(n-1)p^2\binom{n'}{k'}p^{k'}(1-p)^{n'-k'}$$
$$= n(n-1)p^2.$$

$$k' = k-2;\ n' = n-2.$$

Binomial variance

$$\mathbb{E}(X) = np.$$

$$\mathbb{E}(X(X-1)) = \mathbb{E}(X^2) - \mathbb{E}(X) = n(n-1)p^2.$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= n(n-1)p^2 + np - (np)^2 \\
&= np - np^2 = np(1-p).
\end{aligned}$$

Geometric

$T \sim \text{Geometric}(p)$
T is the number of flips to get the first head, if we flip a coin independently and the probability of getting a head in each flip is p.

$$P(T = k) = (1-p)^{k-1}p.$$

e.g., $T = 1$, H
$T = 2$, TH.
$T = 3$, TTH.
$T = 4$, $TTTH$.
Waiting time.

Geometric expectation

$T \sim \text{Geometric}(p)$

$$
\begin{aligned}
\mathbb{E}(T) &= \sum_{k=1}^{\infty} kP(T=k) \\
&= \sum_{k=1}^{\infty} kq^{k-1}p = p\sum_{k=1}^{\infty} \frac{d}{dq} q^k \\
&= p\frac{d}{dq}\sum_{k=1}^{\infty} q^k = p\frac{d}{dq}\left(\frac{1}{1-q} - 1\right) \\
&= p\frac{1}{(1-q)^2} = \frac{1}{p}.
\end{aligned}
$$

Geometric series

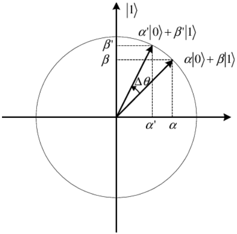
$$
\begin{aligned}
(1-a)(1+a+...+a^m) &= 1+a+...+a^m \\
&\quad -(a+a^2+...+a^m+a^{m+1}) \\
&= 1-a^{m+1}.
\end{aligned}
$$

$$1+a+...+a^m = \frac{1-a^{m+1}}{1-a}.$$

If $|a| < 1$,

$$a^{m+1} \to 0, \ as \ m \to \infty.$$

Quantum bit

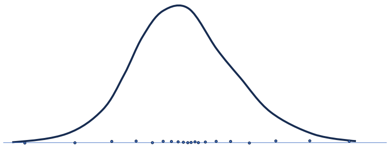


state vector $= \alpha|0\rangle + \beta|1\rangle$.
state vector rotates over time.
squared length $= |\alpha|^2 + |\beta|^2 = 1$ under rotation.
observer: $p(0) = |\alpha|^2$, $p(1) = |\beta|^2$.

Schrodinger cat: P(alive) $= (1/\sqrt{2})^2 = 1/2$.

Continuous random variable: density

Randomly sample a person ω from a population Ω of N people.
$X(\omega)$: height of person ω.
Density or distribution of the N points.

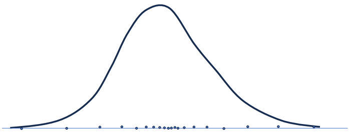


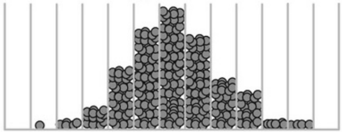
$N(x) =$ number of people in $(x, x + \Delta x)$ (6 ft, 6 ft 1 inch), precision $=$ 1 inch.
Probability density function, probability distribution, law:

$$X \sim f(x) = \frac{P(X \in (x, x + \Delta x))}{\Delta x} = \frac{N(x)/N}{\Delta x}.$$

Mathematical idealization: $N \approx \infty$.

Population scatterplot and histogram





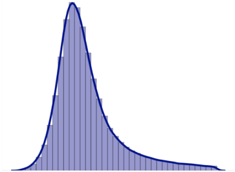
Discretize x-axis into equally spaced bins $(x, x + \Delta x)$, e.g., (6 ft, 6 ft 1 inch), precision = 1 inch.

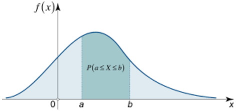
$$P(X \in (x, x + \Delta x)) = \frac{N(x)}{N} = f(x)\Delta x.$$

$f(x)$: height of bin $(x, x + \Delta x)$. $f(x)\Delta x$: area.

$$\sum_x \frac{N(x)}{N} = \sum_x f(x)\Delta x \to \int f(x)dx = 1.$$

Region under curve



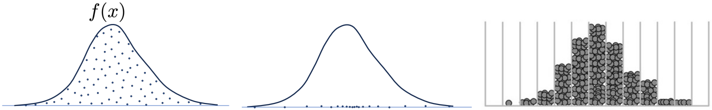


Randomly throw a point ω into the region Ω below curve $f(x)$.
Ω: population of points (tiny squares or balls).
Let $X = X(\omega)$ be the horizontal coordinate of point ω.

$$P(X \in (x, x + \Delta x)) = f(x)\Delta x.$$

$$P(X \in (a, b)) = \sum_{x \in (a,b)} f(x)\Delta x \to \int_a^b f(x)dx.$$

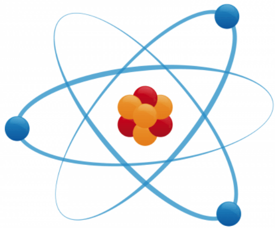
Independent repetitions, sample scatterplot and histogram

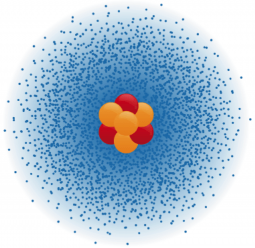


Repeat n times, collapse to x-axis, histogram.
N^n reasoning:
sample scatterplot (random) $\approx$ population scatterplot (fixed),
sample histogram (random) $\approx$ population histogram (fixed).

Point cloud

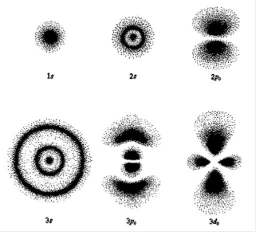
Electron orbits around nucleus: wrong conception
Electron cloud, probability density function, $f(x)$
Wave function $\psi(x)$, evolves over time.
Observer: $f(x) = |\psi(x)|^2$.

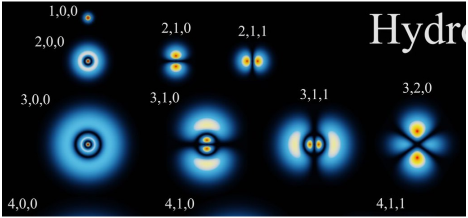




Population

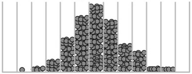
Electron cloud, heat map, prob density

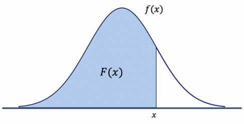


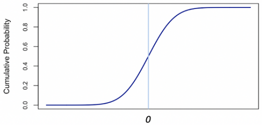


Population of N equally likely possibilities.
Mathematical idealization: $N \approx \infty$.
Prob density = prob mass in the cell / volume of cell.
Observer: $f(x) = |\psi(x)|^2$.

Cumulative density function



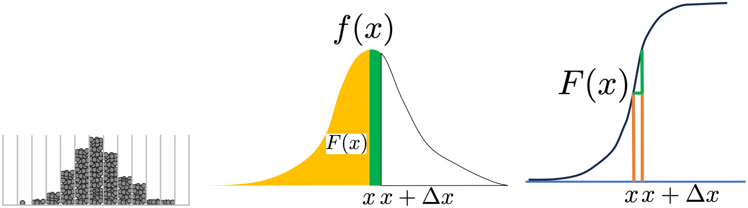




$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx.$$

SAT score $x \rightarrow$ percentile $F(x)$.
Percentage of people below x.

Area and slope



Area:

$$F(x+\Delta x) - F(x) = f(x)\Delta x.$$

Slope:

$$F'(x) = \frac{F(x+\Delta x) - F(x)}{\Delta x} = f(x).$$

Notation:

$$F'(x) = \frac{dF(x)}{dx} = \frac{d}{dx}F(x) = f(x).$$

$$dF(x) = F'(x)dx = f(x)dx$$

Expectation

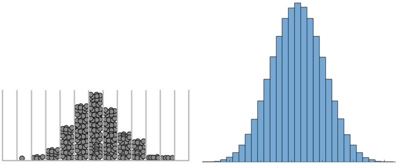
Recall discrete

$$P(X = x) = p(x).$$

$$\mathbb{E}(X) = \sum xP(X = x) = \sum xp(x).$$

Continuous

$$P(X \in (x, x + \Delta x)) = f(x)\Delta x.$$

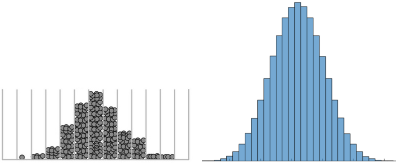


$$\mathbb{E}(X) = \sum xP(X \in (x, x+\Delta x)) = \sum xf(x)\Delta x \to \int xf(x)dx.$$

Population average, long run average, center.

Population average

Population Ω of N people. $X(\omega)$.
$N(x)$: number of people in $(x, x + \Delta x)$.



$$\mathbb{E}(X) = \frac{1}{N} \sum_{\omega} X(\omega) = \frac{1}{N} \sum_{x} xN(x)$$

$$= \sum_{x} x \frac{N(x)}{N} = \sum xP(X \in (x, x + \Delta x))$$

$$= \sum xf(x)\Delta x \to \int xf(x)dx.$$

Expectation of function

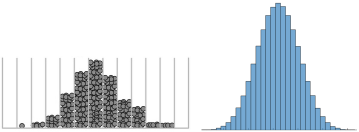
Recall discrete

$$P(X = x) = p(x).$$

$$\mathbb{E}[h(X)] = \sum h(x)P(X = x) = \sum h(x)p(x).$$

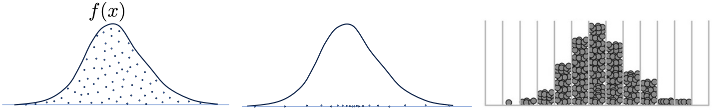
Continuous

$$P(X \in (x, x + \Delta x)) = f(x)\Delta x.$$



$$\begin{aligned}\mathbb{E}[h(X)] &= \sum h(x)P(X \in (x, x + \Delta x)) \\ &= \sum h(x)f(x)\Delta x \to \int h(x)f(x)dx.\end{aligned}$$

Data, long run average



$f(x) \to x_1, ..., x_i, ..., x_n$.
$n(x)$ = number of points in $(x, x + \Delta x)$.

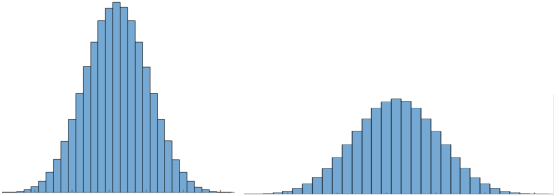
$$\begin{aligned}\bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{x} xn(x) = \sum_{x} x\frac{n(x)}{n} \\ &\to \sum xP(X \in (x, x + \Delta x)) \\ &= \sum xf(x)\Delta x \to \int xf(x)dx = \mathbb{E}(X).\end{aligned}$$

Same logic for $\mathbb{E}(h(X))$.

Variance

Continuous

$$P(X \in (x, x+\Delta x)) = f(x)\Delta x.$$



$$\mathbb{E}(X) = \int x f(x) dx = \mu.$$

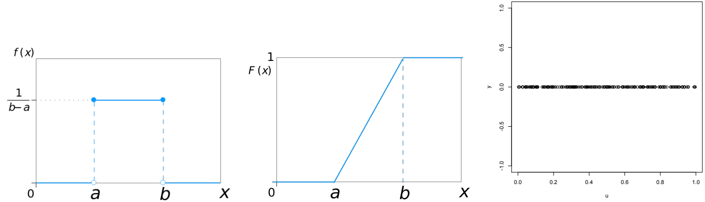
$$\mathrm{Var}(X) = \mathbb{E}[(X-\mu)^2] = \int (x-\mu)^2 f(x) dx.$$

$$\mathrm{Var}[h(X)] = \mathbb{E}[(h(X) - \mathbb{E}(h(X)))^2].$$

Fluctuation, volatility, spread.

Uniform

$U \sim \text{Uniform}[0,1]$, i.e., the density of U is $f(u) = 1$ for $u \in [0,1]$ (or $f(u) = 1/(b-a)$ if $u \in [a,b]$), $f(u) = 0$ otherwise.



$P(U \in (u, u+\Delta u)) = f(u)\Delta u = \Delta u.$

Imagine 1 million points distributed uniformly in [0, 1].

Number of points in $(u, u+\Delta u)$ is Δu million.

e.g., Number of points in $(.3, .31)$ is $.01$ million.

Uniform

$$F(u) = P(U \leq u) = \begin{cases} 0 & 0 < u \\ u & 0 \leq u \leq 1 \\ 1 & u > 1 \end{cases}$$

$F(u)$: proportion of points below u.

$$\mathbb{E}(U) = \int_0^1 u f(u) du = \frac{1}{2}.$$

$$\mathbb{E}(U^2) = \int_0^1 u^2 f(u) du = \frac{1}{3}.$$

$$\text{Var}(U) = \mathbb{E}(U^2) - (\mathbb{E}(U))^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

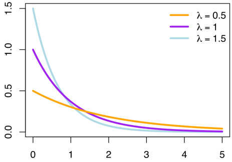
Pseudo-random number generator

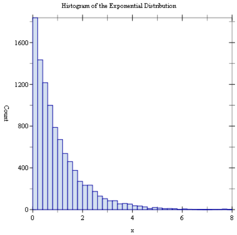
Linear congruential method
Start from an integer X_0, and iterate

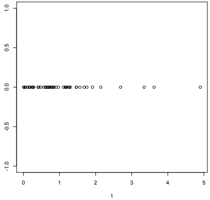
$$X_{t+1} = aX_t + b \bmod M.$$

Output $U_t = X_t/M$. e.g., $a = 7^5$, $b = 0$, and $M = 2^{31} - 1$.
mod: divide and take the remainder, e.g., $7 = 2 \bmod 5$.
e.g., $a = 7$, $b = 1$, $M = 5$, $X_0 = 1$, then
$X_1 = 1 \times 7 + 1 \bmod 5 = 3$.
$X_2 = 3 \times 7 + 1 \bmod 5 = 2$.

Exponential

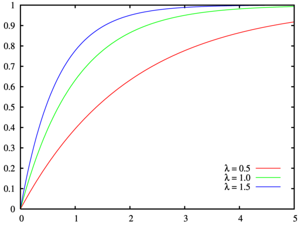


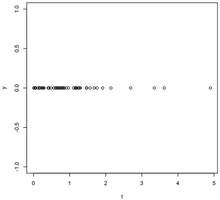




$T \sim \text{Exponential}(\lambda)$,
$f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$,
$f(t) = 0$ for $t < 0$.
$P(T \in (t, t + \Delta t)) = \lambda e^{-\lambda t} \Delta t$.
Imagine 1 million particles, mark the times when they decay.
1 million points on real line. Their distribution is exponential.
Number of points in $(t, t + \Delta t)$ is $\lambda e^{-\lambda t} \Delta t$ million.

Exponential





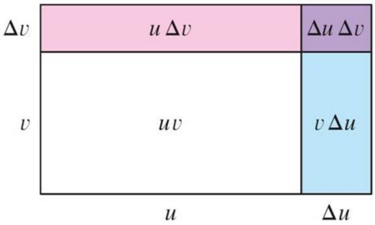
$$F(t) = \int_0^t f(t)dt = \int_0^t \lambda e^{-\lambda t} dt$$
$$= -e^{-\lambda t}|_0^t = 1 - e^{-\lambda t}.$$

$F(t)$: proportion of points below t
Half-life: $F(t_{\text{half}}) = P(T \le t_{\text{half}}) = 1/2$.
1 million particles, by half life, half million will have decayed.

Exponential expectation

$$
\begin{aligned}
\mathbb{E}(T) &= \int_0^\infty t\lambda e^{-\lambda t} dt \\
&= -\int_0^\infty t de^{-\lambda t} \\
&= -(te^{-\lambda t}|_0^\infty - \int_0^\infty e^{-\lambda t} dt) \\
&= -(0 - 0 + \frac{1}{\lambda} e^{-\lambda t}|_0^\infty) = \frac{1}{\lambda}.
\end{aligned}
$$

Integral by parts



$$\frac{d}{dx}u(x)v(x) = u'(x)v(x) + u(x)v'(x).$$

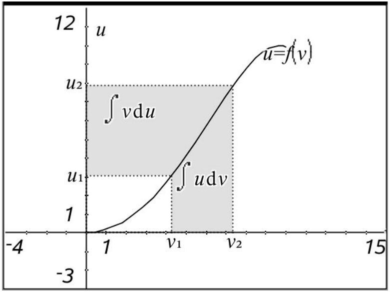
$$duv = udv + vdu.$$

$$\int [u'(x)v(x) + u(x)v'(x)]dx = u(x)v(x).$$

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.$$

$$\int udv = uv - \int vdu.$$

Integral by parts



$$\int u dv = uv - \int v du.$$

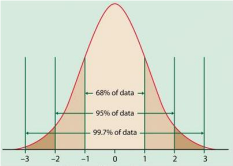
$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.$$

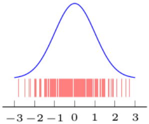
$$\frac{du(x)}{dx} = \frac{d}{dx}u(x) = u'(x);\ du(x) = u'(x)dx.$$

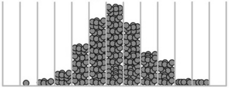
Normal or Gaussian

Let $Z \sim \mathrm{N}(0, 1)$, i.e., the density of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$







$$\int_{-2}^{2} f(z)dz = 95\%.$$

Normal expectation

100A

Ying Nian Wu

Discrete

Continuous

Let $Z \sim \mathrm{N}(0, 1)$, i.e., the density of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

$$\begin{aligned} \mathbb{E}(Z) &= \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} |_{-\infty}^{\infty} \\ &= 0. \end{aligned}$$

The density is symmetric around 0.

Normal variance

Let $Z \sim \mathrm{N}(0,1)$, i.e., the density of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

$$\begin{aligned}
\mathbb{E}(Z^2) &= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-z) de^{-\frac{z^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} (-ze^{-\frac{z^2}{2}} |_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} d(-z)) \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 1.
\end{aligned}$$

$$\mathrm{Var}(Z) = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2 = 1.$$

Variance

For $X \sim f(x)$, let $\mu = \mathbb{E}(X)$.

$$
\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}[(X-\mu)^2] \\
&= \mathbb{E}[X^2 - 2\mu X + \mu^2] \\
&= \mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mu^2 \\
&= \mathbb{E}(X^2) - (\mathbb{E}(X))^2.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[r(X) + s(X)] &= \int [r(x) + s(x)] f(x) dx \\
&= \int r(x) f(x) dx + \int s(x) f(x) dx \\
&= \mathbb{E}[r(X)] + \mathbb{E}[s(X)].
\end{aligned}
$$

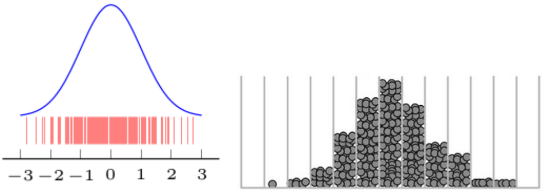
Linear transformation

For $X \sim f(x)$. Let $Y = aX + b$.

$$\begin{aligned}
\mathbb{E}(Y) = \mathbb{E}(aX+b) &= \int (ax+b)f(x)dx \\
&= a\int xf(x)dx + b\int f(x)dx \\
&= a\mathbb{E}(X) + b.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(Y) = \mathrm{Var}(aX+b) &= \mathbb{E}[((aX+b) - \mathbb{E}(aX+b))^2] \\
&= \mathbb{E}[(aX+b-(a\mathbb{E}(X)+b))^2] \\
&= \mathbb{E}[a^2(X-\mathbb{E}(X))^2] \\
&= a^2\mathbb{E}[(X-\mathbb{E}(X))^2] = a^2\mathrm{Var}(X).
\end{aligned}$$

Data



Sampling $f(x) \to x_1, ..., x_i, ..., x_n$
(e.g., random number generator → .22, .31, .92, .45, ...)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \to \mathbb{E}(X) = \mu.$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \to \mathrm{Var}(X) = \sigma^2$$

Data

Sampling $f(x) \to x_1, ..., x_i, ..., x_n$
(e.g., random number generator $\to$ .22, .31, .92, .45, ...)

$$y_i = ax_i + b.$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \to \mathbb{E}(X) = \mu.$$

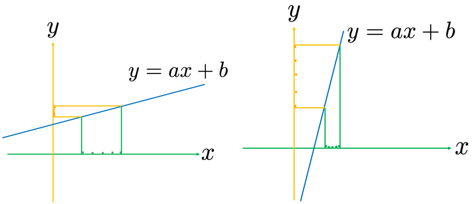
$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{n}\sum_{i=1}^{n}(ax_i + b) = a\frac{1}{n}\sum_{i=1}^{n} x_i + b = a\bar{x} + b.$$

$$\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 = \frac{1}{n}\sum_{i=1}^{n}(ax_i + b - (a\bar{x} + b))^2 = \frac{1}{n}\sum_{i=1}^{n} a^2(x_i - \bar{x})^2.$$

Change of density under linear transformation

Change of variable

$X \sim f(x)$, $Y = aX + b$ $(a > 0)$. $Y \sim g(y)$.



$$y = ax + b,\ x = (y - b)/a.$$

$$P(X \in (x, x + \Delta x)) = P(Y \in (y, y + \Delta y)).$$

$$f(x)\Delta x = g(y)\Delta y.$$

$$g(y) = f(x)\frac{\Delta x}{\Delta y} = f((y - b)/a))/a.$$

Space warping, stretching or squeezing.

Normal or Gaussian

Let $Z \sim \mathrm{N}(0,1)$, i.e., the density of Z is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}.$$

Let $X = \mu + \sigma Z$. $Z = (X - \mu)/\sigma$. Then

$$\mathbb{E}(X) = \mathbb{E}(\mu + \sigma Z) = \mu + \sigma \mathbb{E}(Z) = \mu.$$

$$\mathrm{Var}(X) = \mathrm{Var}(\mu + \sigma Z) = \sigma^2 \mathrm{Var}(Z) = \sigma^2.$$

$$f(z)\Delta z = g(x)\Delta x.$$

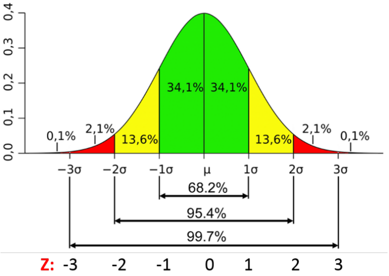
$$\begin{aligned} g(x) &= f(z)\frac{\Delta z}{\Delta x} \\ &= f((x-\mu)/\sigma)/\sigma \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]. \end{aligned}$$

Normal or Gaussian

Let $Z \sim \mathrm{N}(0,1)$. Let $X = \mu + \sigma Z$. $Z = (X-\mu)/\sigma$.
$X \sim \mathrm{N}(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right].$$

(we now use $f(x)$ to denote the density of X.)



$$P(\mu - 2\sigma \le X \le \mu + 2\sigma) = P(-2 \le Z \le 2) = 95\%.$$