Inversion Method

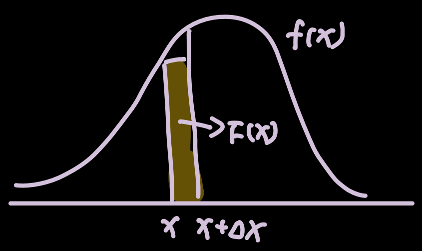
probability density function (pdf)

→ prob mass

$$f(x) = \lim_{\Delta x \to 0} \frac{P(X \in (x, x+\Delta x))}{\Delta x}$$

→ size

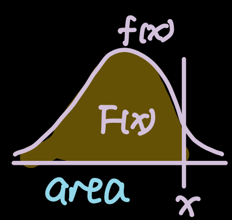
$$P(X \in (x, x+\Delta x)) \doteq f(x)\,\Delta x$$



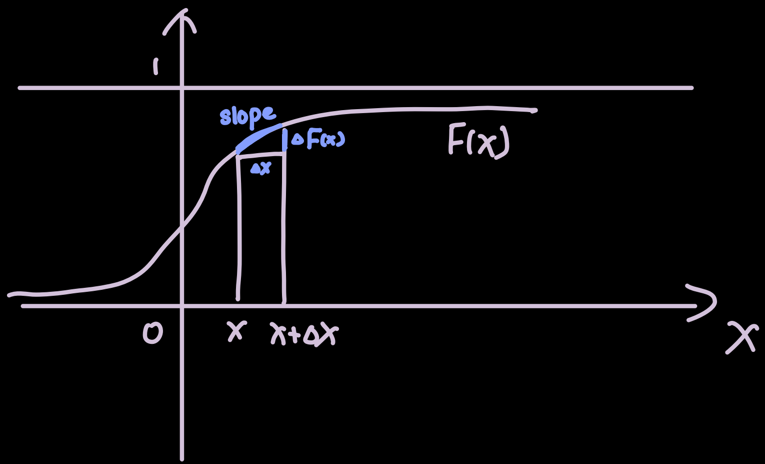
Cumulative Density Function (cdf)

end point
random #

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,d(x) =$$



$$\boxed{\begin{array}{l} \text{Integral} = \text{area} \\ \text{Derivative} = \text{slope} \end{array}}$$



$$f(x) = \frac{dF(x)}{dx}$$

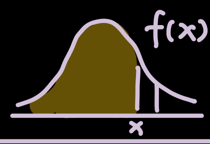
$$dF(x) = f(x)\,dx$$

$$f(x) = F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta F(x)}{\Delta x}$$

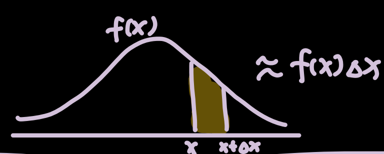
$= \lim_{\Delta x \to 0}$

$-$



Δx

$= \lim_{\Delta x \to 0}$



Δx

$= f(x)$

Inversion :

want $X \sim f(x)$

method $F(x) = \int^{x} f(x)\,dx$

algebraic meaning :

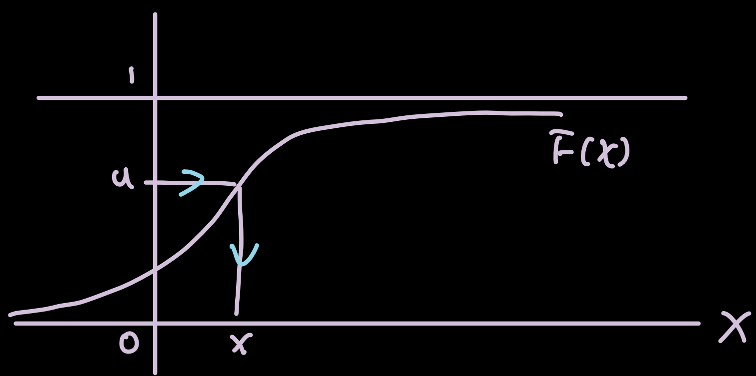
Inverse of $F(\ )$

solution: $x = F^{-1}(u)$ $\quad (\underbrace{F^{-1}F}_{\text{Identity matrix } I}(x) = x \ \forall x)$

$\iff F(x) = u$ equation

$u \in [0, 1]$

Geometric meaning of F^{-1}



Generate random # X

Step 1: $U \sim \text{unif}[0, 1]$

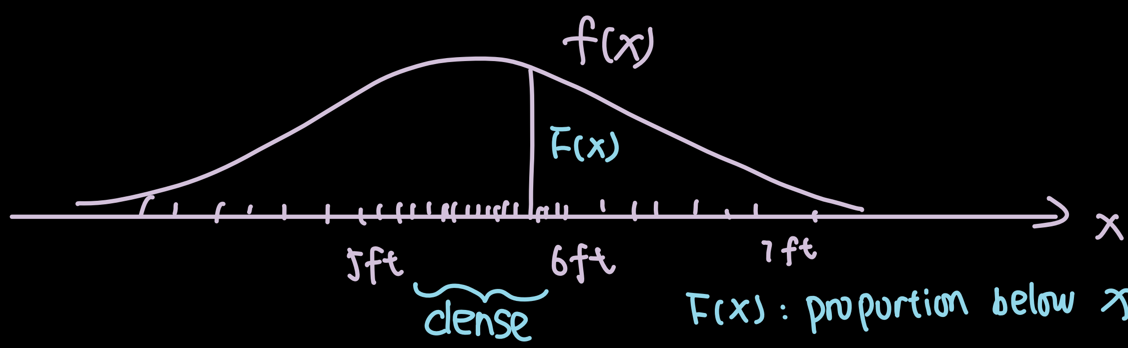
2: $X = F^{-1}(U)$

Pdf f(x) Intuitions

population $\xrightarrow{\text{sampling}}$ person $\longrightarrow$ height $X \sim f(x)$

(300 million)

Population scatter plot (nothing random)



F(x) : proportion below x

$N(x)$ = number of people in $[x, x + \Delta x)$

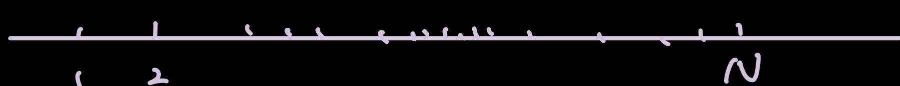
$\doteq N f(x) \Delta x$ = size × population proportion

$$f(x)\Delta x \doteq \frac{N(x)}{N} = \text{population proportion}$$

$$f(x) \doteq \frac{N(x)/N}{\Delta x} = \frac{\text{Population proportion}}{\Delta x}$$

N is essentially infinite.

After ordering by height



Step ①: sample $i \sim \text{Unif}\{1, 2, \ldots, N\}$

②: let X = height of ith person

e.g. SAT :

① $\underbrace{u = \frac{i}{N}}_{\text{percentile of a SAT score}} \sim \text{Unif}[0, 1]$

② population of people below ith percentile

$= u = \frac{i}{N}$

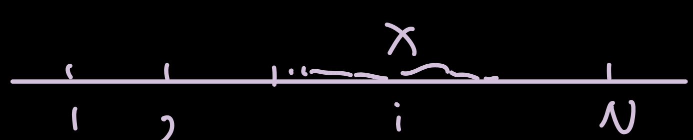
let height of ith person $= x$

③ $F(x)$ = proportion below x

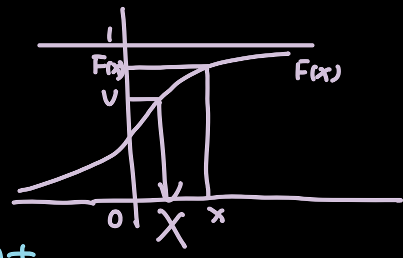
$= \frac{i}{N}$

$= u$

$X = F^{-1}(u)$



Formal Proof



Suppose $\bar{X} = F^{-1}(U)$

equivalent

$$P(\boxed{\bar{X} \le x}) = P(\boxed{U \le F(x)}) = F(x)$$

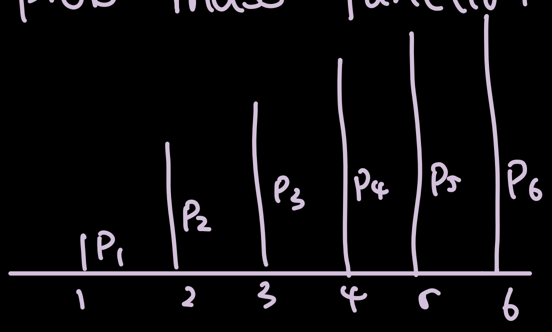
The cumulative density of generated $\bar{X}$ is x.

Another Look

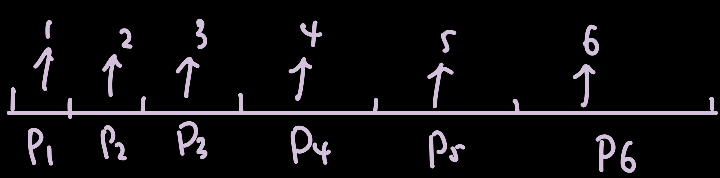
back to discrete random variable

a BIASED die

prob mass function



prob density is infinite

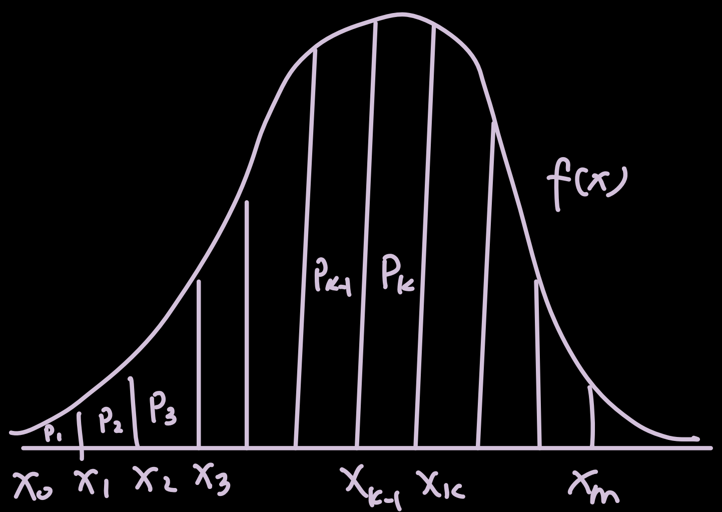


randomly throw a point into $[0,1)$

$\longrightarrow U \sim \mathrm{Unif}[0,1]$

$$X = k \text{ if } p_1 + \dots + p_{k-1} < u \le p_1 + \dots + p_k$$
$$\Downarrow$$
$$F(k-1) < u \le F(k)$$

Continuous random variable

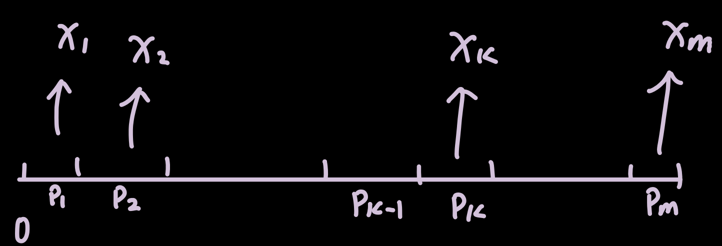


discretive $\{x_1, x_2, \dots x_m\}$

with prob $\{p_1, \dots\dots p_m\}$

concentrate probability mass of each bin to the right end point

"Discretize" according to probabilities



randomly throw a point into $[0,1)$

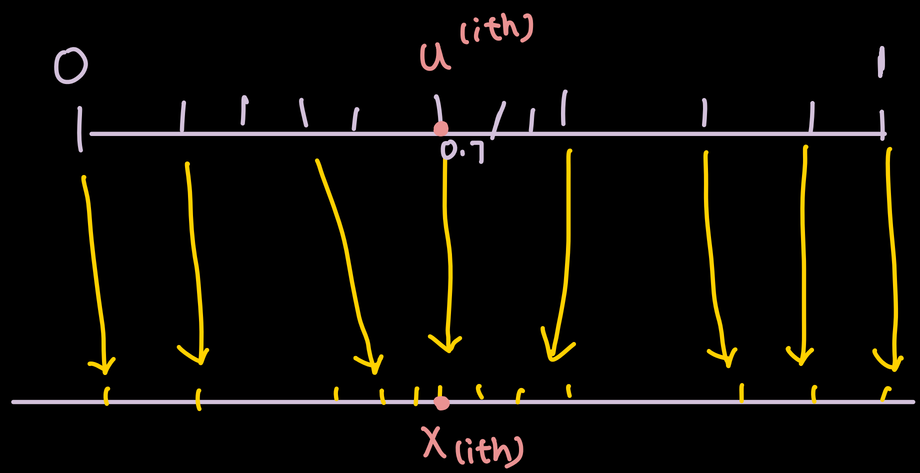
$\longrightarrow U \sim \mathrm{Unif}[0,1]$

$$X = x_k \text{ if } p_1 + \cdots p_{k-1} < u \le p_1 + \cdots + p_k$$

$$F(x_{k-1}) < u \le F(x_k)$$

As $\Delta x \to 0$, $X = x_k$, $u = F(x_k)$

Transformation of continuous random variable



Unif distributed
N points

$f(x)$ N points
↳ $F(x) = P(X \le x)$

order-preserving mapping
(percentile)

map a uniform distribution to arbitrary non-uniform distribution

$$X = G(u)$$

$$P(X \le x) = P(U \le x)$$

$$\| $$

$$F(x) = u$$

$$X = F^{-1}(u)$$

Application : Neural network → Facial recognition

$$X = G(u)$$

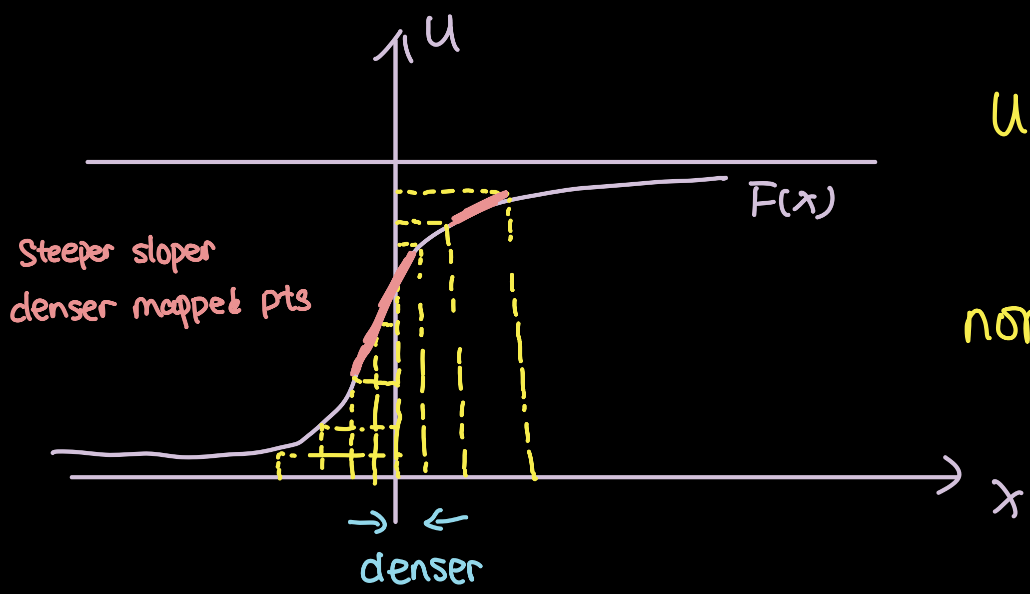
X ↓ face image

G ↓ generator

u ↘ uniform random #'s

Generative Modeling

Inversion Transformation



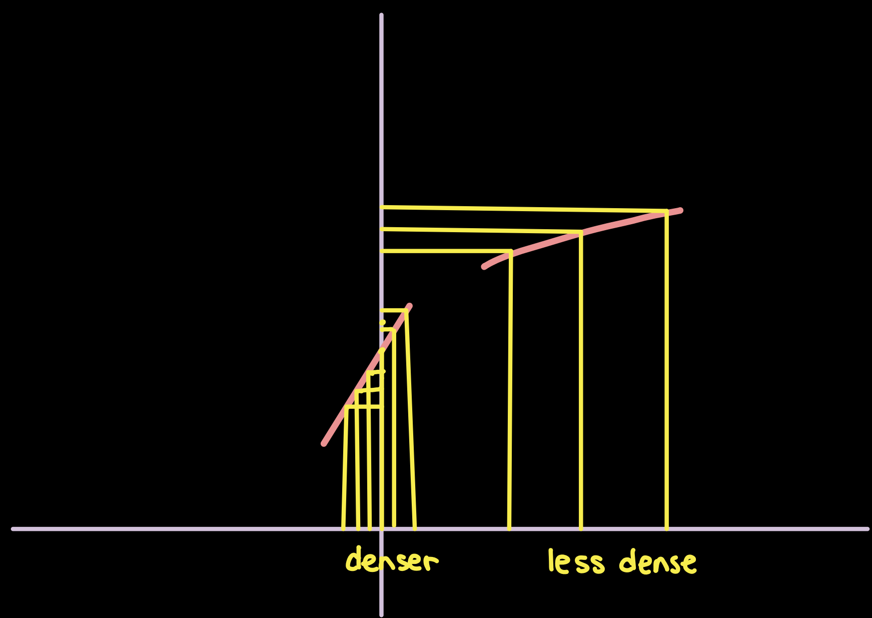
uniform distribution

⇓ transformed

non-uniform distribution

less dense

Slope controls density



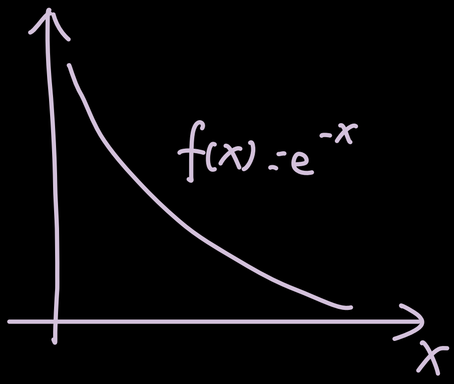
Slope of cdf controls the density of mapped points

$F'(x) = f(x)$ another intuition

Example 1:

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Wait time model



mapping $\Longrightarrow$

population of N points
(decaying time of N particles)

$$F(x) = \int_0^x e^{-x}\,dx$$

$$= -e^{-x}\Big|_0^x$$

$$= 1 - e^{-x}$$

$$F^{-1}(x) = 1 - e^{-x} = u$$

$$1 - u = e^{-x}$$

$$x = -\log(1-u)$$

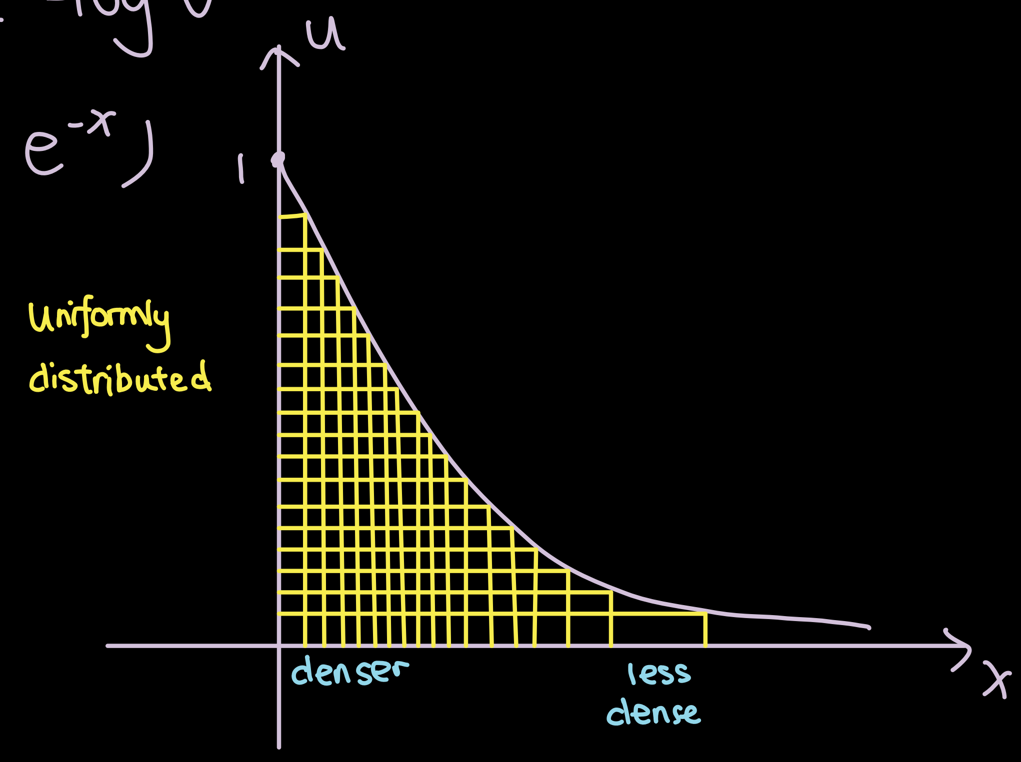
$\Rightarrow$ Inversion method

(1) $U \sim \text{Unif}\,[0, 1]$

(2) $x = -\log(\underbrace{1-u}_{\sim\, \text{unif}\,[0,1]})$

or $X = -\log U$

$(U = e^{-x})$

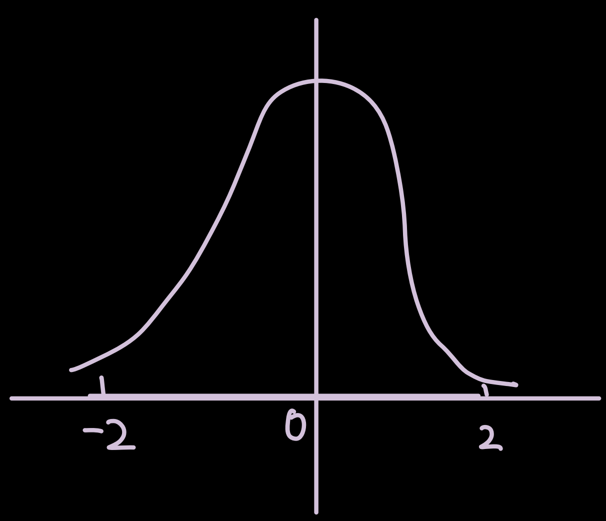


This is a differential transformation of continuous random variable from uniform to non-uniform

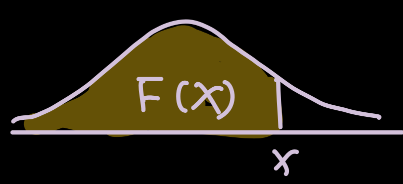
Example 2

$X \sim N(0, 1)$

$f(x) = \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)}$



Need to compute : $F(x) =$



$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$F^{-1}(u)$ = approximation in hw

$$\approx \frac{10}{\log 41} \log\left(1 - \frac{\log((\log u)/\log 2)}{\log 22}\right), \quad 0.5 \le u < 1,$$

and $F^{-1}(1-u) = -F^{-1}(u)$