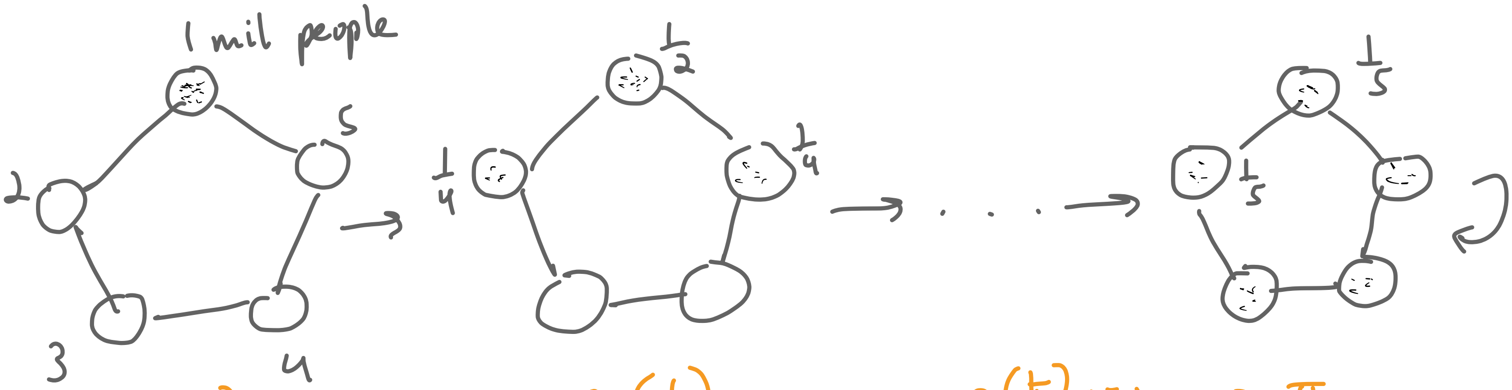
11/22/22

Markov Chain: population migration



$p(0)$ $\quad$ $p(1)$ $\quad \cdots$ $p(t)$ $\cdots \rightarrow \pi$

$$P^{(t)} \xrightarrow{K} P^{(t+1)}$$

transition probability

3 sets of notation

① $P(X_{t+1}) = \underbrace{\sum_{X_t}}_{\text{marginalization}} \underbrace{P(X_t)\, P(X_{t+1} \mid X_t)}_{\text{factorization}}$

② $x \rightarrow y$

$$p^{(t+1)}(y) = \sum_x p^{(t)}(x) K(x,y)$$

Function, operator

③ $i \rightarrow j$

$$p_j^{(t+1)} = \sum_i p_i^{(t)} K_{ij}$$

$$p^{(t+1)} = p^{(t)} K$$

matrix, vector

* HW derive ③ starting from ①

Meaning of K

① noun: transition probability

② verb:

a.) act on right hand side

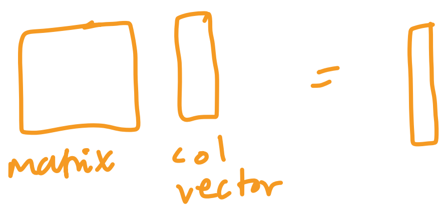
$$p^{(t)} K = p^{(t+1)}$$

ex.) population $p^{(t)}$ takes random walk K, population will be in state $p^{(t+1)}$

"diffusing"

transpose

$$K^T p^{(t)} = p^{(t+1)}$$



act on left hand side

$h =$ 

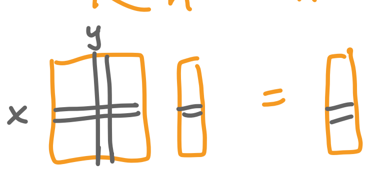
preliminary: $p =$ 

* notation 2 $\quad ph = \sum_x p(x)h(x)$

$$= E_{p}(h(x))$$

x follows distribution p

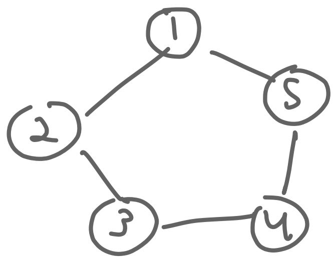
* notation 3 $\quad Kh = \tilde{h}$



* notation 2 $\quad \tilde{h}(x) = \sum_y K(x,y)h(y)$

* notation 1 $\quad = \sum_y P(X_{t+1} = y \mid X_t = x)\, h(y) = E(h(X_{t+1}) \mid X_t = x)$

meaning of $\tilde{h}$



$$\tilde{h}(2) = h(2)\cdot\frac{1}{2} + h(1)\cdot\frac{1}{4} + h(3)\cdot\frac{1}{4}$$

stay w/ prob $\frac{1}{2}$, transition w/ prob $\frac{1}{4}$

local averaging

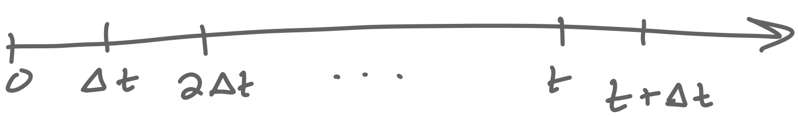
K is smoothing

$$p^{(t)} K h = p^{(t+1)} h$$

$$E_{X_t}\left[E(h(X_{t+1}) \mid X_t)\right] = E(h(X_{t+1}))$$

Continuous Time Process

ex.) Movie Frame example



ex.) bank account

$$X_{t+\Delta t} = (1 + r\Delta t)\cdot X_t$$

amount of money @ time t

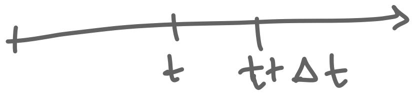
r = interest rate

ex.) Ordinary Differential Equation

$$\frac{X_{t+\Delta t} - X_t}{\Delta t} = rX_t \qquad \frac{dX_t}{dt} = rX_t$$

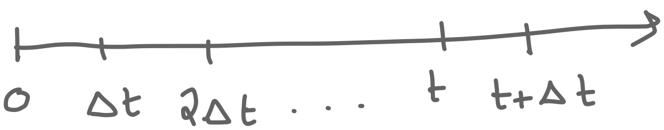
$$X_t = X_0(1+r\Delta t)^{\frac{t}{\Delta t}} = X_0 e^{r\Delta t \frac{t}{\Delta t}} = x_0 e^{rt}$$

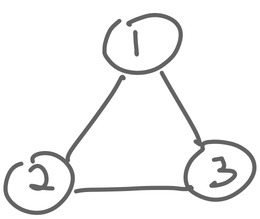
ex.) Poisson Process



$P(\text{head}) = \lambda \Delta t$

Markov jump Process





Transition Matrix

$$K^{(\Delta t)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} & a_{12}\Delta t & a_{13}\Delta t \\ a_{21}\Delta t & & a_{23}\Delta t \\ a_{31}\Delta t & a_{32}\Delta t & \end{bmatrix} \end{array}$$

$a_{12}\Delta t$ interpretation: among all people in state 1 at time t, $a_{12}\Delta t$ is fraction of people who move to state 2 w/in $[t, t+\Delta t]$

$$K^{(\Delta t)} = \begin{bmatrix} 1-(a_{12}+a_{13})\Delta t & a_{12}\Delta t & a_{13}\Delta t \\ a_{21}\Delta t & 1-(a_{21}+a_{23})\Delta t & a_{23}\Delta t \\ a_{31}\Delta t & a_{32}\Delta t & 1-(a_{31}+a_{32})\Delta t \end{bmatrix}$$

$$= I + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Delta t = I + A\Delta t$$

A ↓ transition rate

A : generator

where $a_{ii} = -\sum_{j \neq i} a_{ij}$

$$K^{(t)} = \left(K^{(\Delta t)}\right)^{\frac{t}{\Delta t}} = (I + A\Delta t)^{\frac{t}{\Delta t}} = e^{A\Delta t \cdot \frac{t}{\Delta t}} = e^{At}$$

$$\boxed{K^{(t_1+t_2)} = K^{(t_1)} K^{(t_2)} \quad \{K^{(t)} = e^{At},\ t > 0\} \text{ semigroup}}$$

e^{M} ← matrix

$$e^{M} = 1 + M + \frac{M^2}{2} + \cdots + \frac{M^t}{t!} + \cdots$$

like taylor expansion for

$$e^{x} = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^t}{t!} + \cdots$$

derive differential equation

$$K^{(t+\Delta t)} = K^{\Delta t} \cdot K^{t} \quad \text{step } 1$$

$$= K^{t} \cdot K^{\Delta t} \quad \text{step } 2$$

$$\frac{K^{(t+\Delta t)} - K^{(t)}}{\Delta t} = \frac{K^{(\Delta t)}K^{(t)} - K^{(t)}}{\Delta t} = \frac{(K^{(\Delta t)} - I)K^{(t)}}{\Delta t}$$

$$K^{(\Delta t)} = I + A\Delta t$$

$$\frac{K^{(\Delta t)} - I}{\Delta t} = A$$

$$\Rightarrow \frac{d}{dt} K^{(t)} = AK^{(t)} \longrightarrow \text{backward}$$

similar to bank account $\frac{d}{dt} X(t) = rX(t)$

$$= K^{(t)}A \longrightarrow \text{forward}$$

backward

$$\frac{d}{dt} \overbrace{K^{(t)} h^{(0)}}^{h(t)} = A \overbrace{K^{(t)} h^{(0)}}^{h(t)}$$

$$\frac{d}{dt} h^{(t)} = A h^{(t)}$$

Solution: $h(t) = K^{(t)} h^{(0)}$

massage/smooth $h(0)$ for duration t

↳ similiar to putting money in bank account for time t

Forward

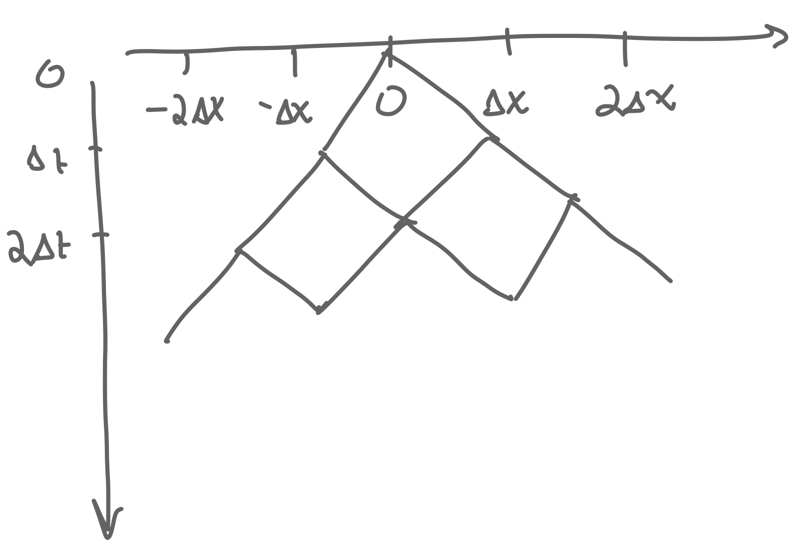
$$\Rightarrow \frac{d}{dt} p^{(0)} K^{(t)} = p^{(0)} K^{(t)} A$$

$$\frac{d}{dt} p^{(t)} = p^{(t)} A$$

Solution: $p^{(t)} = p^{(0)} K^{(t)}$

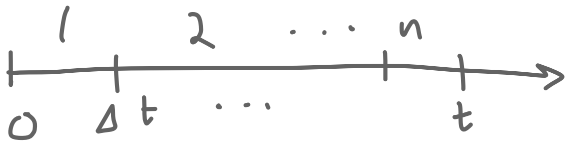
ex.) 1 mil people w/ distribution $p^{(t)}$. Rate of change of dist $\Rightarrow \frac{d}{dt} p^{(t)}$

Diffusion (Brownian Motion, Wiener Process)



$$X_{t+\Delta t} = X_t + \Delta x \cdot \varepsilon_t$$

	-1	$+1$
prob	$\frac{1}{2}$	$\frac{1}{2}$



$$\Delta t = \frac{t}{n}$$

$$X_t = \sum_{i=1}^{n} \Delta x \, \varepsilon_i$$

$$E(\varepsilon_i) = 0 \qquad var(\varepsilon_i) = 1$$

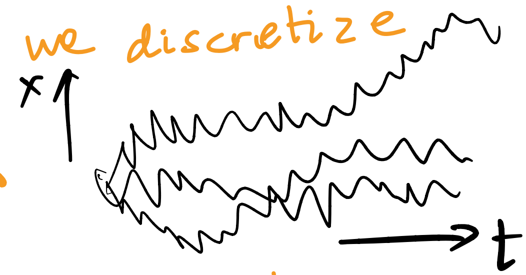
$$E(X_t) = 0 \qquad var(X_t) = \Delta x^2 n = \Delta x^2 \frac{t}{\Delta t}$$

$$\frac{\Delta x^2}{\Delta t} = \sigma^2$$

independent of how we discretize

$$\boxed{\Delta x = \sigma \sqrt{\Delta t}}$$

$$v = \frac{\Delta x}{\Delta t} \to \infty$$



$$X_{t+\Delta t} = X_t + \sigma \sqrt{\Delta t}\, \varepsilon_t$$

simplest case of Brownian motion

ε_t are iid

ε_t	-1	1
prob		

$$K^{(\Delta t)} = \begin{array}{c} \\ x \downarrow \end{array} \overset{y \longrightarrow}{\begin{bmatrix} \ddots & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \frac{1}{2} & 0 & \frac{1}{2} & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & & \ddots \end{bmatrix}}$$

$$K^{(\Delta t)} h(x) = \frac{1}{2}\left(h(x-\Delta x) + h(x+\Delta x)\right)$$

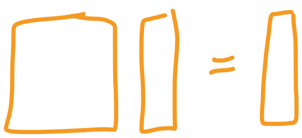
local averaging

$$A = \frac{K^{\Delta t} - I}{\Delta t} = \frac{1}{2}\frac{1}{\Delta t} \underbrace{\begin{bmatrix} 1 & -2 & 1 & & 0 \\ & 1 & -2 & 1 & \\ 0 & & 1 & -2 & 1 \end{bmatrix}}_{D}$$

since
$\Delta x = \sigma\sqrt{\Delta t}$
$\Delta t = \frac{\Delta x^2}{\sigma^2}$

$$= \frac{\sigma^2}{2\Delta x^2} D = \frac{\sigma^2}{2}\frac{d^2}{dx^2}$$

$$\frac{1}{\Delta x^2} D h = \tilde{h}$$



$$\tilde{h}(x) = \frac{h(x-\Delta x) - 2h(x) + h(x+\Delta x)}{\Delta x^2}$$

$$= \frac{\frac{h(x+\Delta x) - h(x)}{\Delta x} - \frac{h(x) - h(x-\Delta x)}{\Delta x}}{\Delta x}$$

$$= \frac{d}{dx^2} h(x)$$

$$(Ah)(x) = \frac{\sigma^2}{2}\frac{d^2}{dx^2} h(x)$$

$$(PA)(x) = \frac{\sigma^2}{2}\frac{d^2}{dx^2} P(x)$$

$$Ah = \frac{K^{(\Delta t)} - I}{\Delta t} h = \frac{K^{(\Delta t)} h - h}{\Delta t} = \tilde{h}$$

$$\tilde{h}(x) = \frac{(K^{(\Delta t)} h)(x) - h(x)}{\Delta t} = \frac{E(h(X_{t+\Delta t} \mid X_t = x)) - h(x)}{\Delta t}$$

$$= \frac{E(h(x + \sigma\sqrt{\Delta t}\,\varepsilon_t)) - h(x)}{\Delta t}$$

$$= \frac{E\left(h(x) + h'(x)\sigma\sqrt{\Delta t}\,\varepsilon_t + \frac{1}{2}h''(x)\sigma^2\Delta t\,\varepsilon_t^2 - h(x)\right)}{\Delta t}$$

$$= \frac{\sigma^2}{2} h''(x)$$

Taylor expansion: throw away terms higher than order Δt

$$(Ah)(x) = \frac{\sigma^2}{2} h''(x)$$

$$A = \frac{\sigma^2}{2}\frac{d^2}{dx^2}$$

general ε_t

$E(\varepsilon_t) = 0 \quad Var(\varepsilon_t) = 1$

forward equation

$$\frac{d}{dt} p^{(t)}(x) = \frac{\sigma^2}{2}\frac{d^2}{dx^2} p^{(t)}(x)$$

"heat equation"

↳ solution is Normal distribution