11/29/22

Final

- like homework
- review posted videos & notes on B-Learn

Markov Chain → Jump Process → Diffusion

	Markov Chain	Jump Process	Diffusion
State:	Discrete	Discrete	Continuous
Time:	Discrete	Continuous	Continuous
Transition:	K	$K = I + A\Delta t$ (generator)	$A = \frac{\sigma^2}{2}\nabla^2$ (operator $\frac{d^2}{dx^2}$)
past:	conditioning	Poisson Process	Binomial/Galton Board/Normal

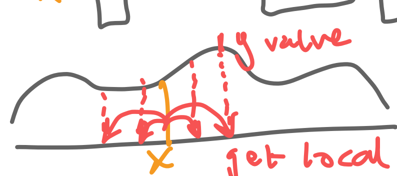
Key Equations:

Markov Chain

Forward equation: $p^{(t+1)} = p^{(t)} K$

Backward equation: $h^{(t+1)} = K h^{(t)}$

local averaging/smoothing



get local neighborhood, get y values

Jump Process

Forward: $p^{(t+\Delta t)} = p^{(t)} K^{(\Delta t)} = p^{(t)}(I + A\Delta t)$

$= p^{(t)} + p^{(t)} A \Delta t$

differential equation

$$\frac{p^{(t+\Delta t)} - p^{(t)}}{\Delta t} = p^{(t)} A$$

Matrix ODE form : $\boxed{\frac{d}{dt} p^{(t)} = p^{(t)} A}$

rate of change of marginal distribution

transition rate

* column version: transpose

$$\frac{d}{dt} p^{(t)} = A^T p^{(t)}$$

ex.) distribution of 1 million particles.

Backward: $h^{(t+\Delta t)} = K^{(\Delta t)} h^{(t)} = (I + A\Delta t) h^{(t)}$

$= h^{(t)} + A h^{(t)} \Delta t$

$$\frac{h^{(t+\Delta t)} - h^{(t)}}{\Delta t} = A h^{(t)}$$

Matrix ODE form: $\boxed{\frac{d}{dt} h^{(t)} = A h^{(t)}}$

Diffusion

$$X_{t+\Delta t} = X_t + \sigma\sqrt{\Delta t}\,\varepsilon_t$$

$E(\varepsilon_t) = 0 \quad Var(\varepsilon_t) = 1$, with all ε_t iid

(so $\varepsilon_t \perp X_t$)

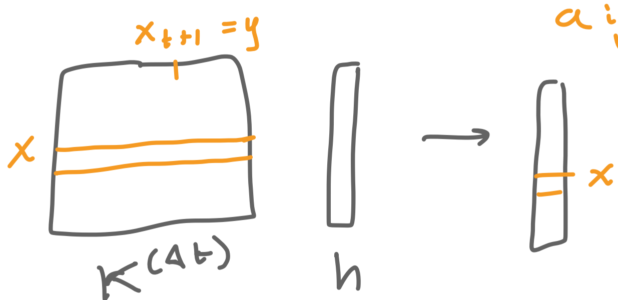
$$K^{(\Delta t)} = I + A\Delta t$$

$$\Rightarrow \quad A = \frac{K^{(\Delta t)} - I}{\Delta t}$$

noun: $K_{ij}^{(\Delta t)} = a_{ij}\Delta t \quad (i \neq j)$

like a Poisson process where $\lambda = a_{ij}$ since a_{ij} is a transition rate

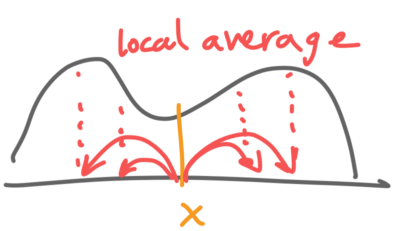
verb: $X_t =$



transitional prob.

$$(K^{\Delta t} h)(x) = \sum_y K^{(\Delta t)}(x, y)\, h(y)$$

$$= \sum_y P(X_{t+\Delta t} = y \mid X_t = x) \cdot h(y)$$



$$= \sum_{X_{t+\Delta t}} P(X_{t+\Delta t} \mid X_t = x)\, h(X_{t+\Delta t})$$

change notation

$$= E(h(X_{t+\Delta t} \mid X_0 = x))$$

$$(Ah)(x) = \frac{(K^{\Delta t} h)(x) - h(x)}{\Delta t} = \frac{E(h(X_{t+\Delta t}) \mid X_t = x) - h(x)}{\Delta t} \Longrightarrow$$

$$\Rightarrow \frac{E(h(X + \sigma\sqrt{\Delta t}\,\varepsilon_t)) - h(x)}{\Delta t}$$

if $\Delta t = .00001$

$\sqrt{\Delta t} = .001$

$$= \frac{E(h(x) + h'(x)\sigma\sqrt{\Delta t}\,\varepsilon_t + \frac{1}{2} h''(x)\sigma^2 \Delta t\, \varepsilon_t^2 - h(x))}{\Delta t}$$

$$= \frac{\sigma^2}{2} h''(x)$$

$$A = \frac{\sigma^2}{2} \nabla^2$$

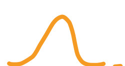
partial differential equation

Backward equation: $\frac{d}{dt} h^{(t)} = \frac{\sigma^2}{2} \nabla_x^2 h^{(t)}$ derivative w/rt x

$$\boxed{\frac{d}{dt} h^{(t)} = A h^{(t)}}$$

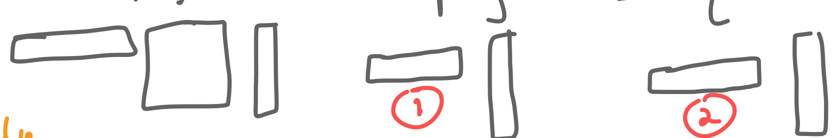
Forward equation:

test function



approaches $-\infty/\infty$ quickly

$$PAh = \overset{Ah}{p\,g} = \overset{PA}{q}\,h$$



① ②

$$g(x) = (Ah)(x) = \frac{\sigma^2}{2} h''(x)$$

$$pg = \int p(x)\, g(x)\, dx$$

$$= \int p(x) \frac{\sigma^2}{2} h''(x)\, dx \quad ①$$

$$= \int q(x)\, h(x)\, dx \quad ②$$

use integration by parts

$$\int p(x)\, h''(x)\, dx = \int p(x)\, d h'(x)$$

$$= p(x)\, h'(x) \Big|_{-\infty}^{\infty} - \int h'(x)\, p'(x)\, dx$$

since $h(x)$ approaches 0 quickly $-\infty/\infty$

$$= -\int p'(x)\, d h(x)$$

$$= -\left[p'(x)\, h(x) \Big|_{-\infty}^{\infty} + \int h(x)\, p''(x)\, dx \right] \quad \left(h(X_{t+\Delta t}) \,\Big|\, X_0 = x \right)$$

$$q(x) = \frac{\sigma^2}{2} \cdot p''(x) = PA$$

Forward: $$\frac{d}{dt} p^{(t)} = \frac{\sigma^2}{2} \nabla_x^2\, p^{(t)}$$ ← heat equation Einstein (1905)

$$\nabla^2 = \frac{1}{\Delta x^2} \begin{bmatrix} 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \end{bmatrix}$$

Solution for $p^{(t)}(x)$:

If $p^{(0)} = \delta_0 \quad (X_0 = 0)$

$$p^{(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{x^2}{2\sigma^2 t}}$$

... Central Limit Theorem

Stochastic differential equation (SDE)

$$X_{t+\Delta t} = X_t + \underbrace{\mu(X_t, t)\Delta t}_{\text{drift}} + \underbrace{\sigma(X_t, t)\sqrt{\Delta t}\,\varepsilon_t}_{\text{diffusion}}$$

differential form:

$$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dB_t$$ ← Brownian motion

$A = ?$ → go through same calculation process as finding $Ah(x)$ on prev. page

ex.)

Bank is described as an ODE:

$$X_{t+\Delta t} = X_t(1+\gamma\Delta t)$$
$$= X_t + \gamma X_t \Delta t$$

stock: $X_t = X_t(1 + \underbrace{\mu}_{\text{interest rate}}\Delta t + \underbrace{\sigma}_{\text{risk/volatility}}\sqrt{\Delta t}\,\varepsilon_t)$

if $\Delta t = .001$, $\Delta t = .01$

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t$$ ← SDE form

$$X_t = X_0(1+\gamma\Delta t)^{\frac{t}{\Delta t}} = X_0 e^{\gamma\Delta t \frac{t}{\Delta t}} = X_0 e^{\gamma t}$$

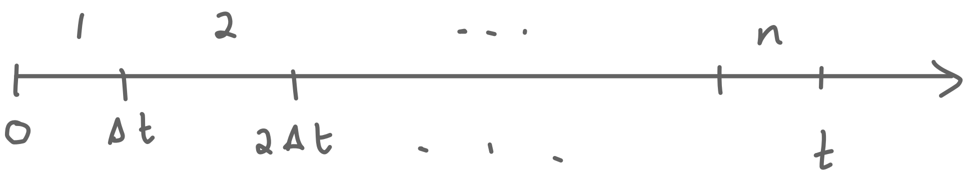
$$\log X_{t+\Delta t} = \log X_t + \log(1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\varepsilon_t) \Longrightarrow$$

$$\log(1+\delta) = \delta - \frac{\delta^2}{2} + o(\delta^2)$$

need 2nd order Taylor expansion because of $\sqrt{\Delta t}$

$$\Rightarrow \log X_t + \mu \Delta t + \sigma\sqrt{\Delta t}\,\varepsilon_t - \frac{\sigma^2 \Delta t\, \varepsilon_t^2}{2}$$

Discretization :



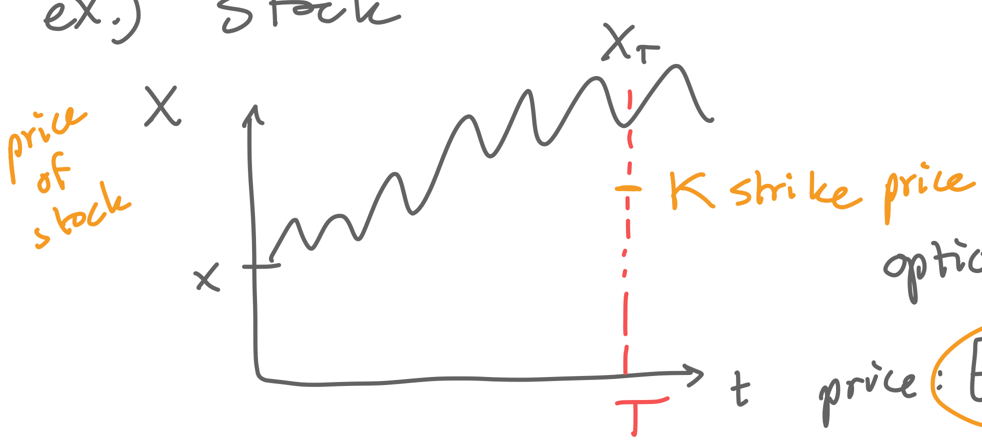
$$\Delta t = \frac{t}{n}$$

$$\log X_t = \log X_0 + \sum_{i=1}^{n}\left[\mu \frac{t}{n} + \sigma\sqrt{\frac{t}{n}}\,\varepsilon_i - \frac{\sigma^2}{2}\frac{t}{n}\varepsilon_i^2\right]$$

$$= \log X_0 + \mu t + \sigma\sqrt{t}\,\underbrace{\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i\right)}_{\substack{n\to\infty \\ Z\sim N(0,1)}} - \frac{\sigma^2 t}{2}\underbrace{\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2}_{\substack{n\to\infty \\ E(\varepsilon_i^2)=1}}$$

$$X_t = X_0 \cdot e^{\mu t + \sigma\sqrt{t}\cdot Z - \frac{\sigma^2 t}{2}}$$ log-normal

ex.) stock



backward eqn

$h(X_T)$

option $(0, X_T - K)$

price: $E(h(X_t) \mid X_0 = x)\, e^{-\gamma T}$

assume $\mu = \gamma$

solution of backward equation

Martingale

Function of $\varepsilon_1, \varepsilon_2, \dots \varepsilon_t$

$$X_t = F(\varepsilon_1, \varepsilon_2, \dots \varepsilon_t)$$

$$E(X_{t+1} \mid \varepsilon_1, \varepsilon_2, \dots, \varepsilon_t) = X_t$$

fair game

Central Limit Theore

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i \xrightarrow{D} N(0,1)$$

Law of Large Numbers

$$\frac{1}{n} \sum_{i=1}^{n} \varepsilon_i^2 \xrightarrow{P} 1$$

Ito Calculus

$$dB_t^2 \overset{\int}{=} dt$$

$$(\sqrt{\Delta t}\, \varepsilon_t)^2 = \Delta t \, \varepsilon_t^2$$

$$= t \frac{1}{n} \varepsilon_t^2$$