9/27/22

Part 1 Random Variables

Ω = population w = random person

$X(w)$ = gender → discrete

$Y(w)$ = height → continuous

Discrete RV

roll fair die → X

pmf (prob mass function): $p(x) = Pr(X = x)$

x	1	2	3	4	5	6	...
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	...		.	$\frac{1}{6}$	

⟹ uniform

$X \sim p(x)$

Doesn't have to be uniform:

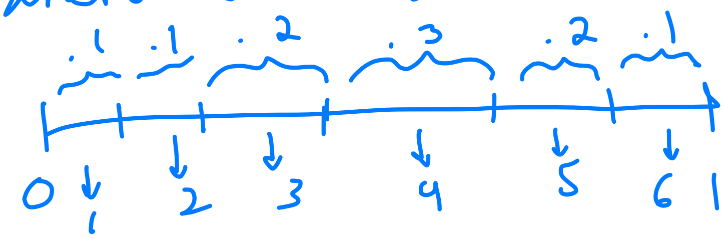
Ex: biased die pmf

x	1	2	3	4	5	6
$p(x)$	.1	.1	.2	.3	.2	.1

property: $\sum_x p(x) = 1$

$$P(X \in (a,b)) = \sum_{\substack{x \in (a,b) \\ x \in A}} p(x)$$

Generation: Inversion method

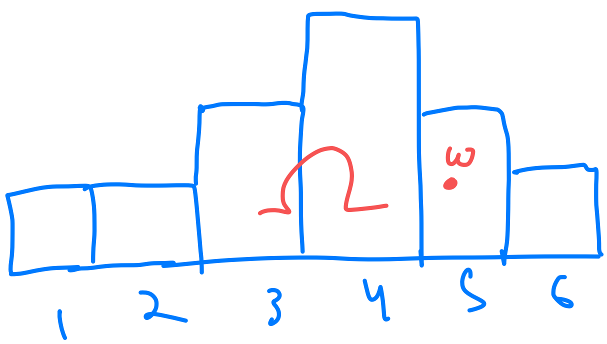


→ can also make a graph

$\omega \sim \text{Unif}(\Omega)$

uniform random number

$X(\omega) := k$ if it falls into the k^{th} interval



$\omega \sim \text{Unif}(\Omega)$

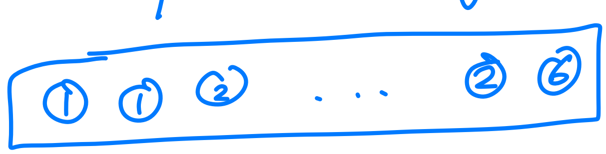
$X(\omega) = k$ if $\omega \in k^{th}$ bin

Summary Statistics

Expectation: $E(X) = \sum_x x \cdot p(x)$ average/mean

(1) population average

Ex.) N balls w/ a designated number



$X(\omega) =$ # carried by ball $\omega \in \Omega$

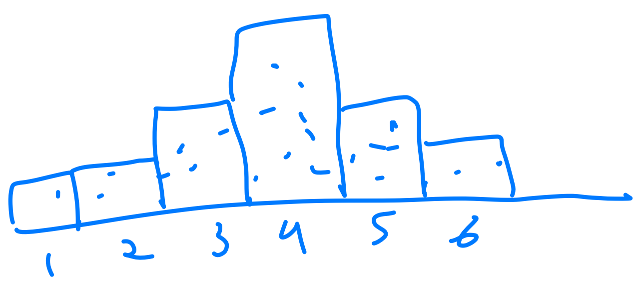
$$P(X = x) = p(x) = \frac{N(x)}{N}$$

$N(x) =$ # of balls carrying x

$$E(X) = \sum_x x\,p(x) = \sum_x x\,\frac{N(x)}{N} = \frac{1}{N}\sum_x x\,N(x)$$

$$= \frac{1}{N}\sum_{\omega \in \Omega} X(\omega) = \text{population average}$$

(2) long run average, repeat n times



$n(x) =$ # of times $X = x$

$$\frac{1}{n}\sum_x x\,n(x) = \sum_x x \cdot \frac{n(x)}{n} \longrightarrow \sum_x x \cdot p(x)$$

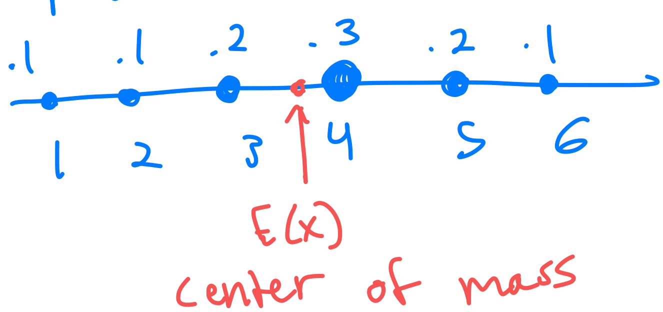
converge

assume $n \to \infty$

$\bar{X} \to E(X)$ in probability

↳ prove later

pmf: physics



Ex.) Casino game

x	1	2	3	4	5	6
$p(x)$	.1	.1	.2	.3	.2	.1
$h(x)$	-20	-10	0	20	30	100

profit in $

Interested in profit average, not expected number drawn

$$E(h(X)) = \sum_x h(x)\,p(x)$$

$$= (-20)(.1) + (-10)(.1) + 0(.2) + \dots$$

Ex.) 2 offers

offer 1:

x	$100
$p(x)$	1

offer 2:

x	$0	$200
$p(x)$	$\frac{1}{2}$	$\frac{1}{2}$

$E(x) = 100 \cdot 1 = 100$

$E(x) = 0 \cdot \frac{1}{2} + 200 \cdot \frac{1}{2} = 100$

value utility

Face value x	0	100	200
Perceived value $h(x)$	0	100	150

offer 1: $E(h(x)) = 100 \cdot 1 = 100$

offer 2: $E(h(x)) = 0 \cdot \frac{1}{2} + 150 \cdot \frac{1}{2} = 75$

Variance: $Var(X) = E(\underbrace{(X - E(X))}_{\text{special function of } X \text{ to reflect fluctuation/variation/volatility/spread}}{}^2)$

Ex: back to offers

Offer 1: $Var(X) = E((100-100)^2) = (100-100)^2 \cdot 1 = 0$

Offer 2: $Var(X) = (0-100)\cdot\frac{1}{2} + (200-100)^2\cdot\frac{1}{2} = \$10000^2$

Standard Deviation: $\sqrt{Var(X)} = \sqrt{10000} = 100$

μ : $E(X)$

σ^2 : $Var(X)$

σ : $SD(X)$

<u>Properties of $E(X)$ and $Var(X)$</u>

1.) If $Y = aX + b$

$$E(Y) = E(aX+b) = \sum_x (ax+b)\,p(x)$$

$$= a\sum_x x\,p(x) + b\sum_x p(x) = aE(X) + b$$

$$Var(Y) = Var(aX+b) = E((aX+b-E(aX+b))^2)$$

$$= E(a^2(X-E(X))^2) = a^2E(X-E(X)^2) = a^2\,Var(X)$$

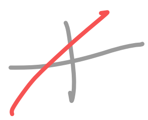
2.) Standardization: $Z = \frac{X-\mu}{\sigma}$

$$E(Z) = \frac{E(X)-\mu}{\sigma} = 0$$

$$Var(Z) = \frac{1}{\sigma^2} Var(X) = 1$$

3.) $E(h(x)+g(x)) = E(h(x)) + E(g(x))$

$$var(X) = E((X-\mu)^2) = E(X^2 - 2\mu X + \mu^2)$$
$$= E(X^2) - 2\mu E(X) + \mu^2$$
$$= E(X^2) - E(X)^2 \geq 0$$

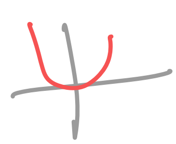
If $h(x) = ax+b$ 

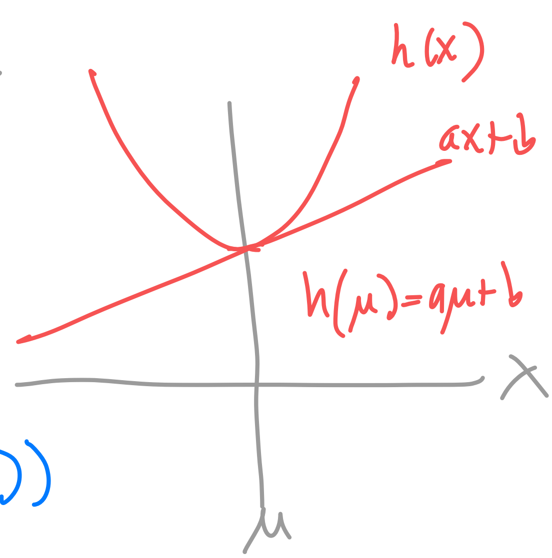
$$E(h(x)) = E(ax+b) = aE(x) + b = h(E(X))$$

can change order w/ linear transformations

If $h(x) = X^2$

$$E(h(X)) = E(X^2) \geq h(E(x)) = E(x)^2$$

If function is convex 



Jensen inequality

$$E(h(x)) \geq h(\overbrace{E(x)}^{\mu})$$

$$E(h(X)) \geq E(aX+b) = aE(X)+b$$
$$= a\mu + b = h(\mu) = h(E(X))$$

Ex.) More offers

offer 1 :

x	μ
$p(x)$	1

$E(X) = \mu$

$E(h(X)) = h(\mu)$

offer 2 :

x	0	100	200
$p(x)$			

$E(X) = \mu$

If $h(x)$ convex

$E(h(X)) \geq h(\mu)$

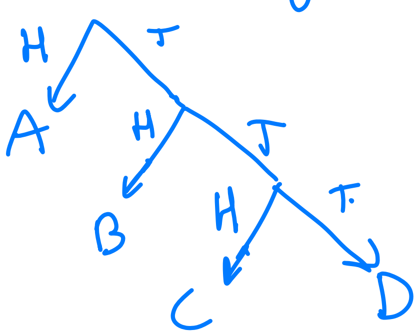
Entropy

interpretation: # of flips or code length

$$E(-\log_2 p(X)) = -\sum_x p(x) \log_2 p(x)$$

X	A	B	C	D
$p(X)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$
$-\log_2 p(X)$	1	2	3	3

Use fair coin to generate:



Ex.) Coding

X	A	B	C	D
code	1	01	001	000
length	1	2	3	3

given: 010011010000

B C A B D

Random sequence $\sim p(x)$

B C A D . . . $\longrightarrow$ 01 1 0 0 1 01 . . .

completely random sequence