9/29/22

Probability Models for Discrete RVs

Bernoulli (p)

Coin Flip : $Z = \begin{cases} 1 & \text{if } H \\ 0 & \text{if } T \end{cases}$

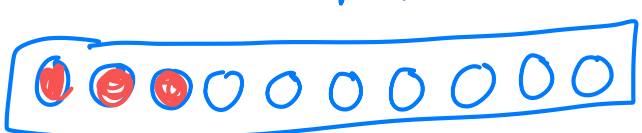
$P(\text{head}) = p$

$Z \sim Ber(p)$

z	0	1
$p(z)$	$1-p$	p

Fair coin $p = \frac{1}{2}$

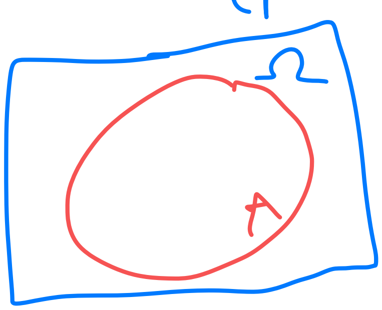
Ex.) Sample a population

10 balls 

3 red 7 blue

$Z = \begin{cases} 1 & \text{red} \\ 0 & \text{blue} \end{cases}$

$Z \sim Ber(p = .3)$

Ex.) 

$Z = 1(\omega \in A) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

$p = \frac{|A|}{|\Omega|}$ $\quad Z \sim Ber\left(p = \frac{|A|}{|\Omega|}\right)$

z	0	1
$p(z)$	$1-p$	p

Return to $z \in \{0, 1\}$

$$E(z) = 0 \cdot (1-p) + 1 \cdot p = p$$

$$E(1(w \in A)) = p(w \in A)$$

$$Var(z) = E(z^2) - E(z)^2$$

$$Var(z) = (0-p)^2 \cdot (1-p) + (1-p)^2 \cdot p$$

prob prob

$$= p^2(1-p) + (1-p)^2 p$$

$$= p(1-p)\left[p + (1-p)\right]$$

$$= p(1-p)$$

$z^2 = z$ so $E(z) - E(z)^2 = p - p^2$

$$= p(1-p)$$

$X \sim Binomial(n, p)$

ex.) Flip coin n times

Independently

$p(H) = p$

X = # of heads

ex.) Sampling a population

$\boxed{\text{red balls } (R) + \text{blue balls } (B)}$

N balls $= R + B$

sample from Ω n times (with replacement)
↳ ind.

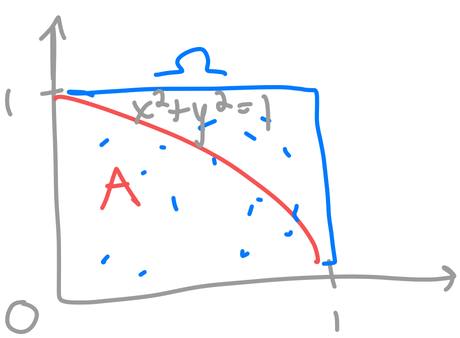
X = # of red balls out of n balls

$X \sim Bin\left(n, p = \frac{R}{N}\right)$

population (N) $\xleftarrow{\text{to estimate}}$ sample (n)

population $\left(\frac{R}{N}\right) = p$ $\xleftarrow{\text{to estimate}}$ proportion $\left(\frac{X}{n}\right) = \hat{p}$

ex.) Region



$P(\omega \in A) = \frac{|A|}{|\Omega|} = \frac{\pi}{4}$

repeat $n = 1$ million times

X = # of points in A

$X \sim Bin\left(n, p = \frac{\pi}{4}\right)$

$\hat{p} = \frac{X}{n}$

$\hat{\pi} = 4\hat{p}$

Monte Carlo Method

in High D, use small cubes to get volume

* Button needle example

Relationship between Bernoulli & Binomial

$$Ber(p) = Bin(1, p)$$

$$X \sim Bin(n, p)$$

$$X = Z_1 + Z_2 + \cdots + Z_n \quad \text{where } Z_i \overset{iid}{\sim} Ber(p)$$

# of 1's

$$E(X) = E\left(\sum_{i=1}^{n} Z_i\right) = \sum_{i=1}^{n} E(Z_i) = np$$

$$Var(X) = Var\left(\sum_{i=1}^{n} Z_i\right) \overset{ind}{=} \sum_{i=1}^{n} Var(Z_i) = np(1-p)$$

$$E\left(\frac{X}{n}\right) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n} \quad \text{because } Var(ax+b) = a^2 Var(x)$$

↳ as $n \to \infty$ $\frac{p(1-p)}{n} = 0$

sample proportion

$$SD\left(\frac{X}{n}\right) = \sqrt{\frac{p(1-p)}{n}}$$

as $n \to \infty$, $\frac{X}{n} \to p$ in probability

$$Pr\left(\left|\frac{X}{n} - p\right| < \varepsilon\right) \longrightarrow 1$$

Back to fair coin ex:

$p = \frac{1}{2}$

$P(X=k) = P(X(\omega)=k) = P(A_k)$

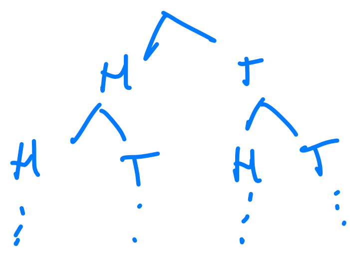
where $A_k = \{\omega : X(\omega) = k\}$

ω: sequence of H, T

$\Omega = \{$all 2^n sequences of H T$\}$

why 2^n?

2 choices, n flips

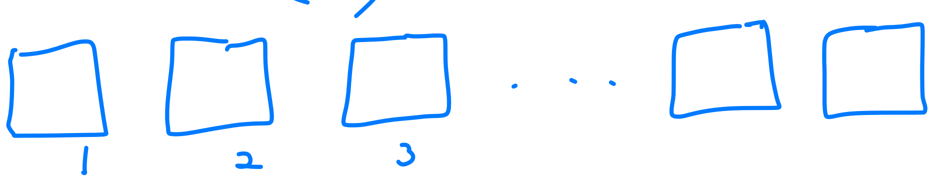


binary tree branches $= 2^n$

$$P(X=k) = P(A_k) = \frac{|A_k|}{|\Omega|}$$

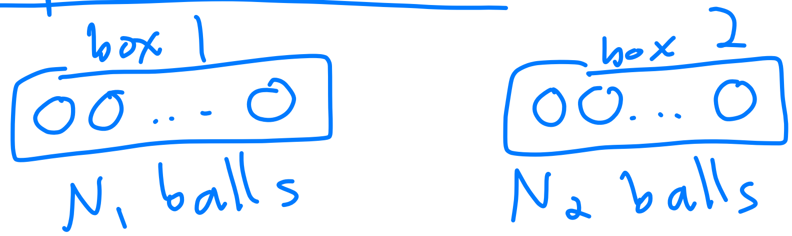
think about doing flips all @ once, not one @ a time

$$|A_k| = \binom{n}{k}$$



choose k blanks, fill in H. Fill the rest of $n-k$ blanks, fill in T

Multiplication rule

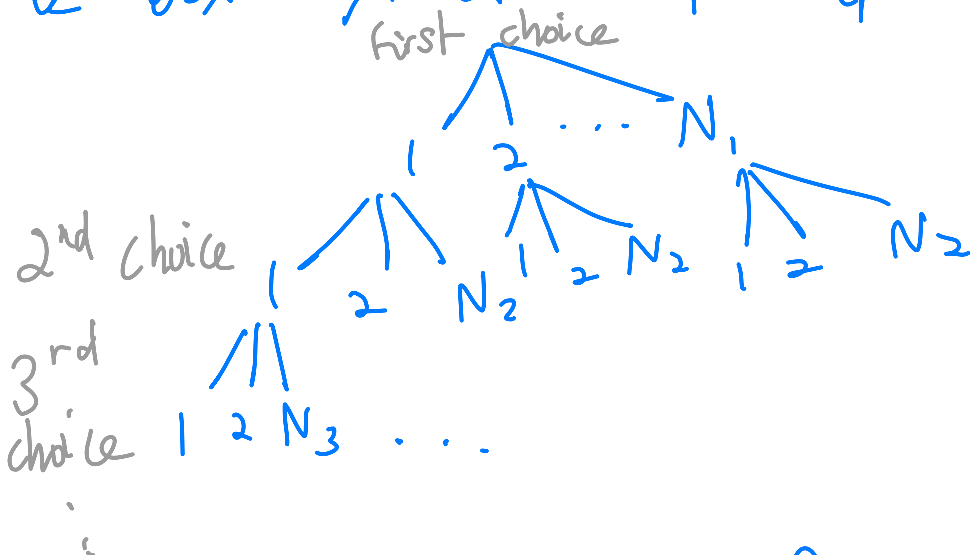


Not sample space

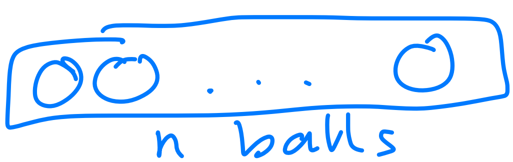
pick a ball from box 1 then pick from box 2

\# of pairs (ordered) $= N_1 \cdot N_2$

k boxes, # of k tuples (pairs & beyond) $= N_1 \cdot N_2 \cdots N_k$



of Permutations $P_{n,k}$



pick k balls sequentially w/out replacement

order matters

$$P_{n,k} = \text{\# of sequences} = n(n-1)\cdots\underbrace{(n-(k-1))}_{n-k+1}$$

$n = k$

$$P_{k,k} = k(k-1)\cdots 1 = k!$$

Combinations

pick k balls w/out replacement. Order does not matter

$\binom{n}{k}$ = # of combinations

$$= \frac{P_{n,k}}{k!} = \frac{n(n-1)\cdots(n-(k-1))(n-k)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!}$$

each combination of k balls = $k!$

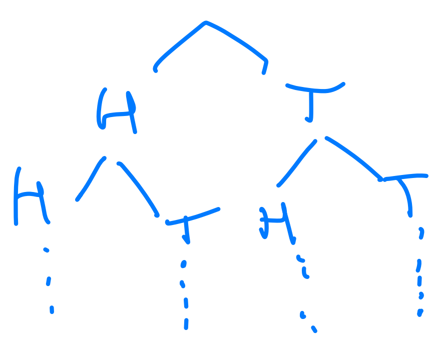
$$P(X=k) = \frac{|A_k|}{|\Omega|} = \frac{\binom{n}{k}}{2^n} \quad \text{for } p = \frac{1}{2}$$

Binomial Formula

$(H+T)^n$

$n=1$ $\quad H+T$

$n=2$ $\quad HH + HT + TH + TT$



$n=3$ $\quad (HH + HT + TH + TT)(H+T)$

$= HH\ HT\ HH\ HT$

sequence of H/T $= \sum_k \binom{n}{k} H^k T^{n-k}$

in other books: $(p+q)^n$

Random Walk on Integers

at each time, flip a fair coin
If H, move forward (+1)
If T, move backward (-1)

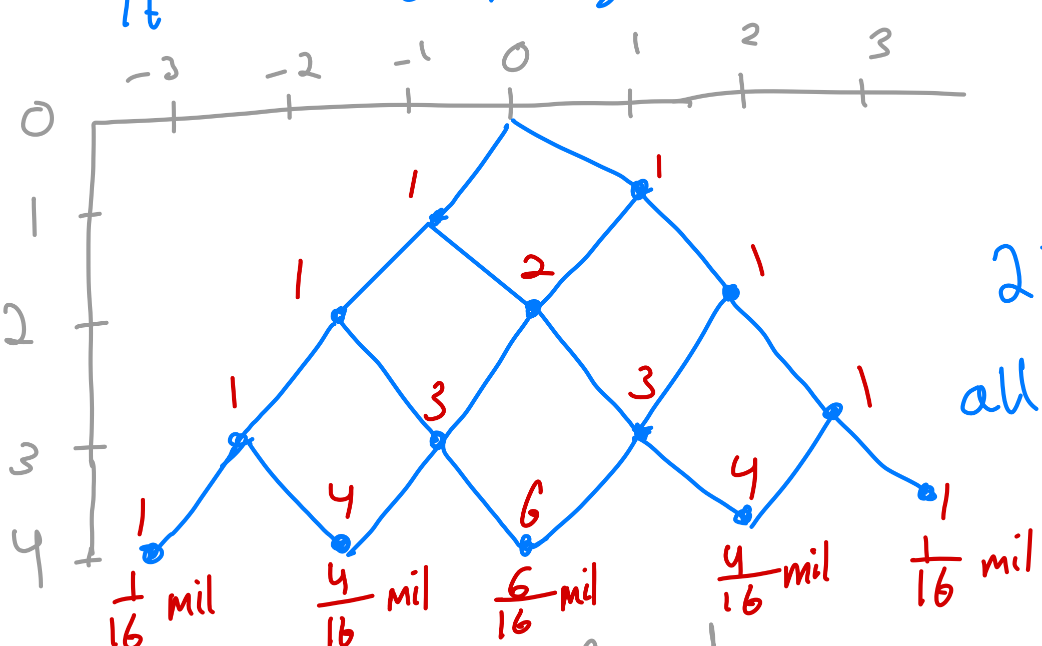
$X_0 = 0$ starting point

$X_t =$ state @ time t

$Y_t =$ # of heads in first t steps

$$X_t = \underbrace{Y_t}_{\#\text{ of H}} - \underbrace{(t - Y_t)}_{\#\text{ of T}} = 2Y_t - t$$

$Y_t \sim Bin(t, \frac{1}{2})$



Galton Board

# of paths is pascal's Δ

2^t paths all equally likely

$$P(X_t = k) = \frac{\#\text{ of paths end at } k}{2^t} =$$

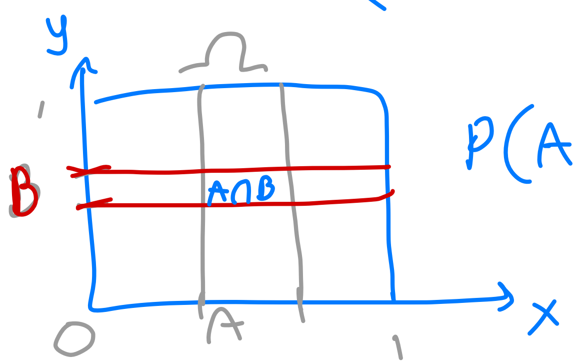
ex $P(X_t = 4)$

General Binomial

$X \sim B(n, p)$ * p may not be $\frac{1}{2}$

Independence: A & are independent if

$$P(A \cap B) = P(A)P(B)$$



$$P(A \cap B) = |A \cap B| = |A|\,|B| = P(A)P(B)$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$