10/6/22

Continuous RV & Properties

$X \sim f(x) \rightarrow$ probability density function (pdf)

$$f(x) = \frac{P(X \in (x, x+\Delta x))}{\Delta x} \quad \Delta x \rightarrow 0$$

$$\underbrace{P(X \in (x, x+\Delta x))}_{\text{basic event}} = f(x)\Delta x$$

$$P(X \in (a,b)) = \int_a^b f(x)\,dx$$

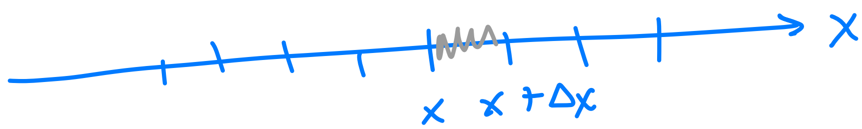
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$E(X) = \int x f(x)\,dx$$

$$E(h(X)) = \int h(x) f(x)\,dx$$

Interpretations

(1) discretization



discrete : $P(X = x) = p(x)$

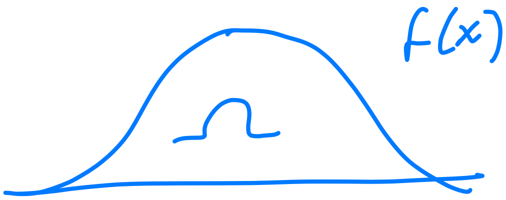
continuous : $P(X \in (x, x+\Delta x)) = f(x)\,\Delta x$

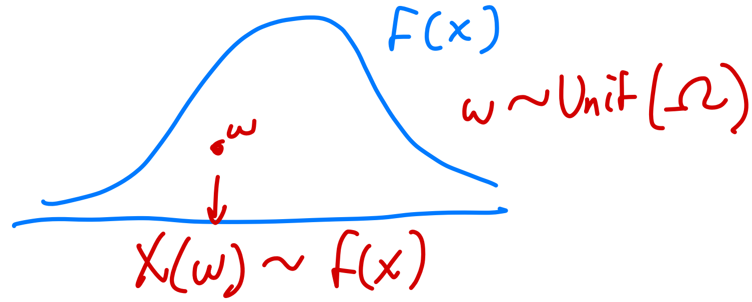
$$P(X \in (a,b)) = \begin{cases} \sum_{x \in (a,b)} p(x) & \text{. . . discrete} \\ \sum_{(a,b)} f(x)\Delta x \rightarrow \int & \text{.. continuous} \end{cases}$$

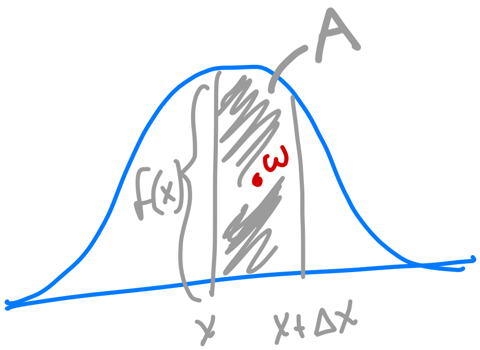
↳ sum over small bins

$$E(X) = \begin{cases} \sum_{x} x\,p(x) & \text{. . . discrete} \\ \sum x\,f(x)\,\Delta x \longrightarrow \int & \text{. . . continuous} \end{cases}$$

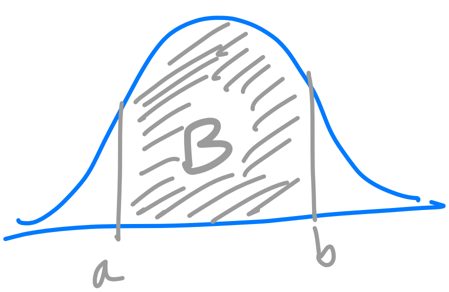
(2) region







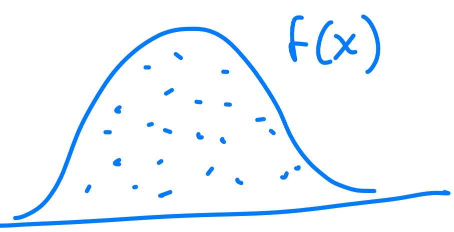
$$P(X(\omega) \in (x, x+\Delta x)) = P(\omega \in A) = \frac{|A|}{|\Omega|} = f(x)\Delta x$$

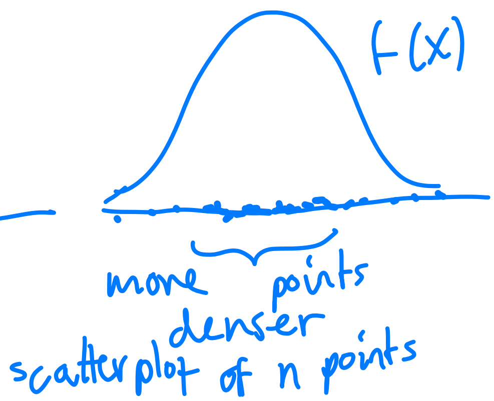


$$P(X(\omega) \in (a,b)) = P(\omega \in B) = \frac{|B|}{|\Omega|} = \int_a^b f(x)\,dx$$

(3) repetition

n points ; $n(x) = \#$ of points in $(x, x+\Delta x)$



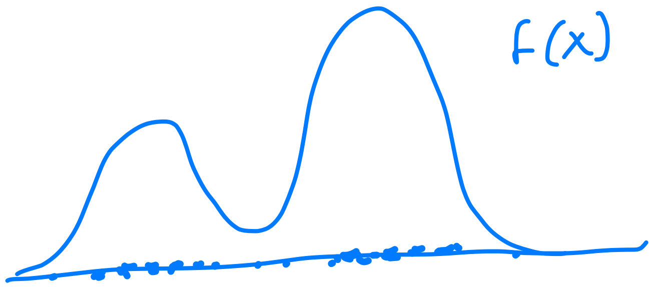


frequency/ sample proportion

$$\frac{n(x)}{n} \xrightarrow[\infty]{n} P(X \in (x, x+\Delta x)) = f(x)\Delta x$$

$$f(x) = \frac{n(x)/n}{\Delta x} \quad \text{as } n \to \infty, \Delta x \to 0$$

$$f(LA) = \frac{\#\text{ people in LA}/\text{total }\#}{\#\text{ people not in LA}}$$



$$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{n}\sum_{\text{bins}} x \cdot n(x)$$

$$= \sum_{\text{bins}} x \cdot \frac{n(x)}{n}$$

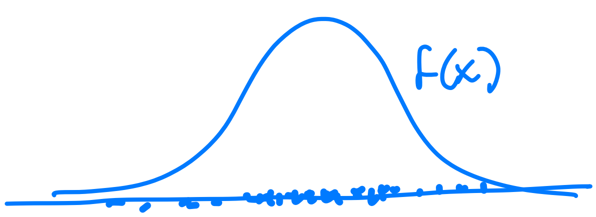
$$= \sum x\, f(x)\Delta x$$

$$\to \int x f(x)\,dx$$

$$\frac{1}{n}\sum_{i=1}^{n} h(X_i) \longrightarrow \int h(x) f(x)\,dx$$

$$E(h(X)$$

(4) population of N people



population scatterplot

$\omega \sim \text{Unif}(\Omega)$

$X(\omega) =$ height of ω

$$f(x) = \frac{N(x)/N}{\Delta x} \qquad N \to \infty, \quad \Delta x \to 0$$

$N(x) = \#$ people in $(x, x+\Delta x)$

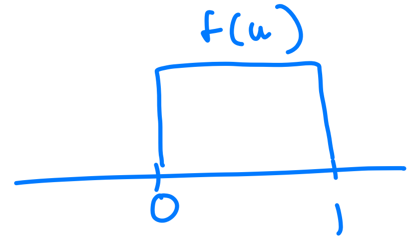
$X \sim f(x)$

$E(X) =$ population average

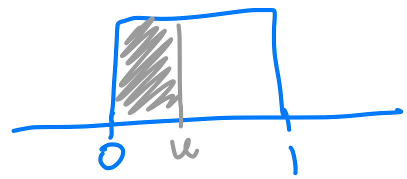
Models

(1) $U \sim \text{Unif}[0,1]$

$$f(u) = \begin{cases} 0 & u < 0 \\ 1 & u \in [0,1] \\ 0 & u > 1 \end{cases}$$



$$F(u) = P(U \leq u) =$$



$$= u$$

cumulative density function

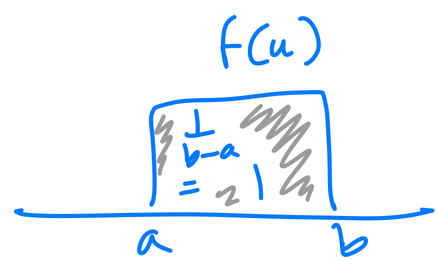
$$E(U) = \int_0^1 u\, f(u)\, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$E(U^2) = \int_0^1 u^2 f(u)\, du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\text{Var}(U) = E(U^2) - E(U)^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

$U \sim \text{Unif}[a,b]$

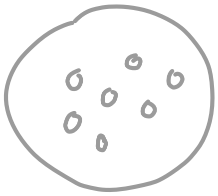
$$f(u) = \frac{1}{b-a} \quad (u \in (a,b))$$



(2) $T \sim \text{Exponential}(\lambda)$

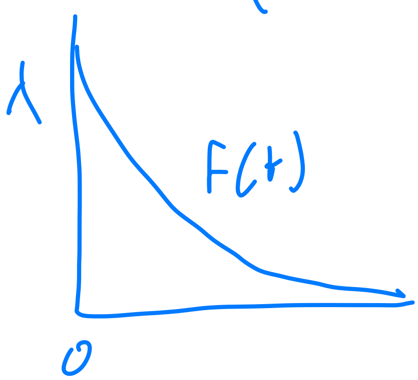
continuous version of geometric

ex.) time until particle decay follows exp. distribution

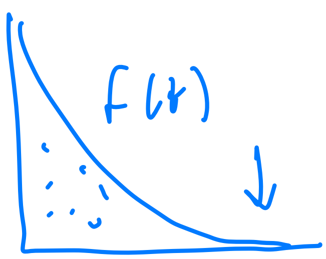
atom 

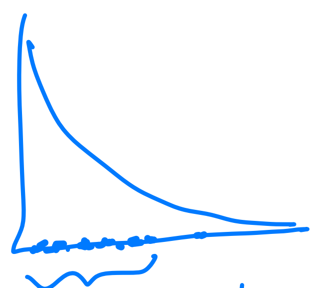
protons neutrons/electrons

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad u \in [0,1])$$



ex.) suppose there are 1 mil particles

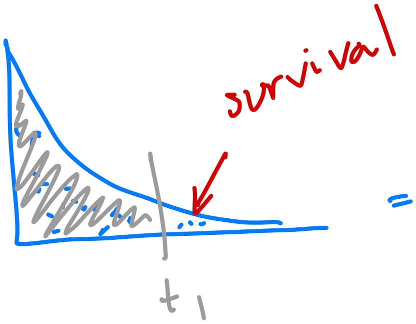




more particles decay here

↳ denser

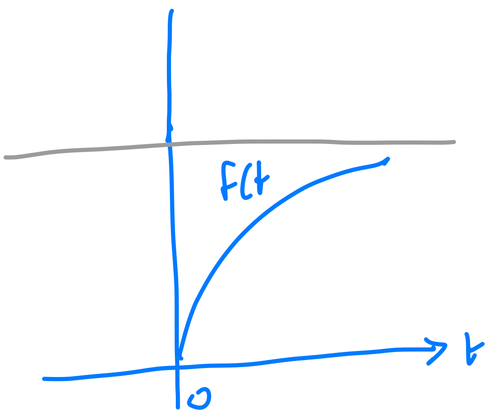
$F(t) = P(T \le t) =$ (proportion) how many particles decay by time t?



$$= \int_0^t f(t)\,dt = \int_0^t \lambda e^{-\lambda t}\,dt$$

$$= -e^{-\lambda t}\Big|_0^t = -e^{-\lambda t} - (-1)$$

$$= 1 - e^{-\lambda t}$$



survival probability $F(t) = P(T > t) = e^{-\lambda t}$

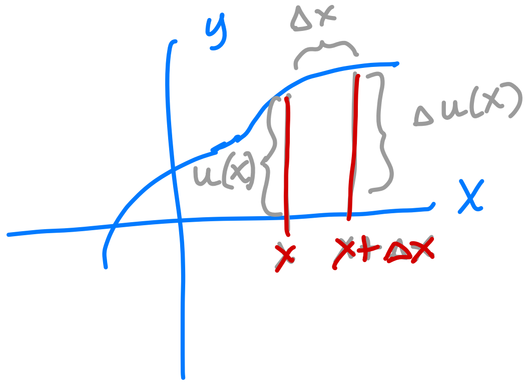
$$E(T) = \int_0^\infty t f(t)\,dt = \int_0^\infty t \lambda e^{-\lambda t}\,dt$$

integration by parts

$$= \int_0^\infty t\, d(-e^{-\lambda t}) = -t e^{-\lambda t}\Big|_0^\infty - \int_0^\infty \left[-e^{-\lambda t}\right] dt$$

$$= \int_0^\infty e^{-\lambda t}\,dt = -\underbrace{\frac{e^{-\lambda t}}{\lambda}}_{0 \text{ when } t \to \infty}\Big|_0^\infty = 0 - \left(-\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

Integral by parts



$$u'(x) = \frac{\Delta u}{\Delta x} = \text{slope (rate of change)}$$

$$u'(x) = \frac{d}{dx}u(x)$$

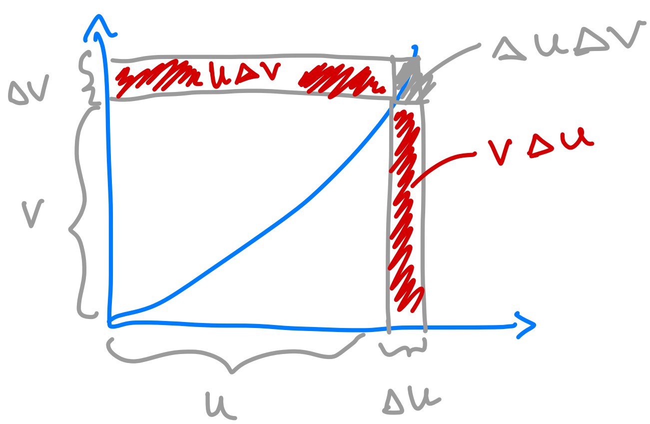
$$du(x) = u'(x)\,dx$$

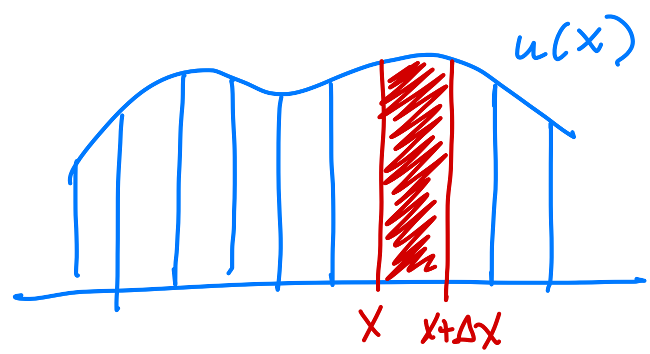
$$d(uv) = u\,dv + v\,du$$

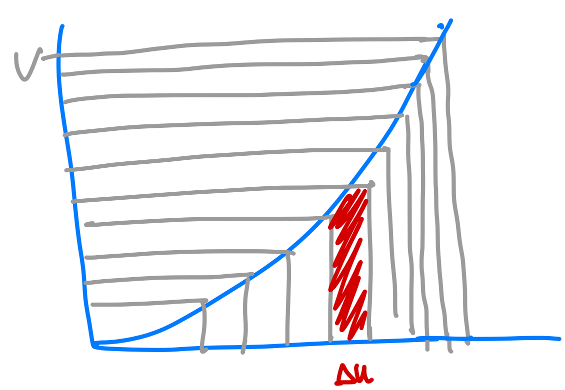
$$\frac{d}{dx}\big(u(x)v(x)\big) = u'(x)v(x) + u(x)v'(x)$$ product rule

$$d\big(u(x)v(x)\big) = u(x)v'(x)dx + v(x)u'(x)dx$$

$$= u(x)\,dv(x) + v(x)\,du(x)$$



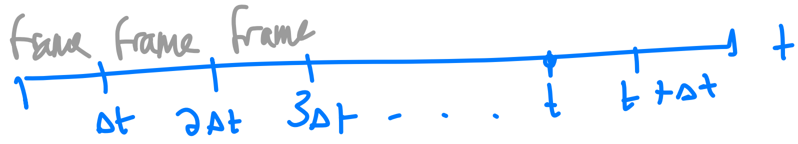




$$\int_u^b u(x)dx = \text{Area}$$

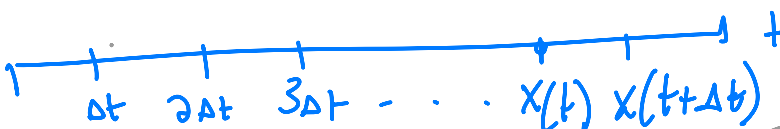
$$uv = \int v\,du + \int u\,dv$$

ex.) make a movie



treating it like discrete time,
but @ end of calculation, let $\Delta t \to 0$

ex.) bank account



$\frac{1}{365}$ if talking about years

$$X(t+\Delta t) = (1 + \underbrace{r}_{\text{interest rate}} \Delta t)\, X(t)$$

$$\frac{X(t+\Delta t) - X(t)}{\Delta t} = r\, X(t)$$

$$\frac{dX(t)}{dt} = r\, X(t)$$

$$dX(t) = r\, X(t)\, dt$$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

$$e^{\Delta x} = 1 + \Delta x + o(\Delta x)$$

small "o"

$$= 1 + \Delta x = \frac{\Delta x^2}{2} + o(\Delta x)^2$$

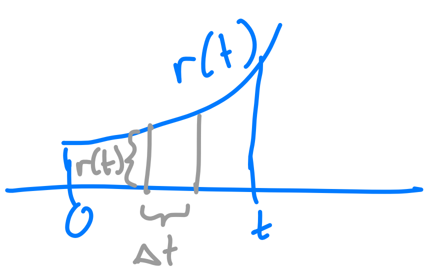
$$X(t) = X(0)\left(1 + r\Delta t\right)^{\frac{t}{\Delta t}}$$

$$= X(0)\, e^{r\Delta t \cdot \frac{t}{\Delta t}} = X(0)\, e^{rt}$$

$$X(t) = X(0) \prod_t \left(1 + r(t)\Delta t\right)$$

$$= X(0) \prod_t e^{r(t)\Delta t}$$

$$= X(0)\, e^{\sum r(t)\Delta t} \rightarrow X(0)\, e^{\int r(t)\,dt}$$



Poisson Process

ex) In each small period, flip a coin independently
(each particle

$$P(\text{head}) = \lambda \Delta t$$

decay   rate of decay

proportion of particles
decay in $(t, t+\Delta t)$

$$\tilde{T} \sim \text{Geometric}(p = \lambda \Delta t)$$

$$T = \tilde{T} \Delta t$$

ex.)

If you flip a head → earthquake

X = # of earthquakes in $[0, t]$

$$X \sim \text{Binomial}\left(n = \frac{t}{\Delta t}, p = \lambda \Delta t\right) \xrightarrow[0]{\Delta t} \text{Poisson}(\lambda t)$$