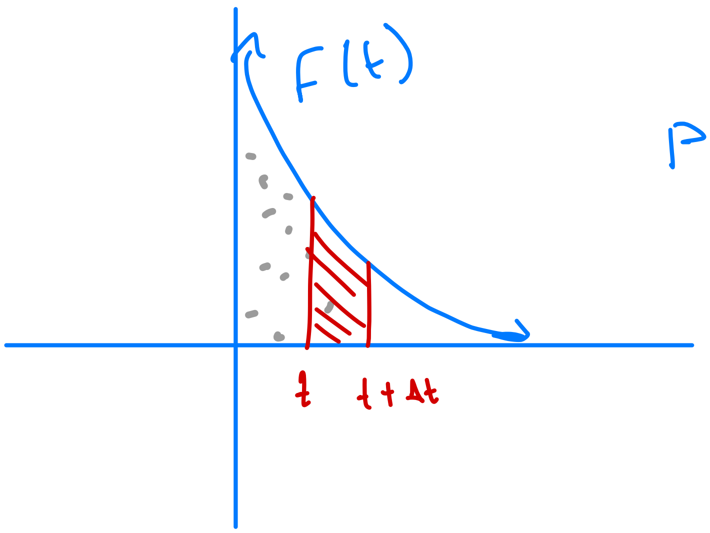
10/11/22

$T \sim \text{Exp}(\lambda)$

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

particle decay, 1 million particles



$$P(T \in (t, t+\Delta t)) = \underbrace{f(t)\,\Delta t}_{\text{area}}$$

Poisson Process

Flip a coin ind. in each period

$$P(H) = \lambda \Delta t$$

$n(t) =$ # of surviving particles

$n(0) = 1$ mil

@ $t+\Delta t$

$$E\left(n(t+\Delta t) \mid n(t)\right) = n(t) - n(t)\underbrace{\lambda \Delta t}_{\text{proportion of particles that will decay}}$$

$$= n(t)(1-\lambda \Delta t)$$

↳ like interest rate

$$P\left(T \in (t, t+\Delta t)\right) = P(\text{Tail}, \text{Tail}, \dots, \text{head})$$

prob of getting tail

$$= (1-\lambda\Delta t)^{\frac{t}{\Delta t}} \lambda \Delta t$$

If $\Delta t \to 0$, $\left(e^{-\lambda \cancel{\Delta t}}\right)^{\frac{t}{\cancel{\Delta t}}} \lambda \Delta t = \lambda e^{-\lambda t} \Delta t$

$$E(T) = \frac{1}{\lambda}$$

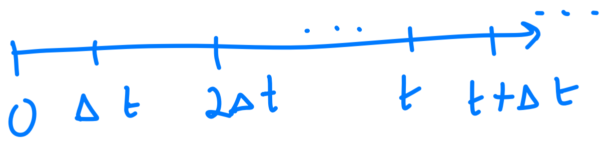
$$\tilde{T} \sim \text{Geo}(p = \lambda \Delta t)$$

↳ prob of flipping until first head

$$E(\tilde{T}) = \frac{1}{p} = \frac{1}{\lambda \Delta t}$$

$T = \underbrace{\tilde{T}}_{\text{period}} \underbrace{\Delta t}_{\text{time}}$ $\quad E(T) = E(\tilde{T})\Delta t = \frac{1}{\lambda}$

ex.) Earthquake



X = # of earthquakes w/in $[0, t]$

$$X \sim \text{Bin}\left(\frac{t}{\Delta t}, p = \lambda \Delta t\right) \xrightarrow[0]{\Delta t} \text{Poisson}$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k-1)}{k!} (\lambda \Delta t)^k (1-\lambda \Delta t)^n (1-\lambda \Delta t)^k$$

$$= \frac{n\Delta t(n\Delta t - \Delta t)\cdots(n\Delta t - (k-1)\Delta t)}{k!} \lambda^k (1-\lambda \Delta t)^n (1-\lambda \Delta t)^k$$

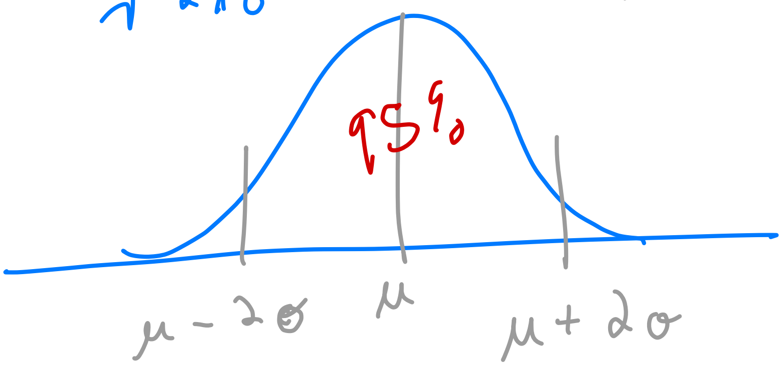
$$\xrightarrow[0]{\Delta t} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$E(X) = np = \frac{t}{\Delta t} \lambda \Delta t = \lambda t$$

$$\lambda = \frac{E(X)}{t}$$ rate ("1 earthquake per 10 yrs")

$X \sim N(\mu, \sigma^2)$ Normal / Gaussian

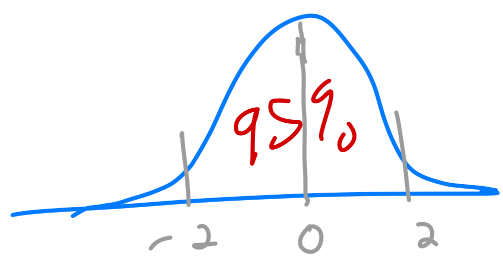
$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$ ... prob. density function



$X \sim N(0, 1)$ Standard Normal

$\mu = 0 \quad \sigma^2 = 1$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$



$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} = 0$$

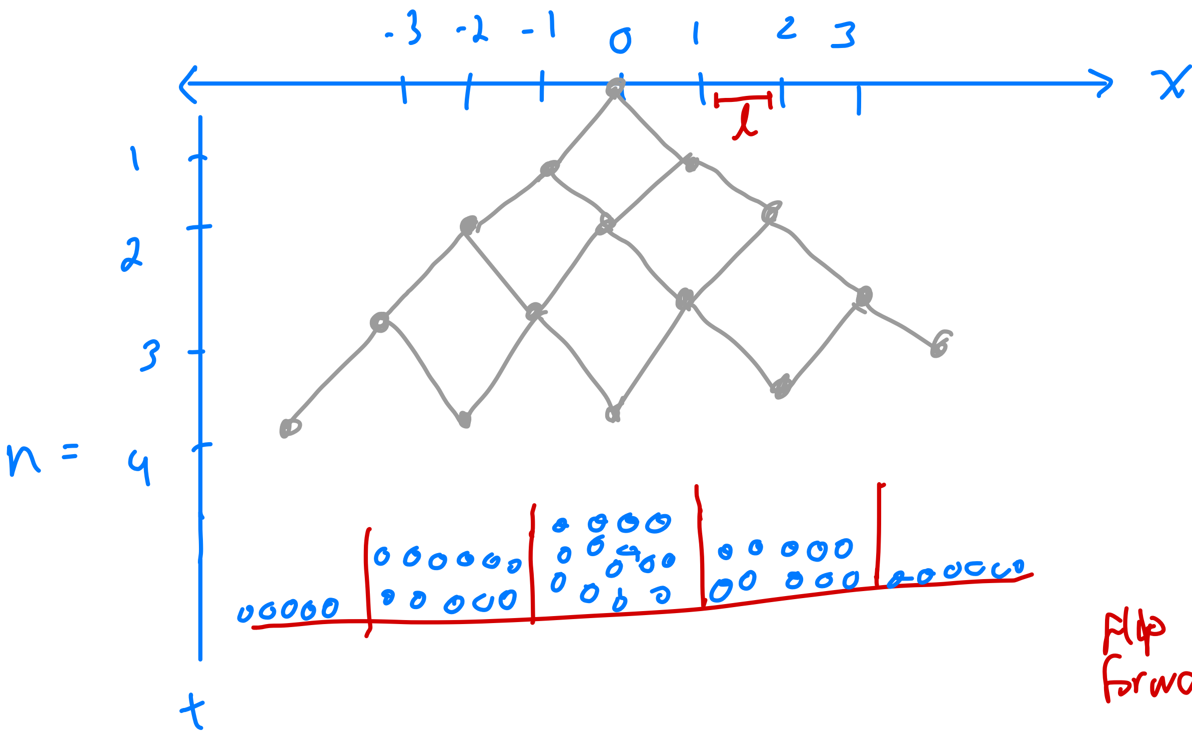
$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = -\int_{-\infty}^{\infty} x \, d f(x)$$

integration by parts

$$= -\left[x f(x)\right]_{-\infty}^{\infty} - \int_{\infty}^{\infty} f(x) dx = 1$$

$$Var(X) = E(X^2) - E(X)^2 = 1$$

random walk on integers



flip H, move forward w/ $p = \frac{1}{2}$.

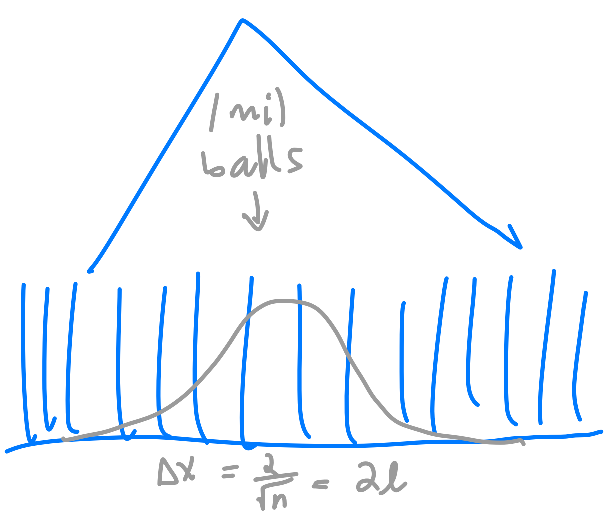
large n — number of steps

$$Y \sim Bin(n, \tfrac{1}{2})$$

$$X = Y \cdot (+1) + (n-Y)(-1)$$
$$= 2Y - n$$

position measured by steps

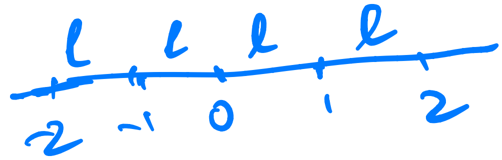
if flip coin 30 x
10 back & 20 forward
then X gives position



Y (steps right)	0	1	2	...	n
X (position of ball)	$-n$	$-n+2$	$-n+4$		n

spacing = 2

with spacing ℓ



$$X = (2Y - n)\,\ell$$

$$E(X) = (2E(Y) - n)\,\ell = \left(2\cdot\frac{n}{2} - n\right)\ell = 0$$

$$Var(X) = (2\ell)^2\, Var(Y) = (2\ell)^2\,\frac{n}{4} = n\ell^2 = 1$$

$$X = (2Y - n)\cdot\frac{1}{\sqrt{2}}$$

so $\ell = \dfrac{1}{\sqrt{n}}$

Spacing of $X = 2\ell = \dfrac{2}{\sqrt{n}} = \Delta X$

$E(\#\text{ of balls in bin } x)$

$\propto$ area of bin $x = f(x)\,\Delta t = p$ ... prob. mass function

$$P(X \in (a,b)) = \sum_{\text{bins in }(a,b)} f(x)\,\Delta X$$ convergence in distribution

$$= \int_a^b f(x)\,dx$$

$f(x) = \dfrac{p(x)}{\Delta t}$ ... probability density function

$\mu = E(Y) = \frac{n}{2}$ $\quad \sigma^2 = Var(Y) = \frac{n}{4}$ $\quad \sigma = \frac{\sqrt{n}}{2}$

$$X = \frac{Y-\mu}{\sigma} = \frac{Y - \frac{n}{2}}{\frac{\sqrt{n}}{2}}$$ standardized

$\leftarrow$ spacing

$$Y = \frac{n}{2} + \underbrace{X \frac{\sqrt{n}}{2}}_{d}$$

$$p(0) = P(X=0) = P\left(Y = \frac{n}{2}\right) = \frac{\binom{n}{\frac{n}{2}}}{2^n} = \frac{n!}{\left(\frac{n}{2}\right)!^2 \cdot 2^n}$$

Sterling formula

$$\sim \frac{\sqrt{2\pi n}\ \cancel{n^n}\ \cancel{e^{-n}}}{\left(\sqrt{2\pi\frac{n}{2}}\ \cancel{\left(\frac{n}{2}\right)^{\frac{n}{2}}}\ \cancel{e^{-\frac{n}{2}}}\right)^2 \cancel{2^n}}$$

double check if correct formula

$$= \frac{\sqrt{n}}{\sqrt{2\pi}\ \frac{n}{2}} = \underbrace{\frac{1}{\sqrt{2\pi}}}_{f(x)}\ \underbrace{\frac{2}{\sqrt{n}}}_{\Delta x}$$

$$p(x) = P(X = x) = P\left(Y = \frac{n}{2} + \underbrace{x\frac{\sqrt{n}}{2}}_{d}\right)$$

$$\frac{p(x)}{p(0)} = \frac{\binom{n}{\frac{n}{2}+d}/2^n}{\binom{n}{\frac{n}{2}}/2^n} = \frac{\frac{\cancel{n!}}{\left(\frac{n}{2}+d\right)!\left(\frac{n}{2}-d\right)!}}{\frac{\cancel{n!}}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}} = \frac{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{\left(\frac{n}{2}+d\right)!\left(\frac{n}{2}-d\right)!}$$

$$= \frac{\left(\frac{n}{2}-d+1\right)\left(\frac{n}{2}-d+2\right)\cdots\left(\frac{n}{2}\right)}{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}+2\right)\cdots\left(\frac{n}{2}+d\right)}$$

d terms

d terms

divide by $\frac{n}{2}$

$$= \frac{1\left(1-\frac{1}{n/2}\right)\left(1-\frac{2}{n/2}\right)\cdots\left(1-\frac{d-1}{n/2}\right)}{\left(1+\underbrace{\frac{1}{n/2}}_{\delta}\right)\left(1+\frac{2}{n/2}\right)\cdots\left(1+\frac{d}{n/2}\right)}$$

$$= \frac{(1-\delta)(1-2\delta)\cdots(1-(d-1)\delta)}{(1+\delta)(1+2\delta)\cdots(1+d\delta)}$$

$$= \frac{e^{-\delta}e^{-2\delta}\cdots e^{-(d-1)\delta}}{e^{\delta}e^{2\delta}\cdots e^{d\delta}}$$

$$= \frac{e^{-\delta(1+2+\cdots+(d-1))}}{e^{\delta(1+2+\cdots+d)}}$$

$$= \frac{e^{-\delta d(d-1)/2}}{e^{\delta d(d+1)/2}} = e^{-\delta\left(d(d-1)/2 + d(d+1)/2\right)}$$

$$= e^{-\delta d^2} = e^{-\frac{1}{n/2}x^2\frac{n}{4}} = e^{-\frac{x^2}{2}}$$

Then

$$p(x) = p(0)\, e^{-\frac{x^2}{2}}$$
$$= \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\, \Delta x$$

$Z_i \sim Ber\left(\frac{1}{2}\right)$

z	-1	1
$p(z)$	$\frac{1}{2}$	$\frac{1}{2}$

$$X \sim \sum_{i=1}^{n} z_i\, l = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} z_i \xrightarrow{\text{according to CLT}} N(0, 1)$$