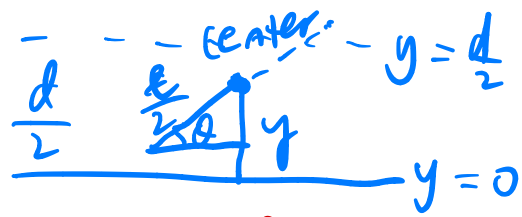
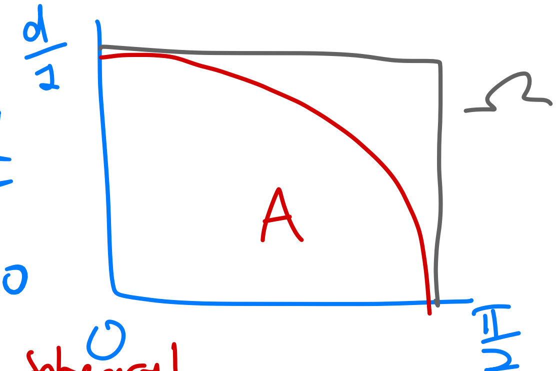
10/13/22

HW 1

Problem 1

a)





compute area of A by integral

$y \le \frac{\ell}{2}\sin\theta$

b) $\frac{1}{\hat{\pi}}$ $\quad E\left(\frac{1}{\hat{\pi}}\right)$ $\quad Var\left(\frac{1}{\hat{\pi}}\right)$ $\quad sd = \sqrt{Var\left(\frac{1}{\hat{\pi}}\right)}$

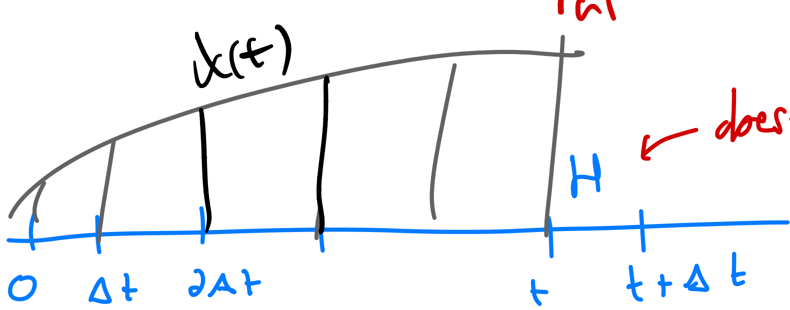
Problem 2

$$P(T \in (t, t+\Delta t)) = \left[\prod_0^t \underbrace{(1-\lambda(t)\Delta t)}_{\text{approximation}}\right]\lambda(t)\Delta t$$

range from 0 to t

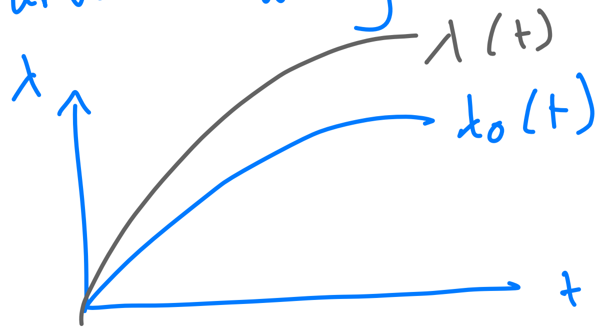
$$= \prod_0^t e^{-\lambda(t)\Delta t}\,\lambda(t)\Delta t = e^{-\sum_0^t \lambda(t)\Delta t}\,\lambda(t)\Delta t$$

$$= e^{-\int_0^t \lambda(t)\,dt}\,\lambda(t)\Delta t$$ intensity function at t



← doesn't happen until after t

survival analysis



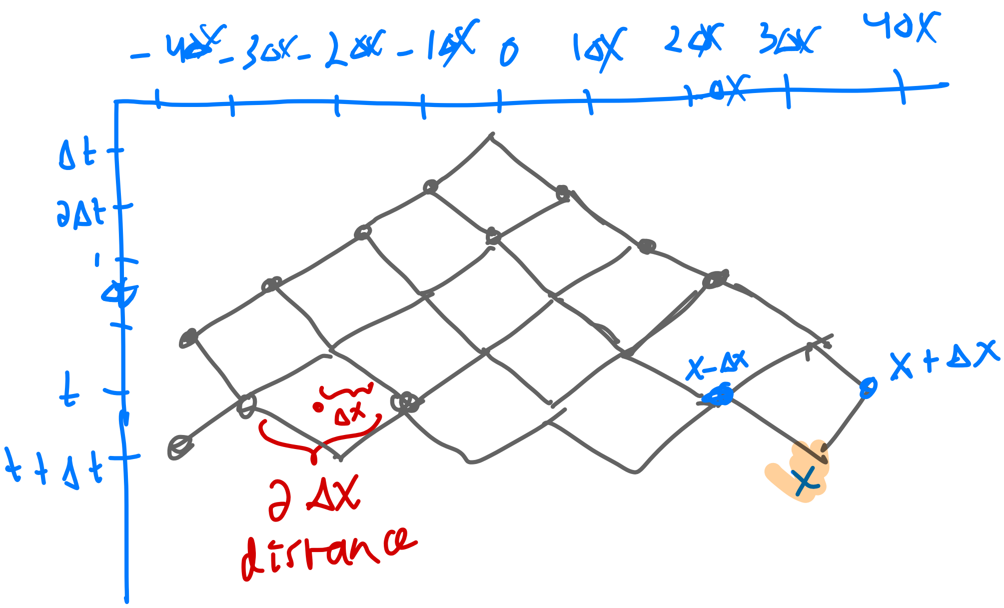
patient rate

baseline rate

$$\log \frac{\lambda(t)}{\lambda_0(t)} = X^T \beta$$

(age, gender, treatment)

Problem 3



n layers

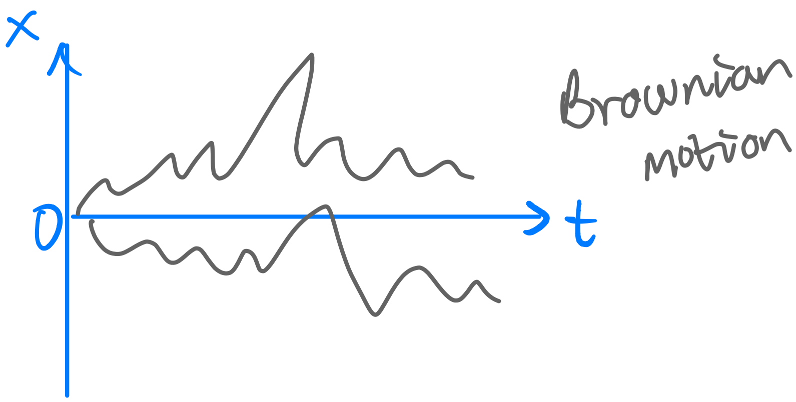
$$X = Y(+1) + (n-Y)(-1)$$
$$= (2Y - n)\Delta x$$

$$E(X) = 0$$

$$\mathrm{var}(X) = 4\Delta x^2 \cdot \mathrm{var}(Y) = n\Delta x^2 = \frac{t}{\Delta t}\Delta x^2$$

$$= t\left(\frac{\Delta x^2}{\Delta t}\right) = \sigma^2 t \qquad \sigma^2 = \frac{\Delta x^2}{\Delta t} \qquad \Delta x = \sigma\sqrt{\Delta t}$$

$$\text{velocity} = \frac{\Delta x}{\Delta t} = \frac{\sigma\sqrt{\Delta t}}{\Delta t} = \frac{\sigma}{\sqrt{\Delta t}} \quad \xrightarrow[\Delta t \to 0]{} \infty$$



$$X(t+\Delta t) = X(t) + \varepsilon(t)\Delta x$$

stochastic differential equation
one particle motion

$\varepsilon(t)$	-1	$+1$
prob	$\frac{1}{2}$	$\frac{1}{2}$

$$P_{t+\Delta t}(x) = \Big(P_t(x-\Delta x) + P_t(x+\Delta x)\Big)/2$$

$$\frac{P_{t+\Delta t}(x) - P_t(x)}{\Delta t} = \frac{\big(P_t(x+\Delta x) - P_t(x)\big) - \big(P_t(x) - P_t(x-\Delta x)\big)}{2\Delta t}$$

$$= \sigma^2 \frac{\frac{P_t(x+\Delta x) - P_t(x)}{\Delta x} - \frac{P_t(x) - P_t(x-\Delta x)}{\Delta x}}{2\Delta x}$$

$$\Delta x = \sigma\sqrt{\Delta t}$$
$$\Delta t = \Delta x^2/\sigma^2$$

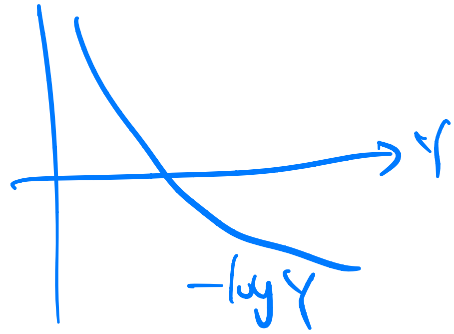
$$\frac{dP_t(x)}{dt} = \frac{\sigma^2}{2}\frac{dP_t(x)}{dx^2}$$

partial differential equation
heat equation
distribution of 1 mil particles

Problem 5

$$E_p\left[\log \frac{p(x)}{q(x)}\right] = \sum_x p(x) \log \frac{p(x)}{q(x)}$$

$$Y = \frac{p(X)}{q(X)}$$



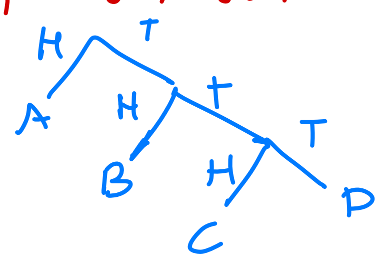
$$\sum_x p(x) \frac{q(x)}{p(x)} = 1$$

$$E_p\left[-\log \frac{q(x)}{p(x)}\right] = E_p[\underbrace{-\log}_{h} Y] \geq -\log(E(Y))$$

ex.)

X	A	B	C	D
p(x)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$
	H	TH	TTH	TTT
	1	0 1	0 0 1	1 1 1

prefix code



$$E(\text{\# of coin flips}) = E_p(-\log_2 p(x))$$
code length

X	A	B	C	D
q(x)	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$

$$E(\text{code length}) = E_p[-\log_2 q(x)]$$

extra length: $D_{KL}(p \| q) = E_p\left[\log_2 \frac{p(x)}{q(x)}\right]$

redundancy

Transformation of continuous RV

$X \sim N(0,1)$

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

density of X

linear transformation

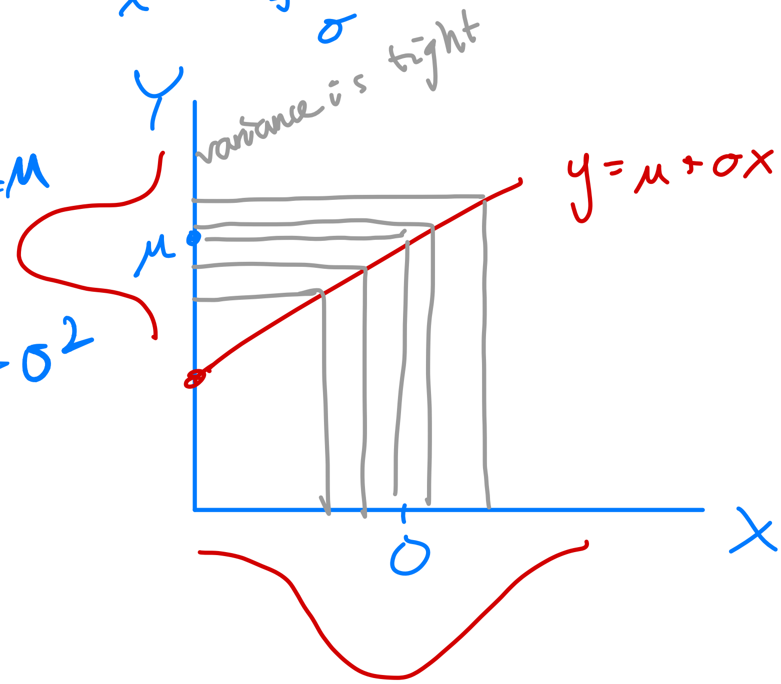
$y = \mu + \sigma x$

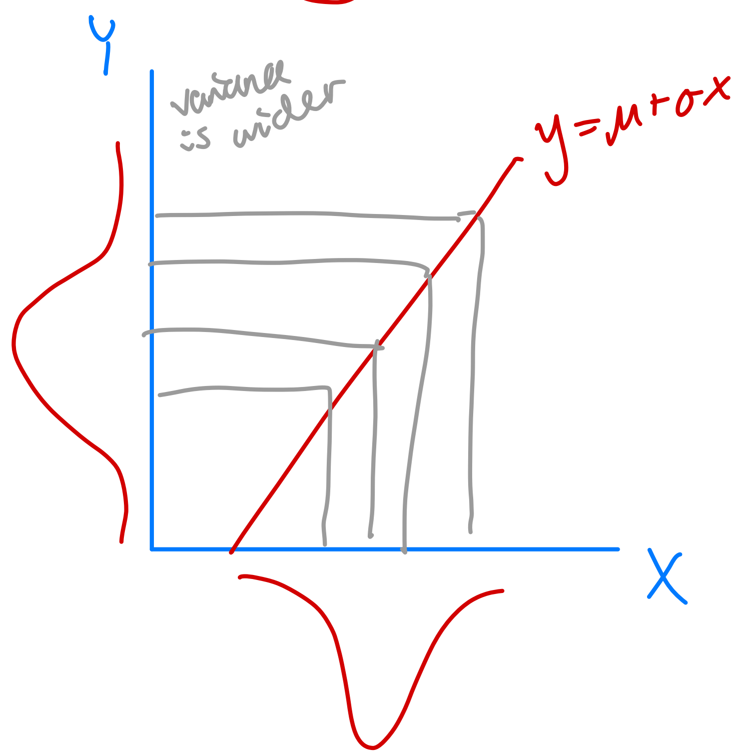
$x = \frac{y-\mu}{\sigma}$

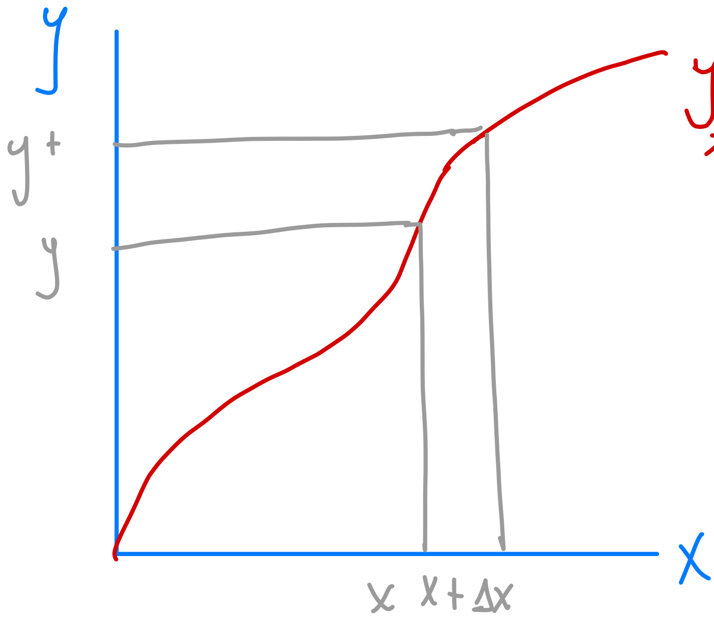
$Y = \mu + \sigma X$

$E(Y) = \mu + \sigma E(X) = \mu$

$Var(Y) = \sigma^2 Var(X) = \sigma^2$







$y = h(x)$

$x = h^{-1}(y) = g(y)$

$$P\big(X \in (x, x+\Delta x)\big) = P\big(Y \in (y, y+\Delta y)\big)$$

$$f_X(x)\,\Delta x = f_Y(y)\,\Delta y$$

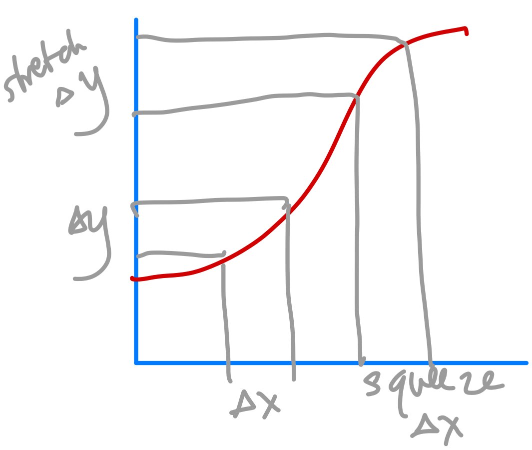
$$dF_X(x) \qquad dF_Y(y)$$

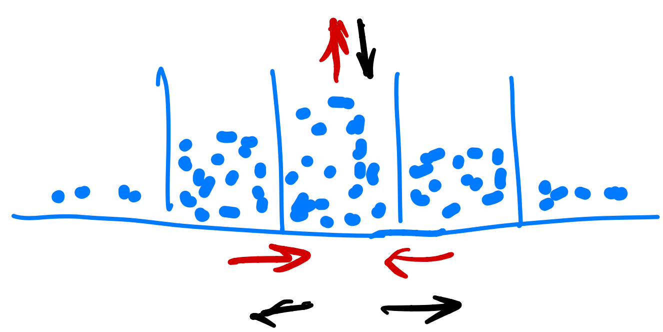
$$f_Y(y) = f_X(x)\,\frac{\Delta x}{\Delta y}$$ ... find density of Y

$$= f_X(g(y))\,|g'(y)|$$

symbolically,

$$X \sim f_X(x)\,dx = f_X(g(y))\,dg(y) = f_X(g(y))\,g(y)\,dy = f_Y(y)\,dy \sim Y$$





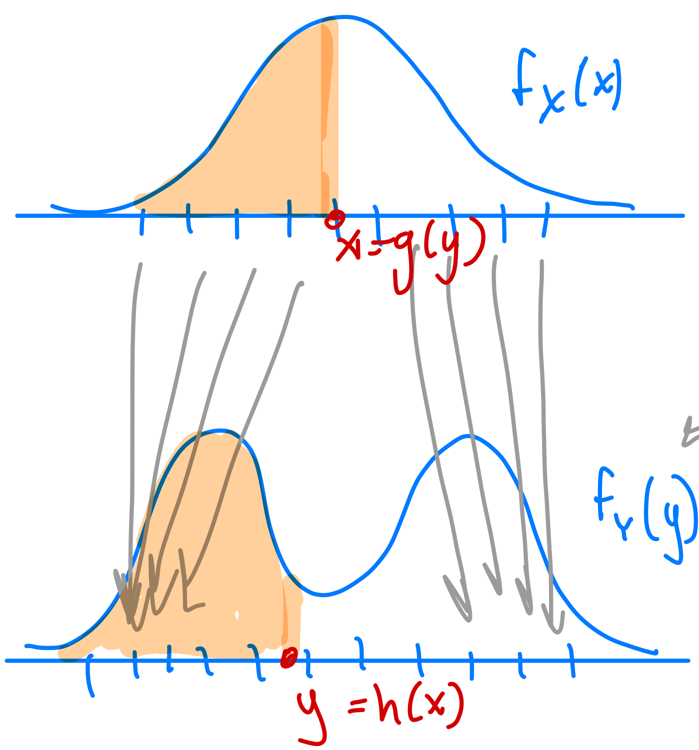
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$X \sim f_X(x)\,dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \underbrace{d\,\frac{y-\mu}{\sigma}}_{\frac{1}{\sigma}dy}$$

$$y = \mu + \sigma$$

$$x = \frac{y-\mu}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$



try to find tranformation to map

Monotone mapping

② $dF_X(x) = d\,F_Y(y)$

①

$$\begin{tikzcd} F_X(x) \arrow[rr, bend left, "h"] & & F_Y(y) \arrow[ll, bend right, "g"'] \end{tikzcd}$$

preserve order

if x is 100^{th} value $\rightarrow$ should be 100^{th} value in Y

same area

you can transform one rv to another rv

$$y = \underbrace{F_Y^{-1}}_{h}(F_X(x))$$

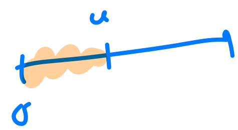
inversion method to generate random variables

$$y = h(x)$$

$X \sim Unif[0, 1]$

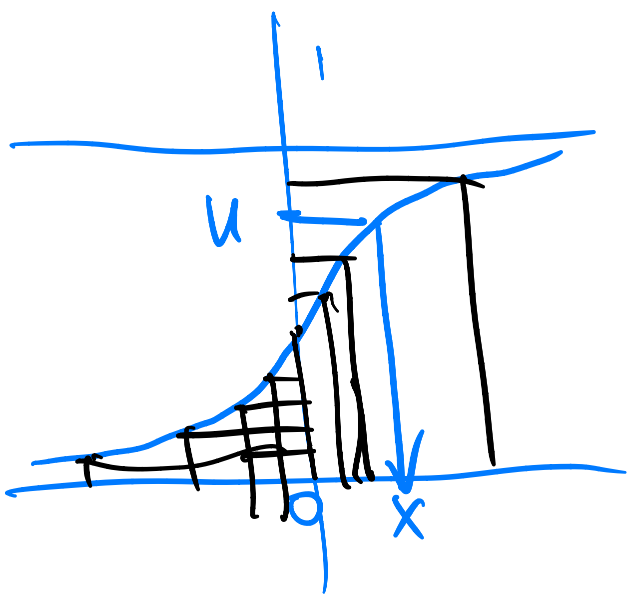
$F_U(u) = u$

prob of $U \le u$



$\Rightarrow \frac{u}{1}$

$$y = F^{-1}(u)$$



sample 0 to 1 vertically and then look horizontal ↳ find u on vertical, then get x

$F'(x) = f(x)$

slope density

gre score	percentile
x	$F(x) = u$
$F^{-1}(u)$	u

given a percentile of ar andom percentile (Uniform)