Lectures on AdS/CFT from the Bottom Up

Jared Kaplan

Department of Physics and Astronomy, Johns Hopkins University

Abstract

AdS/CFT from the perspective of Effective Field Theory and the Conformal Bootstrap.

Contents

1 Introduction

1.1 Why is Quantum Gravity Different?

Fundamental physics has been largely driven by the reductionist program – look at nature with better and better magnifying glasses and then build theories based on what you see. In a quantum mechanical universe, due to the uncertainty relation, we cannot refine our microscope without using correspondingly larger momenta and energies. Hence the proliferation of larger and larger colliders.

Sometimes when you go to higher energies (shorter distances), physics changes radically at some critical energy scale. Theoretical particle physicists get up in the morning to find and understand the mechanisms behind new, frontier energy scales. For example, in Fermi's theory of beta decay there was the dimensionful coupling

$$G_F \sim \frac{1}{(293\ \text{GeV})^2} \sim \left(7 \times 10^{-17}\ \text{m}\right)^2 \tag{1.1}$$

and Fermi's theory was seen to break down at energies somewhere below 293 GeV. Fermi's theory has been replaced by the electroweak theory, where physics does change drastically before this energy scale – one starts to produce new particles, the W and Z bosons. Even more dramatically, when physicists first probed the QCD scale, around a GeV (or 2×10^{-14} m), we found a plethora of new strongly interacting particles; the resulting confusion led to the first S-Matrix program, String Theory, and one of the first uses of Effective Field Theory in particle physics. But we've since learned to describe both the QCD and the Weak scale, and much else, using local quantum field theories, and they no longer remain a mystery.

Gravity appears to differ qualitatively from these examples. The point is that if you were to try to resolve distances of order the Planck length $\ell_{pl} \sim 10^{-35}$ meters, you would need energies of order the Planck mass, $\hbar/\ell_{pl}$, at which point you would start to make black holes. We are fairly certain of this because the universally attactive nature of gravity permits gedanken experiments in which we could make black holes without passing through a regime of physics we don't understand. Pumping up the energy further just results in *larger and larger* black holes, and the naive reductionist program comes to an end. So it seems that there's more to understanding quantum gravity than simply finding a theory of the "stuff" that's smaller than the Planck length – in fact there is no well-defined notion of smaller than ℓ_{pl}.

In hindsight, we have had many hints for how to proceed, and most of the best hints were decades old even back in 1997, when AdS/CFT was discovered. The classic hint comes from Black Hole thermodynamics, in particular the statement that Black Holes have an entropy

$$S_{BH} = \frac{A}{4\ell_{pl}^2} \tag{1.2}$$

proportional to their surface area, not their volume. Since you can throw anything into a black hole, and entropies must increase, this BH entropy formula should be a fundamental feature of the universe, and not just a property of black holes. The largest amount of information you can store in any region in spacetime will be proportional to its surface area. This is radical, and differs from generic non-gravitational systems, e.g. gases of particles.

So spacetime has to die – at short distances it stops making sense, and it doesn't seem to store information in its bulk, but only on its boundary. What can we do with this idea? Well, since the 60s it has been known that gravitational energy is not well-defined locally, but it can be made well-defined at, and measured from, infinity. But energy is nothing other than the Hamiltonian. So in gravitational theories, the Hamiltonian should be taken to live at infinity. Perhaps there's a whole new description of gravitational physics, with not just a Hamiltonain but all sorts of infinity-localized degrees of freedom... a precisely well-defined theory, from which bulk gravity emerges as an approximation?

1.2 AdS/CFT – the Big Picture

AdS/CFT is many things to many people.

For our purposes, AdS/CFT is the observation that any complete theory of quantum gravity in an asymptotically AdS spacetime defines a CFT. For now you can just view AdS as 'gravity in a box'. By 'quantum gravity' we simply mean any theory that is well-approximated by general relativity in AdS coupled to matter, i.e. scalars, fermions, gauge fields, and perhaps more exotic stuff too, such as membranes and strings. AdS/CFT says that the Hilbert spaces are identical

$$\mathcal{H}_{CFT} = \mathcal{H}_{AdS-QG} \tag{1.3}$$

and all (physical, or ‘global’) symmetries can be matched between the two theories. In particular, the spacetime symmetries or isometries of AdS_{d+1} form the group $SO(2, d)$ and these act on the CFT_d as the conformal group[1], containing the Poincaré group as a subgroup. All states in both theories come in representations of this group. We will explain why quantum field theories in AdS produce CFTs, and we will (hopefully) explain which CFTs have perturbative AdS effective field theory duals.

An incomplete theory of quantum gravity in AdS, such as a gravitational effective field theory (e.g. the standard model of particle physics including general relativity), defines an approximate or effective CFT. Finding a complete theory of quantum gravity, a ‘UV completion’ for any given EFT, amounts to finding an exact CFT that suitably approximates the effective CFT. Among other things, this means that we can make exact statements about quantum gravity by studying CFTs. General theorems about CFTs can be re-interpreted as theorems about all possible theories of quantum gravity in AdS.

When placed in AdS, quantum field theories without gravity define what we’ll refer to as Conformal Theories (CTs). These are ‘non-local’ theories that have CFT-like operators with correlators that are symmetric under the global conformal group and obey the operator product expansion (OPE), but that do not have a local stress-energy tensor $T_{\mu\nu}$. By thinking about simple examples of physics in AdS, we can understand old, familiar results in a new, holographic language.

A famous and revolutionary fact is that (large) black holes in AdS are just CFT states. This means that AdS quantum gravity should be a unitary quantum mechanical theory. Roughly speaking, the temperature and entropy of AdS black holes correspond with the temperature of the CFT and the number of CFT states excited at that temperature, respectively. This suggests that the universality of black hole formation in AdS is dual to the universality of finite temperature physics. Another famous and important thought experiment explains how one obtains QCD-like confinement from strings that dip into AdS.

The purpose of these lectures is to explain these statements in detail. In the next two subsections we will make some philosophical and historical comments; hopefully they will provide some perspective on holography and QFT, but don’t feel discouraged if you do not understand them. They can be skipped on a first pass, especially for more practically or technically minded reader.

1.3 Quantum Field Theory – Two Philosophies

Why is Quantum Field Theory the way it is? Does it follow inevitably from a small set of more fundamental principles? These questions have been answered by two[2] very different, profoundly

[1] In $d = 2$ there are many more symmetries (although no more isometries), and these form the full 2-d conformal group, with Virasoro algebra, as originally shown by Brown and Henneaux.

[2] There’s a third and oldest viewpoint which is also important, and plays a primary role in most textbooks. It lacks the air of inevitability, although it’s very practical: one just takes a classical theory and ‘canonically quantizes’ it. This was how QED was discovered – we already knew electrodynamics, and then we figured out how to quantize it. This viewpoint is certainly worth understanding; Shankar’s text gives a quick explanation of classical $\to$ quantum.

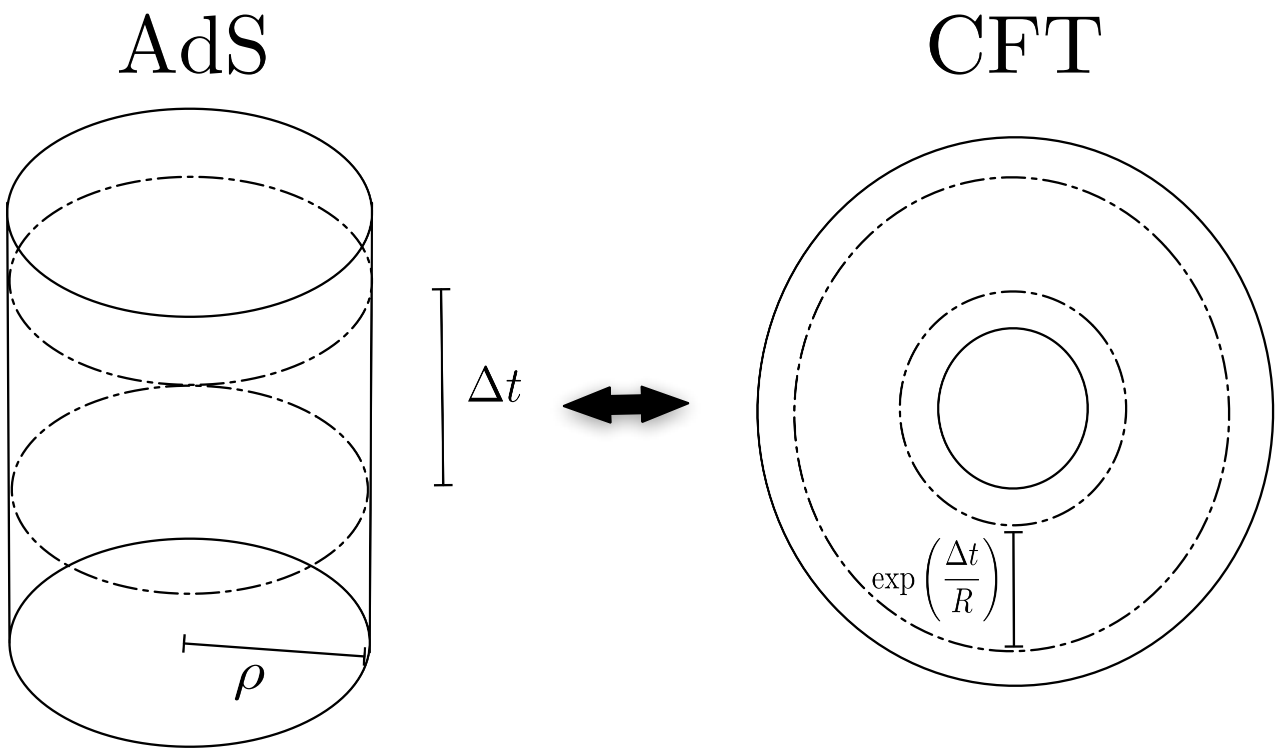

Figure 1: This figure shows how the AdS cylinder in global coordinates corresponds to the CFT in radial quantization. The time translation operator in the bulk of AdS is the Dilatation operator in the CFT, so energies in AdS correspond to dimensions in the CFT. We make this mapping very explicit in section 5.2.2.

compatible philosophies, which I will refer to as *Wilsonian* and *Weinbergian*.

The Wilsonian philosophy is based on the idea of zooming out. Two different physical systems that look quite different at short distances may behave the same way at long distances, because most of the short distance details become irrelevant. In particular, we can think of our theories as an expansion in ℓ_{short}/L, where ℓ_{short} is some microscopic distance scale and L is the length scale relevant to our experiment. We study the space of renormalizable quantum field theories because this is roughly equivalent to the space of universality classes of physical systems that one can obtain by 'zooming out' to long distances. Here are some famous examples

- The Ising Model is a model of spins on a lattice with nearest-neighbor interactions. We can zoom out by 'integrating out' half of the spins on the lattice, leaving a new effective theory for the remainder. However, at long distances the model is described by the QFT with action

$$S = \int d^d x \frac{1}{2} \left((\partial \phi)^2 - \lambda \phi^4 \right) \tag{1.4}$$

The details of the lattice structure become 'irrelevant' at long distances.

- Metals are composed of some lattice of various nuclei along with relatively free-floating electrons, but they have a universal phase given by a Fermi liquid of their electrons. Note that the Fermi temperature, which sets the lattice spacing for the atoms, is around $10,000$ K whereas we are most interested in metals at ~ 300K and below. At these energies metals are very well described by the effective QFT for the Fermi liquid theory. See [1] for a beautiful discussion of this theory and the Wilsonian philosophy. Research continues to understand the effective QFT that describes so-called strange metals associated with high temperature superconductivity.
- Quantum Hall fluids seem to be describable in terms of a single Chern-Simons gauge field; one can show that this is basically an inevitable consequence of the symmetries of theory (including broken parity), the presence of a conserved current, and the absence of massless particles.
- The Standard Model and Gravity. There are enormous hierarchies in nature, in particular from the Planck scale to the weak scale.
- Within the SM, we also have more limited (and often more useful!) effective descriptions of QED, beta decay, the pions and nucleons, and heavy quarks. Actually, general relativity plus 'matter' is another example of an effective description, where the details of the massive matter are unimportant at macroscopic distances (e.g. when we study the motion of the planets, it's irrelevant what they are made of).

So if there is a large hierarchy between short and long distances, then the long-distance physics will often be described by a relatively simple and universal QFT.

Some consequences of this viewpoint include:

- There may be a true physical cutoff on short distances (large energies and momenta), and it should be taken seriously. The UV theory may not be a QFT. Effective Field Theories with a finite cutoff make sense and may or may not have a short-distance = UV completion.
- UV and IR physics may be extremely different, and in particular a vast number of distinct UV theories may look the same in the IR (for example all metals are described by the same theory at long distances). This means that a knowledge of long-distance physics does not tell us all that much about short-distance physics – TeV scale physics may tell us very little about the universe's fundamental constituents.
- Symmetries can have crucially important and useful consequences, depending on whether they are preserved, broken, emergent, or anomalous. The spacetime symmetry structure is essential when determining what the theory can be – high-energy physics is largely distinguished from condensed matter because of Poincaré symmetry.
- QFT is a good approach for describing both particle physics and statistical physics systems, because in both cases we are interested in (relatively) long-distance or macroscopic properties. QFT fails to be a good description when there aren't any interesting degrees of freedom that survive at distances that are long compared to the microscopic scale, e.g. to the lattice spacing.

For a classic review of the Wilsonian picture of QFT see Polchinski [1]. A nice perspective between high-energy and condensed-matter physics is provided by Cardy's book Scaling and Renormalization [2]. Slava Rychkov's notes give a CFT-oriented discussion of some of these ideas.

A natural question: what if we have a theory that does not change when we zoom out? This would be a scale invariant theory. In the case of high energy physics, where we have Poincaré symmetry, scale invariant theories are basically always conformally invariant QFTs,[3] which are called Conformal Field Theories (CFTs). It's easy to think of one example – a theory of free massless particles has this property. One reason why asymptotic freedom in QCD is interesting is that it means that QCD can be viewed as the theory you get by starting with quarks that are arbitrarily close to being free (they have an infinitesimal interaction strength or coupling), and then zooming out. This makes it possible to rigorously define QCD as a mathematical theory.

It seems that all QFTs can be viewed as points along an Renormalization Flow (or RG flow, this is the name we give to the zooming process) from a 'UV' CFT to another 'IR' CFT. Renormalization flows occur when we deform the UV CFT, breaking its conformal symmetry. QCD was an example of this – we took a free theory of quarks and 'deformed it' at high energies by adding a small interaction. This leads to a last implication of the Wilsonian viewpoint as applied to relativistic QFTs:

- Well-defined QFTs can be viewed as either CFTs or as RG flows between CFTs. We can remove the UV cutoff from a QFT (send it to infinite energy or zero length) if it can be interpreted as an RG flow from the vicinity of a CFT fixed point.

So studying the space of CFTs basically amounts to studying the space of all well-defined QFTs. And as we will see, the space of CFTs also includes all well-defined theories of quantum gravity that we currently understand!

The Weinbergian philosophy [3] finds Quantum Field Theory to be the only way to obtain a Lorentz Invariant, Quantum Mechanical (Unitary), and Local (satisfying Cluster Decomposition) theory for the scattering of particles. Formally, a "theory for scattering" is encapsulated by the S-Matrix

$$S_{\alpha\beta} = \langle \alpha_{in} | S | \beta_{out} \rangle \tag{1.5}$$

which gives the amplitude for any "in-state" of asymptotically well-separated particles in the distant past to evolve into any "out-state" of similarly well-separated particles in the future. Some aspects of this viewpoint:

- Particles, the atomic states of the theory, are defined as irreducible representations of the Poincaré group. By definition, an electron is still an electron even if it's moving, or if I rotate around it! This sets up the Hilbert space of incoming and outgoing multi-particle states as a Fock space of free particles. The fact that energies and momenta of distant particles *add* suggests that we can use harmonic oscillators a_p to describe each momentum p, because the harmonic oscillator has evenly spaced energy levels.

[3]This statement is still somewhat controversial, and hasn't been rigorously proven for $d > 2$ dimensions.

- The introduction of creation and annihilation operators for particles is further motivated by the Cluster Decomposition Principle[4]. This principle says that very distant processes don't affect each other; it is the weakest form of locality, and seems necessary to talk sensibly about well-separated particles. Cluster decomposition will be satisfied if and only if the Hamiltonian can be written as

$$H = \sum_{m,n} \int d^3 q_i d^3 k_i \delta \left(\sum_i q_i \right) h_{mn}(q_i, k_i) a^\dagger(q_1) \cdots a^\dagger(q_m) a(k_1) \cdots a(k_n) \tag{1.6}$$

where the function h_{mn} must be a non-singular function of the momenta.

- We want to obain a Poincaré covariant S-Matrix. The S operator defining the S-Matrix can be written as

$$S = T \exp \left(-i \int_{-\infty}^{\infty} dt V(t) \right) \tag{1.7}$$

Note that this involves some choice of t, which isn't very covariant-looking. However, if the interaction $V(t)$ is constructed from a local Hamiltonian density $\mathcal{H}(x)$ as

$$V(t) = \int d^3\vec{x}\, \mathcal{H}(t, \vec{x}) \tag{1.8}$$

where the Lorentz-scalar $\mathcal{H}(x)$ satisfies a causality condition

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0 \quad \text{for} \quad (x-y)^2 \ \text{ spacelike} \tag{1.9}$$

then we will obtain a Lorentz-invariant S-Matrix. How does this come about? The point is that the interactions change the definition of the Poincaré symmetries, so these symmetries do not act on interacting particles the same way they act on free particles. To preserve the full Poincaré algebra with interactions, we need this causality condition.

- Constructing such an $\mathcal{H}(x)$ satisfying the causality condition essentially requires the assembly of local fields $\phi(x)$ with nice Lorentz transformation properties, because the creation and annhiliation operators themselves do not have nice Lorentz transformation properties. The $\phi(x)$ are constructed from the creation and annihilation operators for each species of particle, and then $\mathcal{H}(x)$ is taken to be a polynomial in these fields.

- Symmetries constrain the asymptotic states and the S-Matrix. Gauge redundancies must be introduced to describe massless particles with a manifestly local and Poincaré invariant theory.

- One can prove (see chapter 13 of [3]) that only massless particles of spin ≤ 2 can couple in a way that produces long range interactions, and that massless spin 1 particles must couple to conserved currents J_μ, while massless spin 2 particles must couple to $T_{\mu\nu}$. This obviously goes a long way towards explaining the spectrum of particles and forces that we have encountered.

[4]In AdS/CFT we can *prove* that this principle holds in AdS directly from CFT axioms, but in flat spacetime we have to assume it. This very weak form of AdS locality follows from CFT unitary.

In theories that include gravity, the Weinbergian philosophy accords perfectly with the idea of Holography: that we should view dynamical spacetime as an approximate description of a more fundamental theory in fewer dimensions, which 'lives at infinity'. Holography was apparently not a motivation for Weinberg himself, and his construction can proceed with or without gravity. But the philosophy makes the most sense when we include gravity, in which case the S-Matrix is the only well-defined observable in flat spacetime.

One of the eventual takeaway points from these lectures is that Weinberg's derivation of flat space quantum field theory from desired properties of the S-Matrix can be repeated in the case of AdS/CFT, replacing S-Matrix → CFT correlators. Specifically, quantum field theory in AdS can be derived in an analogous way from properties of correlation functions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle \tag{1.10}$$

of local CFT operators $\mathcal{O}_i$. Specifically:

- Objects in AdS arise as irreducible representations of the conformal group.
- The crossing symmetry of CFT correlators, which gives rise to 'the Bootstrap Equation', can be used to prove the Cluster Decomposition Principle in AdS, guaranteeing long-range locality. AdS cluster decomposition is a consequence of unitary quantum mechanics in CFT.
- If the correlators of low-dimension operators in the CFT are approximately Gaussian (determined entirely by 2-pt correlators), then the AdS/CFT spectrum is approximated by a Fock space[5] of particles in AdS at low energies.
- Crossing symmetric, unitary, interacting (non-Gaussian) CFT correlators built perturbatively on a CFT Fock space will be derivable from an AdS Effective Field Theory description. The cutoff of this effective field theory in AdS can be related to properties of the CFT spectrum. Symmetries play a similarly important role for AdS/CFT.

The key point to notice is that the argument for flat space QFT from the S-Matrix is directly mirrored by the argument for AdS QFT from CFT correlators. The concepts of Poincaré Symmetry, Unitarity, and Cluster Decomposition have been replaced with Conformal Symmetry, Unitarity, and Crossing Symmetry. As we will eventually see, transitioning from flat space to AdS brings additional complications and challenges, but also certain benefits.

The relation between the two approaches is no coincidence – in fact, flat space S-Matrices can be derived from a limit of CFT correlation functions, where the dual AdS length scale $R \to \infty$. From this point of view, AdS space simply serves as an infrared regulator for flat space.

The converse of both Weinbergian arguments is simpler. It is comparatively straightforward (i.e. it's the subject of all standard textbooks) to see that local flat space QFT produces a Poincaré covariant S-Matrix. It is also relatively easy to see why AdS QFTs produces good CFT correlation functions. Also, since we obviously have an interest in UV completing quantum gravity, it's interesting to understand why gravity in AdS *must be* a CFT. So this is the plan of attack for most of these

[5] The Hilbert space formed from any number of non-interacting particles.

lectures – we will start by studying physics in AdS, introducing CFT ideas wherever useful, and show how field theories in AdS naturally produce CFTs[6]. Only at the end will we return to see how AdS EFT actually follows from simple assumptions about the CFT.

1.4 Brief Notes on the History of Holography

Some anachronistic and ideosyncratic comments about the history of Holography:

- In any theory where energy can be measured by a boundary integral, one must have holography, because 'energy' is just the Hamiltonian. The fact that this is the case in general relativity has been known for a long time, dating at least to the 1962 work of Arnowitt, Deser, & Misner [4], see for example the classic paper of Regge and Teitelboim [5] for a thorough treatment. Brice DeWitt and others were aware [6] in the 60s that the only exact observables are associated with the boundary, but they didn't suggest that a different (local!) theory (e.g. a CFT) could live there and compute them.

- Most famously, holography can and usually is motivated via Black Hole thermodynamics, which dates to Bekenstein and Hawking in the 70s. The most important point is that BH's have an entropy proportional to their area, not their volume. The fact that one can throw anything into a BH, combined with the 2nd law of thermodynamics, means that information in a gravitational universe must follow an area law.

- Work studying the asymptotic symmetries of GR proceeded in the 70s and 80s, leading to the famous Brown and Henneaux result [7] that the asymptotic symmetry group in AdS_3 is the infinite dimensional Virasoro algebra = the conformal algebra in 2 dimensions. Note that this work is quite non-trivial because one has to understand the symmetries of spacetime while allowing spacetime to fluctuate. In a certain sense the full Virasoro symmetry of gravitational AdS_3 is always spontaneously broken to the global $SL(2)$ conformal group.

- The Strominger-Vafa results on BH entropy, which used a CFT and some Brown and Henneaux-esque ideas, were also a precursor to AdS/CFT, and are now understood in these terms.

- 't Hooft suggested the idea of taking gravity as a boundary theory seriously, and Susskind subsequently named the idea holography. 't Hooft was unfortunate enough to make a specific and incorrect suggestion for the boundary theory... both suggested the idea that the boundary theory has 'pixels'. I'm not sure if the intention was to have a local boundary theory.

- Anomaly inflow, and the basic point of Witten that certain anomalies can be viewed as being given by $d+1$ dimensional integrals, was also suggestive, although this result only relies on the topology and not the specific geometry of the bulk.

[6]Actually, only UV complete theories in AdS produce exact CFTs. Effective field theories in AdS only produce approximate CFTs, which break down when one studies operators of large scaling dimension. If one side of the duality is incomplete, the other must be as well.

- Polyakov had some ideas about strings propagating in an extra dimension being dual to confinement (see his talk 'String Theory and Quark Confinement' from 1997), and (anecdotally) these ideas inspired Maldacena's discovery.
- Amusingly, Dirac wrote a paper in 1936 on singleton representations in AdS_3 (today we would call this a scalar boson in AdS with a particular negative mass squared, so that it corresponds to a free field in the CFT satisfying $\partial^2\phi = 0$); the paper was titled "A Remarkable respresentation of the 3+2 de Sitter group".

2 Anti-deSitter Spacetime

So what is Anti-deSitter (AdS) spacetime?

AdS_{d+1} is a maximally symmetric spacetime with negative curvature[7]. It is a solution to Einstein's equations with a negative cosmological constant. A particularly useful coordinate system for it, often referred to in the literature simply as 'global coordinates', is given by

$$ds^2 = \frac{1}{\cos^2\left(\frac{\rho}{R}\right)}\left(dt^2 - d\rho^2 - \sin^2\left(\frac{\rho}{R}\right)\, d\Omega_{d-1}^2\right) \tag{2.1}$$

In what follows we will set the AdS length scale $R = 1$. However, it's important to note that we cannot have an AdS spacetime without choosing some particular distance (and curvature) scale. Lengths and energies in AdS can and usually will be measured in these units.

Here the radial coordinate $\rho \in [0, \frac{\pi}{2})$, while $t \in (-\infty, \infty)$, and the angular coordinates Ω cover a $(d-1)$-dimensional sphere. For example, if $d = 3$ then we can write

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \tag{2.2}$$

to cover the familiar S^2. Note that although ρ only runs over a finite range, the spatial distance from any $\rho < \pi/2$ out towards $\pi/2$ diverges, so AdS_{d+1} is not compact. In global coordinates we can picture AdS as the interior of a cylinder, as in figure 2. Note that in these coordinates there is an obvious time translation symmetry, and also an obvious $SO(d)$ symmetry of rotations on the sphere.

By maximally symmetric, we mean that AdS_{d+1} has the maximal number of spacetime symmetries, namely $\frac{1}{2}(d+1)(d+2)$. This is the same number as we have in $d+1$ dimensional flat spacetime, where we have $d+1$ translations, d boosts, and $\frac{1}{2}d(d-1)$ rotations. The easiest way to see the symmetries of AdS is to embed it as the solution of

$$X_A X^A \equiv X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2 \tag{2.3}$$

[7]The use of $d+1$ dimensions is conventional when studying AdS/CFT, because the dual CFT is taken to have d spacetime dimensions.

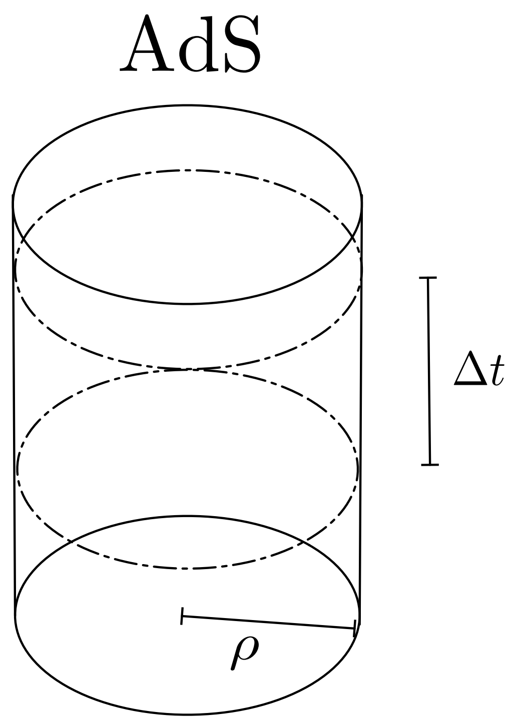

Figure 2: This figure shows AdS in global coordinates. The center is at $\rho = 0$, while spatial infinity is approached as $\rho \to \pi/2$. The global time coordinate t runs from $-\infty$ to ∞.

Note again the appearance of the AdS length R. Using the abstruse equation $\cos^2 t + \sin^2 t = 1$ and the related equation $\sec^2 \rho = \tan^2 \rho + 1$ we can map the global coordinates into the X_A via[8]

$$X_0 = R\frac{\cos t}{\cos \rho} \tag{2.4}$$

$$X_{d+1} = R\frac{\sin t}{\cos \rho} \tag{2.5}$$

$$X_i = R \tan \rho \, \hat{\Omega}_i \tag{2.6}$$

The advantage of the X_A as a presentation of AdS is that all of the symmetries are just the naive rotations and boosts of the X_A. In particular, we have $\frac{1}{2}d(d-1)$ rotations among the X_i with $1 \leq i \leq d$, we have one rotation between the two timelike directions X_0 and X_{d+1}, and then we have $2d$ boosts that mix X_0 and X_{d+1} with the X_i. All of these transformations can be represented by

$$L^A_B = X^A \frac{\partial}{\partial X^B} - X^B \frac{\partial}{\partial X^A} \tag{2.7}$$

[8]To get all of AdS we need to unwrap the X_A to their universal cover, so that t and $t + 2\pi R$ are no longer periodically identified.

which generate the group $SO(2,d)$ of linear transformations of the X_A leaving equation (2.3) invariant. The group $SO(2,d)$ is both the group of isometries of AdS_{d+1} and also the conformal group in d dimensions; so we will often refer to it as the conformal group in what follows. Let's take a look at the 'rotation' in the timeline directions X_0 and X_{d+1}. Note that

$$\frac{\partial}{\partial t} = \frac{\partial X^0}{\partial t}\frac{\partial}{\partial X^0} + \frac{\partial X^{d+1}}{\partial t}\frac{\partial}{\partial X^{d+1}} = L^0_{d+1} \tag{2.8}$$

so in other words, $L_{0,d+1}$ is the generator of time translations in AdS. So it is the Hamiltonian! The purely space-like generators L_{ij} just generate the rotations of $\hat{\Omega}_d$. These are just the isometries of the sphere S^{d-1}, and they form the group $SO(d)$.

2.1 Euclidean Version and the Poincaré Patch

In many cases it will be useful and important to study the Euclidean version of AdS and the Euclidean conformal group, which is $SO(1,d+1)$. The embedding space becomes

$$X_0^2 - \sum_i^{d+1} X_i^2 = R^2 \tag{2.9}$$

In this case the dt^2 term in equation (2.1) flips sign, and we have $\cos t \to \cosh t$ and $\sin t \to \sinh t$ in equation (2.4). This leads to the coordinate identifications in equation (2.10).

There's another coordinate system for AdS, called the Poincaré patch (PP). It's actually used in the literatue more often than the global coordinate system. It's the coordinate system to use when studying RS models, holographic QCD, and basically any theory with broken conformal symmetry. The reason for its importance is that it makes the d-dimensional Poincaré subgroup of the conformal group manifest.

The Poincaré Patch is a bit easier to understand in Euclidean signature. The relationship between Euclidean embedding, global, and Poincaré patch coordinates is

$$\begin{aligned} X_0 &= R\frac{\cosh\tau}{\cos\rho} = \frac{1}{2}\left(\frac{z^2+\vec{r}^2+R^2}{z}\right) \\ X_{d+1} &= R\frac{\sinh\tau}{\cos\rho} = \frac{1}{2}\left(\frac{z^2+\vec{r}^2-R^2}{z}\right) \\ X_i &= R\tan\rho\,\Omega_i = \frac{R}{z}r_i \end{aligned} \tag{2.10}$$

where $\vec{r}$ is a spatial d-vector, and we have written the global coordinate time as τ. Aspects of this relationship are pictured in figure 3. Note that z runs from 0 to ∞; this is necessary so that X_0 has a fixed sign, which is itself required by equation (2.9). In the PP coordinates, dilatations are generated by

$$D = L_{0,d+1} = z\partial_z + r_i\partial_{r_i} \tag{2.11}$$

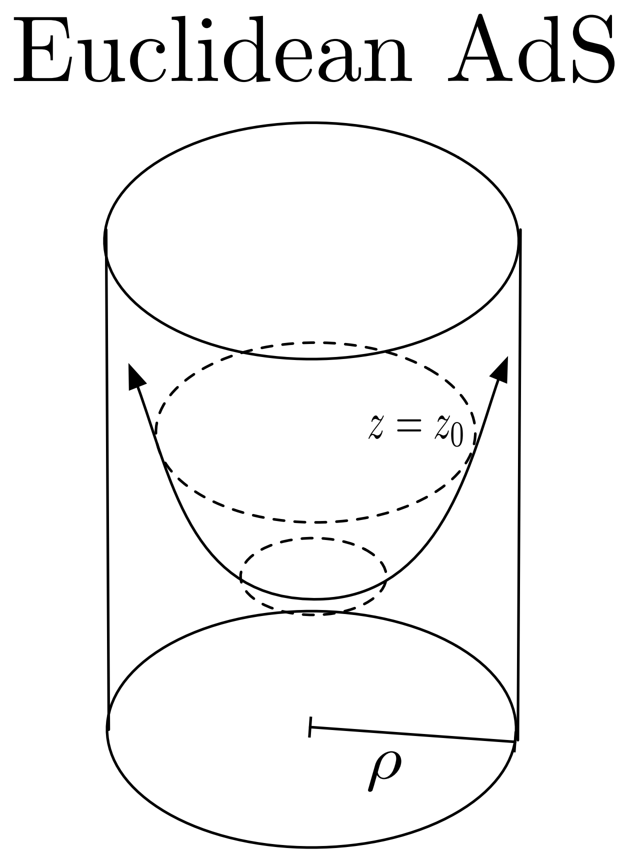

Figure 3: This figure depicts Euclidean AdS. The constant global Euclidean time surfaces are just balls at fixed height on the cylinder. In constrast, the constant z surfaces begin at $\vec{r} = 0$ at some fixed τ with $\rho = 0$, but as $\vec{r} \to \infty$ we simultaneously have $\rho(r) \to \pi/2$ with $\tau \to \infty$, as depicted on the figure. Note that in Euclidean coordinates, both the Poincaré patch and the global coordinates cover all of AdS; in contrast, the PP only covers a portion of AdS in the Lorentzian case.

This operator acts on $\vec{r}$ by stretching the space, as expected. The fact that this is truly $L_{0,d+1}$ can be checked by noting that $(z\partial_z + r_i\partial_{r_i})X_i = 0$, and that

$$z\partial_z + r_i\partial_{r_i} = X^{d+1}\partial_{X^0} - X^0\partial_{X^{d+1}} \tag{2.12}$$

This is easily seen by writing the differential operator $D = z\partial_z + r_i\partial_{r_i}$ as $D(X^A)\partial_{X^A}$.

The relationship in the Lorentzian case is basically given by analytic continuation. The geometry is depicted in figure 4. It's most natural to switch the labels of X_d and X_{d+1} to satisfy equation

(2.3). This gives

$$
\begin{aligned}
X_0 &= R\frac{\cos\tau}{\cos\rho} = \frac{z}{2}\left(1+\frac{R^2+\vec{x}^2-t^2}{z^2}\right) \\
X_{d+1} &= R\frac{\sin\tau}{\cos\rho} = \frac{R}{z}t \\
X_{i<d} &= R\tan\rho\Omega_i = \frac{R}{z}x_i \\
X_d &= R\tan\rho\Omega_d = \frac{z}{2}\left(1-\frac{R^2-\vec{x}^2+t^2}{z^2}\right)
\end{aligned} \tag{2.13}
$$

where $\vec{x}$ is a spatial $d-1$ vector, and we have written the global time as τ to avoid confusion. We can solve for (t,z,x_i) in terms of the global coordinates $(\tau,\rho,\hat{\Omega}_i)$ as

$$
\begin{aligned}
t &= R\frac{\sin\tau}{\cos\tau-\Omega_d\sin\rho} \\
z &= R\frac{\cos\rho}{\cos\tau-\Omega_d\sin\rho} \\
\vec{x}_i &= R\frac{\hat{\Omega}_i\sin\rho}{\cos\tau-\Omega_d\sin\rho}
\end{aligned} \tag{2.14}
$$

It should be clear from this that even when $t\to\pm\infty$ we only cover a finite range in τ, because this limit is only achieved by causing the denominators on the RHS to vanish. In these coordinates we obtain the metric

$$
ds^2 = \frac{1}{z^2}\left(dt^2 - dz^2 - \sum_{i=1}^{d-1} dx_i^2\right) \tag{2.15}
$$

where the z coordinate only covers the range $0<z<\infty$. A very commonly used alternative version re-writes $dy = dz/z$ so that $z=e^y$.

We emphasize that in Euclidean signature, the Poincaré patch covers the entire AdS spacetime, just like global coordinates. However, in Lorentzian signature, the Poincaré patch only covers a small sub-region of the full AdS spacetime, bounded by a causal diamond wrapped around the AdS cylinder.

Generally speaking, the global coordinates make an $SO(2)\times SO(d)$ sub-group of the $SO(2,d)$ conformal group obvious, while the Poincaré patch makes the d-dimensional Poincaré group obvious and dilatations (somewhat) manifest.

2.2 Gauss's Law in AdS and Long Range Interactions

Now let's consider Gauss's law in AdS, in order to understand how electromagnetic and gravitational fields behave at long distances.

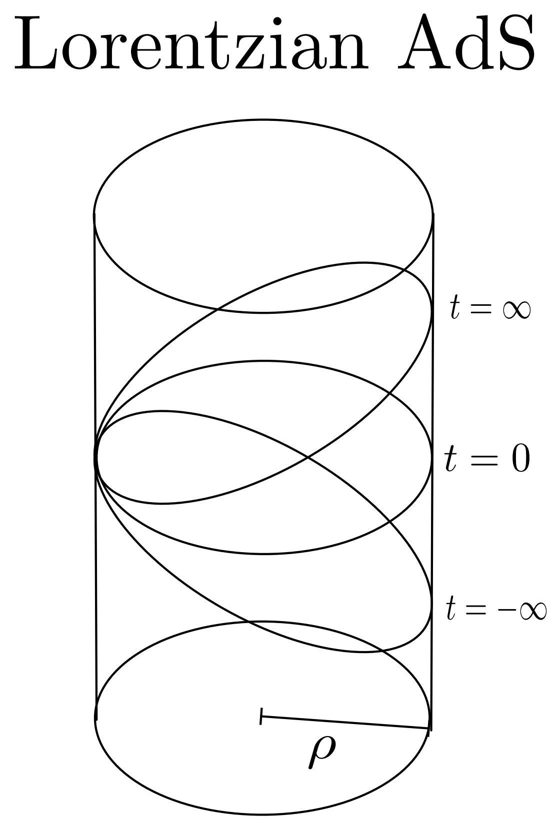

Figure 4: This figure emphasizes the region corresponding to the Poincaré patch, with constant PP times labeled. The PP does not cover all of Lorentzian AdS (in contrast to the Euclidean case).

For this purpose it's natural to transform to a coordinate κ with $d\kappa = \frac{d\rho}{\cos\rho}$, so that $\cosh\kappa = \sec\rho$ and $\sinh\kappa = \tan\rho$. Note that since AdS is maximally symmetric, the point $\kappa = 0$ is equivalent to all other points. In these coordinates the metric takes the form

$$ds^2 = \cosh^2(\kappa) dt^2 - d\kappa^2 - \sinh^2(\kappa)\, d\Omega_{d-1}^2 \tag{2.16}$$

Now consider a constant time surface, such as the surface $t = 0$, and let's imagine we have a charge sitting at $\kappa = 0$. Gauss's law says that the total amount of flux through any sphere around the charge at $\kappa = 0$ will be a constant. Note that since $g_{\kappa\kappa} = 1$ a point with coordinate κ_* is literally a distance κ_* from the point $\kappa = 0$. The surface area of a sphere of (geometrical) radius κ_* is just $[\sinh(\kappa_*)]^{d-1}$ times the area of an S^{d-1} with radius one. This means that the potential due to a point charge at $\kappa = 0$ will fall off with κ as

$$V(\kappa) = \frac{Q}{[\sinh(\kappa)]^{d-2}} \tag{2.17}$$

In the limit that $\kappa \ll 1$, this is just the usual κ^{2-d} potential appropriate to an (approximately) flat

$d+1$ dimensional spacetime. But for $\kappa \gg 1$ this becomes

$$V(\kappa) \approx e^{-(d-2)\kappa} \tag{2.18}$$

We can derive these results more systematically by studying the equation of motion for a spherically symmetric, static electric potential.[9] This follows from studying the action for the electromagnetic field in AdS, namely

$$S = \int d^{d+1}X \sqrt{-g} \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.19}$$

The equation of motion we want can be most easily derived by writing $\vec{A} = 0$ and $A_0 = V$, giving

$$\partial_\kappa \left(g^{tt} g^{\kappa\kappa} \sqrt{-g} \partial_\kappa V \right) = \delta(\kappa) \tag{2.20}$$

or

$$\partial_\kappa^2 V + \left((d-1) \coth \kappa - \tanh \kappa \right) \partial_\kappa V = \delta(\kappa) \tag{2.21}$$

which has the solution given in equation (2.17).

These observations have broader implications. For one thing, they imply that IR divergences are regulated by the AdS geometry. But at a more basic level, there are implications for the relationship between angular momentum and radius of orbit. Since a sphere of radius κ will have an area of order $e^{(d-1)\kappa}$, to cover such a sphere with AdS scale resolution requires spherical harmonics up to $\ell \sim e^{(d-1)\kappa}$ as well. This also means that objects in nearly circular orbits with angular momentum ℓ are only a characteristic distance $\kappa \sim R \log \ell$ from the center of AdS.

As a final point, note that the $\cosh^2(\kappa) dt^2$ term in the metric suggests that (at least in the Newtonian limit) an objects energy must increase as it ventures away from $\kappa = 0$. This means that in effect, there is a gravitational force pushing objects in AdS towards the center at $\kappa = 0$. This makes AdS function like a 'box'. We will see this more directly in the next section, when we study the motion of classical and quantum particles in AdS.

3 A Free Particle in AdS

3.1 Classical Equations of Motion

Now let's study the simplest possible example of AdS physics, a free scalar particle in AdS$_3$. The most straightforward way to work out its behavior is to write down a Lagrangian for the particle, and then study the classical equations of motion and their canonical quantization. The action for a free particle of mass m in any spacetime can be written as

$$S = m \int d\tau = m \int dt \sqrt{g_{\mu\nu}(X(t)) \frac{dX^\mu(t)}{dt} \frac{dX^\nu(t)}{dt}} \tag{3.1}$$

[9] It's worth noting that this differs from the potential for a massless scalar field. In fact, the potential due to a scalar will only agree with this electric potential if the scalar has a slightly negative mass $m^2 = -2(d-2)$ in AdS units, and then it will only agree for $\kappa \gg 1$. Scalars with slightly negative masses can be stable in AdS.

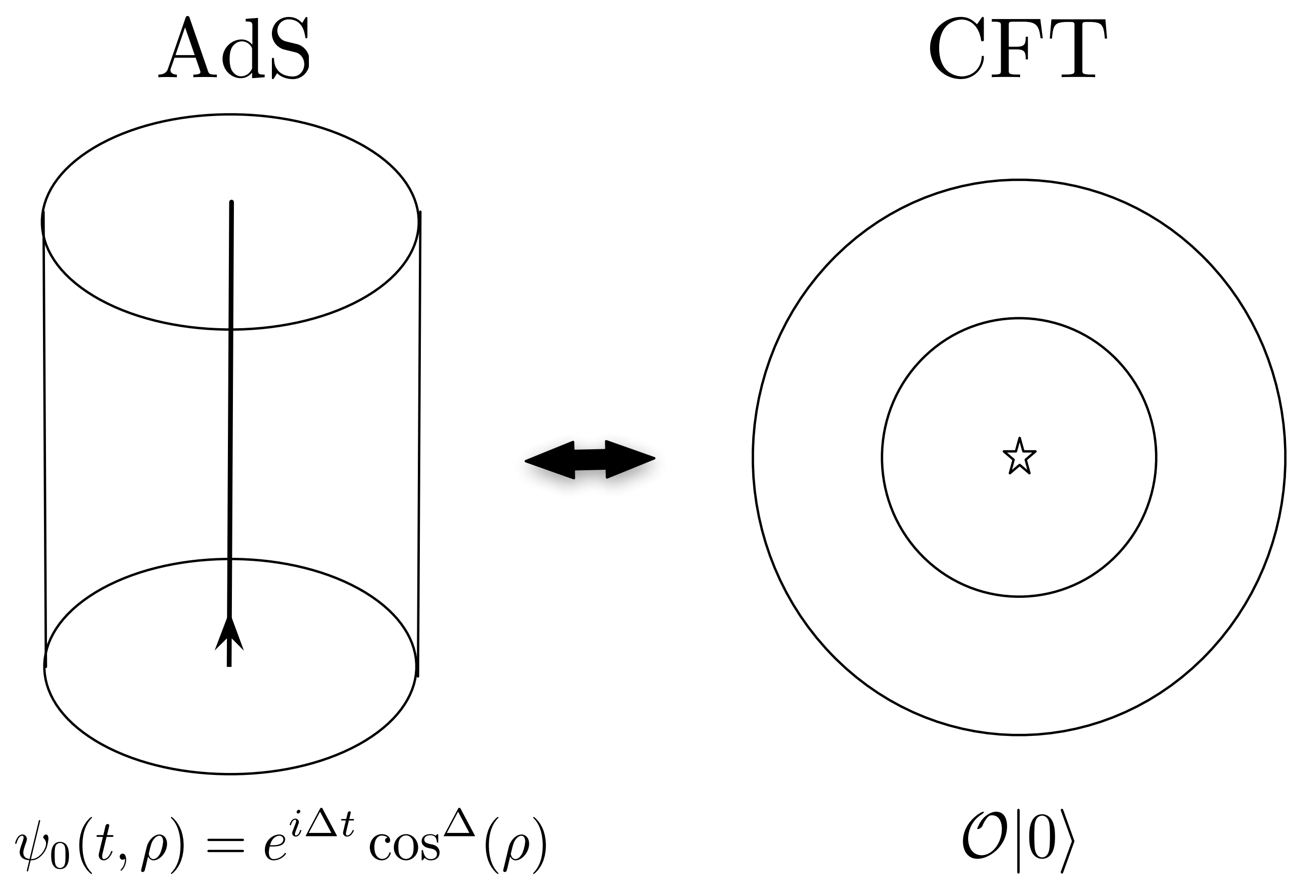

Figure 5: This figure shows an object at rest in AdS and the CFT dual, which is a primary state.

where τ is the proper time, and in the second line $X^\mu(t)$ is the spacetime position of the particle at time t. The fact that this is the correct action for a free particle is equivalent to the statement that free particles move on geodesics. The action computes the geodesic path length in units of $1/m$, because that's the quantum mechanical wavelength of the particle in its own rest frame.

It's worth noting an alternative description (see e.g. [9]). Introducing the lagrange multiplier α, we can write the action as

$$S = \int dt \left(\frac{1}{2\alpha} g_{\mu\nu}(X(t)) \frac{dX^\mu(t)}{dt} \frac{dX^\nu(t)}{dt} + \frac{\alpha}{2} m^2 \right) \tag{3.2}$$

The α equation of motion is trivially solved, and substituting the solution back in gives our first action. But this second version is useful because it eliminates the square root; for example to study massless particles we can just set $m^2 = 0$ from the beginning.

In the simple special case of AdS$_3$, we can take $X^\mu(t) = (t, \rho(t), \theta(t))$ in global coordinates, so we obtain the action

$$S = m \int dt \frac{\sqrt{1 - \dot\rho^2 - \dot\theta^2 \sin^2\rho(t)}}{\cos\rho(t)} \tag{3.3}$$

If we are interested in the classical physics of our free particle, we could proceed to derive the Euler-Lagrange equations for this action, but this gets messy[10].

Instead, let us take a smarter and more elegant approach. We can view AdS as the surface $X_A X^A = R^2$, as defined in equation (2.3). So why not view a particle in AdS as a particle in $d+2$ spacetime dimensions constrained to live on this surface? If we write the action for the coordinates $X_A(\tau)$, where τ is an arbitrary wordline parameter, we find[11]

$$S[X_A, \lambda] = \int d\tau \left[\dot{X}_A \dot{X}^A + \lambda (R^2 - X_A X^A) \right] \tag{3.4}$$

where λ is a Lagrange multiplier that we would also integrate over in the path integral. In fact the Lagrangian formalism was originally invented for exactly this purpose – to describe the motion of objects constrained to live on a surface.

The equations of motion are

$$\ddot{X}_A = -\lambda X_A \tag{3.5}$$

along with the equation of motion from λ, the constraint equation

$$X_A X^A - R^2 = 0 \tag{3.6}$$

The solution to the first set of equations depends on our choice for the variable λ, which is effectively unconstrained. By rescaling the unphysical worldline coordinate τ, we can multiply λ in equation (3.5) by any positive quantity (since the derivative is quadratic). This means that we effectively have three choices: $\lambda = 1, 0$, or -1.

These three choices for λ correspond to timelike, null, or spacelike trajectories (geodesics) in AdS. Let us focus on the case $\lambda = 1$ for now, which corresponds to the motion of a massive particle in AdS. Then, setting $\tau = t$ from now on, we have the solutions

$$X_A = v_A^c \cos t + v_A^s \sin t \tag{3.7}$$

We see that *particles in AdS always have periodic trajectories.* The other constraint equation becomes

$$R^2 \;=\; v_A^c v^{cA} \cos^2 t + v_A^s v^{sA} \sin^2 t + v_A^c v^{sA} \sin(2t) \tag{3.8}$$

This immediately leads to the constraints $v_A^c v^{sA} = 0$ and $v_A^c v^{cA} = v_A^s v^{sA} = R^2$. This leaves us with $2(d+2) - 3 = 2d+1$ solutions for the motion of a particle in a $d+1$ dimensional spacetime. This is one too many solutions; however we can eliminate one by trading the parametric t coordinate for one of the X_A. This gives the correct number of solutions.

The most trivial solution is

$$X_0 = R\cos t \quad \text{and} \quad X_{d+1} = R \sin t \tag{3.9}$$

[10]One can still easily check that circular orbits (those with $\dot{\rho} = 0$) exist iff $\dot{\theta} = 1$.

[11]We should actually be using $\dot{X}^2 \to \frac{1}{2\alpha}\dot{X}^2 + \frac{\alpha}{2}m^2$ in the action. However, if we make a choice for τ so that $m^2\alpha^2 = \dot{X}^2$ is a constant, then the effect of including α is simply to rescale λ. We will make such a choice.

with all the $X_i = 0$. This represents a massive particle at rest at $\rho = 0$. A more interesting solution can be easily found by taking

$$\begin{aligned} X_0 &= \frac{R\cos t}{\cos\rho_*} \\ X_{d+1} &= R\sin t \\ X_1 &= R\tan\rho_*\cos t \end{aligned} \tag{3.10}$$

In this case the particle is oscillating back and forth between $\rho = \pm\rho_*$ in the 1 direction. Another canonical example is

$$\begin{aligned} X_0 &= \frac{R\cos t}{\cos\rho_*} \\ X_{d+1} &= \frac{R\sin t}{\cos\rho_*} \\ X_1 &= R\cos t\tan\rho_* \\ X_2 &= R\sin t\tan\rho_* \end{aligned} \tag{3.11}$$

This represents a particle at fixed $\rho = \rho_*$ making a circular orbit in the 1-2 plane. Other more complicated solutions involving various elliptical orbits can be obtained via some simple algebra. The spacelike and null geodesics of AdS can also be obtained.

One might wonder why we have solutions corresponding to distinct periodic orbits, since AdS is a maximally symmetric spacetime – aren't all trajectories equivalent up to symmetries? In fact they are, but by choosing a specific time coordinate t we have chosen some particular center for AdS, and according to this t-evolution objects move in orbits. Any given orbit can be transformed into the trajectory of a particle at rest via a conformal transformation (an AdS isometry).

A very non-trivial feature of these solutions is that they all have the same frequency with respect to the AdS time t. An immediate consequence of this fact is that when we quantize an AdS free particle, we must have energy levels that are integer spaced. In other words, we must find that all energy levels satisfy

$$E = \Delta + m \tag{3.12}$$

for some Δ and for integers $m = 0, 1, 2, \cdots$. This follows from both the WKB approximation, and from a consideration of the oscillatory behavior of superpositions of wavefunctions. The only way that every semi-classical linear combination of wavefunctions can have equal periodicity is if all energy levels are quantized in this way. Let us now turn to the quantum theory and confirm this prediction.

3.2 Single Particle Quantum Mechanics

Now we would like to study our single particle in AdS quantum mechanically. Note that there are two physical scales in the problem – the AdS length scale R and the mass of the particle m, from

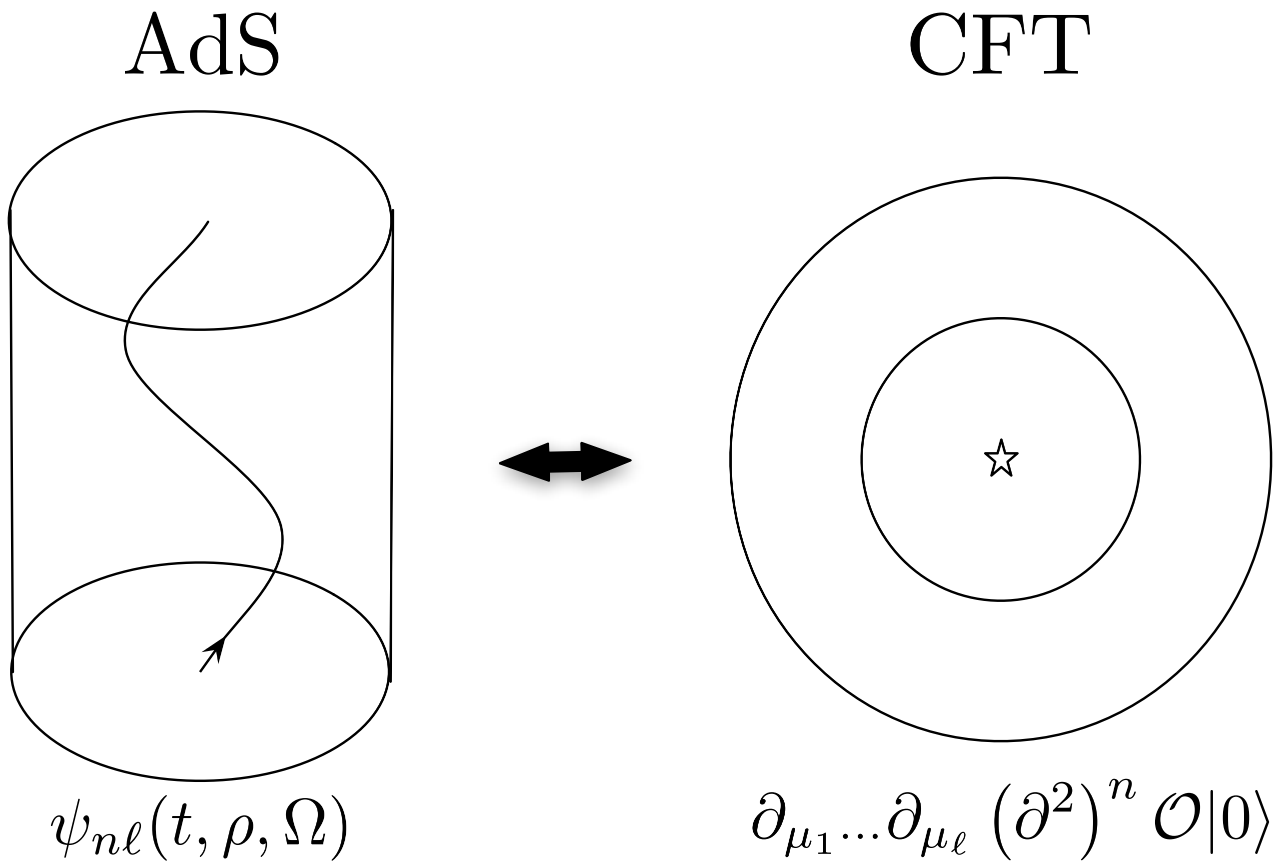

Figure 6: This figure shows an object moving in AdS and the CFT dual, a descendant state.

which we can form the dimensionless number mR on which the wavefunction can depend. This wasn't apparent in the classical analysis because particles move on a timelike geodesics, independent of their masses.

We can straightforwardly derive the AdS Schrodinger equation via canonical quantization of the action for a free particle (there's nothing wrong with describing free relativistic particles using first-quantized quantum mechanics). Let's illustrate this for AdS_2 for simplicity. The action in equation (3.1) in the case of AdS_2 is

$$S = \int dt \frac{m}{\cos\rho}\sqrt{1-\dot{\rho}^2} \tag{3.13}$$

The canonical momentum conjugate to ρ is

$$P_\rho \equiv \frac{\partial L}{\partial \dot{\rho}} = -\frac{m\dot{\rho}}{\cos\rho\sqrt{1-\dot{\rho}^2}} \tag{3.14}$$

Using this we can construct the Hamiltonian, which can be written as

$$H = P_\rho \dot{\rho} - L = \sqrt{\frac{m^2}{\cos^2\rho} + {P_\rho}^2} \tag{3.15}$$

Quantum mechanically, we must impose the canonical commutation relation $[\rho, P_\rho] = i$, which is conveniently implemented by $P_\rho = -i\partial_\rho$ in the ρ-basis of states. The Schrodinger equation then says that $i\partial_t = H$. Applied to the wavefunction for a particle this would say

$$i\frac{d}{dt}\Psi(t,\rho) = \left(\sqrt{\frac{m^2}{\cos^2\rho} - \partial_\rho^2}\right)\Psi(t,\rho) \tag{3.16}$$

It's unclear how to solve this equation, but things become easier if we study instead the square of the time evolution operator, $-\partial_t^2 = H^2$, which produces an equation that must also be satisfied when applied to Ψ. So we would want to solve

$$-\partial_t^2\Psi(t,\rho) = \left(\frac{m^2}{\cos^2\rho} - \partial_\rho^2\right)\Psi(t,\rho) \tag{3.17}$$

to describe a single quantum mechanical particle in AdS_2. We have derived a relativistic version of the single-particle Schrodinger equation for a free particle in AdS_2. We will see later on that this is also compatible with the Klein-Gordon equation in AdS that we obtain from relativistic field theory. In fact, in $d > 2$ there are operator ordering issues that should be resolved to maintain conformal symmetry; the easiest way to do that is simply to use the Klein-Gordon equation $(\nabla^2 + m^2)\psi = 0$.

How do we solve this system? We could just type the differential equation into mathematica, and find a bewildering answer involving hypergeometric functions. But to actually understand the physics, we need to use symmetry. In AdS_2 we actually have 3 symmetry generators, corresponding (in the embedding space) to

$$L_2^0 = \partial_t \tag{3.18}$$

in the language of equation (2.7), and also L_1^0 and L_2^1. Let's look at one of the latter transformations. In the embedding space with $X_0^2 + X_2^2 - X_1^2 = R^2$ we have

$$X_0 = R\frac{\cos t}{\cos\rho} \tag{3.19}$$

$$X_2 = R\frac{\sin t}{\cos\rho} \tag{3.20}$$

$$X_1 = R\tan\rho \tag{3.21}$$

The operator L_1^0 acts as

$$X_0 \to X_0 + \epsilon X_1 \quad \text{and} \quad X_1 \to X_1 + \epsilon X_0 \tag{3.22}$$

We can translate this into a transformation of ρ and t by looking for functions f_t and f_ρ, with $t \to t + \epsilon f_t$ and $\rho \to \rho + \epsilon f_\rho$ such that

$$\begin{aligned}
\frac{\cos(t+\epsilon f_t)}{\cos(\rho+\epsilon f_\rho)} &\approx \frac{\cos t}{\cos\rho} + \epsilon \tan\rho && (3.23)\\
\tan(\rho+\epsilon f_\rho) &\approx \tan\rho + \epsilon\frac{\cos t}{\cos\rho} && (3.24)
\end{aligned}$$

The solution to these equations is

$$\begin{aligned}
f_t &= -\sin t \sin\rho && (3.25)\\
f_\rho &= \cos t \cos\rho && (3.26)
\end{aligned}$$

This tells us that the symmetry generator

$$L_1^0 = -\sin t \sin\rho\partial_t + \cos t\cos\rho\partial_\rho \tag{3.27}$$

Similarly we find that

$$L_2^1 = \cos t\sin\rho\partial_t + \sin t\cos\rho\partial_\rho \tag{3.28}$$

These generators should have the $SO(2,1)$ commutation relations; in fact we find

$$[L_2^0, L_1^0] = -L_2^1, \quad [L_2^0, L_2^1] = L_1^0, \quad [L_1^0, L_2^1] = L_2^0 \tag{3.29}$$

as the commutation relations of these vector fields. This means that if we define $D = L_2^0$, $P = \frac{1}{2}(L_1^0 + iL_2^1)$, and $K = \frac{1}{2}(L_1^0 - iL_2^1)$ then we find

$$[D,P] = iP, \quad [D,K] = -iK, \quad [K,P] = iD \tag{3.30}$$

This is great – it means that we can choose to label our states as eigenstates of D, so that

$$D|\psi\rangle = \Delta|\psi\rangle \tag{3.31}$$

and with respect to this eigenvalue, P acts as a raising operator while K acts as a lowering operator. In particular, the ground state satisfies

$$K|\psi_0\rangle = 0 \tag{3.32}$$

and all of the other states can be built by applying the operator P to this state. Note that, very explicitly, we have

$$K = \frac{1}{2}e^{-it}\left(-i\sin\rho\partial_t + \cos\rho\partial_\rho\right) \tag{3.33}$$

for this operator acting in the (t,ρ) basis of states.

So the ground state of our system must obey the equation

$$-i \sin \rho \partial_t \psi_0(t, \rho) + \cos \rho \partial_\rho \psi_0(t, \rho) = 0 \tag{3.34}$$

Taking $\psi_0 = e^{i\Delta t} \chi(\rho)$ with Δ the eigenvalue of D, we see that

$$\Delta \chi = -\cot \rho (\partial_\rho \chi) \tag{3.35}$$

with the solution

$$\psi(t, \rho) = e^{i\Delta t} \cos^\Delta \rho \tag{3.36}$$

This is the ground state. Now *every other state* in the system can be obtained by acting with the operator

$$P = \frac{1}{2} e^{it} \left(i \sin \rho \partial_t + \cos \rho \partial_\rho \right) \tag{3.37}$$

which is the raising operator for our quantum mechanical system. This explains the fact that all AdS orbits have the same period. Note that flipping $t \to -t$ just exchanges P and K. It turns out that the nth energy level has a wavefunction that can be written explicitly in terms of hypergeometric functions, but we will wait to introduce those until we talk about general AdS$_{d+1}$.

General Dimensions

The fact that all classical solutions oscillate with integer period implies that the energy levels are $E = \Delta + n$ for integers n. Let us understand this fact quantum mechanically. The conformal algebra $SO(d, 2)$ can be re-written in a conventional form as

$$\begin{aligned} &[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu), \qquad [M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu), \qquad [M_{\mu\nu}, D] = 0 \\ &[P_\mu, K_\nu] = -2(\eta_{\mu\nu} D + i M_{\mu\nu}), \qquad [D, P_\mu] = P_\mu, \qquad [D, K_\mu] = -K_\mu \end{aligned} \tag{3.38}$$

where we have the index $\mu = 1, 2, ..., d$. Referring to the generators L_{AB} from equation (2.7), the 'dilatation operator' $D \equiv -iL_{0,d+1}$, while the 'momentum generator' $P_\mu \equiv iL_{d+1,\mu} + L_{0,\mu}$ and 'special conformal generators' $K_\mu \equiv iL_{d+1,\mu} - L_{0,\mu}$. The rotation generators $M_{\mu\nu}$ are just the $SO(d)$ generators $-iL_{\mu\nu}$.

Recall that $D = -iL_{0,d+1} = -i\frac{\partial}{\partial t}$. In other words, the dilatation operator D is actually the Hamiltonian for AdS physics, since it is the generator of time translations. For now you can just think of it as the AdS Hamiltonian; soon we'll learn what role it plays in conformal field theories and thereby understand its name. So what does equation (7.9) mean physically? The first line just indicates that P_μ and K_μ transform as vectors under rotations, while the dilatation operator D is a scalar. Since D commutes with $M_{\mu\nu}$, we can simultaneously diagonalize D and the angular momentum generators. So we can label states by their energy (D eigenvalue) and their angular momentum.

The second line of equation (7.9) gives us even more crucial information. It says that with respect to the Dilatation operator D, P_μ *acts as a raising operator while* K_μ *acts as a lowering* *operator*. This should remind you of the creation and annihilation operators for the harmonic oscillator. It immediately explains the fact that the energy levels of one-particle states are integer spaced (since energies are just the D eigenvalues). Furthermore, since D is the Hamiltonian, there must be one-particle state with minimum eigenvalue Δ. This state cannot be lowered, so it must be annihilated by all d of the K_μ generators. This gives us an equation

$$K_\mu|\psi_0\rangle = 0 \tag{3.39}$$

where ψ_0 is called (again using CFT terminology) a *primary state*. From the point of view of AdS quantum mechanics, ψ_0 is simply the lowest energy state for our particle, and it has energy Δ. Once we have the primary/ground state ψ_0, we can construct general states of the form[12]

$$|\psi_{n,\ell}\rangle = \left(P_\mu^2\right)^n P_{\mu_1} P_{\mu_2} \cdots P_{\mu_\ell}|\psi_0\rangle \tag{3.40}$$

These states have energy $E_{n,\ell} = \Delta + 2n + \ell$ and angular momentum ℓ. These form a basis for all one-particle states in AdS.

We can compute $\psi_{n,\ell}$ as wavefunctions in AdS by writing out K_μ explicitly as a differential operator. This is a straightforward exercise using the coordinate transformations from equation (2.4), the definitions of L_{AB} from equation (2.7), and the fact that $K_\mu \equiv iL_{d+1,\mu} - L_{0,\mu}$ as we discussed above. For example, in AdS$_3$

$$\begin{aligned} K_\pm &= ie^{-it\pm i\theta}\left(\sin\rho\partial_t + i\cos\rho\partial_\rho \mp \frac{1}{\sin\rho}\partial_\theta\right) & (3.41)\\ P_\pm &= ie^{it\pm i\theta}\left(\sin\rho\partial_t - i\cos\rho\partial_\rho \pm \frac{1}{\sin\rho}\partial_\theta\right) & (3.42)\end{aligned}$$

where $K_\pm = K_1 \pm iK_2$ and similarly $P_\pm = P_1 \pm iP_2$. The ground state $\psi_0(t,\rho)$ will be independent of θ so it must satisfy

$$\left(\sin\rho\partial_t + i\cos\rho\partial_\rho\right)\psi_0(t,\rho) = 0 \tag{3.43}$$

and this means that it takes the form

$$\psi_0(t,\rho) = e^{i\Delta t}\cos^\Delta(\rho) \tag{3.44}$$

as expected, with $m^2 = \Delta(\Delta - d)$. All of the higher states can be constructed by acting on this state with the differential operators $P_\pm$. The state and these methods also generalize to higher d. In complete generality, the wavefunctions $\psi_{n,\ell}$ in AdS$_{d+1}$ can be written in terms of a hypergeometric function as

$$\psi_{n,\ell J}(t,\rho,\Omega) = \frac{1}{N_{\Delta n\ell}} e^{-iE_{n,\ell}t} Y_{\ell J}(\Omega)\left[\sin^\ell\rho\cos^\Delta\rho\, {}_2F_1\left(-n, \Delta+\ell+n, \ell+\frac{d}{2}, \sin^2\rho\right)\right] \tag{3.45}$$

[12] This equation is rather schematic for $\ell > 0$, because we need to arrange the indices to form definite irreducible representations of $SO(d)$, but this is the basic idea.

where $N_{\Delta n\ell}$ is a normalization factor and $E_{n,\ell} = \Delta + 2n + \ell$. The J quantum number just encapsulates other angular momentum quantum numbers, e.g. 'm' for AdS_4 with $Y_{\ell m}$ spherical harmonics. In the case of Euclidean AdS, we have $t \to it$.

The easiest way to determine the general solutions in equation (3.45) is to solve a corresponding Schrodinger equation for AdS_{d+1}. We would like to use the symmetries of the problem to our advantage. There are no non-trivial 1st order differential operators that commute with all of the conformal generators $D, K_\mu, P_\mu, M_{\mu\nu}$, so we will not be able to find a fully symmetric 1st order equation. However, just as in the $d = 1$ case we solved explicitly above, we can find a 2nd order differential operator that commutes with all of the conformal generators. This is the *quadratic Casimir* of the conformal group, which takes the form

$$D^2 + \frac{1}{2}(P \cdot K + K \cdot P) + M_{\mu\nu}M^{\mu\nu} = \nabla^2_{AdS} \tag{3.46}$$

when the conformal generators are expressed as isometry generators on AdS. The fact that this is the Laplacian is no surprise – the conformal Casimir is unique, and we know that the laplacian is the only isometry-invariant 2nd order differential operator.

Thus we want to solve $(\nabla^2 + m^2)\psi = 0$ as the Schrodinger equation in AdS_{d+1}. We can immediately separate variables into a radial wavefunction and an angular wavefunction. The latter will just take the form of a d-dimensional spherical harmonic $Y_{\ell J}$, which are the eigenfunctions of the Laplacian on the sphere S^{d-1} with eigenvalue $\ell(\ell + d - 2)$. We can then write a Schrodinger equation for the the ρ-dependent part of the wavefunction $\psi_\ell(\rho)$, which has energy ω^2. It takes the form

$$-\psi''(\rho) + \frac{1-d}{\cos\rho\,\sin\rho}\psi'(\rho) + \left(\frac{\ell(\ell+d-2)}{\sin^2\rho} + \frac{m^2}{\cos^2\rho}\right)\psi(\rho) = \omega^2\psi(\rho) \tag{3.47}$$

We can simplify this equation by defining

$$\psi(\rho) = \chi(\rho)\sin(\rho)[\cot(\rho)]^{\frac{d}{2}} \tag{3.48}$$

to give the equation for $\chi(\rho)$

$$-\chi''(\rho) + \frac{1}{4}\left(\frac{(2\ell-3+d)(2\ell-1+d)}{\sin^2\rho} + \frac{4m^2+d^2-1}{\cos^2\rho}\right)\chi(\rho) = \omega^2\chi(\rho) \tag{3.49}$$

We can interpret this equation using intuition from non-relativistic quantum mechanics. The term in parentheses represents an effective potential with an angular momentum barrier and a potential that diverges as $\rho \to \pi/2$.

It's not especially obvious how to solve this equation exactly. However, it turns out that if we choose the $\ell = 0$ mode for simplicity, and take $m^2 = \Delta(\Delta - d)$ and $\omega = \Delta$, there is a very simple solution for the ground state

$$\psi(\rho) = \cos^\Delta(\rho) \tag{3.50}$$

which we already found. Quantization and the determination of Δ follows as usual from boundary conditions – the wavefunction must vanish as $\rho \to \pi/2$ and it must have $\psi'(0) = 0$ to avoid cusps, since ρ is a radial coordinate. Physically, we see that $\psi_0(\rho)$ has its largest support at small ρ and falls to zero as $\rho \to \pi/2$. For $\Delta \sim 1$ the particle has been confined to a region of order one size R, the AdS length scale. When $\Delta \gg 1$ we have

$$\psi(\rho) \approx e^{-\Delta \frac{\rho^2}{2}} \quad \text{when} \quad \Delta \gg 1 \tag{3.51}$$

so we see that the particle is an approximately Gaussian state so that it is confined to a region of size $R/\sqrt{\Delta}$ due to the AdS curvature.

Actually, there is a slick way of deriving the relation between bulk mass and Δ. We already derived the form of the ground state using the lowering condition $K_\mu \psi_0 = 0$. Once we know its form, we can simply plug it back into the (more complicated) Schrodinger equation, noting that $\omega = \Delta$. A quick computation shows that this equation can then only be satisfied with $m^2 = \Delta(\Delta - d)$.

A Comment on Multiparticle Quantum Mechanics in AdS

Nothing stops us from studying simple examples of multi-particle quantum mechanics in AdS. For example, consider a hydrogen atom. In the usual description in flat space, its wavefunction can be specified in terms of the position of the proton and the position of the electron. Usually it makes more sense to break these up into a center of mass degree of freedom (which is somewhat trivial) and a coordinate representing the relative position between the electron and proton. Then we compute the wavefunctions, energy levels, etc.

We can repeat this same process in AdS, and we could even go on to compute the wavefunctions and binding energies due to the Coulomb attraction. According to what we'll soon learn about AdS/CFT, we could then interpret the results in terms of the scaling dimensions and correlation functions of operators in a putative CFT. We won't take this approach here (although you can read about a very similar approach in [10]), but it's worth emphasizing that even the most elementary results from quantum mechanics can be applied in the AdS/CFT context. Via AdS/CFT we can reinterpret many familiar ideas holographically.

4 Free Fields in AdS

In the previous section we discussed AdS geometry and the physics of a classical or quantum free particle in AdS. Now let us see how to describe any number of quantum free particles in AdS.

The key feature of states constructed from many free particles (Fock space states) is that their quantum numbers are just the sum of the quantum numbers of the individual particles. In particular, their energy (for us the D eigenvalue) is just the sum of the energies of the individual particles. This is why the Hilbert space of free quantum field theory is built using harmonic oscillator creation and annihilation operators – the harmonic oscillator has evenly spaced energy levels, so any two states can be combined by just adding their energies.

For this reason, to describe any number of free particles in AdS we simply introduce the creation and annihilation operators $a_{n\ell J}$ and $a^\dagger_{n\ell J}$ with the usual commutator

$$[a_{n\ell J}, a^\dagger_{n\ell J}] = 1 \tag{4.1}$$

A basis for many particle states can be constructed as usual; a generic k-particle state is

$$a^\dagger_{n_1\ell_1 J_1} a^\dagger_{n_2\ell_2 J_2} \cdots a^\dagger_{n_k\ell_k J_k}|0\rangle \tag{4.2}$$

where $|0\rangle$ is the vacuum, which has no particles. In terms of these operators, the AdS Hamiltonian D is simply

$$D = \sum_{n,\ell J} (\Delta + 2n + \ell) a^\dagger_{n\ell J} a_{n\ell J} \tag{4.3}$$

because it sums the energy of all the particle, and $a^\dagger_{n\ell J} a_{n\ell J}$ is the number operator that counts the number of particles with quantum numbers n, ℓ, J. The other conformal generators can also be expressed in terms of the creation and annihilation operators. So we have an infinite collection of harmonic oscillators in AdS.

It's straightforward to see that this is just what we get from quantizing a free scalar field in AdS, with action

$$S = \int_{AdS} d^{d+1}x\sqrt{-g}\left(\frac{1}{2}(\nabla_A \phi)^2 - \frac{1}{2}m^2\phi^2\right) \tag{4.4}$$

Before we explain why, let's detour to review everything you ever wanted to know about free fields but were afraid to ask.

4.1 Classical and Quantum Fields Anywhere – Canonical Quantization

Here we will review, at a rather formal level, how one goes about solving for the time evolution of a classical field. Then we will explain how this relates to quantization. These ideas are worth understanding for reasons that go beyond AdS – for example, they are an elementary ingredient in Hawking's derivation of black hole evaporation.

Let us study a scalar field theory in a completely general spacetime, with free action

$$S = \int d^d x dt \sqrt{-g}\left(\frac{1}{2} g^{\mu\nu}\nabla_\mu \phi \nabla_\nu \phi - \frac{m^2}{2}\phi^2\right) \tag{4.5}$$

This action implies the usual equations of motion

$$g^{\mu\nu}\nabla_\mu \nabla_\nu \phi + m^2 \phi = 0 \tag{4.6}$$

Let us imagine that we find the general solutions to this equation, functions $f_n(t,x)$ labeled by an index n that may be discrete or continuous. Here 'solving the equation' is a less precise way of saying that we have diagonalized the differential operator $g^{\mu\nu}\nabla_\mu\nabla_\nu + m^2$, obtaining its eigenfunctions and eigenvalues. For example in flat spacetime we usually take[13] $n = \vec{k}$ and $f_k(t,x) = e^{i\omega t}e^{i\vec{k}\cdot\vec{x}}$ with $\omega^2 = \vec{k}^2 + m^2$. In AdS we find solutions labeled by n, ℓ, J.

Even if we are only solving the theory at a classical level, we would still like to be able to write a general solution as (we will get to quantization below)

$$\phi(t,x) = \sum_n \left(c_n f_n(t,x) + c_n^\dagger f_n^\dagger(t,x)\right) \tag{4.7}$$

given some initial data consisting of $\phi_0(0,x)$ and $\dot{\phi}_0(0,x)$ on the $t=0$ surface. The initial data just corresponds to the initial position and velocity of the ϕ field. We need to know both initial conditions to solve for $\phi(t,x)$ because its equation of motion is second order.

We know that $\phi(t,x)$ can be expressed as a sum over the modes $f_n(t,x)$. But given some initial condition, how do we solve for the c_n coefficients? We need an *inner product defined on spacelike surfaces at fixed time.* A natural place to start is with the equations of motion, since these are the equations we have diagonalized. One often obtains inner products by noting that some integral $\int f dg = -\int df g$, and if the operator "d" can be diagonalized, then this integral must vanish unless f and g have the same eigenvalue under the action of d.

Therefore for *any* two functions ψ_1 and ψ_2 on spacetime (vanishing sufficiently fast at spatial infinity) we can define the integral

$$\int_\Omega d^dx dt\sqrt{-g}\left[\psi_1^\dagger\left(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2\right)\psi_2 - \left[\left(g^{\mu\nu}\nabla_\mu\nabla_\nu - m^2\right)\psi_1^\dagger\right]\psi_2\right] \tag{4.8}$$

over some region Ω. Let's imagine that Ω spans all of space, but ends at some finite times t_i and t_f on Cauchy surfaces Σ_i and Σ_f. We can re-write this integral as a boundary integral

$$\int_{\Sigma_f-\Sigma_i} d^dx\sqrt{-g}g^{00}\left[\psi_1^\dagger\nabla_t\psi_2 - \left(\nabla_t\psi_1^\dagger\right)\psi_2\right] \tag{4.9}$$

where we assumed that the t direction is orthogonal to Σ, for simplicity, and we took t pointing forward in time for both surfaces. If both ψ_1 and ψ_2 obey the equations of motion, then our original integral must vanish, so we see that

$$\langle\psi_1,\psi_2\rangle \equiv i\int_{\Sigma_f} d^dx\sqrt{-g}g^{00}\left[\psi_1^\dagger\nabla_t\psi_2 - \psi_2\nabla_t\psi_1^\dagger\right] = i\int_{\Sigma_i} d^dx\sqrt{-g}g^{00}\left[\psi_1^\dagger\nabla_t\psi_2 - \psi_2\nabla_t\psi_1^\dagger\right] \tag{4.10}$$

defines a *conserved* inner product. In particular, if we normalize $\langle f_n, f_m\rangle = \delta_{mn}$ and $\langle f_n^\dagger, f_m\rangle = 0$ at some time then this normalization will be time independent. This gives the usual normalization

[13] Actually, the t dependence should technically be forced to be either $\cos(\omega t)$ or $\sin(\omega t)$ so that the solutions are real, assuming we are studying a real scalar field. Then we would express the full solution as a sum over sines and cosines, instead of complex exponentials with complex coefficients. The real case is simpler, but the complex case is more useful and standard. It's also noteworthy that if the t direction points along a Killing vector, then we have a time translation symmetry, and so the time dependence will always just be $e^{\pm i\omega t}$.

$\frac{1}{\sqrt{2k_0}}e^{ik\cdot x}$ for modes in flat spacetime (noting that in that case, since k is continuous, we get a delta function in place of δ_{mn}).

If we write $\phi(t,x)$ as in equation (4.7) then the c_n will be time independent, as desired. More importantly, it means that we can compute the c_n from

$$c_n = \langle \phi(0,x), f_n \rangle = i \int_{\Sigma_f} d^d x \sqrt{-g} g^{00} \left[\phi_0^\dagger(x) \dot{f}_n(x) - f_n(x) \dot{\phi}_0^\dagger(x) \right] \tag{4.11}$$

using our inner product along with both the $\phi(0,x)$ and $\dot{\phi}(0,x)$ initial conditions. So we see that our inner product automatically knows how initial conditions work in classical theories. At the classical level, if we the kinetic term had been a higher order differential operator, then we would end up with an inner product involving higher time derivatives, in keeping with expectations for the initial value problem.

Now consider the canonical momentum

$$\pi(x) = \frac{\delta L}{\delta \dot{\phi}} = \sqrt{-g(x)} g^{00}(x) \dot{\phi}(x) \tag{4.12}$$

In either the classical or the quantum theory we need to satisfy the canonical commutation relations (or Poisson brackets)

$$[\pi(x), \phi(y)] = -i\delta^d(x-y) \tag{4.13}$$

and as usual, when dealing with the quantum theory we can write

$$\phi(t,x) = \sum_n \frac{1}{N_n} \left[f_n(t,x) a_n + f_n^\dagger(t,x) a_n^\dagger \right] \tag{4.14}$$

in terms of anniliation and creation operators a_n and $a_n^\dagger$, and a normalization factor N_n that we have added in case we need it. When we quantize, the coefficients c_n of the nth classical harmonic mode has been replaced with creation and annihilation operators for quantum versions of that mode.

Now we must demand that the field and its canonical momentum satisfy the canonical commutation relations. A simple computation shows that these will be satisfied if

$$\sqrt{-g(x)} g^{00}(x) \sum_n \frac{1}{N_n^2} \left[\dot{f}_n(0,x) f_n^\dagger(0,y) - \dot{f}_n^\dagger(0,x) f_n(0,y) \right] = -i\delta^d(x-y) \tag{4.15}$$

However, this can be re-interpreted as an expression for the coefficients c_n that would be obtained if the function $\delta^d(x-y)$ (viewed as a function of y, with x serving as a fixed paramter) is expanded in the $f_n(0,y)$ basis. This means we can use the inner product to write

$$\begin{aligned} \sqrt{-g(x)} g^{00}(x) \frac{1}{N_n^2} \dot{f}_n(0,x) &= \int d^d y \sqrt{-g(y)} g^{00}(y) \delta^d(x-y) \dot{f}_n(0,y) \\ &= \sqrt{-g(x)} g^{00}(x) \dot{f}_n(0,x) \end{aligned} \tag{4.16}$$

in this very special case. This determines that all the normalization constants $N_n = 1$ if we work in terms of modes f_n normalized with our inner product as $\langle f_n, f_m \rangle = \delta_{mn}$. So we have obtained a canonically quantized scalar field $\phi(t, x)$ – it obeys the correct EoM, and it has the correct commutation relations.

Note that the choice of time coordinate t played an essential role in our logic. Had we chosen a different notion of time, we would have had different mode functions, a different inner product, and a different quantization. Hawking originally derived black hole radiation by comparing the different natural quantizations that exist before and after black hole formation.

4.2 Canonical Quantization of a Scalar Field in AdS

Let us return and consider the AdS_3 example for concreteness. Then the action is

$$S = \int dt d\rho d\theta \frac{\sin\rho}{\cos\rho} \frac{1}{2} \left(\dot{\phi}^2 - (\partial_\rho \phi)^2 - \frac{1}{\sin^2\rho} (\partial_\theta \phi)^2 - \frac{m^2}{\cos^2\rho} \phi^2 \right) \tag{4.17}$$

The canonical momentum conjugate to ϕ is

$$P_\phi = \frac{\delta L}{\delta \dot{\phi}} = \frac{\sin\rho}{\cos^2\rho} \dot{\phi} \tag{4.18}$$

So we will want to impose the canonical commutations relations

$$[\phi(t, X), P_\phi(t, Y)] = i(2\pi)^2 \delta^2(X - Y) \tag{4.19}$$

We will do this by first solving the equations of motion for ϕ, and then writing ϕ in terms of a sum over harmonic oscillator modes. As we discussed in generality above, the canonical commutation relation then gives a normalization condition for the modes. Note that we are working in global coordinates; we would obtain different modes, with a different interpretation, if we worked in the Poincaré patch using its notion of time.

The equations of motion are

$$\ddot{\phi} - \frac{\cos^2\rho}{\sin\rho} \partial_\rho \left(\sin\rho \cos^{-2}\rho \partial_\rho \phi \right) - \frac{1}{\sin^2\rho} \partial_\theta^2 \phi + \frac{m^2}{\cos^2\rho} \phi = 0 \tag{4.20}$$

Taking $\phi = e^{i\omega t + i\ell\theta} \psi(\rho)$, we have

$$-\psi''(\rho) - \frac{1}{\cos\rho \, \sin\rho} \psi'(\rho) + \left(\frac{\ell^2}{\sin^2\rho} + \frac{m^2}{\cos^2\rho} \right) \psi(\rho) = \omega^2 \psi(\rho) \tag{4.21}$$

which is just the equation for a one-particle wavefunction in $d = 2$ that we obtained previously. The solutions were given in equation (3.45).

The quantized field which obeys canonical commutation relations is

$$\phi(t, \rho, \Omega) = \sum_{n,\ell} \psi_{n\ell J}(t, \rho, \Omega) a_{n\ell} + \psi^*_{nlJ}(t, \rho, \Omega) a^\dagger_{n\ell}, \tag{4.22}$$

with the wavefunctions $\psi_{n\ell}$ that were defined in equation (3.45). To find the normalizations of the $\psi_{n\ell}$ we impose

$$\langle \psi_{n\ell}, \psi_{n'\ell'} \rangle = \delta_{nn'}\delta_{\ell\ell'} = \int_{AdS} d^d x \sqrt{-g} g^{00} (\psi_{n\ell}\partial_t \psi_{n'\ell'} - \partial_t \psi_{n\ell} \psi_{n'\ell'}) \tag{4.23}$$

which translates in AdS_3 to

$$1 = \frac{1}{(N_{\Delta n\ell})^2} \int d\rho d\theta \sin\rho \, 2(\Delta + 2n + \ell) \psi^\dagger_{n\ell}(\rho,\theta) \psi_{n\ell}(\rho,\theta) \tag{4.24}$$

It is easy to see that modes with different ℓ are orthogonal, as the θ integral then vanishes. The orthogonality of modes with different n can be verified directly. One finds that the normalizations of the $\psi_{n\ell J}$ are

$$N_{\Delta n\ell} = (-1)^n \sqrt{\frac{n!\Gamma^2(\ell + \frac{d}{2})\Gamma(\Delta + n - \frac{d-2}{2})}{\Gamma(n + \ell + \frac{d}{2})\Gamma(\Delta + n + \ell)}}. \tag{4.25}$$

by a direct computation. This can be checked by hand for small n and ℓ.

A crucial condition obeyed by $\phi(x)$ is that, as in flat spacetime, $[\phi(x), \phi(y)] = 0$ whenever x and y are spacelike separated in AdS. This means that if we construct local interactions $\mathcal{V}[\phi(x)]$ then we will also have $[\mathcal{V}(x), \mathcal{V}(y)] = 0$ outside the lightcone. Weinberg [3] showed (see section 3.5 of his book) that this condition was necessary for QFT to produce a sensible Poincaré invariant S-Matrix. In section 2.4 of [10] this reasoning was generalized to AdS and the conformal algebra.

We saw in section 3.2 that a single free particle in AdS transforms under an irreducible representation of the conformal group. The particle's ground state is a primary state of the conformal algebra, because it is annihilated by all the 'lowering operators', the special conformal generators K_μ. All the other states can be built with P_μ, the momentum generator or 'raising operator' with respect to D. Soon we will discuss analagous statements for the quantum field $\phi(x)$.

5 Approaching the Boundary of AdS

Now we would like to consider the asymptotic structure of AdS, or the 'boundary at infinity'. First we will discuss what the boundary is and how it inherits structure from the bulk (the inside) of the spacetime. Then we will discuss how the boundary inherits a full fledged theory, with its own correlation functions, from the bulk. This is actually more familiar than one might think – the usual method of obtaining a flat space S-Matrix from a bulk QFT can be interpreted in the same spirit.

5.1 Asymptotia and Penrose Diagrams

The purpose of a Penrose diagram (or 'causal diagram' or 'conformal diagram') is to encapsulate the causal structure of the entire infinite spacetime in a compact picture. To make a Penrose diagram, we (vastly) distort the geometry in order to map the entire spacetime into a finite region, while maintaining the notions of timelike, spacelike, and null separations between points. Penrose diagrams are also useful for understanding the structure of (or 'boundary at') infinity because they turn infinity into a finite codimension one region with a causal structure that is inherited from the bulk.

How can we distort the geometry without affecting causality? An easy way is to multiply the entire metric by a Weyl factor $f(x)$, sending $g_{\mu\nu}(x) \to f(x) g_{\mu\nu}(x)$.

5.1.1 The Penrose Diagram for Flat Spacetime

Let's review how this works in flat spacetime. If we write the flat space metric as

$$ds^2 = dt^2 - dr^2 - r^2 d\Omega^2 \tag{5.1}$$

then we can introduce coordinates $2 \tan U = t + r$ and $2 \tan V = t - r$ so that U and V run from $-\pi/2$ to $\pi/2$. In these coordinates the metric takes the form

$$ds^2 = \frac{4dUdV}{\cos^2 U \cos^2 V} + (\tan U - \tan V)^2 d\Omega^2 \tag{5.2}$$

We can perform a Weyl transformation and multiply by $\cos^2 U \cos^2 V$. Now using the new coordinates $T = U + V$ and $R = U - V$ and a trig identity, we find

$$ds^2 = dT^2 - dR^2 + \sin^2(R) d\Omega^2 \tag{5.3}$$

with T and R only running over a finite diamond. This is the derivation of the usual diamond shaped Penrose diagram for flat spacetime.

The boundary (or 'conformal infinity') is a null diamond. There are special points where time-like and space-like geodesics terminate, and then the null surfaces where all light rays approach infinity. Notice that the boundary is not a nice spacetime; for example, it does not inherit a Lorentzian metric with timelike directions. This explains why it is non-trivial to find a holographic description of flat spacetime – the holographic theory cannot be a conventional QFT, since it would have to live on a bizarre (null) spacetime.

5.1.2 The Penrose Diagram and the Boundary of Global AdS

When we study AdS/CFT we will constantly refer to the boundary of AdS. The factor of $1/\cos^2(\rho)$ multiplies the entire metric in equation (2.1), so we can determine the AdS Penrose diagram by simply dropping it (or more precisely, by multiplying the AdS metric by the Weyl factor $\cos^2(\rho)$). This gives a new metric

$$ds^2 = dt^2 - d\rho^2 - \sin^2(\rho) d\Omega^2 \tag{5.4}$$

Since $0 \leq \rho < \pi/2$, we can now draw a spatially finite Penrose diagram. Time still runs from $-\infty$ to ∞, and we cannot alter this fact while maintaining the feature that light rays move at 45 degree angles in the t-ρ plane.

In terms of the global coordinates that cover the entirety of the AdS manifold, the boundary of AdS is the cylinder $R \times S^{d-1}$ that we obtain by taking $\rho \to \frac{\pi}{2}$, its limiting value at spatial infinity.

We can simply use the coordinates t and Ω to parameterize this boundary cylinder. Note that unlike in the flat case, the boundary of AdS is a nice spacetime with a metric that inherits the signature of the AdS metric.

5.1.3 What About the Poincaré Patch?

Recall that in the Poincaré Patch coordinates the AdS metric is

$$ds^2 = \frac{1}{z^2}\left(dt^2 - dz^2 - \sum_{i=1}^{d-1} x_i^2\right) \tag{5.5}$$

where the z coordinate only covers the range $0 < z < \infty$.

Clearly if we multiply by the Weyl factor z^2 we simply have the metric for a flat $d+1$ dimensional spacetime, so the Penrose diagram must look the same as that for half of Minkowski spacetime. In other words, it will look like a half-diamond.

What does this mean for the boundary of the Poincaré patch? Well, the surface $z = 0$ must be part of the boundary, since the Penrose diagram abruptly ends there. From the identification in equations (2.13) we see that $z \to 0$ corresponds with $\rho \to \pi/2$, as expected. We also see that $z \to \infty$ corresponds to $\rho \to \pi/2$, but only at the unique point on the boundary cylinder where $\Omega_i = 0$ and $\tau = 0$ (in global coordinates). If we take some combinations of $\vec{x}$ and t to infinity we can reach the end of the Poincaré patch, but not the boundary of the spacetime, because we are free to extend the spacetime past this 'infinity', which can actually be reached in a finite proper time.

But the real punchline is that by taking $z \to 0$, we find a different (but only partial) boundary for the AdS Poincaré patch. This boundary is parameterized by $(t, \vec{x})$, and it just looks like it inherits the geometry of a d-dimensional Minkowksi spacetime.

5.1.4 Cosmology and DeSitter Space

Let us make a couple of comments about cosmology, which naturally transpires (during the early epoch of inflation) in quasi-deSitter spacetime. dS_{d+1} is the hyperboloid

$$X_0^2 - \sum_{i=1}^{d+1} X_i^2 = -R^2 \tag{5.6}$$

Setting $R = 1$ (in other words, the Hubble constant is 1) and applying $\sinh^2 t - \cosh^2 t = -1$ we can parameterize dS as

$$ds^2 = dt^2 - \cosh^2(t) d\Omega^2 \tag{5.7}$$

Now using $\tan(\eta/2) = \tanh(t/2)$ we can re-write this metric as

$$ds^2 = \frac{1}{\cos^2\eta}\left(d\eta^2 - d\Omega^2\right) \tag{5.8}$$

Note the similarity to AdS in global coordinates. Now we have a metric that has an obvious Penrose diagram. Note that spatially *there is no boundary*, and the spacetime only ends in time, at $\eta = \pm\pi/2$. This is the origin of the usual statement that deSitter space only has a boundary in the infinite past and infinite future. The absence of any notion of time on the boundary is one (mild) reason to worry about attempts at holography in deSitter spacetime... the deeper problems are that dS is unstable to bubble nucleation, that classically there will be a spacelike singularity in the past (due to the singularity theorems), and that dS behaves like a thermal system, so it is 'asymptotically hot' – fields at infinity never cease their fluctuations.

5.2 General Approaches to the Boundary

Penrose diagrams describe the causal structure of the bulk and boundary. Now let us see how in the case of AdS, we can also ascribe a metric to the boundary. We will see that the boundary metric is not uniquely determined, but can be multiplied by an arbitrary Weyl factor that depends on how we approach infinity.

5.2.1 Lessons from the Poincaré vs Global Approach

It seems we have two different ways to approach the boundary – in global coordinates, we naturally obtain a cylinder parameterized by $(\tau, \hat{\Omega})$ when we take $\rho \to \pi/2$ at the same rate for all $(\tau, \hat{\Omega})$. In contrast, in the Poincaré patch coordinates we find a Minkowksi space parameterized by $(t, \vec{x})$ when we take $z \to 0$ at the same rate for all $(t, \vec{x})$. Actually, there are infinitely more possibilities.

The idea is that in the global coordinate metric of equation (2.1), we can approach different points on the boundary, parameterized by $(t, \hat{\Omega}_i)$, at different rates. This effectively allows us to perform further Weyl transformations as we take $\rho \to \pi/2$ in order to alter the geometry of the boundary. Specifically, we can choose some function $f\left(t, \hat{\Omega}_i\right)$ in the neighborhood $\rho = \pi/2 - \epsilon f(t, \hat{\Omega}_i)$ with small ϵ, and then take $\epsilon \to 0$. This results in an effective metric on the boundary set by

$$\begin{aligned} ds^2 &= \frac{1}{\cos^2(\rho(\epsilon, f))}\left(dt^2 - \sin^2(\rho) d\Omega_i^2\right) \\ &\approx \frac{1}{\epsilon^2[f(t,\Omega)]^2}\left(dt^2 - (1-\epsilon)^2 d\Omega_i^2\right) \\ &\to \left[f\left(t, \hat{\Omega}_i\right)\right]^{-2}\left(dt^2 - d\hat{\Omega}_i^2\right) \end{aligned} \tag{5.9}$$

which is related to the metric on the cylinder by a Weyl transformation. So by taking the limit as we approach the boundary in different ways, we can obtain a CFT on any conformally flat manifold, a.k.a. any manifold related to flat spacetime by a Weyl transformation. Note that a cylinder is conformally flat since we can take $f(t, \hat{\Omega}_i) = e^{-t}$ and switch to the coordinate $r = e^t$ to find a flat space metric (in Euclidean signature).

5.2.2 The Euclidean Version

We parameterized Euclidean AdS_{d+1} using global or Poincaré coordinates as

$$\begin{aligned} X_0 &= R\frac{\cosh\tau}{\cos\rho} = \frac{1}{2}\left(\frac{z^2+\vec{r}^2+R^2}{z}\right) \\ X_{d+1} &= R\frac{\sinh\tau}{\cos\rho} = \frac{1}{2}\left(\frac{z^2+\vec{r}^2-R^2}{z}\right) \\ X_i &= R\tan\rho\Omega_i = \frac{R}{z}r_i \end{aligned} \tag{5.10}$$

where $\vec{r} = (\vec{x}, t)$. Note that this means that $\vec{r}_i = \hat{\Omega}_i e^\tau \sin\rho$ and $z = e^\tau \cos\rho$. A nice thing about the Euclidean version is that as $z \to 0$, we have $\rho \to \pi/2$ with a coordinate mapping

$$\frac{e^{\tau/2}\hat{\Omega}_i}{\sqrt{\frac{\pi}{2}-\rho}} = \frac{r_i}{\sqrt{z}} \tag{5.11}$$

This naturally produces a flat spacetime metric $ds^2 = d\vec{r}^2$ for the boundary.

We can also see this directly in global coordinates without using the Poincaré patch description – as we approach the boundary of Euclidean AdS we naturally obtain a flat space metric in polar coordinates (which we will soon associate with radial quantization of CFT). In particular, with the metric

$$ds^2 = \frac{1}{\cos^2\rho}\left(d\tau^2 + \sin^2\rho d\Omega^2\right) \tag{5.12}$$

we can simply approach the boundary as

$$\rho = \frac{\pi}{2} - \epsilon e^{-\tau} \tag{5.13}$$

with $\epsilon \to 0$. Then we find the effective boundary metric

$$ds^2_{\partial AdS} \to e^{2\tau}(d\tau^2 + d\Omega^2) \to dr^2 + r^2 d\Omega^2 \tag{5.14}$$

with $r = e^\tau$. We see that $r = 0$, the origin of the Euclidean plane, maps to $t = -\infty$, the infinite past in Euclidean AdS, while $r = \infty$ corresponds to the infinite future.

5.2.3 Getting a DeSitter Boundary Geometry

Finally, note that the d-dimensional deSitter metric from equation (5.8) is also conformally flat, so by approaching the boundary of AdS_{d+1} in an appropriate manner, we obtain a deSitter boundary metric. If we choose

$$\rho = \frac{\pi}{2} - \epsilon\cos(t) \tag{5.15}$$

Then we find

$$\frac{1}{\cos^2\rho(\epsilon;t)}(dt^2 - \sin^2\rho d\Omega^2) \to \frac{1}{\cos^2 t}(dt^2 - \sin^2\rho d\Omega^2) \tag{5.16}$$

This is precisely the metric for global deSitter space, where in equation (5.8) we called t the variable η. In particular, now we only have $-\pi/2 < t < \pi/2$. This scaling makes it possible to obtain a CFT living in a deSitter spacetime background! We could use this to study a holographic CFT present during inflation. It is equally easy to find an f that makes the boundary look like an FRW cosmology.

5.2.4 Lorentzian Poincaré Patch Version

Finally, let us see how we could obtain the Lorentzian Poincaré patch from a boundary limit of the global coordinates via an appropriate choice of $f(t,\Omega)$. The boundary of the Poincaré patch is obtained by taking $z \to 0$ with fixed $t, \vec{x}$, so let us translate this into the global coordinates $\rho, \tau, \hat{\Omega}$. Using the relations of equation (2.13), we see that if we take the coordinate ρ towards $\pi/2$ at a rate dictated by

$$\rho(z;\tau,\hat{\Omega}_i) = \frac{\pi}{2} - \frac{z}{R^2}(\cos\theta - \cos\tau) \quad \text{as} \quad z \to 0 \tag{5.17}$$

where θ is the complement of the angle Ω_d. This means that in the limit as we approach the boundary we obtain the metric

$$ds^2 = \frac{1}{(\cos\theta - \cos\tau)^2}(d\tau^2 - d\theta^2 - \sin^2\theta d\Omega_{d-1}^2) \tag{5.18}$$

This is familiar from our discussion of the Penrose diagram for flat spacetime – if we first define $2U = -\theta + \tau$ and $2V = \theta + \tau$ then we obtain a new metric

$$ds^2 = \frac{1}{\sin^2 U \sin^2 V}\left(dUdV - \sin^2(U+V)d\Omega_{d-1}^2\right) \tag{5.19}$$

Now we can define $\cot u = U$ and $\cot v = V$ so that the metric becomes simply

$$ds^2 = dudv - \frac{(u+v)^2}{4}d\Omega_{d-1}^2 \tag{5.20}$$

Finally, we can write this in terms of $2t = u - v$ and $2r = u + v$ and see that it is in fact the flat spacetime metric.

5.2.5 Embedding Space Coordinates and the Projective Null Cone

It is natural to translate all of these statements into the more abstract embedding space coordinates, where the full conformal symmetry is manifest. For convenience, we recall that the identification of coordinates in Euclidean signature is

$$\begin{aligned}
X_0 &= R\frac{\cosh\tau}{\cos\rho} = \frac{1}{2}\left(\frac{z^2+\vec{r}^2+R^2}{z}\right) \\
X_{d+1} &= R\frac{\sinh\tau}{\cos\rho} = \frac{1}{2}\left(\frac{z^2+\vec{r}^2-R^2}{z}\right) \\
X_i &= R\tan\rho\Omega_i = \frac{R}{z}r_i
\end{aligned} \tag{5.21}$$

The limit $\rho = \frac{\pi}{2} - \epsilon f(\tau,\Omega)$ with $\epsilon \to 0$ sends all of the X_A to infinity. We can obtain a finite limit by fixing the *null projective cone* with coordinates

$$P_A \equiv \epsilon X_A \tag{5.22}$$

in the limit $\epsilon \to 0$. Since $X_A X^A$ is fixed, the null projective coordinates satisfy

$$P_A P^A = 0 \quad \text{and} \quad P_A \cong \lambda P_A \tag{5.23}$$

where the second statement means that the P_A and λP_A are identified – the overall positive rescaling is a redundancy of description, or a 'gauge symmetry'. The conformal group $SO(1,d+1)$ acts on the P_A in the obvious way inherited from the X_A.

Note that the redundant rescalings by λ have the effect of sending $\epsilon \to \epsilon/\lambda$. So $P_A \to \lambda P_A$ rescales the boundary metric by an overall factor, enacting a global dilatation. This means that the λ-redundancy can be used to determine the overall scaling of a function $f(P_A)$. If f has overall scaling dimension Δ, then $f(\lambda P_A) = \lambda^{-\Delta} f(P_A)$. If f does not transform homogeneously in this way, then it is not a conformally invariant function. To be concrete, in a conformal theory

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle = \frac{1}{(P_1\cdot P_2)^\Delta} \tag{5.24}$$

scales by $\lambda^{-2\Delta}$, as it should. In this example we also see another very useful feature of the embedding coordinates, namely that the only invariant that can be formed is $P_1 \cdot P_2$.

In the Euclidean case where the conformal group is $SO(1,d+1)$, the P_A can be identified with flat space, or any conformally flat manifold. For example, using our choice

$$\rho = \frac{\pi}{2} - \epsilon e^{-\tau} \tag{5.25}$$

with $\epsilon \to 0$ means that

$$P_A = (P_+, P_-, P_\mu) = (e^{2\tau}, 1, \Omega_i e^\tau) \tag{5.26}$$

where $P_\pm = P_0 \pm P_{d+1}$ and we used the identifications of the X_A above. If we take $r_i = e^\tau \Omega_i$, where r_i is a coordinate in Euclidean flat spacetime, then we obtain flat space as a *section* of the null projective cone. This terminology just means that we are studying the intersection of the hyperplane

$P_- = 1$ with the null projective cone, fixing the projective λ-redundancy and obtaining d-spacetime coordinates.[14] In general we explicitly obtain

$$\begin{aligned} P_0 &= \frac{\cosh\tau}{f(\tau,\Omega)} \\ P_{d+1} &= \frac{\sinh\tau}{f(\tau,\Omega)} \\ P_i &= \frac{\Omega_i}{f(\tau,\Omega)} \end{aligned} \tag{5.27}$$

in the Euclidean case, with the equation $\cos\rho = f$ determining the section of the cone, ie the surface that intersects the cone to fix the geometry.

5.3 Implementing Holography

The basic idea of holography is that physics in $d+1$ dimensions can be reproduced by a 'hologram at infinity', in d or fewer dimensions. It's always a good idea to take ideas very seriously, make them as concrete and specific as possible, and then see where you are led. Now that we have constructed a quantum field $\phi(x)$ depending on the AdS coordinate x, we can do just that – we can study $\phi(x)$ in the limit that x approaches infinity and see what we find. Why is this a reasonable guess? If there exists a holographic dual to effective field theory in AdS, then it must reproduce at least some of the familiar observables in AdS. Correlation functions of operators $\phi(x)$ as $x \to \infty$ would be a reasonable starting point, since they can be easily computed in the AdS QFT, while they are also sensible data 'at infinity'. The S-Matrix is a flat space analog of these observables.

Let us begin with an even broader approach by studying ϕ as an quantum operator with $x \to \infty$. For this purpose we will use the Euclidean version of AdS. We saw in section 5.1 that the boundary of AdS (aka 'infinity') is just the limit $\rho \to \pi/2$ in global coordinates. Being a bit more careful, we saw in section 5.2.2 that we need to take this limit as

$$\rho(\epsilon) = \frac{\pi}{2} - \epsilon e^{-\tau} \tag{5.28}$$

with $\epsilon \to 0$ in a coordinate independent way in order to recover a Euclidean flat spacetime metric on the boundary.

Before we consider $\phi(t,\rho,\Omega)$ in the limit $\rho \to \pi/2$, let us think about how we expect ϕ to behave near infinity. As usual in QFT, we have been assuming that $\phi \to 0$ as $\rho \to \pi/2$ – were this not the case, then ϕ would not be normalizable. Thus naively we expect $\phi = 0$ 'at' $\rho = \pi/2$. However, in the presence of sources with finite energy and charge, we expect ϕ to have a universal asymptotic

[14] Sometimes it is claimed that knowledge of the CFT on one spacetime allows us to 'lift' the correlators to the projective null cone, and then to change to a different section, transforming the CFT to some other manifold related to the original one by a Weyl transformation, $g_{\mu\nu} \to e^{2\Omega(x)} g_{\mu\nu}$. In fact, this procedure is ambiguous for CFTs in $d > 2$ dimensions. Knowledge of the CFT correlators on e.g. flat spacetime does not uniquely determine the correlators on other conformally flat manifolds. General Weyl transformations are not part of the global conformal group, so conformal symmetry is insufficient for determining how the correlators transform.

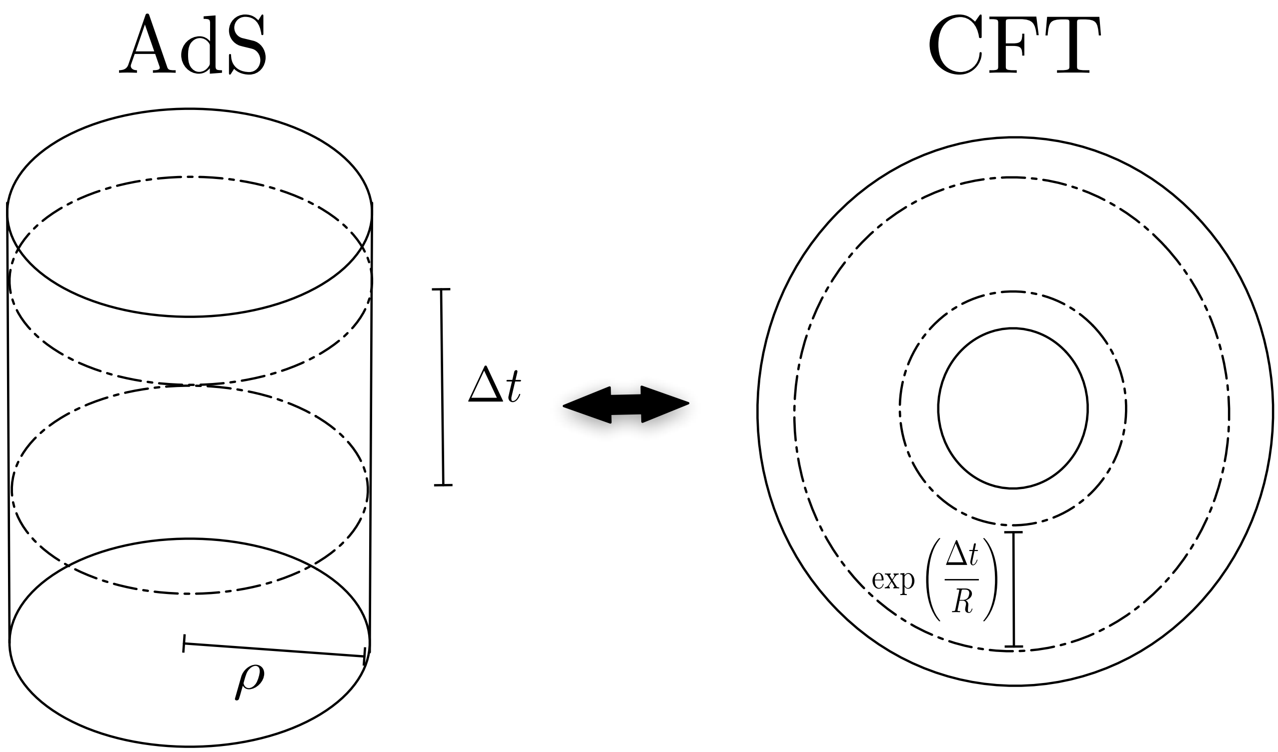

Figure 7: This figure shows how the AdS cylinder in global coordinates corresponds to the CFT in radial quantization. The time translation operator in the bulk of AdS is the Dilatation operator in the CFT, so energies in AdS correspond to dimensions in the CFT. We made this mapping very explicit in section 5.2.2.

behavior *near* infinity. For example, in flat spacetime gauge fields fall off as $1/r$ in $3+1$ dimensions, whereas massive fields fall off exponentially. So we can define a non-vanishing field 'at infinity' by multiplying by a compensating factor, and only then taking the asymptotic limit.

As we can see from the explicit wavefunctions in equation (3.45), as $\rho \to \pi/2$ we have $\psi_{n\ell J} \to 0$ as $\cos^{\Delta} \rho$. One can interpret this as the statement that finite energy particles in AdS have a vanishing probability of making it to a specific point on the boundary, due to the AdS curvature. In other words, the response of the field $\phi(x)$ to finite energy sources/objects/ϕ-particles in the bulk leads to a $\cos^{\Delta} \rho$ profile for ϕ near infinity.

Thus we can define a new quantum operator $\mathcal{O}(t, \Omega)$, 'dual to ϕ', or more accurately, representing ϕ 'at infinity', via[15]

$$\mathcal{O}(t, \Omega) \equiv \lim_{\epsilon \to 0} \frac{\phi(t, \rho(\epsilon), \Omega)}{\epsilon^{\Delta}} \tag{5.29}$$

[15]Note that we are *not* rescaling ϕ with $\cos \rho$, but only with ϵ. This is necessary to preserve the conformal (AdS isometry) transformation properties of ϕ, so that they are inherited by the CFT operator $\mathcal{O}$.

which is finite on the boundary of AdS. Computing this limit for the wavefunction $\psi_{n\ell J}$ we note that

$$\begin{aligned} \sin(\rho) &\rightarrow 1 \qquad (5.30) \\ \cos(\rho) &\rightarrow \epsilon e^{-\tau\Delta} \qquad (5.31) \end{aligned}$$

which means that since

$$\psi_{n\ell J} \propto e^{-E_{n,\ell}\tau} Y_{\ell J}(\Omega) \left[\sin^\ell \rho \cos^\Delta \rho \, {}_2F_1\left(-n, \Delta+\ell+n, \ell+\frac{d}{2}, \sin^2\rho\right)\right] \qquad (5.32)$$

we have

$$\begin{aligned} \psi_{n\ell J} &\rightarrow \frac{1}{N_{\Delta n\ell}} e^{-(2\Delta+2n+\ell)\tau} Y_{\ell J}(\Omega)\, {}_2F_1\left(-n, \Delta+\ell+n, \ell+\frac{d}{2}, 1\right) \qquad (5.33) \\ \psi^\dagger_{n\ell J} &\rightarrow \frac{1}{N_{\Delta n\ell}} e^{(2n+\ell)\tau} Y^\dagger_{\ell J}(\Omega)\, {}_2F_1\left(-n, \Delta+\ell+n, \ell+\frac{d}{2}, 1\right) \qquad (5.34) \end{aligned}$$

Note that the factors of $\cos^\Delta(\rho)$ are responsible for a shift in the t-dependence. Also, a useful fact about hypergeometric functions is that

$${}_2F_1\left(-n, \Delta+\ell+n, \ell+\frac{d}{2}, 1\right) = \frac{\Gamma(\ell+\frac{d}{2})\Gamma(\frac{d}{2}-\Delta)}{\Gamma(n+\ell+\frac{d}{2})\Gamma(\frac{d}{2}-n-\Delta)} \qquad (5.35)$$

This means that the operator $\mathcal{O}$ inherits a normalization

$$\begin{aligned} \frac{1}{N_{\mathcal{O}n\ell}} &\propto \frac{(-1)^n\Gamma(\ell+\frac{d}{2})\Gamma(\frac{d}{2}-\Delta)}{\Gamma(n+\ell+\frac{d}{2})\Gamma(\frac{d}{2}-n-\Delta)} \left(\frac{n!\Gamma^2(\ell+\frac{d}{2})\Gamma(\Delta+n-\frac{d-2}{2})}{\Gamma(n+\ell+\frac{d}{2})\Gamma(\Delta+n+\ell)}\right)^{-1/2} \qquad (5.36) \\ &= \sqrt{\frac{\Gamma(\Delta+n+l)\Gamma\left(\Delta+n-\frac{d-2}{2}\right)\Gamma\left(\frac{d}{2}\right)}{n!\Gamma(\Delta)\Gamma\left(\Delta-\frac{d-2}{2}\right)\Gamma\left(\frac{d}{2}+n+l\right)}} \qquad (5.37) \end{aligned}$$

where in the second line we multiplied by a constant factor of $\sqrt{\Gamma(\Delta+\frac{d-2}{2})/\Gamma(\Delta)}$ so that $\mathcal{O}$ will have a 2-pt correlator with a simple normalization. Now we can write the quantum operator

$$\mathcal{O}(t,\Omega) = \sum_{n,\ell} \frac{1}{N_{\mathcal{O}n\ell}} \left(e^{-(2\Delta+2n+\ell)\tau} Y_{\ell J}(\Omega) a_{n\ell} + e^{(2n+\ell)\tau} Y^\dagger_{\ell J}(\Omega) a^\dagger_{n\ell}\right) \qquad (5.38)$$

living on the boundary of AdS, in the coordinates determined by our specific choice of approach to the boundary. Let us now see why this operator has the correlators of a local CFT operator in flat spacetime.

5.4 Correlators

Now let us look at the correlation function of our new operator $\mathcal{O}$. Then in the next section we will study its conformal transformation properties, the operator state correspondence, and the OPE.

5.4.1 Computation from AdS

In the limit of large distances (which is all we need because we are taking an asymptotic limit in AdS), the 2-pt correlator of ϕ is

$$\langle\phi(X)\phi(Y)\rangle \approx \frac{\Gamma(\Delta)}{\Gamma\left(\Delta+1-\frac{d}{2}\right)} e^{-\Delta\sigma(X,Y)} \tag{5.39}$$

where $\sigma(X,Y)$ is the geodesic distance between X and Y in AdS. One can compute this 2-point correlation function in many ways, e.g. by directly summing over modes created by the oscillators $a_{n\ell J}$. We will give a version of that calculation below. It is also guaranteed to take this form at large distances (up to a normalization) due to the AdS isometries and basic facts about the behavior of solutions to the Klein-Gordon equation.

Geodesic distances in the ρ direction are just given by an integral $\int^{\rho}\sec(\rho)d\rho$. With $\rho=\frac{\pi}{2}-\epsilon e^{-\tau}$, in the $\epsilon\to 0$ limit this is just $\log(\pi/2-\rho(\epsilon))\approx\tau-\log\epsilon$. Without loss of generality we can take X and Y to be on opposite sides of the Euclidean AdS cylinder at some fixed time τ. This gives a 2-pt correlator for $\mathcal{O}$

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{\Gamma(\Delta)}{\Gamma\left(\Delta+1-\frac{d}{2}\right)}\frac{\exp\left[-\Delta(2\tau-2\log\epsilon)\right]}{\epsilon^{2\Delta}} = \frac{\Gamma(\Delta)}{\Gamma\left(\Delta+1-\frac{d}{2}\right)}\frac{1}{e^{2\Delta\tau}} \tag{5.40}$$

Restated in terms of the boundary points $x_1=\hat{\Omega}e^{\tau}$ and $x_2=-\hat{\Omega}e^{\tau}$ and dropping the normalization factor for simplicity, this is

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \propto \frac{1}{(x_1-x_2)^{2\Delta}} = \frac{1}{(2P_1\cdot P_2)^{\Delta}} \tag{5.41}$$

where we have also stated the result in terms of the projective null cone coordinates. Although we computed this result directly, the result actually follows entirely from conformal symmetry. The only conformal invariant that can be made from P_1 and P_2 is $P_1\cdot P_2$, and then the power of Δ follows from the D transformation properties that $\mathcal{O}$ inherits from ϕ.

Note that since we were studying a free theory in AdS, all correlators of the ϕ field can be reduced to Wick contractions leading to products of 2-pt (Gaussian) correlators. So in fact we already know all correlation functions of $\mathcal{O}(x)$ as well, for the same reason.

5.4.2 Computation from the CFT Operator

As we will soon see, $\mathcal{O}$ inherits conformal transformation properties from the AdS isometry transformations of ϕ. In particular, it certainly inherits translations, so WLOG we can compute

$$\langle\mathcal{O}(\vec{r})\mathcal{O}(0)\rangle \tag{5.42}$$

with, as usual $\vec{r}=e^{t}\hat{\Omega}$. However, note that

$$\begin{aligned}\mathcal{O}(0)|0\rangle &= \left(\sum_{n,\ell}\frac{1}{N_{\mathcal{O}n\ell}}e^{-(2\Delta+2n+\ell)t}Y_{\ell J}(\Omega)a_{n\ell}+e^{(2n+\ell)t}Y^{\dagger}_{\ell J}(\Omega)a^{\dagger}_{n\ell}\right)|0\rangle \\ &= a^{\dagger}_{0,0}|0\rangle\end{aligned} \tag{5.43}$$

This follows because all the annihilation operators destroy the vacuum, while the other creation operators are multiplied by positive powers of $\vec{r} = 0$. Now evaluating the correlator is extremely easy, because the only term that survives is proportional to $a_{0,0}a^\dagger_{0,0}$, and we simply find

$$\langle \mathcal{O}(\vec{r})\mathcal{O}(0)\rangle = \frac{1}{r^{2\Delta}} \tag{5.44}$$

which is exactly what we expected. Notice that it was crucial that we scaled ρ appropriately to obtain $\mathcal{O}$ from ϕ – if we had simply taken $\rho \to \pi/2$ uniformly then $\mathcal{O}$ would have been proportional to an additional $e^{-t\Delta}$; this would have given the correct correlators on a cylinder, but not in flat spacetime!

6 Generalized Free Theories and 'AdS/CT'

In this section we will show how our Generalized Free Theory, defined by the holographic limit of the AdS bulk field ϕ, satisfies and in many cases physically illustrates the general abstract properties expected for a local operator in a conformal theory. We refrain from using the words "CFT" because the conformal theory we are discussing lacks a stress-energy tensor.

6.1 The Operator/State Correspondence

First of all, what are these so-called operators?

Clearly we have the object $\mathcal{O}(\vec{r})$ that we defined previously in terms of the AdS field ϕ. To study the question in general, let us specialize for convenience to the case of AdS_3. The reason is that we can significantly simplify the notation in that case by writing

$$\vec{r} = (x, y) \tag{6.1}$$

in terms of

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy \tag{6.2}$$

Since we are studying the boundary of Euclean AdS_3 as the 2-dimensional Euclidean plane, the angular coordinate $\Omega \to \theta$ just parameterizes a single circle. In terms of z and $\bar{z}$ we have

$$r^2 = z\bar{z} \quad \text{and} \quad e^{2i\theta} = \frac{z}{\bar{z}} \tag{6.3}$$

and the $Y_\ell = e^{i\ell\theta}$. Therefore we can write

$$\mathcal{O}(z,\bar{z}) = \sum_{n,\ell} \frac{1}{N^{\mathcal{O}}_{n,\ell}} \left[(z\bar{z})^{-\left(\Delta+n+\frac{|\ell|}{2}\right)} \left(\frac{z}{\bar{z}}\right)^{\frac{\ell}{2}} a_{n\ell} + (z\bar{z})^{n+\frac{|\ell|}{2}} \left(\frac{\bar{z}}{z}\right)^{\frac{\ell}{2}} a^\dagger_{n\ell} \right] \tag{6.4}$$

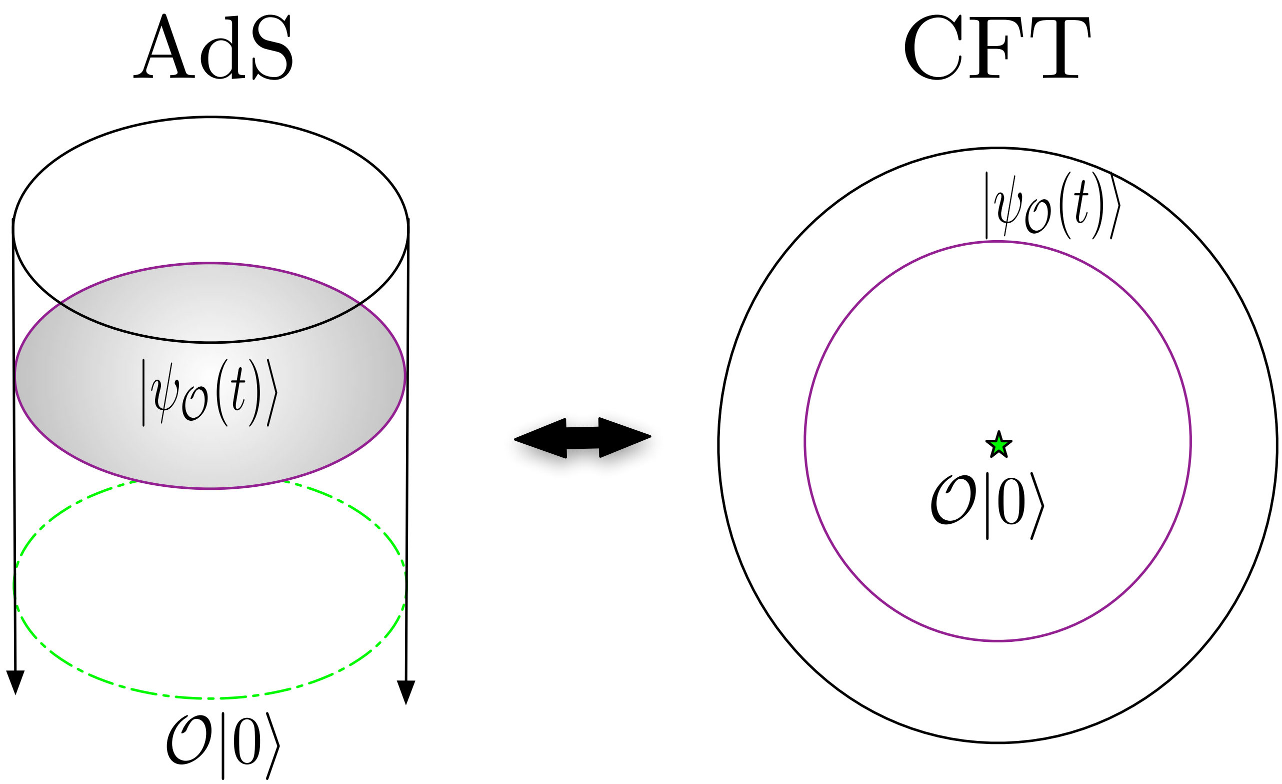

Figure 8: This figure portrays radial quantization in the CFT, while also providing the corresponding picture in AdS. The insertion of an operator at the origin in the CFT defines a specific AdS/CFT state; interpreted in AdS this sets up an initial state in the infinite past that evolves to $\psi_{\mathcal{O}}(t)$ at a finite time. Note that $e^t \sim |\vec{r}|$, so the global coordinate time is the logarithm of the radius of a circle about the origin in the CFT.

When we work in terms of z and $\bar{z}$, it's simpler to define the operator in terms of integers h and $\bar{h}$ with

$$\mathcal{O}(z, \bar{z}) = \sum_{h,\bar{h}=0}^{\infty} \frac{1}{N^{\mathcal{O}}_{h,\bar{h}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h,\bar{h}} + z^h \bar{z}^{\bar{h}} a^\dagger_{h,\bar{h}} \right] \tag{6.5}$$

where $2n + |\ell| = h + \bar{h}$ and $\ell = h - \bar{h}$, so that $n = \min(h, \bar{h})$.

Above we saw that when we act $\mathcal{O}(0)$ on the vacuum, we obtain the state

$$\mathcal{O}(0)|0\rangle = a^\dagger_{0,0}|0\rangle \tag{6.6}$$

Now let us consider what we get from P_μ acting on $\mathcal{O}$. We can simply write the momentum operator in terms of a basis as $P_z = \partial_z$ and $P_{\bar{z}} = \partial_{\bar{z}}$ (we will derive the fact that P_μ acts on $\mathcal{O}$ in this way in the next section). Thus we find

$$(P_z \mathcal{O})(0) = \frac{1}{N^{\mathcal{O}}_{h=1,\bar{h}=0}} a^\dagger_{h=1,\bar{h}=0}|0\rangle \tag{6.7}$$

Similarly, we can act with any number of P_z and $P_{\bar{z}}$ on $\mathcal{O}$. So we have a whole host of operators of the form

$$\left((\partial_z)^h(\partial_{\bar{z}})^{\bar{h}}\mathcal{O}\right)\Big|_{z,\bar{z}=0} \tag{6.8}$$

These create all of the states $a^\dagger_{h,\bar{h}}$! Crucially, these are just *the one-particle states in AdS_3 labeled by n and ℓ*. So we see that all one-particle states can be created by some particular CT operator! This is the beginning of the *operator/state correspondence*; it's not an abstract idea but a very concrete concept that can be visualized directly in AdS.

What about multipartcle states? We would like to define an operator like

$$\mathcal{O}^2(0) \tag{6.9}$$

and we would expect it to create 2 AdS particles, both in their ground state. But if we evaluate this directly, there appears to be a singularity. For example, if we look at

$$\begin{aligned}\mathcal{O}(z,\bar{z})\mathcal{O}(w,\bar{w}) &= \left(\sum_{h,\bar{h}=0}^{\infty}\frac{1}{N^{\mathcal{O}}_{h,\bar{h}}}\left[(z\bar{z})^{-\Delta}z^{-h}\bar{z}^{-\bar{h}}a_{h,\bar{h}}+z^h\bar{z}^{\bar{h}}a^\dagger_{h,\bar{h}}\right]\right)\\ &\quad\times\left(\sum_{h,\bar{h}=0}^{\infty}\frac{1}{N^{\mathcal{O}}_{h,\bar{h}}}\left[(w\bar{w})^{-\Delta}w^{-h}\bar{w}^{-\bar{h}}a_{h,\bar{h}}+w^h\bar{w}^{\bar{h}}a^\dagger_{h,\bar{h}}\right]\right)\end{aligned} \tag{6.10}$$

then we have singularities from the limit of $\omega\to 0$ in the second operator, as well as singularities that arise when the annhilation operators in $\mathcal{O}(z,\bar{z})$ hit the creation operators in $\mathcal{O}(w,\bar{w})$. Thus we can define $\mathcal{O}^2(0)$ as the normal ordered object (annihilation operators artificially moved to the right, without using the commutation relations) with these singularities removed, so that we get

$$\left(\mathcal{O}^2\right)(0)\equiv\left(\frac{1}{N^{\mathcal{O}}_{0,\bar{0}}}\right)^2 a^\dagger_{0,0}a^\dagger_{0,0} \tag{6.11}$$

Simillarly, we can define

$$\left(\mathcal{O}^3\right)(0)\equiv\left(\frac{1}{N^{\mathcal{O}}_{0,\bar{0}}}\right)^3 a^\dagger_{0,0}a^\dagger_{0,0}a^\dagger_{0,0} \tag{6.12}$$

and also the operator

$$\left(\mathcal{O}P_z\mathcal{O}\right)(0)\equiv\left(\frac{1}{N^{\mathcal{O}}_{0,\bar{0}}N^{\mathcal{O}}_{1,\bar{0}}}\right) a^\dagger_{0,0}a^\dagger_{1,0} \tag{6.13}$$

We can proceed to define operators with an arbitrarily long string of P_z, $P_{\bar{z}}$, and $\mathcal{O}$s. Again, the crucial point is that by acting with general linear combinations of such operators on the vacuum $|0\rangle$, we can obtain every single state in the theory – in particular, *there is a one-to-one correspondence between AdS/CFT states in the Hilbert space and local operators acting at the origin, $z = 0$. This is the operator/state correspondence for the CT defined by a free field in AdS.* From the CT point of view, this system is called a Generalized Free Theory, a Gaussian CFT, a CFT at infinite N, or a mean field theory. I will usually stick with the first term.

In closing, we should note that throughout this section we have used *radial quantization* without drawing much attention to it. This is the natural quantization scheme in mapping from AdS in global coordinates to a CFT on a plane.

6.2 Conformal Transformations of Operators

We obtained the Conformal Theory operator $\mathcal{O}(\vec{r})$ from a limit of $\phi(t, \rho, \Omega)$, a free field in AdS with an action symmetric under the AdS isometries aka conformal transformations. This means that $\mathcal{O}$ naturally inherits conformal transformation properties from AdS - so what are they? The intelligent way to approach this question would be to use the embedding space coordinates. However, we can also be somewhat less clever and see how it works directly in the global coordinate system.

We already computed the dilatation operator D as the AdS Hamiltonian, it is

$$D = \sum_{n,\ell J} (\Delta + 2n + \ell) a^\dagger_{n\ell J} a_{n\ell J} \tag{6.14}$$

and by definition, since in AdS there was no explicit time dependence, we have that

$$[D, \phi(t, \rho, \Omega)] = \partial_t \phi \tag{6.15}$$

in Euclidean signature. Let's compute its action on $\mathcal{O}$ in two different ways. Note that if we had defined $\mathcal{O}$ by simply taking $\rho \to \pi/2$ in a t-independent way, then we would find that $\mathcal{O}$ obeys the same relation as ϕ. That would be the correct result for a CFT living on the cylinder $R \times S^{d-1}$. However, in order to obtain a CT in flat Euclidean space, we took $\rho = \pi/2 - \epsilon e^{-t}$, producing an extra explicit t-dependence.

In particular, an infinitesimal shift $t \to t + a$ leads to

$$\begin{aligned} [D, \mathcal{O}(t, \Omega)] &= \lim_{\epsilon \to 0} \left(\frac{d}{dt} \frac{\phi(t, \rho(\epsilon; t), \Omega)}{\epsilon^\Delta} - \frac{\partial \rho}{\partial t} \frac{\partial}{\partial \rho} \frac{\phi(t, \rho(\epsilon; t), \Omega)}{\epsilon^\Delta} \right) \\ &\approx \frac{d}{dt} \mathcal{O}(t, \Omega) - \lim_{\epsilon \to 0} \frac{\partial \rho}{\partial t} \partial_\rho \frac{\phi(t, \rho, \Omega)}{\epsilon^\Delta} \\ &\approx \left(\Delta + \frac{\partial}{\partial t} \right) \mathcal{O}(t, \Omega) \end{aligned} \tag{6.16}$$

with $\rho(\epsilon; t) = \pi/2 - \epsilon e^{-t}$ as usual. The crucial step comes in the first line, where we recognize that time derivatives acting on $\phi(t, \rho(\epsilon; t), \Omega)$ will also act on $\rho(\epsilon; t)$, and so we have to subtract the second term to compensate. A similar procedure will be necessary for P_μ and K_μ. The final result can be written in $\vec{r} = e^t \hat{\Omega}$ coordinates as

$$[D, \mathcal{O}(\vec{r})] = \left(\Delta + r \frac{\partial}{\partial r} \right) \mathcal{O}(\vec{r}) \tag{6.17}$$

This is the correct conformal transformation rule for a CFT operator of dimension Δ living on the plane. The lesson is that radial time derivatives on $\mathcal{O}$ act both explicitly (via partial derivative) and with a shift of Δ in order to cancel the t dependence that arises through $\rho = \pi/2 - \epsilon f(t,\Omega)$. We can also compute this a different way, using our explicit formula for $\mathcal{O}$ and D in terms of oscillators in $\mathrm{AdS}_3/\mathrm{CFT}_2$. We find

$$\begin{aligned}
&\left[\sum_{h,\bar{h}}(\Delta+h+\bar{h})a^\dagger_{h,\bar{h}}a_{h,\bar{h}},\left(\sum_{h,\bar{h}=0}^{\infty}\frac{1}{N^{\mathcal{O}}_{h,\bar{h}}}\left[(z\bar{z})^{-\Delta}z^{-h}\bar{z}^{-\bar{h}}a_{h,\bar{h}}+z^h\bar{z}^{\bar{h}}a^\dagger_{h,\bar{h}}\right]\right)\right] \\
&= \sum_{h,\bar{h}=0}^{\infty}\frac{1}{N^{\mathcal{O}}_{h,\bar{h}}}\left[-(\Delta+h+\bar{h})(z\bar{z})^{-\Delta}z^{-h}\bar{z}^{-\bar{h}}a_{h,\bar{h}}+(\Delta+h+\bar{h})z^h\bar{z}^{\bar{h}}a^\dagger_{h,\bar{h}}\right] \\
&= (\Delta+z\partial_z+\bar{z}\partial_{\bar{z}})\mathcal{O}(z,\bar{z})
\end{aligned} \tag{6.18}$$

as promised, so the two methods agree.

In AdS_3 there is only one angular momentum generator, proportional to ∂_θ. This does not change when we take it to the boundary, and so we just find

$$L = -i\partial_\theta = \frac{1}{2}\left(z\partial_z - \bar{z}\partial_{\bar{z}}\right) \tag{6.19}$$

in the 2-dimensional plane.

In AdS_3 we looked at the momentum operator $P_\pm$. Converting to the Euclidean case, it is

$$P_\pm = e^{-t\pm i\theta}\left(\sin\rho\partial_t + \cos\rho\partial_\rho \pm \frac{i}{\sin\rho}\partial_\theta\right) \tag{6.20}$$

We cannot immediately drop $\cos\rho\partial_\rho$ because it scales as a constant as $\rho\to\pi/2$. This term just produces $-\Delta$ when acting on ϕ. This cancels with a shift of Δ from converting ∂_t from an action on ϕ to an action on $\mathcal{O}$, equivalent to that which we observed when we examined the dilatation operator. So as we approach the boundary, we simply find that it acts on $\mathcal{O}$ as

$$\begin{aligned}
P_\pm &\to e^{t\pm i\theta}\left(-\partial_t \pm i\partial_\theta\right) \\
&= \frac{1}{\sqrt{z\bar{z}}}\left(\frac{z}{\bar{z}}\right)^{\pm\frac{1}{2}}\left[\frac{1}{2}(z\partial_z+\bar{z}\partial_{\bar{z}}) \mp \frac{1}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})\right] \\
&= \partial_{\bar{z}/z}
\end{aligned} \tag{6.21}$$

This just translates into the statement that $P_- = P_z$ and $P_+ = P_{\bar{z}}$. So the P_μ isometry of AdS just turns into the translation operator, as we would expect.

Finally, we can consider the special conformal generators, which take the form

$$K_\pm = e^{t\pm i\theta}\left(\sin\rho\partial_t - \cos\rho\partial_\rho \mp \frac{i}{\sin\rho}\partial_\theta\right) \tag{6.22}$$

in AdS$_3$. Here the contribution of $\cos\rho\partial_\rho$ does not cancel, instead it contributes additively with the shift in ∂_t. Otherwise this operator behaves similarly to the translation operator, and we find

$$\begin{aligned} K_\pm &\to e^{t\pm i\theta}\left(\partial_t + 2\Delta \mp i\partial_\theta\right) \\ &= \sqrt{z\bar{z}}\left(\frac{z}{\bar{z}}\right)^{\pm\frac{1}{2}}\left[\frac{1}{2}(z\partial_z + \bar{z}\partial_{\bar{z}}) + 2\Delta \pm \frac{1}{2}(z\partial_z - \bar{z}\partial_{\bar{z}})\right] \end{aligned} \tag{6.23}$$

This leads to

$$K_+ = z^2\partial_z + 2z\Delta \tag{6.24}$$

which is the correct result, up to an overall normalization. Note that the holomorphic and anti-holomorphic parts of the algebra can be decoupled from each other. We have derive the action of all of the conformal generators on our CT operator $\mathcal{O}(z,\bar{z})$.

To summarize, we have found that the 2d global conformal algebra $SO(1,3)$ acts as

$$[D,\mathcal{O}(x)] = \left(\Delta + \frac{1}{2}z\partial_z + \frac{1}{2}\bar{z}\partial_{\bar{z}}\right)\mathcal{O}(z,\bar{z}) \tag{6.25}$$

$$[P_\mu,\mathcal{O}(x)] = \partial_\mu\mathcal{O}(z,\bar{z}) \tag{6.26}$$

$$[K_z,\mathcal{O}(x)] = \left(2z\Delta + z^2\partial_z\right)\mathcal{O}(z,\bar{z}) \tag{6.27}$$

$$[K_{\bar{z}},\mathcal{O}(x)] = \left(2\bar{z}\Delta + \bar{z}^2\partial_{\bar{z}}\right)\mathcal{O}(z,\bar{z}) \tag{6.28}$$

$$[L,\mathcal{O}(x)] = \frac{1}{2}\left(z\partial_z - \bar{z}\partial_{\bar{z}}\right)\mathcal{O}(z,\bar{z}) \tag{6.29}$$

We derived all of these relations from AdS. When we study general CFTs in the next section, these relations will take on a fundamental character – in fact, one way of *defining* a CT is as a theory of abstract operators $\mathcal{O}(z,\bar{z})$ transforming in exactly this way. But we have understood these transformation properties as an inevitable consequence of AdS physics.

We can also write these symmetry generators in terms of $a_{h,\bar{h}}$, just as we did for D. There is an algorithm to do this – we just need to find the conformal generators in terms of the AdS field ϕ, and then use the expansion of that field in creation and annhilation operators. We can also try to get the answer by guessing. For example, to obtain P I find we need to take

$$\begin{aligned} P_z &\equiv \sum_{h,\bar{h}} -\sqrt{\min(h,\bar{h})(\Delta + \min(h,\bar{h}) - 1)}\,a^\dagger_{h,\bar{h}}a_{h-1,\bar{h}} \\ P_{\bar{z}} &\equiv \sum_{h,\bar{h}} -\sqrt{\min(h,\bar{h})(\Delta + \min(h,\bar{h}) - 1)}\,a^\dagger_{h,\bar{h}}a_{h,\bar{h}-1} \end{aligned} \tag{6.30}$$

Note that (crucially!) these operators also satisfy the commutation relations $[D,P_\mu] = P_\mu$ and $[P_z,P_{\bar{z}}] = 0$. It's fairly trivial to find an operator P_z that acts on $\mathcal{O}(z,\bar{z})$ as ∂_z, the property that actually makes this a Conformal Theory is that the operators D, P_μ, K_μ, and $M_{\mu\nu}$ that act appropriately on $\mathcal{O}(x)$ also satisfy the commutation relations of the conformal algebra.

Finally, note that in Lorentzian AdS we had the relation $P_\mu^\dagger = K_\mu$. This has a simple meaning in AdS – it just says that since complex conjugation sends $it \to -it$, conjugation exchanges the past and future. This is what we are used to in Quantum Mechanics; for example when considering scattering we have

$$(|\text{in}\rangle)^\dagger = \langle\text{out}| \tag{6.31}$$

as usual. However, when we go from AdS $\to$ CFT, we went to Euclidean space and took $r^2 = z\bar{z} = e^{2t}$. Thus in the CFT, $t \to -t$ means

$$r \to \frac{1}{r} \tag{6.32}$$

or in other words, *complex conjugation corresponds to performing a spacetime inversion about the origin*. The states built at the origin transform like

$$\mathcal{O}(0)|0\rangle \to \lim_{r\to\infty} r^{2\Delta}\langle 0|\mathcal{O}(r) \tag{6.33}$$

under this transformation. More generaly, for a primary operator $\mathcal{O}$ we have

$$[\mathcal{O}(\vec{r})]^\dagger = r^{-2\Delta}\mathcal{O}^*\left(\frac{\vec{r}}{r^2}\right) \tag{6.34}$$

where the $*$ means that we take $a \to a^\dagger$. The reason for the factor of r can be seen explicitly in equation (5.38). There we see that the $a_{n\ell J}$ annihilation operators are accompanied by this factor, whereas the creation operators are not, so we need to multiply by $r^{-2\Delta}$ to insure that the operator is Hermitian, ie that $[\mathcal{O}(r)]^\dagger = \mathcal{O}(r)$.

6.3 Primary and Descendant Operators

In our earlier discussions we noted that states in a theory with conformal symmetry can be classified as *primary or descendant*. Primary states have the property that

$$K_\mu|\psi_0\rangle = 0 \tag{6.35}$$

so they are annihilated by all the special conformal (or lowering) operators. One example of a primary states is $a_{0,0}^\dagger|0\rangle$; this corresponds to a free particle in its quantum mechanical ground state in AdS$_3$. States formed by acting on this state with P_μ are descendants.

But there are many many more primaries. One example is

$$\mathcal{O}^k(0)|0\rangle = \left(a_{0,0}^\dagger\right)^k|0\rangle \tag{6.36}$$

but this is just the beginning. This can be interpreted as a condensate of k particles in AdS$_3$, all in their ground state.

Let us pause to recall why this is a primary operator. In AdS, note that the CoM wavefunction is

$$\begin{aligned}\psi_k(t_{CoM}, \rho_{CoM}) &= \langle \phi^k(t_{CoM}, \rho_{CoM}) \mathcal{O}^k(0) \rangle &(6.37)\\ &= e^{-k\Delta t_{CoM}} \cos^{k\Delta} \rho_{CoM} &(6.38)\end{aligned}$$

So we see it has the correct form for a primary wavefunction with dimension $= k\Delta$. In particular, it will be annihilated by all the SCT K_μ in AdS, which act as differential operators in the variables t_{CoM} and ρ_{CoM}. In the CFT, we can test if this state is primary by computing

$$K_\mu \mathcal{O}^k(0)|0\rangle = 0 \tag{6.39}$$

We can calculate this by commuting K_μ through so that it hits the vacuum, where we obtain 0. So in other words, this state will be a primary if

$$\left[K_\mu, \mathcal{O}^k(0)\right] = 0 \tag{6.40}$$

This is the condition for an *operator to be a primary.* Note that the operator must be evaluated at the origin. A primary operator is just an operator that creates a primary state when it acts at the origin. The converse is also true, namely if $\mathcal{O}(0)|0\rangle$ is a primary state, then we must have $[K_\mu, \mathcal{O}(0)] = 0$, because this commutator must have vanishing correlators with other local operators.

A more interesting and non-trivial example is the state

$$\left[\partial_z \mathcal{O} \partial_{\bar{z}} \mathcal{O} - \mathcal{O} \partial_z \partial_{\bar{z}} \mathcal{O}\right] |0\rangle \tag{6.41}$$

where the operators $\mathcal{O}$ are normal ordered. One can check that this is always a primary state by using the conformal commutation relations. Note that

$$\partial_z \mathcal{O} = [P_z, \mathcal{O}] \tag{6.42}$$

and so this state takes the form

$$\left([P_z, \mathcal{O}][P_{\bar{z}}, \mathcal{O}] - \mathcal{O}[P_z, [P_{\bar{z}}, \mathcal{O}]]\right) |0\rangle \tag{6.43}$$

To check that it is a primary we need to show

$$K_{z/\bar{z}} \left([P_z, \mathcal{O}][P_{\bar{z}}, \mathcal{O}] - \mathcal{O}[P_z, [P_{\bar{z}}, \mathcal{O}]]\right) |0\rangle = 0 \tag{6.44}$$

and this will be true iff

$$\left[K_z, \left([P_z, \mathcal{O}][P_{\bar{z}}, \mathcal{O}] - \mathcal{O}[P_z, [P_{\bar{z}}, \mathcal{O}]]\right)(0)\right] = 0 \tag{6.45}$$

and similarly for $z \to \bar{z}$, since the conformal generators act on commutators via the Lie algebra commutator. Note that the operator we wish to show is primary is evaluated at the origin. We could also compute this directly in AdS using the known wavefunctions.

Carrying out the computations using $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$, we find that

$$[K_\mu, [P_z, \mathcal{O}][P_{\bar{z}}, \mathcal{O}]] = 2\Delta \mathcal{O} \left(\delta_{z\mu}[P_{\bar{z}}, \mathcal{O}] + \delta_{\bar{z}\mu}[P_z, \mathcal{O}]\right) \tag{6.46}$$

while a similar computation gives

$$[K_\mu, \mathcal{O}[P_z, [P_{\bar{z}}, \mathcal{O}]] = 2\Delta \mathcal{O} \left(\delta_{z\mu}[P_{\bar{z}}, \mathcal{O}] + \delta_{\bar{z}\mu}[P_z, \mathcal{O}] \right) \tag{6.47}$$

so the linear combination we wrote down is in fact a primary operator. Note that it was crucial for these computations that we are dealing with a generalized free (or quadratic) theory. This assumption was used whenever we assumed that

$$\lim_{x\to 0}[A, \mathcal{O}(x)\mathcal{O}(0)] = [A, \mathcal{O}(0)]\mathcal{O}(0) + \mathcal{O}(0)[A, \mathcal{O}(0)] \tag{6.48}$$

an identity we used implicitly above. We also implicitly normal ordered the operators to subtract off the identity contribution.

This state corresponds to 2 particles in AdS_3 in an excited state with $\ell = 0$ and with their center of mass degree of freedom in its ground state. However, this state has a non-trivial s-wave momentum, so neither particle is expected to be in its ground state. Because the center of mass degree of freedom in AdS_3 is at rest, this state is primary – from the AdS point of view, that's the physical definition of primary-ness. Acting on this state with P_μ will generate descendant 2-particle states.

We already established a one-to-one operator/state correspondence. So we see that not only states, but also *operators can be classified as either primaries or descendants.* Usually this classification is stated in terms of the conformal transformation properties of operators, but the operator/state correspondence and AdS provide an alternate physical picture.

Primary operators transform according to the rules similar to those we derived above for $\mathcal{O}(z, \bar{z})$; we will give the general story below. Descendants do not transform as simply because they take the form

$$\mathcal{O}_{desc} \sim [P_{\mu_1}, [P_{\mu_2}, [\cdots, [P_{\mu_x}, \mathcal{O}_{prim}(x)] \cdots] \tag{6.49}$$

and so the conformal generators act on them in a relatively complicated way. In general CFTs primary operators are often defined as the operators with the 'best' or 'simplest' transformation properties.

In a unitary quantum mechanical theory, all states must have positive normalization. However, since different states are connected by raising and lowering using P and K, the norm of one state can be computed in terms of the norm of another, and this puts restrictions on the theory. Unitarity bounds follow from evaluating the matrix element (for a scalar state/operator)

$$\langle \Delta | K_\alpha K^\alpha P_\mu P^\mu | \Delta \rangle \tag{6.50}$$

and demanding that it is positive definite, since it must be the norm of a physical state. Note that this follows because $P_\mu^\dagger = K_\mu$; this relation must hold so that e.g.

$$(P_\mu |\psi\rangle)^\dagger = \langle \psi | K_\mu \tag{6.51}$$

and raising and lowering work correctly. Let us derive the unitarity relation. First we consider

$$\begin{aligned} \langle \Delta | K_\alpha P^\mu | \Delta \rangle &= \langle \Delta | P^\mu K_\alpha + 2\eta_{\alpha\mu} D + 2M_{\alpha\mu} | \Delta \rangle & (6.52) \\ &= 2\Delta \langle \Delta | \Delta \rangle & (6.53) \end{aligned}$$

and from this we learn something – namely that $\Delta \geq 0$ is required in order to have a positive (or zero) norm descendant. Note that in fact $\Delta = 0$ is allowed – this is the case of the identity operator 1, which clearly does not have any descandents at all, since $[P_\mu, 1] = \partial_\mu 1 = 0$. The identity transforms in the trivial representation of the conformal group.

We will now see that if $\Delta \neq 0$ then we must have a stricter bound. Let us go on and evaluate the norm of a 2nd level descendant. This is

$$\begin{aligned}
\langle \Delta | K_\alpha K^\alpha P_\mu P^\mu | \Delta \rangle &= \langle \Delta | K_\alpha P_\mu K^\alpha P^\mu + 2K^\alpha (\eta_{\mu\alpha} D + M_{\mu\alpha}) P^\mu | \Delta \rangle && (6.54)\\
&= \langle \Delta | 2K^\alpha P^\mu (\eta_{\mu\alpha} D + M_{\mu\alpha}) + 2K \cdot P(\Delta + 1 + (1-d)) | \Delta \rangle && (6.55)\\
&= 2(2\Delta + 2 - d)\langle \Delta | K \cdot P | \Delta \rangle && (6.56)
\end{aligned}$$

by assumption the matrix element in the last line must be positive, and we assume that $\langle \Delta | K \cdot P | \Delta \rangle$ is positive since if it was not then the theory already would not have been unitary at the first level. This means that we must have

$$\Delta \geq \frac{d-2}{2} \tag{6.57}$$

as promised in order to have a unitary representation. For operators/states with spin ℓ a similar analysis shows that

$$\Delta \geq d - 2 + \ell \tag{6.58}$$

for unitarity.

6.4 The Operator Product Expansion

Finally, we are ready to discuss one of the other key attributes of a conformal theory – the Operator Product Expansion (OPE), as applied to generalized free theories. The idea behind the OPE in AdS/CFT is pictured in figure 9.

Let us examine the general operator quantity

$$\begin{aligned}
\mathcal{O}(z,\bar{z})\mathcal{O}(w,\bar{w}) &= \left(\sum_{h,\bar{h}=0}^{\infty} \frac{1}{N^{\mathcal{O}}_{h,\bar{h}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h,\bar{h}} + z^h \bar{z}^{\bar{h}} a^\dagger_{h,\bar{h}} \right] \right) \\
&\quad \times \left(\sum_{h,\bar{h}=0}^{\infty} \frac{1}{N^{\mathcal{O}}_{h,\bar{h}}} \left[(w\bar{w})^{-\Delta} w^{-h} \bar{w}^{-\bar{h}} a_{h,\bar{h}} + w^h \bar{w}^{\bar{h}} a^\dagger_{h,\bar{h}} \right] \right) \qquad (6.59)
\end{aligned}$$

We cannot say much about this quantity that is useful, in general. However, if it is the case that *between these two operators the CFT is in its vacuum state*, then we can translate $w \to 0$ for simplicity and note that this operator product creates the linear combination of states

$$\mathcal{O}(z,\bar{z})\mathcal{O}(0)|0\rangle = \left(\sum_{h,\bar{h}=0}^{\infty} \frac{1}{N^{\mathcal{O}}_{h,\bar{h}}} \left[(z\bar{z})^{-\Delta} z^{-h} \bar{z}^{-\bar{h}} a_{h,\bar{h}} + z^h \bar{z}^{\bar{h}} a^\dagger_{h,\bar{h}} \right] \right) a^\dagger_{0,0}|0\rangle \tag{6.60}$$

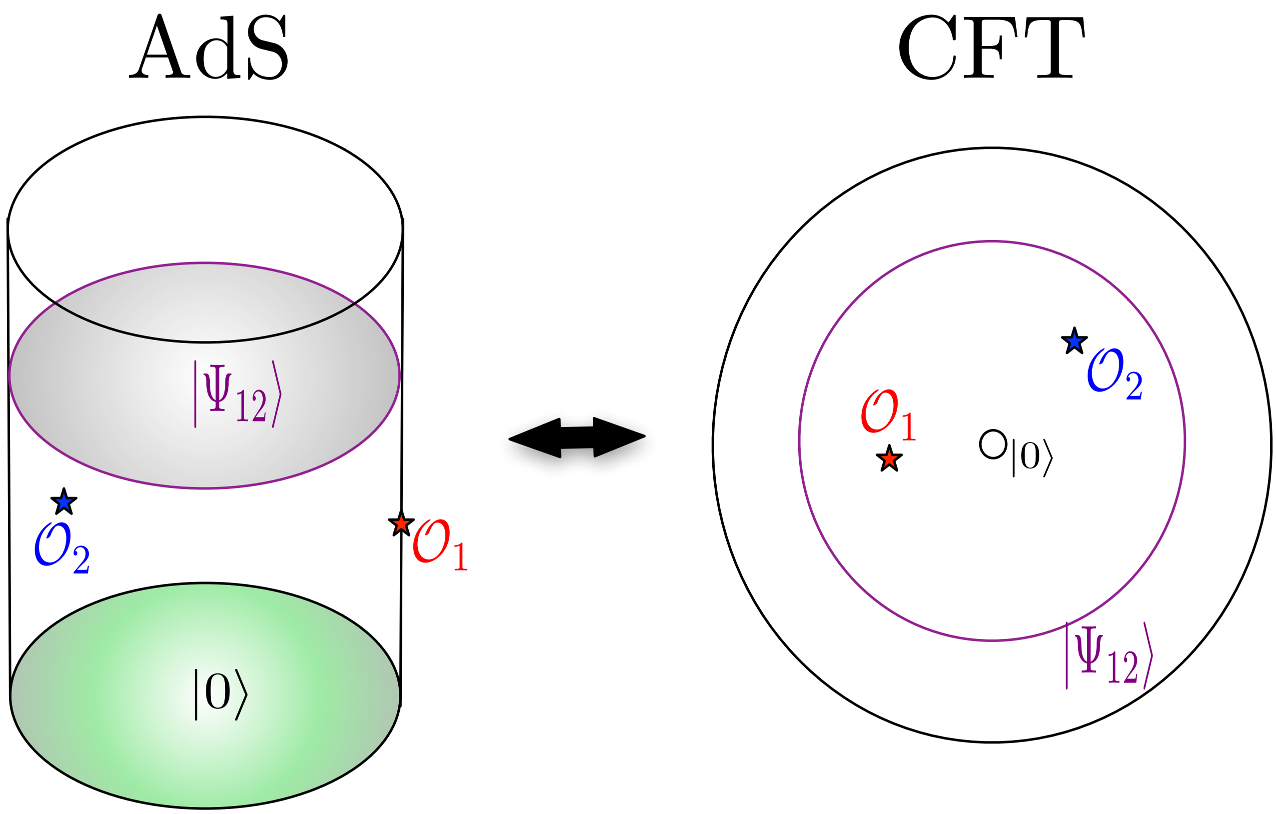

Figure 9: This figure shows how the OPE can be derived in the CFT, and how the derivation is mirrored in AdS. The final point left off the diagram is that the state ψ_{12} can be created by a local operator acting at the origin in the CFT, corresponding to an operator in the infinite past in global coordinate AdS. This follows from the operator/state correspondence of radial quantization.

That's quite a bit simpler!

The AdS picture is the following: assume that the theory is in its vacuum state in the infinite past. Then if we insert two operators before some time $t = \log(z\bar{z})$ we obtain a state $|\psi_{12}(t)\rangle$. But then we can evolve that state all the back to $t = -\infty$, replacing $|\psi_{12}(t)\rangle$ with $|\psi_{12}(-\infty)\rangle$. All matrix elements of the operator product with local operators that appear at times $t > \log(z\bar{z})$ can be computed as matrix elements with this new initial state $|\psi_{12}(-\infty)\rangle$.

Now we can ask what linear combination of local operators, acting at the origin, creates the state $|\psi_{12}(-\infty)\rangle$? We already saw that every CFT state is associated with an operator, so can write the answer to this question in terms of a sum over primary and descendant operators. Since all descendants are just ∂_z or $\partial_{\bar{z}}$ acting on a primary, we must have that

$$\mathcal{O}(z,\bar{z})\mathcal{O}(0) = \sum_{\text{primary } n,\ell} \left(C_{n,\ell}(z,\bar{z},\partial_z,\partial_{\bar{z}})\mathcal{O}_{n,\ell}\right)(0) \tag{6.61}$$

as an operator statement, for some coefficient function $C_{n\ell}$. The $\mathcal{O}_{n,\ell}$ are just the primary operators made from two of the fundamental $\mathcal{O}$ operators. This is the OPE in a generalized free theory. The coefficients $C_{n,\ell}$ can be computed once and for all, as they are determined by conformal symmetry, up to a constant (which can also be determined once and for all in generalized free theories).

Let us go back and look at the generalized free theory result more explicitly. We can write

$$\begin{aligned}
\mathcal{O}(z,\bar{z})\mathcal{O}(0)|0\rangle &= \left(\sum_{h,\bar{h}=0}^{\infty} \frac{1}{N^{\mathcal{O}}_{h,\bar{h}}}\left[(z\bar{z})^{-\Delta} z^{-h}\bar{z}^{-\bar{h}} a_{h,\bar{h}} + z^h \bar{z}^{\bar{h}} a^\dagger_{h,\bar{h}}\right]\right) a^\dagger_{0,0}|0\rangle \\
&= \left[\frac{1}{(z\bar{z})^\Delta} + \left(\sum_{h,\bar{h}=0}^{\infty} \frac{1}{N^{\mathcal{O}}_{h,\bar{h}}} z^h \bar{z}^{\bar{h}} a^\dagger_{h,\bar{h}} a^\dagger_{0,0}\right)\right]|0\rangle \\
&= \left[\frac{1}{(z\bar{z})^\Delta} + \frac{1}{N^{\mathcal{O}}_{0,\bar{0}}}\mathcal{O}^2(0) + \frac{\bar{z}}{N^{\mathcal{O}}_{0,\bar{1}}}\mathcal{O}\partial_{\bar{z}}\mathcal{O}(0) + \frac{z}{N^{\mathcal{O}}_{1,\bar{0}}}\mathcal{O}\partial_z\mathcal{O}(0) + \cdots\right]|0\rangle
\end{aligned} \tag{6.62}$$

so we can explicitly see various operators appearing. Note also the appearance of the identity operator $\equiv 1$, from the Wick contraction.

In fact, the OPE has a much more precise structure determined by conformal symmetry. The point is that once we write down the OPE as a sum over all of the various operators that can appear on the RHS, including both primaries and descendants, we can then act with conformal symmetry generators on both sides to find relations among the coefficients. In fact, the only free coefficients are those in front of the primary operators! We will return to explain this fact in detail when we discuss general CFTs.

6.5 Conformal Symmetry of Correlators in GFT

Recall that in d Euclidean dimensions, a free scalar boson has the 2-pt function

$$\langle\phi(x)\phi(0)\rangle = \frac{1}{x^{d-2}} \tag{6.63}$$

All of the other correlators are just obtained from this one by Wick contraction. Another way to discover generalized free theories is to define (really just make up!) a theory that generalizes this in a simple and conformally invariant way. It will be a theory whose entire Hilbert space is generated by the action of a local operator $\mathcal{O}(x)$ that has purely 'Gaussian' correlators – in other words, all correlation functions of $\mathcal{O}(x)$ will be determined by its 2-pt function via Wick contraction. To be specific, the 2-pt function has the form

$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{x^{2\Delta}} \tag{6.64}$$

where $\Delta \geq \frac{d}{2} - 1$ is a number that we are free to specify. The 3-pt correlator of $\mathcal{O}$ vanishes, as do all correlators with an odd number of $\mathcal{O}$s. The 4-pt correlator is

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta}}\frac{1}{x_{34}^{2\Delta}} + \frac{1}{x_{13}^{2\Delta}}\frac{1}{x_{24}^{2\Delta}} + \frac{1}{x_{14}^{2\Delta}}\frac{1}{x_{23}^{2\Delta}} \tag{6.65}$$

Working out any n-pt correlator is just an exercise in combinatorics. These correlators show why this is a 'generalized free theory' – in a free theory we would have $\Delta = \frac{d}{2} - 1$, but here we are letting the exponent be a general parameter.

So how do we see the conformal symmetry of these correlators? Obviously these correlators are invariant under translations and rotations. To study the other transformations, we can use the conformal algebra directly, recalling that $[D, \mathcal{O}] = (\Delta + r\partial_r)\mathcal{O}$ so that we have

$$\begin{aligned} 0 &= \langle D\mathcal{O}(r_1)\mathcal{O}(r_2)\rangle & (6.66)\\ &= \langle [D, \mathcal{O}(r_1)]\mathcal{O}(r_2)\rangle + \langle \mathcal{O}(r_1)[D, \mathcal{O}(r_2)]\rangle & (6.67)\end{aligned}$$

This means that the 2-pt correlator must obey

$$(r_1 \cdot \partial_{r_1} + r_2 \cdot \partial_{r_2}) \langle \mathcal{O}(\vec{r}_1)\mathcal{O}(\vec{r}_2)\rangle = -2\Delta \langle \mathcal{O}(\vec{r}_1)\mathcal{O}(\vec{r}_2)\rangle \tag{6.68}$$

which is exactly what we find. Special conformal symmetries can be checked in the same way. In the next section we will use the projective null cone coordinates to make conformal invariance manifest.

6.6 Connection to Infinite N

Many physicists would refer to a Generalized Free Theory as 'a CFT at infinite N'. There are a whole host of examples. For instance, we might have

- A large N gauge theory with gauge field A^μ_{ab}, as well as matter in the adjoint such as ψ_{ab} and Φ_{ab} fields, or vector-like matter ψ_a, perhaps with flavor quantum numbers. The classic AdS/CFT example is the maximally supersymmetric $\mathcal{N} = 4$ SYM Theory.
- A vector-like theory such as the $O(N)$ model, with scalar fields ϕ_a and an $O(N)$ symmetry.
- Any other theory with a large N Lie Group symmetry forcing the observables to take a symmetric form, such as $\phi_a\phi^a$ or $\mathrm{Tr}\left[(F^{ab}_{\mu\nu})^k\right]$.

Let us understand why large N relates to AdS/CFT. Consider a free theory of an $SO(N)$ matrix valued field ϕ_{ab} where these indices a, b run from $a = 1, 2, \cdots, N$. The action will simply be

$$\begin{aligned} S &= \int d^dx \frac{1}{2} \sum_{a,b} \partial_\mu \phi_{ab} \partial^\mu \phi_{ab} \\ &= \int d^dx \frac{1}{2}\, \mathrm{Tr}\left[\partial_\mu \phi \partial^\mu \phi\right] & (6.69)\end{aligned}$$

Thus the ϕ_{ab} fields have extremely simple correlation functions, namely

$$\langle \phi_{ab}(x)\phi_{cd}(y)\rangle = \frac{\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}}{(x-y)^{d-2}} \tag{6.70}$$

or it can be written as

$$\langle \phi_A(x)\phi_B(y)\rangle = \frac{\delta_{AB}}{(x-y)^{d-2}} \tag{6.71}$$

where A indexes the generators T^A_{ab} of $SO(N)$, since ϕ is in the adjoint representation. None of these details matter much for our purposes here. All of the n-pt correlators of ϕ are just built on these via Wick contraction, since this is just a free theory with N^2 free fields.

The interesting point is that now we can consider more general operators made out of ϕ. For example, consider

$$\mathcal{O}_2(x) = \frac{1}{N\sqrt{2}} \text{ Tr}\left[\phi^2(x)\right] = \frac{1}{N\sqrt{2}} \sum_{ab} \phi_{ab}(x)\phi^{ba}(x) \tag{6.72}$$

What is the 2-pt correlator of $\mathcal{O}_2$? It is easy to compute

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{(x-y)^{2(d-2)}} \tag{6.73}$$

This is the form of a generalized free field with dimension $\Delta = (d-2)$. If we were to consider $\mathcal{O}_k = \text{ Tr}\left[\phi^k\right]$ then we could get any $\Delta = k(d-2)/2$.

But now let's consider a 4-pt correlator

$$\begin{aligned}\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle = &= \frac{1}{x_{12}^{2\Delta}}\frac{1}{x_{34}^{2\Delta}} + \frac{1}{x_{13}^{2\Delta}}\frac{1}{x_{24}^{2\Delta}} + \frac{1}{x_{14}^{2\Delta}}\frac{1}{x_{23}^{2\Delta}} \\ &\quad + \frac{1}{N}(\cdots)\end{aligned} \tag{6.74}$$

In the limit that $N \to \infty$, the 4-pt correlator of $\mathcal{O}_2$, and in fact of all properly normalized $\mathcal{O}_k$, will be given entirely by evaluating pairs of 2-pt functions. Thus in the large N limit, this theory reproduces what we wanted – a generalized free theory with many choices for Δ, the dimension of the operator. It's that last point that differentiates perturbation theory in $1/N$ from the more familiar examples of perturbation theory in a small coupling. For further reading, Witten's 'Baryons and the Large N Expansion' [11] is a great intro to the $1/N$ expansion.

An interesting point to note here is that the quantum mechanical expansion of ϕ_{ab} is in terms of free field oscillator modes, which are not much like the AdS oscillator modes of a generalized free field. So the relationship between an operator like $\text{tr}[\phi^k]$ and $\mathcal{O}$ isn't straightforward quantum mechanically.

7 General CFT Axioms from an AdS Viewpoint

Now we will discuss general CFTs from a bottom-up, or 'axiomatic' point of view. The reader should compare and contrast the properties of general CFTs with those of the generalized free theories analyzed in the previous section. Here is a list of the main ideas:

- Conformal transformations consist of the Poincaré group of transformations, plus scale transformations and special conformal transformations. The extra symmetries can be most quickly derived by demanding Poincaré invariance plus an additional symmetry under inversions about a point, where $x^i \to x^i/x^2$. They are best understood as coordinate transformations that do not leave the metric invariant, but rescale it by an overall spacetime-dependent factor, ie the symmetries are conformal killing vectors.

- CFTs can be quantized in the standard way, along flat spacelike surfaces that evolve with time. However, an alternative *radial quantization* in Euclidean space is often more useful, and will play a large role in what follows; in this quantization we start at a point and evolve outwards on expanding spheres. This idea is pictured in figure 7.
- Radial quantization associates the entire Hilbert space of CFT states with an arbitrarily small region about a point. Associating states on arbitrarily small circles with operators at the center of these circles leads to the *operator-state correspondence*. This can be made very explicit when there is a path integral description of the CFT, but the existence of a path integral is not necessary.
- Using conformal symmetry we can classify states according to irreducible representations of the conformal group, $SO(1, d+1)$ in Euclidean space, and therefore describe every state as a linear combination of primaries and their descendants; the latter are obtained from primaries by acting with the momentum generators.
- The conformal symmetry allows us to move these operators around, so that from $\mathcal{O}(0)$ at the origin we obtain $\mathcal{O}(x)$ at any point x. The correlators of any 2 or 3 CFT operators are entirely fixed by symmetry, up to a finite set of constants.
- As usual in a quantum mechanical theory, we can multiply any two operators to obtain some other (new) operator. So what about operators at different points in space? The operator-state correspondence applied to a product $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ leads us to the *Operator Product Expansion* (OPE), which has a finite radius of convergence in any CFT. This can be made very explicit when there is a path integral description of the theory (see e.g. Weinberg Volume 2), but the existence of a path integral is not necessary. Operator products and OPEs in general CFTs are more subtle than in the generalized free theories we studied in the previous section.
- Local conserved currents are special and extremely important. Spin-1 currents J_μ satisfying $\partial^\mu J_\mu = 0$ generate global symmetries, while the spin-2 conserved current $T_{\mu\nu}$ is even more special. Conventionally a theory is only defined to be a CFT if a conserved $T_{\mu\nu}$ exists; if there is no conserved spin 2 current the lore holds that the theory is 'non-local',[16] as we will discuss.

7.1 What is a C(F)T?

Although later on we will take a more abstract point of view, let's review the standard approach to CFTs. Conventionally, conformal isometries are defined as a change of coordinates or diffeomorphism $x_\mu \to x'_\mu(x)$ such that the metric

$$dx^2 \to dx'^2 = \Omega^2(x) dx^2 \tag{7.1}$$

where $\Omega(x)$ is an arbitrary function of the coordinates. This specification preserves angles, but not distances. This means that in the Lorentzian case, it always preserves the causal structure of the spacetime.

[16] in quotes because this phrase is vague... different physicists use it to mean very different things.

Thinking infinitesimally, we have $x'_\mu = x_\mu + v_\mu(x)$, where we treat v_μ as small. The new metric is

$$dx'_\alpha dx'^\alpha = dx^\mu dx^\nu \left(\eta_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu + \cdots\right) \tag{7.2}$$

and so we must have $\partial_\mu v_\nu + \partial_\nu v_\mu = \omega(x)\eta_{\mu\nu}$, where $\Omega = 1 + \omega/2$. This immediately gives the equation

$$\partial_\mu v_\nu + \partial_\nu v_\mu - \frac{2}{d}\partial_\alpha v^\alpha \eta_{\mu\nu} = 0 \tag{7.3}$$

A well-known fact is that in 2-dimensions there are infinitely many solutions, whereas in $d > 2$ dimensions there are only a finite number of solutions, the transformations that form the conformal group $SO(2, d)$ (in Lorentzian signature).

Note that there is a significant difference between a Weyl transformation $g_{\mu\nu}(x) \to W(x)g_{\mu\nu}(x)$, where we are changing the physical metric, and a conformal isometry, which is a change of coordinates (diffeomorphism) $x_\mu \to x'_\mu(x)$ such that in the new coordinates the metric takes the form of the old metric multiplied by some function $\Omega^2(x)$.

To make this all clearer, let's consider why a free massless scalar field theory is invariant under a conformal isometry. We have $\phi(x) \to \phi(x + v) \approx \phi(x) + v_\mu(x)\partial^\mu\phi(x)$. Then the free Lagrangian transforms as

$$\begin{aligned} \delta L &= \partial_\mu\phi(x)\partial^\mu(v_\alpha(x)\partial^\alpha\phi(x)) \\ &= \partial_\mu v_\alpha \left(\partial^\mu\phi\partial^\alpha\phi - \frac{1}{2}\eta_{\mu\alpha}(\partial\phi)^2\right) = \partial_\mu v_\alpha T^{\mu\alpha} \end{aligned} \tag{7.4}$$

When we take $v_\mu = \lambda x_\mu$, corresponding to a dilatation, or the more complicated $v_\mu = b_\mu x^2 - 2x_\mu b\cdot x$ for a special conformal transformation, we find that $\delta L \propto T^\mu_\mu$, the trace of the energy momentum tensor, as expected.

You might notice that $T_{\mu\nu}$ for the scalar is only traceless in $d = 2$ spacetime dimensions; this is because we have automatically (and in general incorrectly) assigned ϕ a scaling dimension of 0 under dilatations. This can be fixed by adding a local improvement term to $T_{\mu\nu}$, namely some multiple of the conserved total derivative $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$. The point is that the scale and special conformal transformations can involve a combination of $T_{\mu\nu}$ and other local conserved currents; see [12] for more details. For example, for the case of dilatations as long as we can define

$$S_\mu = x^\nu T_{\mu\nu} - V_\mu \tag{7.5}$$

with $T^\mu_\mu = \partial^\mu V_\mu$ then the current S_μ is a scale current, which generates dilatations as we desire. In the case of a free theory we have $V_\mu \propto \partial_\mu\phi^2$.

This can be understood more directly. When we computed $T_{\mu\nu}$ above we assumed that $\phi(x) \to \phi(x+v) \approx \phi(x) + v_\mu(x)\partial^\mu\phi(x)$, but this is not correct for e.g. dilatations for any choice of v_μ. We know that ϕ will be a primary operator of dimension $\Delta = \frac{d}{2} - 1$, and so it should transform as $\delta\phi(x) \to (\Delta + x\cdot\partial_x)\phi(x)$. But taking $v_\mu = \epsilon x_\mu$ does not lead to this transformation rule. To obtain a scale current we therefore want to take $\delta\phi \to \epsilon(x)(\Delta + x\cdot\partial_x)\phi$. This leads to a scale current

$$S_\mu = x^\nu \left(\partial^\mu\phi\partial^\alpha\phi - \frac{1}{2}\eta_{\mu\alpha}(\partial\phi)^2\right) - \Delta\phi\partial_\mu\phi \tag{7.6}$$

which accords with the form we found above by the inclusion of an explicit improvement term into $T_{\mu\nu}$. The effect of the improvement term is simply to shift the scaling dimension of ϕ without altering the conservation or symmetry (as a matrix) of the stress-energy tensor.

It's worth noting that in the theories we are used to studying, e.g. $\lambda\phi^4$ theory, gauge theories, etc., the fact that conformal symmetry is broken quantum mechanically means that $T^\mu_\mu \propto \beta(\lambda)$, where β is the usual beta function for the running coupling. This means that these classically conformal theories have an anomaly.

The preceding remarks were an example of how CFTs are usually introduced. We will take a rather different and more abstract point of view that follows more naturally from thinking about AdS theories. Conformal Theories (CTs) are defined as quantum mechanical theories where

- All states fall into representations of the d-dimensional conformal algebra:

$$
\begin{aligned}
&[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \qquad [M_{\mu\nu}, D] = 0 \\
&[P_\mu, K_\nu] = -2(\eta_{\mu\nu}D + iM_{\mu\nu}), \qquad [D, P_\mu] = P_\mu, \qquad [D, K_\mu] = -K_\mu
\end{aligned} \tag{7.7}
$$

- There are local operators $\mathcal{O}_i(x)$ called primaries that transform covariantly. We already saw what this means for scalar operators in Generalized Free Theories in the previous section. As we will see again in the next section, there is a one-to-one correspondence between all the operators at any point x and all of the states in the Hilbert space.

- We can multiply operators $\mathcal{O}_1(x)\mathcal{O}_2(y)$. If the CFT is in its vacuum state throughout the space intervening betwen these operators, then the product can be re-written as a convergent sum over all the operators in the theory at any point between x and y. This is called the Operator Product Expansion (OPE), to be derived in general below. The coefficients of all descendant are determined by conformal symmetry and the coefficients of the primaries.

From these properties many others follow, for example that the correlators must take a very constrained form, and for the theory to be unitary all operators have to have scaling dimensions (eigenvalues of the dilatation operator D) greater than the unitarity bound, as derived in the previous section.

For a theory to be a Conformal *Field* Theory (CFT):

- The theory must contain in the spectrum of operators a spin 2 tensor $T_{\mu\nu}$ with dimension $\Delta = d$, the spacetime dimension, or equivalently, this stress tensor must be conserved. This stress tensor certainly was *not* present in the Generalized Free Theory we studied in the last section, except when $\Delta = \frac{d}{2} - 1$, in which case the GFT reduces to an ordinary free scalar theory.

We will discuss below why $T_{\mu\nu}$ is so uniquely important, and how its special properties relate to the universality of gravity. For now it should at least be apparent from our discussion above that $T_{\mu\nu}$ is related to implementing the conformal symmetries in a way that is manifestly local in spacetime ('local' because the symmetries are expressed in terms of $T_{\mu\nu}(x)$, which is localized *at the point* x).

7.2 Radial Quantization and Local Operators – A General Story

Let us start with some general comments on the Hilbert space construction in QFT.[17] This procedure is linked to the choice of foliation of spacetime with fixed time surfaces. Time evolution connects states on one surface to states on the other surfaces.

Each leaf of the foliation becomes endowed with its own Hilbert space. We create in states ψ_{in} by inserting operators or directly specifying a wavefunction in the past of a given surface. Analogously we deal with out states by inserting operators or directly specifying the wavefunction in the future. The overlap of an in and out states living on the same surface $\langle \psi_{out}|\psi_{in}\rangle$ is equal to the correlation function of operators which create these in and out states (or to the S-matrix element if the in and out states consist of well-separated particles).

In standard textbook discussions of QFT, states are defined on flat spacelike surfaces in Minkowski spacetime. The time evolution operator is 'P_0', the momentum generator in the time direction. Since the momenta all commute, states can be classified according to their total momentum and energy. To summarize

- States are defined on flat spacelike surfaces in Minkowski spacetime
- Since the momenta all commute

  $$[P_\mu, P_\nu] = 0 \tag{7.8}$$

  states can be classified according to their total momentum and energy.
- The time evolution operator is 'P_0', the momentum generator in the time direction. Via Lorentz transformations we can consider other compatible (boosted) notions of P_0.

In CFTs we additionally have the symmetries D = dilatations and K_μ = special conformal transformations. Could still use standard picture, but more natural to think about states in a new way – Radial Quantization. CFT algebra:

$$\begin{gathered}[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \qquad [M_{\mu\nu}, D] = 0 \\ [P_\mu, K_\nu] = -2(\eta_{\mu\nu}D + iM_{\mu\nu}), \qquad [D, P_\mu] = P_\mu, \qquad [D, K_\mu] = -K_\mu\end{gathered} \tag{7.9}$$

Some implications

- Simultaneously diagonalize D and $M_{\mu\nu}$ = angular momentum. Label states by Δ or $\tau = \Delta - \ell$ and angular momentum ℓ, so have

  $$D|\tau, \ell J\rangle \;\; = \;\; (\tau + \ell)|\tau, \ell J\rangle \tag{7.10}$$

- P_μ is raising, K_μ lowering operator wrt D.
- Define primary as state killed by K_μ. Get discrete spectrum.

[17] One can find a very similar discussion in Slava's notes.

- Radial quantization with Hamiltonian is D, unitarty evolution via $e^{iD\tau}$ with $\tau = \log r$.

The vacuum $|0\rangle$ is annihilated by all the conformal generators, since we assume conformal symmetry is unbroken. Associated to each *primary state* is a *primary operator* $\mathcal{O}(0)$ with twist τ and angular momentum ℓ, so that acting on the vacuum

$$\mathcal{O}(0)|0\rangle = |\tau, \ell\rangle \tag{7.11}$$

This is the local operator/state correspondence (isomorphism or identification).

Note that radial quantization and the operator/state correspondence can be treated in a natural way in the path integral formalism. In that case we define states by wave functionals on Cauchy surfaces, and the path integral allows us to evolve from one Cauchy surface to another. Radial quantization is then immediate. The operator/state correspondence follows if we evolve back in time towards a point. Then the wave functional at the origin is equivalent to some insertion of a functional of the fields directly into the path integral, and this insertion is itself the desired operator.

Let us now see how the conformal transformation properties of $\mathcal{O}(0)$ completely determine the transformation properties of $\mathcal{O}(x)$, and therefore define the operator everywhere. This is interesting because it means that once we know about the primary state $\mathcal{O}(0)|0\rangle$ we can derive everything we would like to know about the corresponding local operator at any point in spacetime. The conformal generators act on $\mathcal{O}(0)$ according to

$$\begin{aligned}
[K_\mu, \mathcal{O}(0)] &= 0 && (7.12)\\
[D, \mathcal{O}(0)] &= -i\Delta\mathcal{O}(0) && (7.13)\\
[M_{\mu\nu}, \mathcal{O}(0)] &= -i\Sigma^\ell_{\mu\nu}\mathcal{O}(0) && (7.14)
\end{aligned}$$

Along with the assumption that

$$[P_\mu, \mathcal{O}(x)] = -i\partial_\mu\mathcal{O}(x) \tag{7.15}$$

this entirely determines the action of the conformal algebra on the local operator $\mathcal{O}(x)$, assuming the commutation relations of the conformal group. At a general point in flat Euclidean space, we have the commutators

$$\begin{aligned}
[D, \mathcal{O}(x)] &= -i(\Delta + x^\mu\partial_\mu)\mathcal{O}(x) && (7.16)\\
[P_\mu, \mathcal{O}(x)] &= -i\partial_\mu\mathcal{O}(x) && (7.17)\\
[K_\mu, \mathcal{O}(x)] &= -i(2x_\mu\Delta + 2x^\alpha\Sigma_{\alpha\mu} + 2x_\mu x^\alpha\partial_\alpha - x^2\partial_\mu)\mathcal{O}(x) && (7.18)\\
[M_{\mu\nu}, \mathcal{O}(x)] &= -i(\Sigma_{\mu\nu} + x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{O}(x) && (7.19)
\end{aligned}$$

where Δ is the scaling dimension of $\mathcal{O}$ and $\Sigma_{\mu\nu}$ is a finite dimensional spin matrix encoding the angular momentum representation of $\mathcal{O}$.

For fun let us derive the first infinitesimal approximation of the first equation above. We know from equation (7.15) that we can write

$$\begin{aligned}
\mathcal{O}(\epsilon) &\approx \mathcal{O}(0) + \epsilon^\mu\partial_\mu\mathcal{O}(0) + \cdots \\
&= \mathcal{O}(0) + i\epsilon^\mu[P_\mu, \mathcal{O}(0)] + \cdots && (7.20)
\end{aligned}$$

Now if we act the dilation operator on $\mathcal{O}(\epsilon)$ we find

$$\begin{aligned}
[D, \mathcal{O}(\epsilon)] &\approx [D, \mathcal{O}(0)] + i\epsilon^\mu [D, [P_\mu, \mathcal{O}(0)]] \\
&= -i\Delta\mathcal{O}(0) + i\epsilon^\mu \left([[D, P_\mu], \mathcal{O}(0)] - [P_\mu, [\mathcal{O}, D]]\right) \\
&= -i\Delta\mathcal{O}(0) + i\epsilon^\mu \left([-iP_\mu, \mathcal{O}(0)] - i\Delta[P_\mu, \mathcal{O}(0)]\right) \\
&= -i\Delta\mathcal{O}(0) + i\epsilon^\mu \left(-\partial_\mu \mathcal{O}(0) - \Delta\partial_\mu \mathcal{O}(0)\right) \\
&\approx -i(\Delta + \epsilon^\mu \partial_\mu)\mathcal{O}(\epsilon)
\end{aligned} \tag{7.21}$$

so we have derived the first of the equations (7.16) when $x = \epsilon$ is near to 0.

The AdS dual of radial quantization is pictured in figure 7 – it's just ordinary quantization in AdS global coordinates. Specifically, 'ordinary quantization' in global coordinates means that we use the global time t to define our clocks, and so the dilatation operator D is the Hamiltonian (the generator of time translations). The Cauchy surfaces in AdS on which states are defined are just the constant t surfaces, parameterized by the coordinates ρ and Ω.

7.3 General CFT Correlators and the Projective Null Cone

We already saw some non-trivial examples of the way that conformal symmetry acts on and constrains CFT correlators. In general, all two-point correlators are completely determined up to a normalization, while 3-pt correlators are determined up to a finite number of coefficients (in the case of three scalar operators, there is only one overall coefficient). Higher point correlators can only depend on the conformal cross ratios.

We can study the conformal transformation properties in an elegant fashion by using the projective null cone coordinates P_A, which satisfy $P_A P^A = 0$ and are identified under $P_A \sim \lambda P_A$. Since we have been thinking about CFTs from the AdS point of view, it is natural to understand CFT correlators as expressed in terms of the P_A with reference to AdS.

Recall that we defined a CFT operator $\mathcal{O}$ in terms of an AdS field $\phi(X_A)$ as

$$\mathcal{O}(t, \Omega) = \lim_{\epsilon \to 0} \frac{\phi(t, \rho(\epsilon; t, \Omega), \Omega)}{\epsilon^\Delta} \tag{7.22}$$

where for e.g. scalars $m^2 = \Delta(\Delta - d)$ relates the AdS mass and CFT dimension, and $\rho(\epsilon; t, \Omega) = \frac{\pi}{2} - \epsilon f(t, \Omega)$ controls the way we approach the boundary of AdS. Furthermore, using our coordinate identifications we can write

$$\begin{aligned}
X_0 &= R\frac{\cosh t}{\cos\rho} \to \frac{\cosh t}{\epsilon f(t, \Omega)} = \frac{1}{\epsilon} P_0 \\
X_{d+1} &= R\frac{\sinh t}{\cos\rho} \to \frac{\sinh t}{\epsilon f(t, \Omega)} = \frac{1}{\epsilon} P_{d+1} \\
X_i &= R\tan\rho\,\Omega_i \to \frac{\Omega_i}{\epsilon f(t, \Omega)} = \frac{1}{\epsilon} P_i
\end{aligned} \tag{7.23}$$

The choice of function $f(t,\Omega)$ here represents a choice of a section of the null cone. Note that with our favorite $f = e^{-t}$ we obtain

$$(P_0, P_{d+1}, P_i) = \left(\frac{e^{2t}+1}{2}, \frac{e^{2t}-1}{2}, e^t\Omega_i\right) \tag{7.24}$$

as usual for Euclidean flat space.

Now in general, instead of taking $\rho = \frac{\pi}{2} - \epsilon f(t,\Omega)$ we could have equivalently taken $\rho(\epsilon; t, \Omega) \to \frac{\pi}{2} - \frac{1}{\lambda}\epsilon f(t,\Omega)$ for a constant $\lambda > 0$. This has precisely the effect of sending $P_A \to \lambda P_A$ in our definition. But we also know from the definition of $\mathcal{O}$ that this would transform

$$\mathcal{O}(\lambda P_A) = \lambda^{-\Delta}\mathcal{O}(P_A) \tag{7.25}$$

for a conformal primary operator obtained from an AdS theory. Note that we could use a different λ for each operator in a correlator – we do not have to rescale them all in the same way, since we can use a different ϵ for each operator.

Furthermore, one can reverse the logic, and show that this transformation rule must obtain for any CFT primary expressed in terms of the null projective cone, regardless of whether it comes from AdS. It accords with all the results we have derived before by other means.

This puts powerful constraints on CFT correlation functions when we combine it with the fact that only $P_i \cdot P_j$ are conformally covariant objects. In particular, it enforces that

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\rangle = 0 \quad \text{if} \quad \Delta_1 \neq \Delta_2 \tag{7.26}$$

and that this correlator takes the conventional form otherwise. Furthermore, in the case of 3-pt correlators of scalars it implies

$$\langle \mathcal{O}_1(P_1)\mathcal{O}_2(P_2)\mathcal{O}_3(P_3)\rangle = \frac{C_{123}}{(P_1 \cdot P_2)^{\alpha_{12,3}}(P_2 \cdot P_3)^{\alpha_{23,1}}(P_1 \cdot P_3)^{\alpha_{13,2}}} \tag{7.27}$$

where we must have

$$\alpha_{ij,k} = \frac{\Delta_i + \Delta_j - \Delta_k}{2} \tag{7.28}$$

to achieve the correct scalings under $P_i \to \lambda_i P_i$ for each $i = 1, 2, 3$.

In general, it's much easier to work using the projective null cone coordinates. Conformally invariant correlators can only be a function of the conformal cross ratios

$$u_{ijkl} \equiv \frac{(P_i \cdot P_j)(P_k \cdot P_l)}{(P_i \cdot P_k)(P_j \cdot P_l)} \tag{7.29}$$

that are invariant under scalings of the individual P_i. In particular, when we study the 4-pt correlator, we only have two cross ratios, conventially called u and v

$$u = \frac{(P_1 \cdot P_2)(P_3 \cdot P_4)}{(P_1 \cdot P_3)(P_2 \cdot P_4)} \quad \text{and} \quad v = \frac{(P_1 \cdot P_4)(P_2 \cdot P_3)}{(P_1 \cdot P_3)(P_2 \cdot P_4)} \tag{7.30}$$

Note that in terms of these variables, our generalized free theory correlator can be written as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle_{GFT} = \frac{1}{(P_1 \cdot P_3)^\Delta (P_2 \cdot P_4)^\Delta} \left(u^{-\Delta} + 1 + v^{-\Delta}\right) \tag{7.31}$$

The factor out front just encodes the necessary kinematical data associated with the scaling dimension of $\mathcal{O}$, while the terms in parentheses contain all of the dynamical information about the theory. In a general theory this correlator would take the form

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{(P_1 \cdot P_3)^\Delta (P_2 \cdot P_4)^\Delta} \mathcal{A}(u,v) \tag{7.32}$$

where the function $\mathcal{A}(u,v)$ cannot be determined from symmetry alone.

7.4 Deriving the OPE from the CFT, and from an AdS QFT

A key property of all local quantum field theories is the existence of an Operator Product Expansion. In CFTs this expansion converges in a finite region, and so it can be used to make various exact statements. Let us first understand[18] what the OPE is and what it has to do with locality. Then we will show that any field theory in global AdS gives rise to CFT correlators that satisfy an OPE. The idea of the derivation is pictured in figure 9.

The OPE says that for any local operators ϕ_1 and ϕ_2 we have

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O}} C(x,\partial)\mathcal{O}(0) \tag{7.33}$$

This relation is derived via path integral methods in Weinberg V2 [13], although it can be easily derived without any explicit action for the CFT. Let us review it, and then we will show that any local QFT in AdS will give rise to CFT correlators that satisfy an OPE. As we will see, the derivation of the OPE in AdS is very similar to the derivation in the CFT.

To derive the OPE in the CFT, we consider two operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$. Now draw a circle around x_1 and x_2, centered at some point x – the circle should have a radius r larger than $|x - x_1|$ and $|x - x_2|$ so that it contains both points, but does not contain an insertion of any other operators. Choose to radially quantize the CFT about x. If we imagine the radial evolution from x outwards, we start with the vacuum and then we eventually hit the operators $\mathcal{O}_1$ at x_1 and $\mathcal{O}_2(x_2)$. Thus on the circle of radius r, we have some state

$$|\psi_{12}(r)\rangle = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle \tag{7.34}$$

because the CFT was in its vacuum at x. This state will be some linear combination of all of the states in the Hilbert space. By the operator state correspondence, there exists some local operator $\mathcal{O}_{\psi_{12}}(x)$ such that

$$\mathcal{O}_{\psi_{12}}(y)|0\rangle = |\psi_{12}(r)\rangle = \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|0\rangle \tag{7.35}$$

[18] In Weinberg V.2 the OPE is derived from the path integral, and in Slava's notes its derived from radial quantization.

The operator $\mathcal{O}_{\psi_{12}}$ will certainly not have a definite dimension or spin – it's an extremely abstract object. However, we can express $\mathcal{O}_{\psi_{12}}$ as a sum over all of the primary operators of the theory and their descendants, and categorized according to their dimension and spin. This gives the OPE

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\Delta,\ell} \lambda_{\Delta,\ell} C_{\Delta,\ell}(x_1 - y, x_2 - y, \partial_y)\mathcal{O}_{\Delta,\ell}(y) \tag{7.36}$$

Note that we are free to use translation symmetry to take $x_1 = y = 0$ for convenience, in which case the function on the RHS just becomes $f_{\Delta,\ell}(x_1, \partial)$. In fact, this function is entirely determined by conformal symmetry, so the only non-trivial information in the OPE is the value of the OPE coefficients $c_{\Delta,\ell}$, which are labeled entirely by the dimension and spin of the primary operators in the theory. However, there are still an infinite number of these OPE coefficients. If this is your first sight of the OPE, then it probably seems very abstract, and in fact there's much more to say about and do with it, but before any of that let's discuss how it's derived from QFT in AdS.

In fact, the AdS derivation mostly copies from the CFT version, as pictured in figure 9. The key point is that the Hilbert space of the theory lives on complete spacelike Cauchy surfaces in AdS. As $t \to -\infty$ in global AdS coordinates, the theory is in the vacuum. If we insert some operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ at some finite times, then we can always study a time t in the future of x_1 and x_2. On this time slice we will find a wavefunction $\psi_{12}(t)$, where the radius of the corresponding CFT circle is just $r = e^t$ in some units. But this state $\psi_{12}(t)$ could just as well have been created by setting it up at some point arbitrarily far into the past. In the limit that we setup ψ_{12} in the infinite past, we can just create it by acting with a local operator at the origin in the CFT. This local operator can be written as a sum over primary operators of definite dimension (AdS energy) and spin, and this expansion reproducies the OPE for the CFT.

Now let us look at the OPE in a more refined way, in order to relate the OPE coefficients of descendants to the coefficients of their associated primary operator. For example, let us consider some particular term associated with a primary operator $\mathcal{O}$

$$\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle = \frac{c}{x^p}\left[\mathcal{O}_{\Delta,\ell}(0) + \cdots\right]|0\rangle + \text{other primaries} \tag{7.37}$$

One might immediately guess that the power p is fixed the dimensions Δ_1, Δ_2, and Δ. We can verify this by acting on both sides with the dilatation operator (and ignoring other contributions), giving

$$\begin{aligned} D\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle &= (\Delta_1 + x\cdot\partial_x)\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle + \Delta_2\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle \\ &= (\Delta_1 + \Delta_2 - p)\frac{c}{x^p}\left[\mathcal{O}_{\Delta,\ell}(0) + \cdots\right]|0\rangle \end{aligned} \tag{7.38}$$

However, we can also compute this directly as an action on $\mathcal{O}$, giving

$$D\frac{c}{x^p}\left[\mathcal{O}_{\Delta,\ell}(0) + \cdots\right]|0\rangle = \frac{c}{x^p}\left[\Delta\mathcal{O}_{\Delta,\ell}(0) + \cdots\right]|0\rangle \tag{7.39}$$

so we find that $p = \Delta_1 + \Delta_2 - \Delta$ for consistency. More interesting relations can be obtained by considering not just the primary $\mathcal{O}_{\Delta,\ell}$ but also the descendants that make up the ellipsis. The next term will be

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \frac{c}{x^p}\left[\mathcal{O}_{\Delta,\ell}(0) + \kappa x^\mu \partial_\mu \mathcal{O}_{\Delta,\ell}(0) + \cdots\right]|0\rangle \tag{7.40}$$

and we can determine the coefficient c using conformal symmetry. Let us act on this equation with K_μ. Note that this SCT annhilates $\mathcal{O}_2(0)$ because its primary, so we find

$$\begin{aligned} K_\mu \mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle &= (2x_\mu \Delta_1 + 2x^\alpha \Sigma_{\alpha\mu} + 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu)\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle \qquad (7.41) \\ &= (2x_\mu \Delta_1 + 2x^\alpha \Sigma_{\alpha\mu} + 2x_\mu x^\alpha \partial_\alpha - x^2 \partial_\mu)\frac{c}{x^p}\left[\mathcal{O}_{\Delta,\ell}(0) + \kappa x^\mu \partial_\mu \mathcal{O}_{\Delta,\ell}(0) + \cdots\right]|0\rangle \end{aligned}$$

This must match with the action of K_μ directly on the RHS, but this is

$$K_\mu \frac{c}{x^p}\left(\mathcal{O}_{\Delta,\ell}(0) + \kappa x^\alpha \partial_\alpha \mathcal{O}_{\Delta,\ell}(0) + \cdots\right)|0\rangle = \frac{c}{x^p}\left(\kappa x^\alpha [K_\mu, [P_\alpha, \mathcal{O}_{\Delta,\ell}]](0) + \cdots\right)|0\rangle \qquad (7.42)$$

where we note that K_μ annihilates the primary operator $\mathcal{O}(0)$, and we have written the last part very explicitly in terms of the conformal generators to emphasize that it can be evaluated just using the algebra.

To match the two sides, note that in equation (7.41) the first two terms in K_μ actually cancel against the action of the derivative operators on x^{-p}, so that all terms proportional to $\mathcal{O}_{\Delta,\ell}(0)$ drop out. Then for $\ell = 0$ we are left with the first term as

$$2\kappa \Delta x^\mu \mathcal{O}_{\Delta,0} = (2\Delta_1 - p)x^\mu \mathcal{O}_{\Delta,0} \qquad (7.43)$$

so we find that

$$\kappa = \frac{1}{2} + \frac{\Delta_1 - \Delta_2}{2\Delta} \qquad (7.44)$$

for the first coefficient. In principle we can compute all subsequent coefficients this way, although it would be rather laborious.

Finally, now that we have demonstrated that the appearance of descendant operators in the OPE is entirely determined by the appearance of primaries, let us consider how to compute the primary OPE coefficients. Actually, this is easy, and also provides a way to compute $C(x, \partial)$. The method is to take the OPE expansion and compute its correlator with a single operator $\mathcal{O}_{\Delta,\ell}$, giving

$$\langle \mathcal{O}_{\Delta,\ell}(z)\mathcal{O}_1(x)\mathcal{O}_2(y)\rangle = \left\langle \mathcal{O}_{\Delta,\ell}(z) \sum_{primary\ \mathcal{O}} \lambda^{12}_{\Delta,\ell} C(x-y, \partial_y)\mathcal{O}(y)\right\rangle \qquad (7.45)$$

where in our current notation $\lambda^{12}_{\Delta,\ell}$ is the OPE coefficient. Now note that CFT 2-pt functions always vanish unless the operators have the same dimension, charge, and spin, so on the RHS we have projected out the 2-pt function of a single primary operator $\mathcal{O}_{\Delta,\ell}$. On the LHS we just have a CFT 3-pt function, and these are fixed up to a finite number of constants (in the case of a scalar-scalar-spin ℓ correlator there is a unique answer kinematically, so we only have one overall constant). Considering the scalar case for simplicity, we find that

$$c^{12}_{\Delta,\ell=0}\frac{1}{(x-y)^{\alpha_{12\Delta}}(z-y)^{\alpha_{2\Delta 1}}(x-z)^{\alpha_{1\Delta 2}}} = \lambda^{12}_{\Delta,0} C(x-y, \partial_y)\frac{1}{(z-y)^{2\Delta}} \qquad (7.46)$$

So we see that there is a very simple relationship between the coefficient $c^{12}_{\Delta,0}$ of the 3-pt function and the OPE coefficient $\lambda^{12}_{\Delta,0}$. Furthermore, we can also determine the function $C(x-y, \partial_y)$ by matching the kinematics on both sides of this equation.

7.5 The Conformal Partial Wave Expansion and the Bootstrap

We derived the OPE in the case of generalized free theory, and then we explained that in fact the OPE exists with a finite radius of convergence in any CFT. Crucially, we found that all coefficients for descendant operators in the OPE are determined kinematically in terms of the coefficients of primary operators. We also saw how one can derive the OPE directly by thinking about the AdS field theory description.

Let us now see what happens when we apply the OPE to a 4-pt CFT correlator. This is straightforward; we see that

$$\begin{aligned}
\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \sum_{\tau,\ell}\lambda_{\tau,\ell}C_{\tau,\ell}(x_{12},\partial_1)\langle\mathcal{O}_{\tau,\ell}(x_2)\phi(x_3)\phi(x_4)\rangle \\
&= \sum_{\tau,\ell}\lambda_{\tau,\ell}C_{\tau,\ell}(x_{12},\partial_2)\sum_{\tau',\ell'}\lambda_{\tau',\ell'}C_{\tau',\ell'}(x_{34},\partial_4)\langle\mathcal{O}_{\tau,\ell}(x_2)\mathcal{O}_{\tau',\ell'}(x_4)\rangle \\
&= \sum_{\tau,\ell}\lambda_{\tau,\ell}^2\left[C_{\tau,\ell}(x_{12},\partial_2)C_{\tau',\ell'}(x_{34},\partial_4)\langle\mathcal{O}_{\tau,\ell}(x_2)\mathcal{O}_{\tau,\ell}(x_4)\rangle\right] \\
&= \frac{1}{(x_1-x_3)^{2\Delta}(x_2-x_4)^{2\Delta}}\sum_{\tau,\ell}\lambda_{\tau,\ell}^2 g_{\tau,\ell}(u,v) \qquad (7.47)
\end{aligned}$$

In the first line we applied the OPE in 12, while in the second we applied it in 34. In the third line we simplified by noting that

$$\langle\mathcal{O}_{\tau,\ell}(x_2)\mathcal{O}_{\tau',\ell'}(x_4)\rangle \propto \delta_{\tau\tau'}\delta_{\ell\ell'} \qquad (7.48)$$

so this term is only present when $\tau=\tau'$ and $\ell=\ell'$. In the last line we noted that

$$\frac{1}{(x_1-x_3)^{2\Delta}(x_2-x_4)^{2\Delta}}g_{\tau,\ell}(u,v) \equiv (C_{\tau,\ell}(x_{12},\partial_2)C_{\tau',\ell'}(x_{34},\partial_4)\langle\mathcal{O}_{\tau,\ell}(x_2)\mathcal{O}_{\tau,\ell}(x_4)\rangle) \qquad (7.49)$$

is entirely determined by conformal symmetry – in other words, this *conformal partial wave* is a purely kinematic quantity. The conformal block coeffs are squares of OPE coeffs:

$$\lambda_{\tau,\ell}^2 > 0 \qquad (7.50)$$

This is how *Unitarity* shows up.

Another equivalent way to think about it is via the insertion of complete set of states in the middle. Organize according to irreps of conformal symmetry. This is the same thing as using the OPE, as is obvious from the radial quantization derivation. Contribution of a single irrep is a *conformal partial wave* or *conformal block*:

$$\frac{g_{\tau,\ell}(u,v)}{(x_{13}^2x_{24}^2)^{\Delta_\phi}} = \langle\phi(x_1)\phi(x_2)\left(\sum_{\tau,\ell\ irrep}|\alpha\rangle\langle\alpha|\right)\phi(x_3)\phi(x_4)\rangle \qquad (7.51)$$

This is an even easier way of explaining these conformal partial waves – it is just what we get when we insert **1** into the correlator as a sum over a particular set of states.

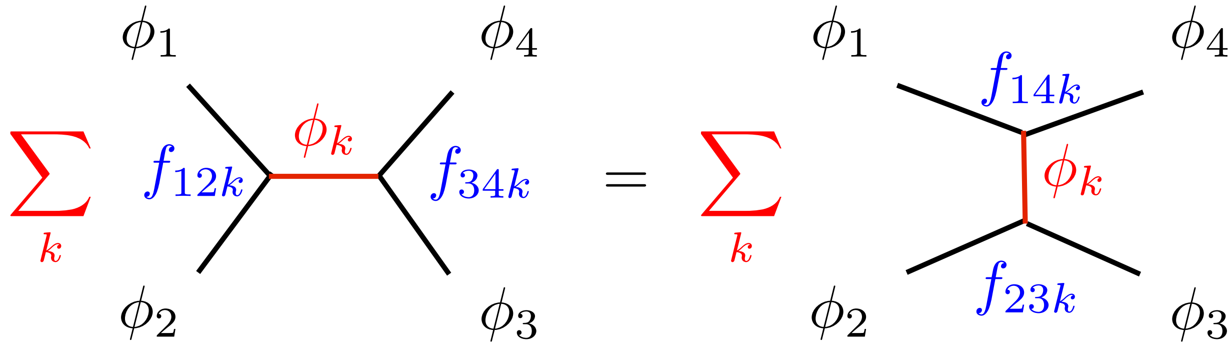

Figure 10: This figure illustrates the bootstrap equation relating the conformal partial wave expansion in two different channels. We see the explicit appearance of varoius OPE coefficients in the different channels.

The conformal partial wave expansion is in all respects analgous to the partial wave expansion of scattering amplitudes. The analogy connects angular momentum to angular momentum, while the twist τ or dimension $\Delta = \tau + \ell$ plays the role of the center of mass energy.

The conformal blocks are known (complicated) functions in many, but not all cases. For example, for CFTs in $d = 2$ dimensions the conformal partial waves are

$$g_{\tau,\ell}(z, \bar{z}) = k_{\tau+2\ell}(z)k_\tau(\bar{z}) + k_{\tau+2\ell}(\bar{z})k_\tau(z) \tag{7.52}$$

where $u = z\bar{z}$ and $v = (1 - z)(1 - \bar{z})$ and

$$k_{2\beta}(x) = x^\beta {}_2F_1(\beta, \beta, 2\beta; x) \tag{7.53}$$

where this is the hypergeometric function. The reason for the notation of z and $\bar{z}$ is that if we take the correlator

$$\langle \phi(0)\phi(z)\phi(1)\phi(\infty)\rangle \tag{7.54}$$

then we see that

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = \frac{(z\bar{z})\infty^2}{1^2\infty^2} \to z\bar{z} \tag{7.55}$$

and we obtain a similarly simple result with v. So z and $\bar{z}$ are literally complex coordinates in a two dimensional plane.

Recall that in the previous section we noted various identities between CFT correlators when we exchanged $2 \leftrightarrow 3$ and $2 \leftrightarrow 4$. The bootstrap equation uses this *crossing symmetry* to equate the conformal partial wave expansions in different channels:

$$u^{-\Delta_\phi} + u^{-\Delta_\phi} \sum_{\tau,\ell} P_{\tau,\ell}\, g_{\tau,\ell}(u, v) = v^{-\Delta_\phi} + v^{-\Delta_\phi} \sum_{\tau,\ell} P_{\tau,\ell}\, g_{\tau,\ell}(v, u) \tag{7.56}$$

This is pictured in figure 10. Let us now see how the conformal partial wave decomposition works in our favorite example, that of a Generalized Free Theory. In the next section we will study what happens when we add AdS interactions.

The GFT the 4-pt correlator can be written as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{(x_{13}^2 x_{24}^2)^\Delta}\left(\frac{1}{(z\bar{z})^\Delta} + 1 + \frac{1}{(1-z)^\Delta(1-\bar{z})^\Delta}\right) \tag{7.57}$$

We want to equate this with the conformal block decomposition, so in other words we are looking for

$$\frac{1}{(z\bar{z})^\Delta}\sum_{\tau,\ell} P_{\tau,\ell}\, g_{\tau,\ell}(z,\bar{z}) = 1 + \frac{1}{(1-z)^\Delta(1-\bar{z})^\Delta} \tag{7.58}$$

where we have already canceled off the contribution of the identity operator on both sides. Now let us expand the RHS at small z and $\bar{z}$; we find

$$1 + \frac{1}{(1-z)^\Delta(1-\bar{z})^\Delta} \approx 2 + \Delta(z+\bar{z}) + \frac{\Delta(\Delta+1)}{2}(z^2+\bar{z}^2) + \Delta^2(z\bar{z}) + \cdots \tag{7.59}$$

Notice that all of the powers that appear are integral. However, note that the terms on the LHS with the conformal partial waves behave as

$$\frac{1}{(z\bar{z})^\Delta} g_{\tau,\ell}(z,\bar{z}) = (z\bar{z})^{\frac{\tau}{2}-\Delta}\left(z^\ell + \bar{z}^\ell + \cdots\right) \tag{7.60}$$

So matching both sides around $z,\bar{z} \sim 0$ immediately implies that we must always have $\tau = 2\Delta + 2n$ for a generalized free theory. We have just re-discovered the fact that the only operators that can appear in the OPE of $\mathcal{O}(x)\mathcal{O}(0)$ are

$$\mathcal{O}(x)\mathcal{O}(0) \sim 1 + a\mathcal{O}^2(0) + b\mathcal{O}\partial^2\mathcal{O} + \cdots \tag{7.61}$$

so that they involve $\mathcal{O}\partial^{2n}\partial_{\mu_1}\cdots\partial_{\mu_\ell}\mathcal{O}$ and have dimension $2\Delta + m$. By matching both sides of equation (7.60) we can compute the conformal partial wave coefficients (and therefore the OPE coefficients) for a generalized free theory. When we add AdS interactions in the next section we will see that the various operators get anomalous (shifted) dimensions proportional to the coupling constants. Note that shifting τ produces $(z\bar{z})^\gamma \approx 1 + \gamma\log(z\bar{z})$, so anomalous dimension appear as logarithms in CFT correlators.

8 Adding AdS Interactions

Let us compare the conformal partial wave expansion with various perturbative methods. The reader is likely already familiar with Feynman diagram perturbation theory in QFT, and perhaps also with 'Old Fashioned Perturbation Theory'. Let us compare these descriptions of correlators to the Conformal Partial Wave (aka Conformal Block) decomposition and the bootstrap equation.

Feynman Diagram Perturbation Theory aka Witten Diagrams in an AdS context, provide a perturbative expansion that keeps the spacetime symmetries (conformal symmetry) manifest at every step of the calculation. However, Feynman diagrams obscure which physical states contribute to a given process.

Specifically, both single and multi-particle states are exchanged even in tree-level diagrams. This feature of perturbation theory is not very prominent in flat spacetime, where partial waves are continuous functions of the scattering energy, but it is manifest in AdS/CFT.

Old Fashioned Perturbation Theory is another perturbative method that's closely connected to conventional quantum mechanical perturbation theory. It requires an explicit choice of time slicing, with an accompanying specification of the Hilbert space of states on those constant time surfaces. As a consequence, OFPT breaks spacetime symmetries that explicitly involve time. However, OFPT has the advantage of being formulated directly in terms of physical states.

The Conformal Partial Wave Expansion aka the conformal block decomposition of a CFT correlation function – this is a non-perturbative expansion that keeps all spacetime symmetries manifest, and that only involves physical states. When combined with crossing symmetry, the conformal block decomposition leads to the bootstrap equation, which provides powerful constraints on correlators, but unlike the previous methods, it is not a systematic technique for solving the theory order-by-order in an expansion parameter.

However, given a correlator computed in perturbation theory, one can always decompose it into conformal partial waves. This leads to a clear specification of which states (operators) appear in various OPEs. Note that the identity operator, which is dual to 'free propagation' in AdS, has a very non-trivial conformal partial wave decomposition in other channels. This means that it's not always easy to isolate 'free propagation' from 'interactions' in the conformal block decomposition.

Now let us see how an interacting QFT in AdS spacetime give rise to an interacting C(F)T. Then we will see how abstract CFT data like OPE coefficients and anomalous dimensions can be computed in AdS/CFT, and then we will translate these ideas into the much more mundane-sounding physics of wavefunction overlaps and the binding energies between particles.

8.1 Anomalous Dimensions from Basic Quantum Mechanics

In the following sections we will discuss AdS Feynman diagrams and perform some computations in $\lambda\phi^4$ theory. For now we will discuss a simpler, old-fashioned perturbation theory approach [10] that basically amounts to an application of undergraduate quantum mechanics. This approach is actually easier and more efficient than Feynman diagrams in many cases.

Consider a situation where we have a free Hamiltonian $D^{(0)}$ that we have diagonalized, and we add a small interaction Hamiltonian D^I. Everyone knows that to first order in perturbation theory, the energy of a $D^{(0)}$ eigenstate $|\psi\rangle$ simply changes by

$$\gamma_\psi = \left\langle \psi | D^I | \psi \right\rangle \tag{8.1}$$

We can apply this formula directly in the case of $\lambda\phi^4$ theory in AdS in order to compute the energy shifts of 2-particle states in AdS. Recall that 2-particle states in AdS are created by the CFT operators $[\mathcal{O}\mathcal{O}]_{n\ell}$ where $\mathcal{O} = \lim_{\rho\to\pi/2}\phi$ as usual. So computing the energy shifts of these states is equivalent to computing the 'anomalous dimensions' of these operators.

Let us consider $\lambda\phi^4$ theory in AdS$_3$ for simplicity. The (Lorentzian) action is

$$S[\phi] = \int_{AdS} d^3x\sqrt{-g}\left(\frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right) \tag{8.2}$$

We derive the Hamiltonian as usual, by defining canonical momenta and computing $D = P\dot{q} - L$. This gives the Hamiltian (Dilatation operator) for the free theory plus an interaction part, which is

$$D^I(t) = \frac{\lambda}{4!}\int_{AdS} d^2x\sqrt{-g}\phi^4(t,x) \tag{8.3}$$

This interaction vanishes except for the scalar or $\ell = 0$ partial wave, so $[\mathcal{O}\mathcal{O}]_{n\ell}$ do not receive anomalous dimensions from $\lambda\phi^4$ for $\ell > 0$. This is directly analogous to the fact that $\lambda\phi^4$ interaction only contributes to the $\ell = 0$ partial wave when we study scattering in flat spacetime. Specializing to the case of $\ell = 0$ we want to compute the matrix element

$$\begin{aligned}\gamma(n) &= \langle n,0|D^I|n,0\rangle \\ &= \frac{\lambda}{4!}\int_{AdS} d^2x\sqrt{-g}\ \langle n,0|\phi^4(x)|n,0\rangle\end{aligned} \tag{8.4}$$

where

$$|n,\ell\rangle = [\mathcal{O}\mathcal{O}]_{n\ell}|0\rangle \tag{8.5}$$

is the primary state corresponding to two particles in AdS. Now we can evaluate this matrix element directly. In order for it to be non-vanishing, two of the $\phi(x)$ operators must destroy the initial state particles while two destroy the final state particles, so we find

$$\gamma(n) = \frac{\lambda}{4}\int_0^{2\pi} d\theta\int_0^{\frac{\pi}{2}} d\rho\frac{\sin\rho}{\cos^3\rho}\langle n,0|\phi^2(\rho,\theta)|0\rangle\langle 0|\phi^2(\rho,\theta)|n,0\rangle \tag{8.6}$$

However, note that these states are scalar primaries, and $\phi^2(x)$ is a scalar operator, so by symmetry we must have

$$\langle 0|\phi^2(\rho,\theta)|n,0\rangle = \frac{1}{\sqrt{2\pi}}(e^{it}\cos\rho)^{2\Delta+2n} \tag{8.7}$$

since these are primary wavefunctions that are annhilated by K_μ. Note that this matrix element would vanish for $\ell > 0$ – this is how one can see that $\lambda\phi^4$ does not produce anomalous dimensions at first order for $\ell > 0$ operators. The correct normalization can be computed [10], although the calculation is a bit non-trivial. I have simply written the correct result for AdS$_3$. Performing the integrals gives

$$\gamma(n) = \frac{\lambda}{8\pi(2\Delta + 2n - 1)} \tag{8.8}$$

so we have computed the anomalous dimension of all $[\mathcal{O}\mathcal{O}]_{n\ell}$ in $\lambda\phi^4$ theory in AdS. As we have emphasized before, anomalous dimensions are just AdS energy shifts due to interactions.

8.2 Perturbative Interactions and $\lambda\phi^4$ Theory

Interacting QFTs in AdS can be studied in exactly the same way that we study such theories in flat spacetime. In particular, we can expand perturbatively in the coupling constants and compute using AdS Feynman diagrams. To derive the Feynman rules we can use either an operator or a path integral formalism, as usual; the result are rules for propagators, which are just the 2-pt functions of AdS fields, and vertices, which follow as direct generalizations of those of flat spacetime. A significant technical difference is that in AdS spacetime we lack a set of commuting translation operators, and so we cannot use the Fourier transform to momentum space to simplify computations. Conceptually this may almost be an advantage, but computationally it means that AdS Feynman diagrams are much more cumbersome to calculate.

Let us consider a standard example, that of $\lambda\phi^4$ theory in AdS. We have an AdS path integral

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} \tag{8.9}$$

where the Euclidean action is

$$S[\phi] = \int_{AdS} d^{d+1}x\sqrt{-g}\left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right) \tag{8.10}$$

The bulk-to-bulk propagators have a simple expression in terms of the geodesic distance $\sigma(X,Y)$ between X and Y:

$$G_\Delta(X,Y) \;=\; \mathcal{C}_\Delta z^{\Delta/2}{}_2F_1(\Delta, h, \Delta+1-h, z), \tag{8.11}$$

where $z = e^{-2\sigma}$.[19] The factor $\mathcal{C}_\Delta = \Gamma(\Delta)/(2\pi^h\Gamma(\Delta+1-h))$ with $h = d/2$. This propagator can be obtained as usual as the solution to the Klein Gordon equation in Euclidean AdS with a delta function source, namely

$$\left(\nabla_X^2 + m^2\right) G_\Delta(X,Y) = (2\pi)^{d+1}\delta^{d+1}_{AdS}(X-Y) \tag{8.14}$$

It is also given as a sum over modes if we compute

$$\langle\phi(X)\phi(Y)\rangle = G_\Delta(X,Y) \tag{8.15}$$

using the expansion of $\phi(X)$ in terms of creation and annihilation operators with corresponding wavefunctions.

[19]This normalization of the bulk-to-bulk propagator differs from that in e.g. [10] by a factor of $\mathcal{C}_\Delta$. Other useful representations of the bulk-to-bulk propagator are

$$\begin{aligned} G_\Delta(X,Y) &= \mathcal{C}_\Delta\frac{y^{\Delta/2}}{2^\Delta}{}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta}{2}+\frac{1}{2}, \Delta+1-h, y\right) && (8.12)\\ &= \frac{\mathcal{C}_\Delta}{u^\Delta}{}_2F_1\left(\Delta, \Delta-h+\frac{1}{2}, 2\Delta-2h+1, -\frac{4}{u}\right), && (8.13)\end{aligned}$$

where $u = (X-Y)^2$ and $y^{-\frac{1}{2}} = \cosh\frac{\sigma}{R} = 1 + \frac{u}{2R^2}$.

The Feynman rules for the theory are identical to the position space Feynman rules for $\lambda\phi^4$ theory in flat spacetime, except that we replace flat space propagators with $G_\Delta(X,Y)$, and we integrate over AdS. For example, the 4-pt correlator of ϕ to first order in λ is

$$\langle\phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4)\rangle = \lambda\int_{AdS} d^{d+1}Y\sqrt{-g}G_\Delta(X_1,Y)G_\Delta(X_2,Y)G_\Delta(X_3,Y)G_\Delta(X_4,Y) \tag{8.16}$$

If we had a $g\phi^3$ theory instead we would have

$$\begin{aligned}\langle\phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4)\rangle &= g^2\int_{AdS} d^{d+1}Y_1\sqrt{-g(Y_1)}G_\Delta(X_1,Y_1)G_\Delta(X_2,Y_1) \\ &\quad\times\int d^{d+1}Y_2\sqrt{-g(Y_2)}G_\Delta(Y_1,Y_2)G_\Delta(X_3,Y_2)G_\Delta(X_4,Y_2)\end{aligned} \tag{8.17}$$

Loops can be calculated in the same way, with an AdS position-space integral for every vertex, and a propagator for each line.

Now that we know how to compute correlators of fields in AdS, we can apply our usual dictionary to compute the correlation functions of CFT operators. We need only apply our formula

$$\mathcal{O}(t,\Omega) = \lim_{\epsilon\to 0}\frac{\phi\left(t,\frac{\pi}{2}-\epsilon f(t,\Omega),\Omega\right)}{\epsilon^\Delta} \tag{8.18}$$

for each of the AdS fields in an AdS correlator. Applying this formula to any of the examples above, we immediately notice the appearance of the function

$$G_{B\partial}(t,\Omega;Y) = \lim_{\epsilon\to 0}\frac{1}{\epsilon^\Delta}G_\Delta\left(t,\frac{\pi}{2}-\epsilon f(t,\Omega),\Omega;Y\right) \tag{8.19}$$

which we refer to as the *bulk-boundary propagator*. For example, to compute the CFT 4-pt correlator at tree level in $\lambda\phi^4$ theory in AdS, we use the formula

$$\langle\mathcal{O}(\vec{r}_1)\mathcal{O}(\vec{r}_2)\mathcal{O}(\vec{r}_3)\mathcal{O}(\vec{r}_4)\rangle = \lambda\int_{AdS} d^{d+1}Y\sqrt{-g}G_{B\partial}(\vec{r}_1,Y)G_{B\partial}(\vec{r}_2,Y)G_{B\partial}(\vec{r}_3,Y)G_{B\partial}(\vec{r}_4,Y) \tag{8.20}$$

This is an example of how to use AdS/CFT to compute non-trivial CFT correlation functions.

Fortunately, the bulk-boundary propagator $G_{B\partial}$ is much simpler than the bulk-bulk propagator G_Δ. In terms of the projective null cone coordinates P, we simply have

$$G(P,X) = \lim_{\epsilon\to 0}\frac{\mathcal{C}_\Delta e^{-\sigma(X,Y(\epsilon))}{}_2F_1(\Delta,h,\Delta+1-h,e^{-2\sigma(X,Y(\epsilon))})}{\epsilon^\Delta}, \tag{8.21}$$

where in this limit $Y\to\frac{1}{\epsilon}P$ becomes the boundary point. In this limit the geodesic distance goes to infinity, and we simply obtain

$$G_{B\partial}(P,X) = \frac{\mathcal{C}_\Delta}{(2P\cdot X)^\Delta} = \frac{1}{\Gamma(\Delta-h+1)}\int_0^\infty\frac{dt}{t}t^\Delta e^{-2tP\cdot X} \tag{8.22}$$

where as usual

$$\mathcal{C}_\Delta = \frac{\Gamma(\Delta)}{2\pi^h \Gamma(\Delta - h + 1)} \tag{8.23}$$

The last formula in terms of a t integral is useful for computations.

This formula for $G_{B\partial}$ was essentially forced on us by conformal symmetry – the bulk boundary propagator could only depend on the conformal invariant $P \cdot X$, and the scaling relation $\mathcal{O}(\lambda P) = \lambda^{-\Delta}\mathcal{O}(P)$ forces the power-law relation that we derived. In this normalization the 2-pt function is

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle = \frac{\mathcal{C}_\Delta}{(2P_1 \cdot P_2)^\Delta} \tag{8.24}$$

for an operator of dimension Δ. This could be derived by taking the point $X \to P_2$ on the boundary of AdS.

Let us now try to compute the 4-pt correlator to first order in $\lambda\phi^4$ theory. This is

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\mathcal{O}(P_4)\rangle = \frac{\lambda}{[2\pi^h\Gamma(\Delta - h + 1)]^4} \int_{AdS} d^{d+1}Y \prod_{i=1}^4 \int_0^\infty \frac{dt_i}{t_i} t_i^\Delta e^{-2t_i P_i \cdot Y} \tag{8.25}$$

Now we can exchange the orders of integration to write

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\mathcal{O}(P_4)\rangle = \frac{\lambda}{[2\pi^h\Gamma(\Delta - h + 1)]^4} \prod_{i=1}^4 \int_0^\infty \frac{dt_i}{t_i} t_i^\Delta \int_{AdS} d^{d+1}Y e^{-2\left(\sum_i t_i P_i\right)\cdot Y} \tag{8.26}$$

We can do the integral over AdS most easily using the Euclidean Poincaré patch coordinates. We can use conformal symmetry to rotate so that the only non-vanishing component of Q is Q_0, and we have $Y_0 = \frac{1}{2}\left(\frac{z^2+\vec{r}^2+R^2}{z}\right)$, so that in this frame $2Q \cdot Y = |Q|(1 + z^2 + r^2)/z$, leading to

$$\begin{aligned} \int_{AdS} d^{d+1}Y e^{2Q\cdot Y} &= \int_0^\infty \frac{dz}{z} z^{-d} \int d^d\vec{r} e^{-(1+z^2+r^2)|Q_0|/z} \\ &= \pi^h \int_0^\infty \frac{dz}{z}(z|Q|)^{-h} e^{-(1+z^2)|Q_0|/z} \\ &= \pi^h \int_0^\infty \frac{dz}{z} z^{-h} e^{-z+Q^2/z} \end{aligned} \tag{8.27}$$

where we note that $Q_0^2 = Q^2$ and we have now restored manifest conformal symmetry. This means that

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\mathcal{O}(P_4)\rangle = \frac{\lambda}{[2\pi^h\Gamma(\Delta - h + 1)]^4} \prod_{i=1}^4 \int_0^\infty \frac{dt_i}{t_i} t_i^\Delta \int_0^\infty \frac{dz}{z} z^{-h} e^{-z+\frac{1}{z}\sum_{i,j=1}^4 t_i t_j P_i \cdot P_j} \tag{8.28}$$

Now we can rescale $t_i \to t_i\sqrt{z}$ and factor out the z dependence completely. Then we find that

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\mathcal{O}(P_4)\rangle \propto \int_0^\infty \prod_{i=1}^4 \frac{dt_i}{t_i} t_i^\Delta \, e^{-\sum_{i,j=1}^4 t_i t_j P_i \cdot P_j} \tag{8.29}$$

This last integral can be written using the Symanzik 'star formula'. This formula arises by noting

$$e^{-x} = \int_{-i\infty}^{i\infty} d\delta\, \Gamma(\delta) x^{-\delta} \tag{8.30}$$

Inserting this identity for each of the $e^{-t_i t_j P_i \cdot P_j}$ term and performing the t_i integrals leads to

$$\int_0^\infty \prod_{i=1}^n \frac{dt_i}{t_i} t_i^\Delta \, e^{-\sum_{i,j=1}^n t_i t_j P_i \cdot P_j} = \frac{1}{2} \int_{-i\infty}^{i\infty} \prod_{i<j}^n \frac{d\delta_{ij}}{2\pi i} \Gamma(\delta_{ij}) (2P_i \cdot P_j)^{-\delta_{ij}} \tag{8.31}$$

where the δ_{ij} are constrained by

$$\sum_{j \neq i}^n \delta_{ij} = \Delta_i \tag{8.32}$$

although in this case all $\Delta_i = \Delta$, the dimension of $\mathcal{O}$. Note that if we imagine that $\delta_{ij} = p_i \cdot p_j$, then these constraints are just what we would find from momentum conservation plus the on-shell condition $p_i^2 = \Delta_i$. One should think of the δ_{ij} as analogous to the Mandelstam invariants s_{ij} for a scattering amplitude.

This last form for the correlator has transformed it into 'Mellin space', where the δ_{ij} variables parameterize the Mellin space. The Mellin amplitude is defined through

$$\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\mathcal{O}(P_4)\rangle = \int \prod_{i<j}^4 \frac{d\delta_{ij}}{2\pi i} \Gamma(\delta_{ij}) (2P_i \cdot P_j)^{-\delta_{ij}} M(\delta_{ij}) \tag{8.33}$$

where $M(\delta_{ij})$ is the Mellin Amplitude. Here we have found that a $\lambda\phi^4$ interaction in AdS space leads to a Mellin amplitude

$$M(\delta_{ij}) = \lambda \tag{8.34}$$

so it is simply a Mellin space-independent constant. This should remind you of the simplicity of momentum space scattering amplitudes.

We can also re-write the Mellin form by eliminating the extra δ_{ij} variables, giving the non-trivial part of the correlator as

$$\mathcal{A}(u,v) = \int_{-i\infty}^{i\infty} \frac{d\delta_{12} d\delta_{14}}{(2\pi i)^2} M(\delta_{12}, \delta_{14}) \Gamma^2(\delta_{12}) \Gamma^2(\delta_{14}) \Gamma^2(\Delta - \delta_{12} - \delta_{14}) u^{\Delta - \delta_{12}} v^{-\delta_{14}} \tag{8.35}$$

We see that the Mellin amplitude is just the transform in the conformally invariant cross-ratios. For general reference, the Mellin transform of a function $f(x)$ is

$$M(s) = \int_0^\infty x^{s-1} f(x) dx \tag{8.36}$$

$$f(x) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} x^{-s} M(s) \tag{8.37}$$

with the inverse also listed. Note that this is basically just a Fourier transform in the logarithm of the usual variables. So for the 4-pt correlator the Mellin amplitude is just a Mellin transform in the cross-ratios u and v.

What about the position-space correlator? In general, it is a special function referred to in the literature as a 'D-function', which are usually written as $D_{\Delta_1\Delta_2\Delta_3\Delta_4}$ as they depend on the external dimensions. The dependence on the spacetime dimension d only comes about through an overall normalization factor. The choice

$$D_{1111}(z,\bar{z}) = \frac{2\mathrm{Li}_2\left(\frac{1}{\bar{z}}\right) - 2\mathrm{Li}_2\left(\frac{1}{z}\right) + \log(z\bar{z})\log\left(\frac{\bar{z}(z-1)}{z(\bar{z}-1)}\right)}{z-\bar{z}} \tag{8.38}$$

can be expressed in terms of logarithms and dilogarithms. The other D-functions with integer Δ_i can be computed in terms of this one by taking derivatives, but in general for non-integral Δ_i there is no such simple expression. Here $z\bar{z} = u$ and $(1-z)(1-\bar{z}) = v$ as usual.

8.3 Conformal Partial Waves and $\lambda\phi^4$

Let us consider $\lambda\phi^4$ theory in AdS, and see how the interaction effects the CFT. For concreteness, let us imagine that we are working in AdS$_4$, so $d = 3$, and we have fixed the squared mass of the ϕ field to be $m^2 = \Delta(\Delta - 3) = -2/R^2$, so that $\Delta = 1$. Then the correlator of the operator $\mathcal{O}$ obtained from taking ϕ to the boundary is

$$\begin{aligned} \langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \;=\;& \frac{1}{x_{13}^2x_{24}^2}\left(1 + \frac{1}{z\bar{z}} + \frac{1}{(1-z)(1-\bar{z})}\right) \\ & -\lambda\frac{\mathcal{N}}{x_{13}^2x_{24}^2}\left(\frac{2\mathrm{Li}_2\left(\frac{1}{\bar{z}}\right) - 2\mathrm{Li}_2\left(\frac{1}{z}\right) + \log(z\bar{z})\log\left(\frac{\bar{z}(z-1)}{z(\bar{z}-1)}\right)}{z-\bar{z}}\right) \end{aligned} \tag{8.39}$$

where $\mathcal{N}$ is a normalization constant that just depends on our choice $\Delta = 1$ and $d = 3$. We chose these parameters so that we could utilize the simplest non-triival D-function.

In section 7.5 we discussed the conformal partial wave expansion of correlators, and the specific partial waves that appear in the case of a generalized free theory. Now we would like to see how the conformal partial wave expansion changes due to the $\lambda\phi^4$ interaction. We are interested in this because it tells us about the change in the OPE coefficients that are proportional to 3-pt functions such as

$$\langle\mathcal{O}(x_1)\mathcal{O}(x_3)[\mathcal{O}\mathcal{O}]_{n,\ell}(y)\rangle \tag{8.40}$$

where $[\mathcal{O}\mathcal{O}]_{n,\ell}$ is the primary operator that would have dimension $2\Delta + 2n + \ell$ and spin ℓ in the generalized free theory. It can be viewed schematically as

$$[\mathcal{O}\mathcal{O}]_{n,\ell} \sim \mathcal{O}\left(\overleftrightarrow{\partial}\right)^{2n}\overleftrightarrow{\partial}_{\mu_1}\cdots\overleftrightarrow{\partial}_{\mu_\ell}\,\mathcal{O} \tag{8.41}$$

although there are a lot of combintorial coefficients and traces to remove to get the precise primary operator.

Furthermore, the correlator also tells us about the *anomalous dimensions* $\gamma(n, \ell)$ of these operators – this is the shift in their dimension due to the interaction. Recall that these operator simply correspond to 2-particle states in AdS, and that the dilatation operator is the AdS Hamiltonian, so an anomalous dimension in this context is simply the interaction energy (attractive or repulsive) between the two particles. As we will see, this can be computed directly using old-fashioned quantum mechanical perturbation theory.

Let us expand the correlator in $z, \bar{z}$ near 0, since this is the OPE limit. Ignoring the normalization factor and just keeping track of the parametric dependence on λ, we find

$$\mathcal{A}(z, \bar{z}) = \frac{1}{z\bar{z}} + 2 + z + \bar{z} - \frac{\lambda}{2}(4 + z + \bar{z} - (2 + z + \bar{z}) \log(z\bar{z})) + \cdots \tag{8.42}$$

where the ellipsis denotes higher order terms in $z, \bar{z}$. Recall that conformal partial waves behave as

$$\frac{1}{(z\bar{z})^{\Delta_\phi}} g_{\tau,\ell}(z, \bar{z}) = (z\bar{z})^{\frac{\tau}{2} - \Delta_\phi} \left(z^\ell + \bar{z}^\ell + \cdots\right) \tag{8.43}$$

The terms that are independent of λ are just what we have in a generalized free theory. But let us examine the order λ terms more closely. The correlator contains

$$\mathcal{A}(z, \bar{z}) \supset (2 - 2\lambda + \lambda \log(z\bar{z})) \tag{8.44}$$

Recall that the OPE in a generalized free theory begins (at small $z, \bar{z}$) with

$$\mathcal{O}(z, \bar{z})\mathcal{O}(0) = \left[\frac{1}{(z\bar{z})^\Delta} + \frac{1}{N^{\mathcal{O}}_{0,\bar{0}}}\mathcal{O}^2(0) + \frac{\bar{z}}{N^{\mathcal{O}}_{0,\bar{1}}}\mathcal{O}\partial_{\bar{z}}\mathcal{O}(0) + \frac{z}{N^{\mathcal{O}}_{1,\bar{0}}}\mathcal{O}\partial_z\mathcal{O}(0) + \cdots\right] \tag{8.45}$$

The terms we have found in the correlator $\mathcal{A}$ are of order $z^0\bar{z}^0$, so we can identify them as corresponding to the operator $\mathcal{O}^2(0)$. In other words, this is the operator $[\mathcal{O}\mathcal{O}]_{n,\ell}$ with $n = \ell = 0$. We see that in the generalized free theory the conformal block coefficient was simply 2, but we have found that this has shifted from $2 \to 2 + 2\lambda$. This also means that the OPE coefficient has shifted – since the conformal block coefficient is the square of the OPE coefficient, we have

$$c_{0,0} = \sqrt{2} - \frac{\sqrt{2}}{2}\lambda \tag{8.46}$$

so that $(c_{0,0})^2$ gives $2+2\lambda$, the conformal block coefficient (although recall that the precise numerical coefficient isn't meaningful since we didn't keep track of the normalization $\mathcal{N}$).

Now let us consider the $\lambda \log(z\bar{z})$ term in $\mathcal{A}$. Note that

$$\lambda \log(z\bar{z}) \approx (z\bar{z})^\lambda - 1 \tag{8.47}$$

in perturbation theory in λ. In other words, *we should interpret this logarithm as a shift in the scaling dimension of* $\mathcal{O}^2(0)$. From this logarithm we can compute the anomalous dimension $\gamma(n, \ell)$ for $n = \ell = 0$, namely

$$\gamma(0, 0) = (z\partial_z + \bar{z}\partial_{\bar{z}}) \left[\lambda \log(z\bar{z})\right] = 2\lambda \tag{8.48}$$

Note that the anomalous dimension is positive for positive λ, so the dimension of $\mathcal{O}^2$ has increased. This is due to the fact that the corresponding 2-particle state in AdS is experiencing a repulsive force.

Let us now revisit the utility of the Mellin amplitude representation of the correlator. We noted that for the 4-pt correlator we can write the contour integral

$$\mathcal{A}(u,v) = \int_{-i\infty}^{i\infty} \frac{d\delta_{12} d\delta_{14}}{(2\pi i)^2} M(\delta_{12},\delta_{14})\Gamma^2(\delta_{12})\Gamma^2(\delta_{14})\Gamma^2(\Delta - \delta_{12} - \delta_{14}) u^{\Delta-\delta_{12}} v^{-\delta_{14}} \tag{8.49}$$

We have just seen that perturbative AdS interactions shift the OPE coefficients and operator/state dimensions in the CFT, and that these shifts can be extracted by looking at the coefficients of power-law monomials in $\mathcal{A}(u,v)$. Furthermore, we have seen that anomalous dimensions are associated with logarithms.

Note that a single pole in an integral of the form

$$\int_{-i\infty}^{i\infty} \frac{d\delta}{2\pi i} \frac{1}{\delta+\Delta} u^{-\delta} = u^{\Delta} \tag{8.50}$$

while a double pole gives

$$\int_{-i\infty}^{i\infty} \frac{d\delta}{2\pi i} \frac{1}{(\delta+\Delta)^2} u^{-\delta} = \frac{d}{d\Delta} u^{\Delta} = u^{\Delta}\log u \tag{8.51}$$

So the OPE coefficients and anomalous dimensions in a CFT are associated with the *residues of poles* in the Mellin integrand. Much of the power of Mellin space is that it allows us to use complex analysis for the study of AdS/CFT correlators.

8.4 Generalities About AdS Interactions and CFT

We will discuss expectations for the behavior of AdS interactions, a formal proof that (appropriate) perturbative interactions in AdS do not destroy conformal symmetry, and the meaning of EFT in AdS for the CFT.

8.4.1 Physical Discussion of AdS/CFT with Interactions

In perturbation theory the free propagator plays a crucial role. Let us examine various physical limits of the bulk-to-bulk propagator $G_\Delta(X,Y)$. In the short distance limit, it must agree with the flat spacetime propagator in $d+1$ dimensions. This is the limit where $z = e^{-2\sigma(X,Y)} \approx e^{-2r/R_{AdS}}$ where $r \ll R_{AdS}, 1/m$ is the distance between X and Y. Note that

$${}_2F_1(\Delta, h, \Delta+1-h, e^{-2r/R_{AdS}}) \propto \frac{1}{\Gamma(\Delta)}\left(\frac{R_{AdS}}{r}\right)^{d-1} \tag{8.52}$$

where I have dropped some factors of 2 and π, and approximated the hypergeometric function near 1. Thus we find that in the very short distance limit

$$G_{\text{short}}(r) \propto \left(\frac{R_{AdS}}{r}\right)^{d-1} \tag{8.53}$$

which is what we would expect for a field in $d+1$ dimensional flat spacetime at very short distances. At very long distances $z \ll 1$, and we find that

$$G_{\text{long}}(r) \propto e^{-\Delta \frac{r}{R_{AdS}}} \tag{8.54}$$

which falls off very quickly at long-distances in AdS, due to the mass and AdS curvature. There is also a more delicate intermediate regime, where $\Delta \gg \frac{R_{AdS}}{r} \gg 1$ in which the usual Yukawa potential is reproduced. This can be obtained from a saddle-point approximation to the hypergeometric function at large Δ.

Let us consider whether the interactions disturb the limit where we take AdS QFT operators towards the boundary. There is an analogous question in flat spacetime. When we construct the flat spacetime S-Matrix, a key physical point is that we can study initial and final states composed of asymptotically well-separated particles. This is actually rather subtle, and in fact there are well-known IR divergences in four and fewer dimensions that must be treated with great care.

In AdS/CFT the question is whether when we take bulk operators $\phi(t, \rho, \Omega)$ to the boundary in an interacting theory, the interactions effectively shut off. The answer is yes – in fact, interactions shut off in AdS much more robustly than in flat spacetime, due to the AdS curvature. Recall that in section 2.2 we showed that even massless fields in AdS decrease exponentially fast at large distances from their sources. This means that well-separated sources in AdS decouple very quickly at large distances, i.e. as we approach the boundary of AdS.

In particular, we would like to see that as we take our AdS fields to the boundary, their interactions are dominated by points deep within the bulk (i.e. not arbitrarily close to the boundary). Note that when we approach the boundary we take $\phi(t, \rho(\epsilon; t, \Omega), \Omega)$ with $\epsilon \to 0$ but with fixed t, Ω. This means that when we have a correlator

$$\langle \phi_1(t_1, \rho(\epsilon; t_1, \Omega_1), \Omega)\phi_2(t_2, \rho(\epsilon; t_2, \Omega_2), \Omega_2) \cdots \phi_n(t_n, \rho(\epsilon; t_n, \Omega_n), \Omega_n)\rangle \tag{8.55}$$

as we take $\epsilon \to 0$, the geodesic distance $\sigma(X_i, X_j)$ between X_i and X_j will always go to infinity. In other words, any two distinct points on the boundary of AdS are infinitely far from each other. This by itself does not prevent us from having interactions near the boundary, since we are rescaling the ϕ_i by $\epsilon^{-\Delta_i}$ to obtain CFT operators.

However, consider some interaction vertex (from a Feynman diagram) localized at Y in AdS. If Y follows X_i as $X_i \to P_i$ on ∂AdS, then the vertex at Y will be further and further from all of the other vertices, and so bulk-to-bulk propagators connecting it to the rest of the AdS Feynman diagram will be exponentially suppressed. This is the reason that non-trivial interactions do not follow our bulk fields out to the boundary. Note that if all interaction vertices move together out towards the boundary then there is no suppression, but this uniform displacement does not lead to non-trivial interactions at infinity. This is analogous to the fact that in flat spacetime, the integration over the center of mass coordinate just leads to an overall momentum conserving delta function enforcing translation invariance.

8.4.2 A Weinbergian Proof of Conformal Symmetry

We saw by explicit computations that free fields in AdS give rise to Generalized Free Theories, which are conformal theories. Now we are studying AdS field theories with interactions, so how can we be sure that the boundary theory is a conformal theory?

This is not so obvious. In a free theory we have the conformal symmetry generators $D^{(0)}$, $P_\mu^{(0)}$, $K_\mu^{(0)}$, and $M_{\mu\nu}^{(0)}$. But when we add interactions we are changing the dilatation operators = bulk Hamiltonian by

$$D = D^{(0)} + D^I \tag{8.56}$$

where D^I represents the effects of interactions. The free generators may have satisfied the conformal algebra, but now the presence of D^I will (naively) ruin it. In order to restore conformal symmetry we need to deform the other generators.

If we are dealing with a local relativistic QFT in AdS spacetime, then the preservation of the conformal algebra including interactions is actually pretty easy to see. The point is that the AdS isometries[20] are explicitly a symmetry of the action, and so we can use Noether's theorem to compute the symmetry generators as classical or quantum operators. Then these operators also act naturally on the fields when we take them to the boundary of AdS, $\phi \to \mathcal{O}$, and the discussion of section 8.4.1 shows that in fact the interactions decouple so that the CFT operators $\mathcal{O}$ transform just as we found in the free AdS theory.

Let us make this argument in a more general and formal way. We expect the interaction to satisfy

$$[D^I, M_{\mu\nu}] = 0 \tag{8.57}$$

so that we do not need to deform the AdS rotations, and we can continue to use angular momentum quantum numbers. This should not be surprising, since angular momentum is a discrete quantum number, so it would be difficult to imagine deforming it continuously. However, the generators $P_\mu^{(0)}$ and $K_\mu^{(0)}$ do not commute with $D^{(0)}$ and so we should expect that they will need to be deformed by P_μ^I and K_μ^I in the presence of the interaction D^I. Specifically, if we consider K_μ we must have

$$\left[D^{(0)} + D^I, K_\mu^{(0)} + K_\mu^I\right] = -K_\mu^{(0)} - K_\mu^I \tag{8.58}$$

so we need to define K_μ^I and ensure that this commutation relation holds. We already know that the zeroth order commutation relations work, so we need

$$\left[D^{(0)}, K_\mu^I\right] + \left[D^I, K_\mu^{(0)}\right] + \left[D^I, K_\mu^I\right] = -K_\mu^I \tag{8.59}$$

Let us assume that

$$D^I(t) = \int_{AdS} d^d x \sqrt{-g}\,\mathcal{V}(t,x) \tag{8.60}$$

where we have included the t dependence to emphasize that the integral is a spatial integral at some fixed time t. In the interaction picture, the time evolution of $\mathcal{V}$ is given by

$$\partial_t \mathcal{V}(t,x) = -i[D^{(0)}, \mathcal{V}(t,x)] \tag{8.61}$$

[20]Note that in the case of AdS_3 in the presence of gravity, we have more symmetries than just the AdS_3 isometries, but this more subtle point can only be seen by studying the asymptotic symmetry group of the spacetime.

So now let us look at the commutator of $K_\mu^{(0)}$ with D^I. We worked these generators out explicitly for AdS_3, so specializing to that case note that

$$[K_\pm^{(0)}, D^I] = -i \int d^2x \frac{\sin\rho}{\cos^3\rho} e^{-it\pm i\theta} \left(\sin\rho \partial_t + i\cos\rho\partial_\rho \mp \frac{1}{\sin\rho}\partial_\theta \right) \mathcal{V}(t,\rho,\theta) \tag{8.62}$$

We can replace the first term with the commutator, while integrating the last two terms by parts gives

$$[K_\pm^{(0)}, D^I] = -\int d^2x \sqrt{-g} \sin\rho \, e^{-it\pm i\theta} \left([D^{(0)}, \mathcal{V}(t,x)] + \mathcal{V}(t,x) \right) \tag{8.63}$$

This means that if we define

$$K_\pm^I = \int d^2x \sqrt{-g} \sin\rho \, e^{-it\pm i\theta} \mathcal{V}(t,x) \tag{8.64}$$

then we will satisfy the desired commutation relations if the term $[D^I, K_\mu^I]$ vanishes. This will be possible if

$$[\mathcal{V}(t,x), \mathcal{V}(t,y)] = 0 \tag{8.65}$$

This is the condition we wanted to derive – we can preserve the conformal symmetry if *the interaction density commutes with itself outside the lightcone*. Weinberg derived precisely this condition in order to ensure the Poincaré invariance of the flat spacetime S-Matrix; here we see that it is necessary to preserve the conformal invariance of a theory living in AdS spacetime. The same condition is sufficient to preserve the interacting commutators of the P_μ generators.

8.4.3 Effective Conformal Theory and the Breakdown of AdS EFT

Thus far we have been talking about quantum field theory in AdS as though it is immutable and UV complete. However, in most cases[21] it is more sensible to view QFT as an effective field theory that breaks down at an energy scale Λ, or (more physically) at a short-distance scale $1/\Lambda$. So it becomes natural to ask how the cutoff scale Λ in AdS translates to the CFT. Roughly speaking, one gets the answer immediately by noting that an AdS energy scale Λ is an eigenvalue of the AdS Hamiltonian D, so it must translate into a *dimension* ΛR_{AdS} in the CFT. An AdS field theory with a UV cutoff Λ corresponds to a CFT where we only keep states and operators with dimension $\Delta < \Delta_\Lambda$ with $\Delta_\Lambda = \Lambda R_{AdS}$.

As a more detailed comment – to discuss EFT in an invariant way, we would like to talk about integrating out states with large invariant mass. This means that if we have some AdS/CFT state $|\psi\rangle$ and some AdS operator that creates it, then we have a wavefunction

$$\nabla^2_{AdS} \langle 0|\Phi_{AdS}(X)|\psi\rangle \propto M^2 \langle 0|\Phi_{AdS}(X)|\psi\rangle \tag{8.66}$$

[21]In the case of AdS/CFT this is always true, because once we involve gravity the QFT description can only be an approximation.

But the ∇^2_{AdS} operator is just a combination of conformal generators, in fact it is the conformal Casimir operator

$$\nabla^2_{AdS} = D^2 + \frac{1}{2}(K \cdot P + P \cdot K) + M_{\mu\nu}M^{\mu\nu} \tag{8.67}$$

and so states of definite invariant mass have definite conformal Casimir, where the eigenvalues of this operator are $M^2 = \Delta(\Delta - d) + \ell(\ell + d - 2)$ for general spins. So integrating out the high mass states means integrating out operators/states with large conformal Casimir. Since single particles typically have small spin, large Casimir naturally corresponds to large dimension Δ.

We know from experience that EFTs neglecting higher dimension operators are a good systematic approximation for lower-energy physics. This translates into the statement that corrections to correlators involving operators of dimension Δ from neglected higher dimension effects should be suppressed by powers of Δ/Δ_Λ. In fact, some recent results show that the approximation can be even better than this. For example, consider the conformal partial wave expansion of a correlator

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{(x_{13})^{2\Delta}(x_{24})^{2\Delta}} \sum_{\Delta,\ell}^{\Delta_\Lambda} \lambda^2_{\Delta,\ell} g_{\Delta,\ell}(u,v) \tag{8.68}$$

where instead of summing over all Δ, ℓ, we have restricted the sum to $\Delta < \Delta_\Lambda$. One can prove [14] that in the Euclidean region this results in an *exponentially* good approximation to the CFT correlator. This is better than one might have expected from effective field theory in AdS, where corrections are generally power-laws. The reason it is better is that we are assuming that the OPE coefficients and operator dimensions are exactly correct for $\Delta < \Delta_\Lambda$, and that all corrections come from dropping the conformal partial waves with larger dimension. In an EFT description we do not simply leave out the states with large energy – we also miss power-law corrections to the low-energy states. This is (most-likely) why dropping high dimension partial waves gives a better approximation than one might guess from EFT.

One can see AdS EFT directly by studying theories with massive particles, moving the cutoff, and integrating these heavy states out. Then the CFT correlators, OPE coefficients, and anomalous dimensions can be matched between the high-energy and low-energy description. This analysis was carried out explicitly in [10].

8.5 Gravitational Interactions in AdS$_3$

We studied a $\lambda\phi^4$ interaction in the previous sections. Unfortunately, studying even that simple interaction required an involved calculation in AdS resulting in a not-very-illuminating special function. To get more intuition, let us study an important example that's also soluble – gravitational interactions between a heavy source and a light probe in AdS$_3$.

We will be studying a 4-pt correlator of two CFT$_2$ operators $\mathcal{O}_H$ and $\mathcal{O}_L$, which we write as

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle \tag{8.69}$$

Conceptually, we will be computing the effects pictured in figure 11. But instead of using Feynman diagrams, we will simplify the calculation in a series of steps.

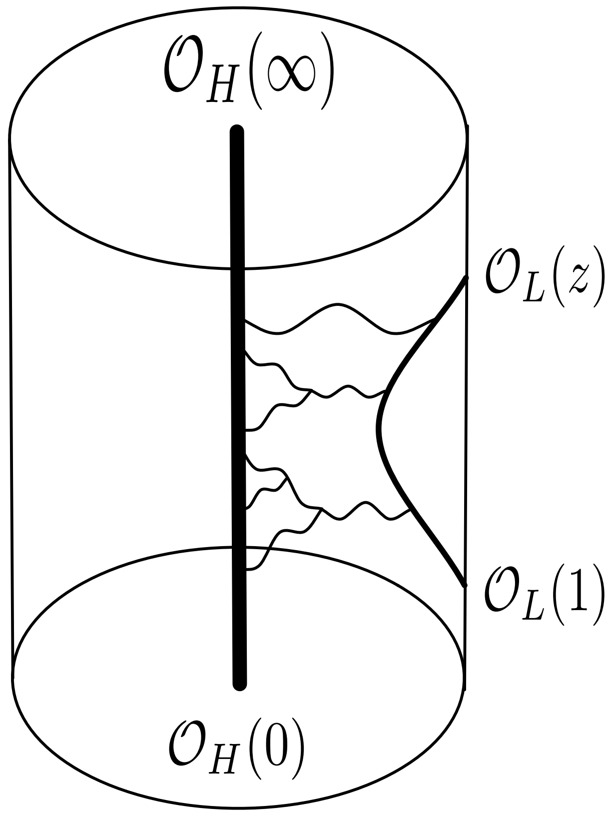

Figure 11: This figure provides a suggestive depiction of how graviton exchanges in AdS build up the classical field experienced by a light probe.

First of all, note that $|\mathcal{O}_H(0)\rangle$ and $\langle\mathcal{O}_H(\infty)|$ correspond to primary states, which means that they represent an object at rest in the center of global AdS. So we can interpet our calculation as the evaluation of the $\mathcal{O}_L(1)\mathcal{O}_L(z)$ 2-pt correlator in a background determined by $\mathcal{O}_H$. We are allowing $\mathcal{O}_H$ and $\mathcal{O}_L$ to interact gravitationally, but that just means that $\mathcal{O}_H$ will create a gravitational field, and then the AdS dual of $\mathcal{O}_L$ will move in that non-trivial field. In other words, we can compute the correlator by solving for the gravitational field created by $\mathcal{O}_H$, and then solving for the 2-pt function of $\mathcal{O}_L$ in that perturbed geometry.

There are many strategies that we can use for the second step, where we compute $\mathcal{O}_L(1)\mathcal{O}_L(z)$ in the gravitational background. One method would be to quantize a field $\phi_L(X)$ in the appropriate geometry, compute its 2-pt correlator, and then extrapolate to the boundary. But since we are treating ϕ_L as a free field, its 2-pt correlator is simply the solution to a differential equation

$$\nabla^2 K(X,Y) = \delta^{d+1}(X-Y) \tag{8.70}$$

So a natural method is to just solve this equation in the appropriate geometry, taking care of the boundary conditions. Then the correlator will just be (we re-labeled $\mathcal{O}_L \to \mathcal{O}$)

$$\langle \mathcal{O}_H(\infty)\mathcal{O}(1)\mathcal{O}(z)\mathcal{O}_H(0)\rangle = \lim_{\epsilon\to 0} \frac{K(z,\rho_X(\epsilon);1,\rho_Y(\epsilon))}{\epsilon^{2\Delta}} \tag{8.71}$$

Another method follows from the fact that we are neither creating nor destroying ϕ particles, so we can use a first quantized action for ϕ. As we discussed for the pure AdS case, the first quantized action is just $m\int d\tau$, the mass times the proper time in the particle's frame. In the semi-classical limit that $\Delta_L \sim (mR_{AdS}) \gg 1$ the correlator will be dominated by a saddle point, giving

$$\langle \mathcal{O}_H(\infty)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle = e^{-mS[z,1]} \tag{8.72}$$

where $S[z,1]$ is the length of a geodesic stretching through the bulk from the boundary points z and 1. Thus the computation of the correlator can be reduced to that of a geodesic in the background created by $\mathcal{O}_H$.

Let's proceed with the calculation; we will first find the background, and then we will just compute $K(X,Y)$ by using the method of images. The state $|\mathcal{O}_H(0)\rangle$ has an AdS wavefunction centered on the origin. It will give rise to a spherically symmetric vacuum solution of Einstein's equations, the AdS-Schwarzschild solution. In AdS_3 this is simply

$$ds^2 = -\left(r^2+1-\mu\right)dt^2 + \frac{dr^2}{r^2+1-\mu} + r^2 d\phi^2 \tag{8.73}$$

where the case $\mu = 0$ corresponds to the AdS vacuum. We can write

$$\mu = 8G_N M_H = 12\frac{\Delta_H}{c} = \frac{24h_H}{c} \tag{8.74}$$

where c is the central charge of the dual 2d CFT and Δ_H is the dimension of $\mathcal{O}_H$. We will discuss central charges like c and their relationship with gravity in later sections; for now it's enough to know that c sets the strength of AdS_3 gravity via $c = \frac{3}{2G_N}$.

Note that for $\mu > 1$ the geometry changes dramatically, as there is an event horizon at $r^2 = \mu - 1 \equiv r_+^2$. This geometry represents a BTZ black hole. In the case $\mu < 1$ there is a deficit angle in ϕ; this can be seen by studying the spatial geometry in the vicinity of $r = 0$, where there is a conical singularity.

Now we need to compute the correlators in this background. It turns out that locally the BTZ metric is just AdS_3; it only differs from BTZ by global identifications. This means that it can be computed via the method of images. Furthermore, the metric and the relevant results are analytic functions r_+ or equivalently, μ. For $\mu < 1$ we can re-write the metric as

$$ds^2 = -(1-\mu)\left(r^2+1\right)dt^2 + \frac{dr^2}{r^2+1} + (1-\mu)r^2 d\phi^2 \tag{8.75}$$

which makes it manifest that the deficit angle is $\sqrt{1-\mu}$. Thus we can convert from AdS_3 to the deficit angle background by using the *method of images*. This works just as in electromagnetism –

we take the free field propagator and add shifted copies of it to itself in order to guarantee that we have smooth boundary conditions.

In fact, we can write the entire BTZ geometry in terms of the hyperoloid coordinates $X_A X^A = 1$, by making the identification

$$X_3 \pm X_2 \sim e^{2\pi r_+}(X_3 \pm X_2) \tag{8.76}$$

where we identify our friendly X_A with the usual coordinates via

$$\begin{aligned} X_3 \pm X_2 &= \frac{r}{r_+} e^{\pm r_+ \phi} \\ X_1 \pm X_0 &= \frac{\sqrt{r^2 - r+^2}}{r_+} e^{\pm r_+ t} \end{aligned} \tag{8.77}$$

The limit of pure AdS_3 is just $r_+ = i$. Deficit angle metrics have $0 < -ir_+ < 1$ while real r_+ corresponds to BTZ. So the propagators are obtained by adding images in order to ensure continuity across these identifications. The result for the bulk-to-boundary correlator (from hep-th/0212277) is

$$K(X, b) = \sum_{n=-\infty}^{\infty} \frac{1}{\left(-\frac{\sqrt{r^2 - r_+^2}}{r_+} \cosh(r_+ \delta t) + \frac{r}{r_+} \cosh(r_+(\delta\phi + 2\pi n))\right)^{\Delta}} \tag{8.78}$$

where X is in the bulk and b is a boundary point, and the sum on n fixes the boundary conditions.

We want to compute the CFT 2-pt correlator, which comes from the $r \to \infty$ limit of this bulk correlator, after a rescaling by r^{2h}. In fact, that's not quite what we want, because we would like to study the correlator *on the plane*, not on the cylinder, and so we also need to multiply by an overall factor of $e^{-\Delta\tau}$. The result is

$$\langle \mathcal{O}(\tau, \phi)\mathcal{O}(0, 0)\rangle_{BTZ} = \sum_{n=-\infty}^{\infty} \frac{e^{\Delta\tau}}{(\cosh(r_+(\phi + 2\pi n)) - \cos(r_+\tau))^{\Delta}} \tag{8.79}$$

where we switched to $\tau = it$, the Euclidean time. We also assumed that the operator is a scalar so that $h = \bar{h} = \Delta/2$. Now we see that the sum over images simply guarantees that the correlator will be periodic in ϕ. You should also note that the Hawking temperature is

$$T_H = \frac{r_+}{2\pi} = \frac{\sqrt{\frac{24h_H}{c} - 1}}{2\pi} \tag{8.80}$$

as can seen via the periodicity in Euclidean time.

We would like to expand this correlator in the $\mathcal{O}(z)\mathcal{O}(0)$ OPE channel in order to understand which operators are being exchanged between $\mathcal{O}$ and the heavy operator that has created the black hole. For this purpose it's useful for write

$$\langle \mathcal{O}(\tau, \phi)\mathcal{O}(0, 0)\rangle_{BTZ} = \sum_{n=-\infty}^{\infty} \frac{e^{\Delta\tau}}{(2\sin(\pi T_H(\tau + i\phi + 2\pi in))\sin(\pi T_H(\tau - i\phi - 2\pi in)))^{\Delta}} \tag{8.81}$$

Now we can write

$$1 - z = e^{-\tau + i\phi}, \quad 1 - \bar{z} = e^{-\tau - i\phi} \tag{8.82}$$

to translate this into $z, \bar{z}$ coordinates, where we used $1 - z$ so that $z \to 0$ corresponds with the OPE limit. Note that now each term has an independent z and $\bar{z}$ dependence, so to understand the OPE it suffices to look at

$$\frac{e^{-h \log(1-z)}}{\left[\sin\left(\pi T_H (\log(1-z) + 2\pi i n)\right)\right]^{2h}} \tag{8.83}$$

where for convenience we write $\Delta = 2h$. The cases where $n = 0$ and $n \neq 0$ are very different. The series expansions are

$$\frac{1}{(\pi T_H z)^{2h}} \left(1 + \frac{h}{12}\left(4\pi^2 T_H^2 + 1\right) z^2 + \cdots\right) \tag{8.84}$$

for the $n = 0$ case versus

$$\frac{1}{\left(\sin^2\left(2\pi^2 n i T_H\right)\right)^h} \left(1 + hz - 4\pi h z T_H \cot\left(2\pi^2 i n T_H\right) + \cdots\right) \tag{8.85}$$

for the $n \neq 0$ case. Note that final term cancels between n and $-n$ when we sum, so that we would be left with $1 + hz + \cdots$ as an overall coefficient.

We see that there is a clear physical distinction between the two cases. When $n = 0$, we have a singularity $\frac{1}{z^\Delta}$, signalling the presence of the $\mathbf{1}$ operator in the $\mathcal{O}(z)\mathcal{O}(0)$ OPE. The first correction, at order z, vanishes, because identity does not have any global conformal descendants, and the probe operator $\mathcal{O}$ and the black hole do not exchange any fields of dimension 1. However, there is a contribution at order z^2, which corresponds to the exchange of *one graviton*, ie the stress tensor quasi-primary state $T(0)|0\rangle$, between the black hole and the light probe. As expected, the graviton has dimension 2 in a CFT_2. Also note that

$$\frac{h}{12}\left(4\pi^2 T_H^2 + 1\right) = \frac{2 h h_H}{c} \tag{8.86}$$

which is what we would expect from the various OPEs – the graviton couples to the product of h and h_H, divided by $c/2$. Further multigraviton states can be seen if we expand the $n = 0$ case to higher orders.

In contrast, the $n \neq 0$ terms begin with a constant – they are not singular in the OPE limit. This means that they involve the exchange of operators of dimension 2Δ. These are precisely the 'double-trace' primary operators $\mathcal{O}^2$ that we discovered when we studied generalized free theories. These operators have descendants, and so we have corrections at every integer order z^k. In particular, notice that ratio of coefficients between the order z term and the order 1 term is h, which is exactly what we should expect from the ratio of first descendant and primary for an operator with holomorphic dimensio $2h$, ie for $\mathcal{O}^2$.

Furthermore, if we looked at higher orders we could also identify other operators corresponding to 2-particle states in AdS_3, such as $\mathcal{O}\partial^2\mathcal{O}$ etc. Matching to the (global) conformal block expansion in this channel would allow us to identify the OPE coefficients of general $\mathcal{O}\partial^k\mathcal{O}$ primary operators with the operator $\mathcal{O}_H$ that creates and destroys the black hole. We would also find the contributions from operators like $\mathcal{O}\mathcal{O}T$, $\mathcal{O}\mathcal{O}TT$, $\mathcal{O}\partial^r\mathcal{O}T\partial^s T$, etc involving two $\mathcal{O}$ particles accompanied by any number of gravitons.

9 On the Existence of Local Currents J_μ and $T_{\mu\nu}$

When we first learn QFT (or perhaps just classical mechanics), one of the earliest lessons is the beautiful and important Noether's theorem, which connects symmetries to conservation laws. When we have a Lagrangian description, Noether's theorem allows us to connect three different ideas:

1. the existence of a symmetry of the theory
2. the existence of a local current $J_\mu(x)$ (so called because it depends on a point in spacetime, and because we usually implicitly assume that it is insensitive to very distant physics) that is conserved, so that $\partial^\mu J_\mu(x) = 0$
3. the existence of a charge (no spacetime dependence) Q that measures the conserved quantity associated with the symmetry; if J_μ and Q both exist then we can usually write $Q = \int d^3x J^0(x)$

Global symmetry currents J_μ and the stress-energy tensor $T_{\mu\nu}$ form an extremely important class of operators in CFTs. As we mentioned above, conventionally CFTs are *defined* to have a conserved $T_{\mu\nu}$, although one can study 'CTs' or 'almost-CFTs' that have all the other features of a CFT. So let us discuss these operators in more detail in order to understand their physical properties and to see why they play such an important role. In particular, we will discuss

1. The difference between a simple conserved charge Q versus a local symmetry current. It is possible to have J_μ without Q or a charge Q without a corresponding J_μ.
2. How local currents such as $J_\mu(x)$ and $T_{\mu\nu}(x)$ generate symmetry transformations that produce new states, thereby telling us something about the Hilbert space of the theory.
3. The fact that when we *gauge* local symmetries, we identify all these different states, thereby 'mod-ing out' by the symmetry. Gauge symmetries are really redundancies of the description.

Although we will discuss physics quantum mechanically in this section, all of our statements also pertain to the classical phase space of classical theories, and in fact might be more natural and elementary in that context.

9.1 An $SU(2)$ Example

To understand these issues very concretely, let's study a very simple free theory of a pair of complex bosons, (Φ_1, Φ_2) with an $SU(2)$ symmetry. The action is just

$$S = \int d^d x \sum_{i=1}^{2} \left(\partial_\mu \Phi_i^\dagger \partial^\mu \Phi_i - m^2 \Phi_i^\dagger \Phi_i \right) \tag{9.1}$$

where $i = 1, 2$. The local conserved current associated to this $SU(2)$ global symmetry is the operator

$$J_a^\mu(x) = i\sigma_a^{ij} \left(\Phi_i^\dagger(x) \partial^\mu \Phi_j(x) - \partial^\mu \Phi_i^\dagger(x) \Phi_j(x) \right) \tag{9.2}$$

where σ_a^{ij} are the $SU(2)$ generators (Pauli matrices). Let's check that it is conserved

$$\begin{aligned} \partial_\mu J_a^\mu(x) &= i\sigma_a^{ij} \left(\partial_\mu \Phi_i^\dagger(x) \partial^\mu \Phi_j(x) + \Phi_i^\dagger(x) \partial_\mu \partial^\mu \Phi_j(x) - \partial_\mu \partial^\mu \Phi_i^\dagger(x) \Phi_j(x) - \partial^\mu \Phi_i^\dagger(x) \partial_\mu \Phi_j(x) \right) \\ &= i\sigma_a^{ij} m^2 (\Phi_i^\dagger(x) \Phi_j(x) - \Phi_i^\dagger(x) \Phi_j(x)) = 0 \end{aligned} \tag{9.3}$$

Note that the momentum conjugate to Φ_i is

$$\Pi_i(x) \equiv \frac{\delta \mathcal{L}}{\delta \dot{\Phi}_i} = \dot{\Phi}_i^\dagger \tag{9.4}$$

so that the time component of the current can also be written as

$$J_a^0(x) = i\sigma_a^{ij} \left(\Phi_i^\dagger(x) \Pi_j^\dagger(x) - \Pi_i(x) \Phi_j(x) \right) \tag{9.5}$$

This means that the commutators of $J_a^0(x)$ can be computed easily using the canonical commutation relations, giving

$$[J_a^0(t, x), \Phi^i(t, y)] = i\delta^{d-1}(\vec{x} - \vec{y}) \sigma_a^{ij} \Phi_j(t, x) \tag{9.6}$$

so as we would expect, the current locally rotates the Φ_i field in $SU(2)$ space. We can measure a total charge using the integrated quantity

$$Q_a = \int d^{d-1} x J_a^0(x) \tag{9.7}$$

and note that it is time-independent using the conservation of J_a^μ and the vanishing of fields at infinity.

Although Q_a can be used to measure the total charge (in this case, an $SU(2)$ representation) associated with a given state, it always adds up the contribution of the entire universe. This severely limits the utility of Q_a by itself.

However, the local current J_a^μ is much more powerful, because we can use it to turn one state into another. The point is that we can smear J^0 over a small region surrounding some isolated sub-system, so that it only transforms that sub-system, but leaves the rest of the universe alone. When you pick up an object and spin it around, you are (in some sense) taking advantage of the existence of local translation and rotation currents. We will come back to this point directly below when we discuss $T_{\mu\nu}$, but for now let us stick with our $SU(2)$ current.

For example, say we have a state $|\Psi_K\rangle$composed of K well-separated Φ_1 particles, located around $x_1, x_2, \cdots x_K$. Then we can immediately construct an $SU(2)$ matrix operator R_a via

$$R_{ij}(t) = \exp\left[i\frac{\pi}{2}\int_{B_i\supset x_i} d^{d-1}y J_1^0(t,y)\sigma^1\right] \tag{9.8}$$

where the region B_i is some compact ball containing only the ith particle, and σ_1 is the 1th or xth Pauli matrix (chosen as a random eaxmple). Note that $[R_{ij}(t), R_{kl}(t)] = 0$ since they are compactly supported away from each other. Acting with R_{ij} on the state $|\psi_K\rangle$ will take the ith particle from $(1,0)$ to $(0,1)$, but leave the others alone. In other words, out of the state $|\Psi_K\rangle$ with all the bosons in the $(1,0)$ configuration, we can generate all $SU(2)$ configurations for the isolated bosons! This obviously would not be possible with only the global charge Q_a. Note that if we had only studied an abelian current then all we could do is multiply the state with a phase, which is rather boring – this is why we chose an $SU(2)$ current as our example.

An important feature of R_{ij} is that it takes eigenstates of the Hamiltonian to other eigenstates. In other words, it guarantees that if a particle in the $(1,0)$ state is an eigenstate, then $(0,1)$ is an eigenstate as well. The conservation of J_a^μ is crucial for this, because it means that the time derivative of the R_i is

$$\begin{aligned}
\partial_t R_{ij} &= R_{ij}\frac{i\pi}{2}\int_{B_i\supset x_i} d^{d-1}y\partial_t J_1^0(y)\sigma^1 \\
&= R_{ij}\frac{i\pi}{2}\int_{B_i\supset x_i} d^{d-1}y\partial_j J_1^j(y)\sigma^1 \\
&= R_{ij}\frac{i\pi}{2}\int_{\partial B_i} d^{d-2}s\,\hat{n}_j J_1^j(s)\sigma^1
\end{aligned} \tag{9.9}$$

and this last quantity vanishes when it acts on a state where the particles are well-separated, because the current will vanish on the surface of the ball B_i. This means that operators like R_{ij} are the exponential of local charges that are conserved in states where the particles are well-separated.

9.2 A Very Special Current – $T_{\mu\nu}$

The conserved energy-momentum tensor $T_{\mu\nu}$ plays a special role because it generates the spacetime symmetry transformations, including the conformal transformations in CFTs.

To understand more concretely what we can do with $T_{\mu\nu}(x)$, let's revisit the ultra-familiar symmetry of rotations in the case of a free QFT. If we take a free ϕ field in $2+1$ dimensions, we have the action

$$S = \int dt dr r d\theta \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right) \tag{9.10}$$

In these specially chosen coordinates, rotations are just $\phi \to \phi + \epsilon\partial_\theta\phi$, and we find that the 0 component of the current is $T_\theta^0 = r\dot{\phi}\partial_\theta\phi$. Note that since the canonical conjugate to ϕ is $\pi(x) = \dot{\phi}(x)$, we can also write this as

$$T_\theta^0(t,x) = r\partial_\theta\phi(t,x)\pi(t,x) \tag{9.11}$$

so that it's clear what the commutation relations of $T_\theta^0(x)$ are with $\phi(y)$, namely

$$[T_\theta^0(t,x), \phi(t,y)] = i\delta^2(x-y)\,[r\partial_\theta\phi(t,x)] \tag{9.12}$$

so clearly T_θ^0 generates rotations, as desired. The global charge is

$$Q = \int drd\theta\, r\dot{\phi}\partial_\theta\phi \tag{9.13}$$

However, this charge simply rotates the entire universe. So it isn't very useful physically – we cannot observe that we have rotated the entire universe.

Fortunately, we can make a local version using T_θ^μ. Using a radius dependent function $\eta(r)$, we can now consider the rotation operator

$$R[\eta] = \int drd\theta\, \eta(r)(r\dot{\phi}\partial_\theta\phi) \tag{9.14}$$

As an example, we might have a gas of particles well-inside some sphere of radius R, along with some far away gas. So if we choose $\eta(r) = \Theta(R-r)$ then we have a rotation that turns off at R – we are just rotating the region inside the sphere while holding the outside universe constant.

We would now like to check that by performing this transformation we have not changed the energy of our state. Since $\partial_\mu J^\mu = 0$, we see that

$$\begin{aligned}
\partial_t R[\eta] = i\,[H, Q[\eta]] &= \int drd\theta\, \eta(r)\partial_i J^i(r,\theta) \\
&= \int drd\theta\, [\partial_r\eta(r)]\, J^r(r,\theta) \\
&= \int d\theta\, J^r(R,\theta)
\end{aligned} \tag{9.15}$$

where the last line follows because the r-derivative of $\eta(r)$ is simply a delta function. But this result makes a lot of sense – we see that $R[\eta]$ is time independent, and therefore commutes with the Hamiltonian, as long as we are working in a state where J^r doesn't see any particles at the surface of the sphere. So we can rotate everything inside a ball separately from the rest of the universe and obtain an equivalent but new state as long as there aren't any particles on the surface of the ball.

So this is just a spacetime version of the $SU(2)$ operator we defined in equation (9.8). Instead of performing an internal $SU(2)$ rotation on everything within a ball, it performs a spacetime rotation on the contents of the ball, creating a physically distinct state in the Hilbert space.

9.3 Separating Q and J_μ – Spontaneous Breaking and Gauging

We mentioned above that there are situations where a current J_μ exists but Q does not. This can happen when the integral defining Q is not well-defined, because it doesn't converge in infinite volume. A very physically important example of this phenomenon is when the symmetry is *spontaneously broken*. In that case the current J^μ still exists and is conserved, so $\partial_\mu J^\mu = 0$, however J_μ does not annihilate the vacuum:

$$J_\mu(x)|0\rangle = |\pi_{J_\mu(x)}\rangle \neq 0 \tag{9.16}$$

This makes it obvious that if we try to integrate over all of (an infinite) spacetime, the result will be ill-defined, assuming we have not also broken translational symmetry. This state $|\pi_{J_\mu(x)}\rangle$ is a one-goldstone boson state. A more physical state is

$$\frac{p_\mu}{F_\pi\sqrt{2p_0}}|\pi_p\rangle = \int d^{d-1}x\, e^{ip\cdot x} J_\mu(x)|0\rangle \tag{9.17}$$

and up to a normalizaiton, this is just a state consisting of a single goldstone boson with momentum $\vec{p}$. The non-existence of Q corresponds with the fact that a goldstone boson state with $\vec{p} = 0$ is not normalizable. Note that current conservation implies that

$$\frac{-ip^\mu p_\mu}{F_\pi}|\pi_p\rangle = \int d^{d-1}x\, e^{ip\cdot x}\partial^\mu J_\mu(x)|0\rangle = 0 \tag{9.18}$$

so that we must have

$$p^2 = 0 \tag{9.19}$$

for a one-goldstone boson state. In other words, we learn that goldstone bosons must be massless. This basically shows that $J_\mu(x)$ acts on the vacuum to create a one-particle state, so goldstone bosons really are single particles. One can read more about all of this in Weinberg V2 [13].

There are also situations where Q exists but J_μ does not. The example that is most relevant for these notes is the 'CT' dual[22] of non-gravitational $d + 1$ dimensional field theories in AdS. In these d-dimensional boundary theories we have global symmetry charges associated with the full conformal group, enlarging the Poincaré generators, but these do not arise from the integral of a local current (these local currents would be made out of $T_{\mu\nu}$ if it existed, but it does not).

An alternate way to have a charge Q without J_μ is to 'gauge' the symmetry corresponding to J_μ. This means that we *identify* all of the states related by the action of J_μ, turning the *symmetry into a redundancy* and eliminating many degrees of freedom. To be precise, in a gauge theory the various states related by the action of operators such as the R_i of equation (9.8) would all be viewed as the same state. As an example where degrees of freedom really are eliminated from a low-energy theory, consider a complex field

$$\phi(x) = (v + h(x))e^{i\theta(x)} \tag{9.20}$$

[22] A theory with conformal symmetry but not a 'Field Theory'.

If $v \neq 0$ then $\theta(x)$ is a true goldstone boson degree of freedom. However, if we gauge the symmetry under which $\theta(x) \to \theta(x) + \epsilon$, then we elimnate this degree of freedom, as it is 'eaten' by the abelian gauge field $A_\mu(x)$.

In the case of gravity, when we gauge spacetime symmetries we remove local spacetime observables from the exact quantum mechanical theory; the only well-defined observables are 'holographic' – they are associated with infinity. Let us be a bit more precise about this. The energy-momentum tensor $T_{\mu\nu}$ is really the current for the d different charges P_μ, which correspond to translations (not rotations, or dilatations and special conformal transformations – these are associated with currents like $x^\nu T_{\mu\nu}$). When we place a theory with energy-momentum tensor $T_{\mu\nu}$ in a dynamical spacetime, we are therefore gauging, or redundantly identifying, the spacetime translations. Local translations occur when one object in spacetime moves with respect to another object. In general relativity, distances between objects are gauged, although obviously just as with spin one gauge theories, the gravitational field also changes under these gauge transformations (diffeomorphisms).

These ideas see a concrete realization when one studies membranes (or actually just particles) in spacetime. Particles, strings, and membranes can all be thought of as objects that spontaneously break translation symmetry! If we have a $d-1$ dimensional membrane in d-dimensional spacetime, then we can parameterize it using a goldstone boson field

$$\pi_d(t, x_1, \cdots, x_{d-1}) \tag{9.21}$$

that tells us that the spacetime position of the membrane is $(t, x_1, \cdots, x_{d-1}, \pi_d)$. There is a universal low-energy action for π_μ called the DBI action

$$T \int d^{d-1}x \sqrt{1 - \partial_\mu \pi_d \partial^\mu \pi_d} \tag{9.22}$$

which just measures the world-volume of the membrane. The parameter T is the tension of the membrane. But the interesting point for us is that when we turn on gravity, thereby 'gauging' the translation symmetry, the goldstone boson π_d gets 'eaten' by the gravitational field, just like a standard goldstone boson in a spin one gauge theory. How do we see this? Actually, it's trivial – we can always just choose a spacetime coordinate system built on top of the membrane so that the membrane is at $\pi_d = 0$ by definition. With this gauge choice, obviously the membrane fluctations have been absorbed into fluctuations of the spacetime metric!

These statements are also directly related to the well-known Weinberg-Witten theorem [15], which says that theories with massless spin-1 particles cannot have an associated conserved current, and that theories with massless spin 2 particles cannot have a conserved stress-energy tensor transforming appropriately under spacetime symmetries. The point with this theorem is that gauging the symmetry in order to include a spin one or spin two massless particle necssarily alters the Poincaré transformation properties of the associated current.

The statement that gauge theories have fewer degrees of freedom may seem counter-intuitive, because the gauge theories we usually study have more degrees of freedom – either additional gauge bosons or gravitons (viewing gravity as a gauging of the spacetime symmetries). Essentially, this is the case because we need to introduce a fluctuating gauge field A_μ or gravitational field $g_{\mu\nu}$ in order to identify states related by symmetry without sacrificing locality.

Finally, we should mention that in the case of theories such as QED and non-abelian gauge theories, the existence of Q without J_μ is fairly trivial (sometimes physicists simply neglect its existence entirely). The reason is that this global charge is conserved, so once the universe exists in an eigenstate of Q the eigenvalue can never change. Thus we can view universes with different values of Q as different super-selection sectors, or as different theories entirely – they can never 'talk to each other'. Also, in theories such as Yang-Mills or QCD where charge is confined, a non-zero charged state in an infinite spacetime would have infinite energy, so if we are only interested in finite energy states, then we must always have $Q = 0$, rendering Q completely trivial.

10 Universal Long-Range Forces in AdS and CFT Currents

The central point of this section is that *spin one gauge fields A_a in AdS are dual to spin one conserved currents J_μ in the CFT, while the spin two graviton in AdS is dual to the spin two energy momentum tensor $T_{\mu\nu}$ of the CFT.* So we will begin by explaining how this correspondence works. Then our major theme will be that the universality of the interactions of gauge bosons and gravitons in AdS translates into the universality of the correlators of J_μ and $T_{\mu\nu}$ in the CFT, due to Ward identities.

Some of the ideas from the previous section can be stated more formally in terms of Ward identities for conserved currents in CFTs. We will derive these CFT identities and then we will explain how they relate to the universality of long-range forces in AdS.

10.1 CFT Currents and Higher Spin Fields in AdS

Before we discuss symmetries and dynamics, let us note a few kinematic features of conserved currents in CFTs and massless particles in AdS. The basic point will be to note how the counting of degrees of freedom match between AdS and the CFT. This is essentially a quick version of the analysis we went through in the first several sections of these notes for a scalar particle, where we showed that one-particle scalar states in AdS correspond to the states created by $\mathcal{O}(0)$ acting on the AdS/CFT vacuum. Here the point is that $J_\mu(0)|0\rangle$ corresponds to a vector particle in AdS in its ground state, while $T_{\mu\nu}(0)|0\rangle$ corresponds to a spin two particle in AdS in the ground state. Conservation of the currents corresponds to massless particles in AdS, which have fewer degrees of freedom due to a gauge redundancy. If the AdS theory is free (corresponding to coupling $g = 0$ or $G_N = 0$) then we just have Gaussian correlators for J_μ or $T_{\mu\nu}$. Let us give a quick explanation of some of these statements.

Conserved currents satisfy

$$\partial^\nu J_{\nu\mu_2\cdots\mu_\ell} = 0 \quad \text{or} \quad [P^\nu, J_{\nu\mu_2\cdots\mu_\ell}] = 0 \tag{10.1}$$

We will assume that these are symmetric traceless tensors. At the level of the CFT states we have

$$P^\nu J_{\nu\mu_2\cdots\mu_\ell}|0\rangle = 0 \tag{10.2}$$

In other words, this descendant state and all others that normally would have followed from further applications of P^μ vanish. This means that a large number of states that would have been present for a current that wasn't conserved have been eliminated from the Hilbert space of the theory.

For a spin 1 current J_μ, we have $d-1$ independent primary DoFs, where we are comparing to the case of a scalar operator with 1 DoF. For a traceless spin 2 current $T_{\mu\nu}$, we naively have $d(d+1)/2-1$ degrees of freedom, but another d are eliminated by the conservation condition, leaving us with only $d(d-1)/2-1$ DoFs. Similar counts can be applied to higher spin currents.

A simple calculation determines that $\Delta = d-2+\ell$ for conserved currents. This is the CFT scaling dimension of a massless spin ℓ field in AdS. Our prior derivation of the unitarity bound proves that if $\Delta = d-2+\ell$ then the current must be conserved. For the converse, note that since J is primary we have

$$\begin{aligned} 0 &= [K_\alpha, [P_\beta, J_{\mu_1\cdots\mu_\ell}]] + [J_{\mu_1\cdots\mu_\ell}, [K_\alpha, P_\beta]] \\ &= [K_\alpha, [P_\beta, J_{\mu_1\cdots\mu_\ell}]] + 2\Delta_J J_{\mu_1\cdots\mu_\ell}\eta_{\alpha\beta} + 2\sum_i \eta_{\alpha\mu_i} J_{\mu_1\cdots\beta\cdots\mu_\ell} - \eta_{\beta\mu_i} J_{\mu_1\cdots\alpha\cdots\mu_\ell} \end{aligned} \tag{10.3}$$

Now if we take the trace via $\eta^{\beta\mu_1}$ then the first commutator term vanishes by conservation, giving

$$0 = \Delta_J J_{\alpha\mu_2\cdots\mu_\ell} - (d+\ell-2) J_{\alpha\mu_2\cdots\mu_\ell} \tag{10.4}$$

where the -2 comes from one missing term when $i=1$ and a single non-vanishing contribution from the first term in the summand, also when $i=1$. So we have shown that $\Delta_J = d-2+\ell$.

The point of these observations is that they agree with the DoF counting for massless fields with spin – specifically, a massless gauge boson A_A in AdS$_{d+1}$ containts the same degrees of freedom as a conserved current J_μ in a CFT$_d$, and a massless graviton h_{AB} in AdS$_{d+1}$ has the same degrees of freedom as the stress-energy tensor $T_{\mu\nu}$ in a CFT$_d$. In the case of gravitons and gauge bosons, the extra degrees of freedom between the massive and massless case are eliminated by the gauge redundancy of either a gauge theory with Lie Group symmetry or the diffeomorphism redundancy of General Relativity. So to summarize, *conserved spin ℓ CFT currents are dual to massless AdS fields of spin ℓ whose description necessarily involves extra redundancy in order to eliminate unphysical degrees of freedom.*

Now let us see more explicitly how the mapping works for a simple example. Chapter 7 of Raman's notes [8] also give a nice discussion. First consider a free massive spin one boson with action

$$S = -\int_{AdS} d^{d+1}X\sqrt{-g}\left(\frac{1}{4}F_{AB}F^{AB} + \frac{1}{2}m^2 A_B A^B\right) \tag{10.5}$$

where $F_{AB} = \nabla_A A_B - \nabla_B A_A$ as usual, and indices are raised and lower with the AdS metric. This theory does not have a gauge symmetry. Alternatively, one can view this as a gauge theory that has been higgsed and written in unitarity gauge.

However, as in any theory where the kinetic term comes from F^2, the component A_0 does not have a kinetic term (there is no term of the form $\partial_t A_0$ anywhere in the action). This means that A_0 is non-dynamical, so we can use its equations of motion to eliminate it, after which point we will have $d+1-1=d$ separate degrees of freedom. Concretely, the A_B field can create d distinct states corresponding to a single particle at rest; these are the d states created by a general current $J_\mu(0)$ in the CFT when it acts on the CFT vacuum.

The equation of motion for A_B is

$$\nabla^M F_{MN} - m^2 A_N = 0 \tag{10.6}$$

as usual. In the Poincaré patch coordinates for e.g. AdS_5 from the equations of motion we obtain

$$z^3 \partial_z \left(\frac{A_z}{z^3} \right) = \partial_\mu A^\mu \tag{10.7}$$

so that if we wish we can express the A_z component in terms of the others, instead of eliminating the A_0 component as is conventional. Explicitly, we have

$$A_z(z, x) = z^3 \int_0^z dw \frac{\partial_\mu A^\mu(x, w)}{w^3} \tag{10.8}$$

The utility of this method is that we can write

$$\lim_{z \to 0} \frac{A_\mu(z, x)}{z^{\Delta - 1}} = J_\mu(x) \tag{10.9}$$

and so these components become the (not conserved) CFT current J_μ of dimension Δ. Note that to get a finite result we scale A_μ by its twist $\tau = \Delta - \ell$, not its dimension, as we approach the boundary. One can obtain similar results for massive tensor fields. But we also see what happened to A_z. We have

$$\lim_{z \to 0} \frac{A_z(z, x_\mu)}{z^\Delta} = \lim_{z \to 0} \frac{\partial^\mu A_\mu(z, x_\mu)}{z^{\Delta - 1}} = \partial^\nu J_\nu(x_\mu) \tag{10.10}$$

A_z is a redundant degree of freedom which becomes the (also redundant) descendant operator $\partial^\nu J_\nu$ in the CFT.

Now let us consider the case of a massless gauge boson in AdS. In this case the action has the gauge redundancy under

$$A_M(X) \to A_M(X) + \nabla_M \Lambda(X) \tag{10.11}$$

for any scalar function $\Lambda(X)$ on AdS. However, the field strength F_{MN} is gauge invariant in the abelian case, and it transforms as an ordinary adjoint field in the non-abelian case. Thus we can use its components $F_{\mu z}$ to directly define

$$J_\mu(x) = \lim_{z \to 0} \frac{F_{\mu z}(z, x)}{z^{d-3}} \tag{10.12}$$

Note that here we have that $\tau = \Delta - \ell = d - 2$, and we removed an additional power of z from the denominator in this definition because of the derivatives in $F_{\mu z}$.

Another natural way to obtain this result is to fix the gauge $A_z(z,x) = 0$. There is a residual gauge symmetry – we can shift A_μ by $\partial_\mu \Lambda(x)$ independent of z – and so we can also take $\partial_\mu A^\mu = 0$ in the directions orthogonal to z. This immediately implies that if we take

$$\begin{aligned} J_\mu(x) &= \lim_{z\to 0} \frac{F_{\mu z}(z,x)}{z^{d-3}} \\ &= \frac{\partial_z A_\mu(z,x)}{z^{d-3}} \\ &= \frac{A_\mu(z,x)}{z^{d-2}} \end{aligned} \tag{10.13}$$

and furthermore that we will have $\partial_\mu J^\mu(x) = 0$ from our residual gauge condition, so $J_\mu(x)$ will be a conserved current, as expected.

A similar analysis applies to massive and massless linearized gravitational fields h_{MN} in AdS. See Raman's notes [8] for the details.

10.2 CFT Ward Identities and Universal OPE Coefficients

Ward identities are a formal consequence of symmetries as applied to correlation functions in QFTs. They play an especially important role in CFTs because they give us a lot of information about the interactions of currents J_μ and especially the energy momentum tensor $T_{\mu\nu}$ with other operators in the theory. In particular, Ward identities for the CFT currents

$$\langle J_\mu(y)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \quad \text{and} \quad \langle T_{\mu\nu}(y)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \tag{10.14}$$

directly relate to the universality of gauge and gravitational forces in AdS.

The Ward identity states that

$$\frac{\partial}{\partial y^\mu}\langle J^\mu(y)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = -i\sum_{m=1}^{n}\delta^d(y-x_m)\langle\mathcal{O}_1(x_1)\cdots[G_m\mathcal{O}_m(x_m)]\cdots\mathcal{O}_n(x_n)\rangle \tag{10.15}$$

where G_m enacts (represents in the group theory sense) the symmetry transformation corresponding to J_μ on a given operator $\mathcal{O}_m$. If $\mathcal{O}_m$ does not transform under this symmetry, then it does not contribute to the right hand side. Let us give the path integral derivation of this Ward identity for the conserved current $J_\mu(x)$.

To derive the identity, consider the path integral computation of a correlator

$$\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \int [\mathcal{D}\phi]\,\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{-S[\phi]} \tag{10.16}$$

where the operators $\mathcal{O}_i$ are some general operators in the theory with fields represented by ϕ. Now if we perform an infinitesimal symmetry transformation under which

$$\delta_\epsilon\mathcal{O}_i = -i\epsilon(x)G_i\mathcal{O}_i \tag{10.17}$$

then the action and the operators appearing within the path integral will both transform. By assumption, the action would be invariant if $\epsilon(x)$ were a constant, and the measure of integration is invariant, so we must have

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = \int [\mathcal{D}\phi]\,(1+\delta_\epsilon)\,(\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n))\, e^{-S[\phi]-\int d^dx\partial_\mu J^\mu(x)\epsilon(x)} \tag{10.18}$$

Expanding to first order in ϵ tells us that

$$\langle \delta_\epsilon\,(\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n))\rangle = \int d^dx\,\epsilon(x)\partial_\mu\langle J^\mu(x)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \tag{10.19}$$

In order for this identity to hold for any infinitesimal function $\epsilon(x)$, we must have the Ward identity of equation (10.15).

If it exists, the conserved charge associated with J_μ is

$$Q = \int d^{d-1}x J^0(t,x) \tag{10.20}$$

This charge will be time-independent because $\partial_\mu J^\mu = 0$. We can also use the Ward identity to give a formal proof that

$$[Q,\mathcal{O}_m] = -iG_m\mathcal{O}_m \tag{10.21}$$

for Lorentzian correlators (where it is more obvious what 'time' means). For this purpose let $\mathcal{O}_m(t_m,x_m)$ have t_m distinct from the other t_i in any n-pt correlator. Now let us take the Ward identity and integrate over the spatial y and between times $t_y \pm \epsilon$ for an infinitesimal ϵ. Let us choose $m=1$ for convenience. This gives

$$\langle[Q(t_1+\epsilon)\mathcal{O}_1(x_1)]\cdots\mathcal{O}_n(x_n)\rangle - \langle[Q(t_1-\epsilon)\mathcal{O}_1(x_1)]\cdots\mathcal{O}_n(x_n)\rangle = -i\langle G_1\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \tag{10.22}$$

Since these are Lorentzian correlators, analytically continued from Euclidean spacetime, they will be time ordered. Since the other operators appearing in the correlator were completely arbitrary, this identity can be interpreted as equation (10.21).

Our proof of the Ward identity relied on a path integral for the CFT. One can also derive the Ward identity more directly from the CFT axioms [16, 17] by expanding the OPE of $J_\mu(x)\mathcal{O}_i(0)$ and constraining the possible terms that appear using conformal invariance.

Now let us consider the Ward identity for the correlator between an abelian spin one current and any number of operators. If we integrate the identity over all y, we find that the LHS vanishes, and so we must have that

$$\sum_i q_i = 0 \tag{10.23}$$

for the correlator to be non-vanishing, where $G_i = q_i$ for an abelian current. This also follows from the fact that the vacuum is uncharged, so $Q|0\rangle = 0$. If we apply the Ward identity to a non-vanishing 3-pt correlator with the current, we see

$$\partial_\mu\langle J^\mu(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle = -iq\left[\delta^d(y-x_1)\langle\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle - \delta^d(y-x_2)\langle\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle\right] \tag{10.24}$$

Recall that 3-pt functions such as $\langle J\mathcal{O}\mathcal{O}^\dagger\rangle$ are completely determined by conformal invariance up to some overall constant. Once the operator $\mathcal{O}$ is normalized, the Ward identity completely fixes this constant to be the charge q of $\mathcal{O}$. In other words, the Ward identity tells us that *conserved currents must have a 3-pt coupling proportional to charge*. Of course this 3-pt coupling also sets the OPE coefficient for the appearance of J_μ in the OPE of $\mathcal{O}(x)\mathcal{O}^\dagger(0)$. The Ward identity is the CFT version of the statement that massless gauge bosons must couple to a conserved charge.

Now let us consider the analogous statements for the energy momentum tensor $T_{\mu\nu}$. In this case we have the 'obvious' Ward identites associated with the conservation condition $\partial_\mu T^{\mu\nu} = 0$ which give information about translations. However, there are also various non-obvious identities associated with rotations, dilatations, and special conformal transformations. The first also follows from conservation of $T^{\mu\nu}$, but the latter are associated with the tracelessness condition $T^\mu_\mu = 0$. This is the conservation condition for the scale current

$$S_\mu(x) = x^\nu T_{\mu\nu} \tag{10.25}$$

We already know that D is the charge associated with the scale current, and so it acts as $[D, \mathcal{O}(x)] = -i(\Delta + x\cdot\partial_x)\mathcal{O}(x)$. Thus we see that

$$\begin{aligned}
\partial_\mu\langle S^\mu(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle &= \langle T^\mu_\mu(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle + \langle y_\nu\partial_\mu T^{\mu\nu}(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle \\
&= -i\Delta\left[\delta^d(y-x_1)\langle\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle + \delta^d(y-x_2)\langle\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle\right] \\
&\quad -i\left[\delta^d(y-x_1)x_1\cdot\partial_{x_1}\langle\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle + \delta^d(y-x_2)x_2\cdot\partial_{x_2}\langle\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle\right]
\end{aligned} \tag{10.26}$$

We see that the final term on the first line is equal to the terms on the last line – this identity is just the Ward identity for translations. This means that

$$\langle T^\mu_\mu(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle = -i\Delta\left[\delta^d(y-x_1)\langle\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle + \delta^d(y-x_2)\langle\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle\right] \tag{10.27}$$

This identity immediately tells us that the 3-pt correlation function $\langle T_{\mu\nu}(y)\mathcal{O}(x_1)\mathcal{O}^\dagger(x_2)\rangle$ has *a universal coefficient proportional to* Δ. Note that this 3-pt function exists for *all primary operators in the theory*, since $T_{\mu\nu}$ generates the conformal transformations on all operators/states. This contrasts with the abelian current J_μ we considered above, which only acts on the sector of charged states. The Ward identity we have derived is the CFT version of the statement that gravity couples universally to energy-momentum, whose role is played here by the dimension-charge Δ.

As a final comment, let us consider the 3-pt correlator

$$\langle T_{\mu_1\nu_1}(x_1)T_{\mu_2\nu_2}(x_2)T_{\mu_3\nu_3}(x_3)\rangle \tag{10.28}$$

In general d this correlator can involve various tensor structures, and it needs to be normalized. It is natural to normalize the 2-pt function of $T_{\mu\nu}$ via

$$\langle T_{\alpha\beta}(x)T_{\mu\nu}(0)\rangle = \frac{C_T}{x^{2d}} I_{\alpha\beta,\mu\nu}(x) \tag{10.29}$$

with

$$I_{\alpha\beta,\mu\nu}(x) \equiv \frac{1}{2}\left(I_{\alpha\mu}I_{\beta\nu} + I_{\alpha\nu}I_{\beta\mu}\right) - \frac{1}{d}\eta_{\alpha\beta}\eta_{\mu\nu} \quad \text{with} \quad I_{ab} \equiv \eta_{ab} - 2\frac{x_a x_b}{x^2} \tag{10.30}$$

A natural normalization of $T_{\mu\nu}$ occurs because various integrals of $T_{\mu\nu}$ give the conformal generators; e.g. P_μ which must act on the conformal primaries in a specific way. This fixes C_T, a *central charge of the CFT*. Roughly speaking, one can think of C_T as counting the number of degrees of freedom of the theory. This follows intuitively because the direct product of two CFTs with central charges C_1 and C_2 yields a new CFT with central charge $C_1 + C_2$, since the energy-momentum tensors in the two CFTs are decoupled.

Once the energy-momentum tensor is normalized the coefficient of each tensor structure in its 3-pt correlator gives dynamical information about the theory. Since all $\langle T_{\mu\nu}\mathcal{O}\mathcal{O}^\dagger\rangle$ correlators must be proportional to $\Delta_{\mathcal{O}}$, by looking at the universal energy-momentum 3-pt correlator we can fix the proportional constant once and for all for a given theory. For more details of this see [17]. This proportionality constant is the inverse square root of a central charge of the CFT. This central charge is proportional to a power of $M_{pl}R_{AdS}$ in the AdS theory.

In general d there are several central charges, and they are more conventially discussed as coefficients of the expansion of the 'trace anomaly', which is $\langle T^\mu_\mu\rangle$ evaluated when the CFT is placed in a general curved spacetime. For example, in $d = 4$ we have

$$\langle T^\mu_\mu\rangle = \frac{c}{16\pi^2}W_{abcd}W^{abcd} - \frac{a}{16\pi^2}\left(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2\right) - \frac{a'}{16\pi^2}\nabla^2 R \tag{10.31}$$

where the terms on the right hand side are various curvature tensors for the 4-d spacetime in which the CFT is living. Note that $W_{abcd}W^{abcd} = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + R^2/4$ is the square of the Weyl tensor, and the terms proportional to a are the 4-d Euler density. Also, the quantity a' is scheme dependent, but both a and c are physically meaningful, and a is known to decrease under RG flows. These coefficients can also be computed directly from the 3-pt correlator of $T_{\mu\nu}$ in flat spacetime, see [17]. In AdS$_5$ theories where gravity has only an Einstein-Hilbert action term, it turns out that $a = c$, and so both are proportional to $(M_5R_{AdS})^3$, where M_5 is the 5-d Planck scale.

10.3 Ward Identites from AdS and Universality of Long-Range Forces

In the previous section we saw how CFT Ward identities imply that conserved spin one currents must couple to conserved charges, while the stress-energy tensor must have couplings proportional to the dimension Δ. In both cases by a 'coupling' we meant the magnitude of the universal 3-pt correlators

$$\langle J_\mu(y)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \quad \text{and} \quad \langle T_{\mu\nu}(y)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \tag{10.32}$$

which also set the OPE coefficients for the appearance of J_μ and $T_{\mu\nu}$ in the OPE of $\mathcal{O}_1(x)\mathcal{O}_2(0)$. Note that these operators $\mathcal{O}_1$ and $\mathcal{O}_2$ were *any* operators in the theory, not just the operators dual to single-particle states in AdS. However, in the case where we have some AdS field theory involving a gauge boson A_μ or a graviton $h_{\mu\nu}$ coupling to some other field ϕ, we will have some linearized couplings in AdS of the schematic form

$$q\int d^{d+1}X\sqrt{-g}(\phi^\dagger\partial_\mu\phi - \partial_\mu\phi^\dagger\phi)A^\mu \quad \text{or} \quad \sqrt{G_N}\int d^{d+1}X\sqrt{-g}\partial_\mu\phi\partial_\nu\phi h^{\mu\nu} \tag{10.33}$$

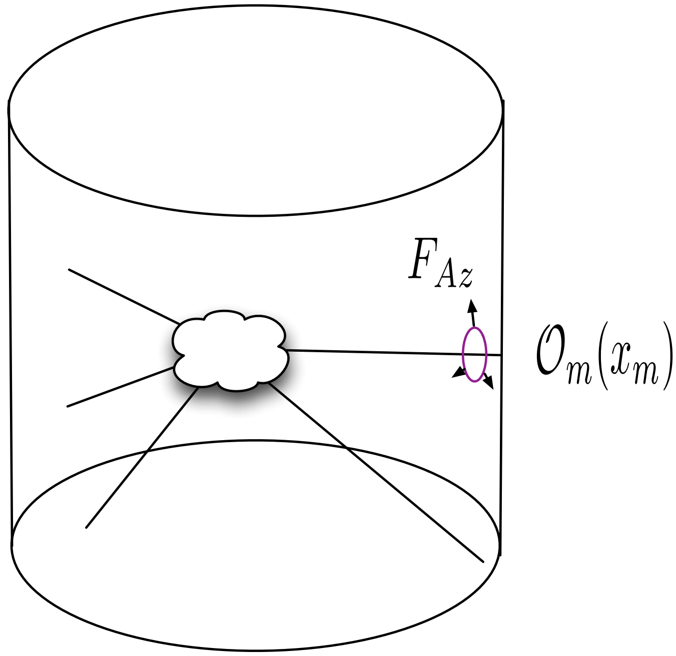

Figure 12: This figure illustrates the derivation of the Ward identity for a CFT current from Gauss's Law in AdS. Taking the surface integral towards the boundary of AdS gives the integral form of the Ward Identity given in equation (10.34).

If we compute the AdS correlator $\langle \phi(X_1)\phi(X_2)A_\mu(X_3)\rangle$ or $\langle \phi(X_1)\phi(X_2)h_{\mu\nu}(X_3)\rangle$ and then send $X_i \to P_i$ on the boundary of AdS as usual, then we will recover the 3-pt CFT correlators of a current or the stress-energy tensor from equation (10.32). The normalization of these 3-pt correlators will then either be the charge or the energy in Planck units, for A_μ and $T_{\mu\nu}$, respectively.

Let us give a quick derivation/interpretation of the CFT Ward identity for a gauge field A_M in AdS in order to better understand the physics. The Ward identity can be re-expressed as the statement that

$$\int_{B_m} d^d y \frac{\partial}{\partial y^\mu} \langle J^\mu(y)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = -i\langle \mathcal{O}_1(x_1)\cdots[G_m\mathcal{O}_m(x_m)]\cdots\mathcal{O}_n(x_n)\rangle \quad (10.34)$$

where B_m is a d-dimensional ball containing the point x_m and none of the others; we can make the ball B_m as small as we like. Now the left hand side can be re-expressed as a surface integral

$$\int_{\partial B_m} d^{d-1}s_\mu \langle J^\mu(y)\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = -i\langle \mathcal{O}_1(x_1)\cdots[G_m\mathcal{O}_m(x_m)]\cdots\mathcal{O}_n(x_n)\rangle \quad (10.35)$$

where the normal to the surface of B_m is contracted with J_μ. Now let us use the definition of the current from the bulk gauge field, equation (10.13), to write this as

$$\lim_{z\to 0} z^{3-d}\int_{\partial B_m} d^{d-1}s_\mu \langle F_{\mu z}(y,z)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = -i\langle \mathcal{O}_1(x_1)\cdots[G_m\mathcal{O}_m(x_m)]\cdots\mathcal{O}_n(x_n)\rangle \quad (10.36)$$

Finally, let us interpret this equation by viewing all of the operators as $z \to 0$ limits of bulk fields $\phi_i(x, z)$. In that case, the surface integral, pictured in figure 12, is just

$$\int_{\partial B_m} d^{d-1} s_A E^A(s, z) = q_m \tag{10.37}$$

where the integral is being performed at fixed small z in AdS. This is the integral of the AdS 'electric field' $E^A = F^{Az}$ around the state created by the mth operator. The other particles/blobs/states are irrelevant because they are infinitely far away in the limit that $z \to 0$. So the Ward identity in the CFT just corresponds to Gauss's law in AdS!

Now we will review an old argument of Weinberg that proves that in flat spacetime, massless spin one particles must couple to conserved charges while massless spin two particles must couple to energy-momentum. The argument also shows that higher-spin massless particles cannot couple in a way that would lead to long-range forces. This classic theorem goes a long way towards explaining why the spectrum of elementary particles we observe ends at the spin two graviton. The argument itself is independent of AdS/CFT, but it is probably the best general argument one can give in order to explain why massless spin one and spin two particles couple in a universal way at long distances.

We begin by considering some scattering amplitude $\mathcal{M}_n(p_i)$ for n particles in flat spacetime. Given this n-particle process, what is the amplitude for emitting an additional very soft photon with momentum q, so that $q \cdot p_i \ll p_j \cdot p_k$? This question is of great importance because the exchange of particles with small energy and momentum is what leads to long-range forces.

If the soft photon is emitted from any external line in the scattering amplitude then we find a contribution

$$\mathcal{M}_n(p_i) \sum_{m=1}^{n} \frac{e_m p_m^\mu}{p_m \cdot q - i\epsilon} \epsilon_\mu \tag{10.38}$$

where ϵ_μ is the polarization vector for the soft photon, and e_m is the coupling constant for the mth particle. One might also wonder if the photon can couple to vectors made from the spin of the mth particle, but this will also not give a pole as $q \to 0$ since multipole moments cannot give long-distance effects. There will be other contributions from emission of the photon from internal parts of the diagram, but these will not have a pole as $q \to 0$, so they will be subdominant in the soft limit. The soft emission amplitude is dominated by contributions from long time periods of order $1/q$ associated with the nearly free propagation of the initial and final state particles.

The pole we have found literally corresponds to a sum over contributions of the form

$$\begin{aligned} e^{ie_m \int dt \hat{n}_m^\mu A_\mu(x(t))} - 1 &\approx ie_m \int dt\, \hat{n}_m^\mu A_\mu(x(t)) + \cdots \\ &= ie_m\, \epsilon \cdot \hat{n} \int dt \hat{e}^{iq \cdot x(t)} \end{aligned} \tag{10.39}$$

which is just the amplitude for a charged particle to source an electromagnetic field, with $x(t) = t\hat{n}$, because the action for such charged particle is just the integral of $A_\mu \dot{x}^\mu(t)$ along the world-line of the particle. This gives the contribution

$$\sum_{m=1}^{n} e_m \, \epsilon \cdot \hat{n}_m \int_0^\infty dt e^{it\hat{n}_m \cdot q} \tag{10.40}$$

where the integral over time proceeds from the scattering process out to infinity, and $\hat{n}_m$ is a unit vector pointing in the direction of p_m. Performing the integral gives the soft emission amplitude.

Consider the Lorentz transformation properties of this scattering amplitude. In particular, note that the polarization tensor ϵ_μ for a massless particle does not transform as a vector, instead it transforms as

$$\epsilon_\mu \to \Lambda^\nu_\mu \epsilon_\nu + \alpha(q, \Lambda) q_\mu \tag{10.41}$$

where α is some function. So in other words, the polarization tensor shifts by q_μ under general Lorentz transformations. This means that to recover a Lorentz invariant S-Matrix we must have

$$\mathcal{M}_n(p_i) \sum_{m=1}^{n} \frac{e_m p^\mu_m}{p_m \cdot q - i\epsilon} q_\mu = \mathcal{M}_n(p_i) \sum_{m=1}^{n} e_m = 0 \tag{10.42}$$

The sum of the charges of the scattering particles must vanish. In other words, *a massless spin one particle must couple to a conserved charge.*

Consider the Lorentz transformation properties of this object. In particular, note that the polarization tensor ϵ_μ for a massless particle does not transform as a vector, instead it transforms as

$$\epsilon_\mu \to \Lambda^\nu_\mu \epsilon_\nu + \alpha(q, \Lambda) q_\mu \tag{10.43}$$

where α is some function. So in other words, the polarization tensor shifts by q_μ under general Lorentz transformations. This means that to recover a Lorentz invariant S-Matrix we must have

$$\mathcal{M}_n(p_i) \sum_{m=1}^{n} \frac{e_m p^\mu_m}{p_m \cdot q - i\epsilon} q_\mu = \mathcal{M}_n(p_i) \sum_{m=1}^{n} e_m = 0 \tag{10.44}$$

The sum of the charges of the scattering particles must vanish. In other words, *a massless spin one particle must couple to a conserved charge.*

In the case of a massless spin two particle, the soft emission amplitude must be

$$\mathcal{M}_n(p_i) \sum_{m=1}^{n} \frac{G_m p^\mu_m p^\nu_m}{p_m \cdot q - i\epsilon} \epsilon_{\mu\nu} \tag{10.45}$$

where $\epsilon_{\mu\nu}$ is the polarization tensor of the graviton, and G_m are its coupling constants. The polarization tensor transforms as

$$\epsilon_{\mu\nu} \to \Lambda^\alpha_\mu \Lambda^\beta_\nu \epsilon_{\alpha\beta} + \alpha_\mu q_\nu + \alpha_\nu q_\mu \tag{10.46}$$

and so in order to have a Lorentz invariant S-Matrix, we must have

$$\sum_{m=1}^{n} G_m p_m^\mu q_\mu = 0 \tag{10.47}$$

But the only way this can be zero for all q_μ without putting a restriction on the allowed momenta p_μ for scattering is if $G_m = G$ is a universal constant, so that the equation is satisfied due to momentum conservation. Thus *a massless spin two particle must couple universally to energy-momentum.* Repeating this argument for higher spin massless particles shows that they cannot couple in a way that will give rise to long-range forces, which necessarily involve propagators with poles in the soft $q \to 0$ limit.

As promised, we have shown that massless particles must couple to conserved charges or momenta, meaning that they couple to both fundamental and composite particles/objects/states with universal 3-pt vertices. Interpreted in AdS, these universal 3-pt vertices are just Feynman diagrams that give rise to universal correlators for J_μ and $T_{\mu\nu}$ with any operators $\mathcal{O}_1$ and $\mathcal{O}_2$.

11 AdS Black Holes and Thermality

Let us begin with a thought experiment. Consider a CFT living on the Lorentzian cylinder $R \times S^{d-1}$, and let us slowly heat it up. Since the dilatation operator serves as the Hamiltonian, as we increase the temperature the CFT will be in a state characterized by larger and larger operator/state dimensions.

Since the AdS Hilbert space is identical to that of the CFT, we can interpret our hot CFT as a thermal state in AdS. But what will this state consist of? At low temperatures we will just have a thermal gas made up of the light particles in AdS. Due to the AdS geometry, these particles will mostly move around near the center of AdS, with only occasional excursions further away. This means that as we increase the temperature, we will be cramming more and more energy into a region of roughly fixed size. In the presence of dynamical gravity, this cannot go on forever – eventually, at some critical temperature T_c, the hot gas will collapse to form a black hole in AdS. *Our thought experiment shows that black holes in AdS must correspond to a hot CFT!*

Now let us consider the thermodynamics more carefully. We will use the microcanonical ensemble, fixing the total energy of the system, as we will see that the microcanonical and canonical ensembles look somewhat different in equilibrium, due to the existence of a phase with negative specific heat.

Let us nevertheless use the notation T to denote the average energy of a graviton in the gas of gravitons. In that case the occupation number of the n, ℓ mode will be

$$\frac{1}{1 - e^{-\beta(\Delta+2n+\ell)}} \approx \frac{R_{AdS} T}{\Delta + 2n + \ell} \tag{11.1}$$

when this ratio is large. For gravitons $\Delta = d$, and there will be of order T^d such modes with $\Delta + 2n + \ell < T$, so we expect that

$$E_{tot} \sim T^{d+1} R_{AdS}^d \tag{11.2}$$

for a gas of gravitons with $T \gg 1/R_{AdS}$. This is what we would expect for a box of size R_{AdS} in $d+1$ dimensions. As we increase T, E_{tot} will eventually grow so large that its corresponding Schwarzschild radius will be of order R_{AdS}, at which point the gas will collapse to form a large black hole. In $d+1$ dimensional flat spacetime the Schwarzschild radius would be

$$R_S = (G_{d+1} E_{tot})^{\frac{1}{d-2}} \tag{11.3}$$

where $G_{d+1} = (M_{d+1})^{1-d}$ is the Newton constant and M_{d+1} is the $d+1$ dimensional Planck mass. If we are studying a homogeneous gas, collapse will first occur on the largest scales, because gravitational effects are cumulative with energy. This suggests that a black hole only forms once $R_S \sim R_{AdS}$, when the energy E_{tot} is within its own Schwarzschild radius, so that

$$T_{collapse} \sim \left(G_{d+1} R_{AdS}^2\right)^{-\frac{1}{d+1}} \tag{11.4}$$

For example, in the case of AdS$_4$ this would be a temperature $\sqrt{M_4/R_{AdS}}$, at an intermediate energy between the AdS scale and the Planck scale.

However, this analysis in the microcanonical ensemble has been rather misleading. First of all, in most examples considered in the literature there are other regimes aside from the gas of gravitons and the large AdS black hole – there is also a Hagedorn stringy regime, and a regime where small black holes (much smaller in size than R_{AdS}) can be produced. Furthermore, the parametric relationships change due to the presence of extra dimensions of R_{AdS} size. Finally, this analysis was misleading because in actuality, the large AdS black holes have such a large entropy that in the canonical ensemble they start to dominate already at $T \sim 1/R_{AdS}$. In other words, the huge number of different AdS black hole states is able to make up for their $e^{-\beta E}$ Boltzmann factor.

11.1 The Hagedorn Temperature and Negative Specific Heat

Let us briefly discuss the Hagedorn phenomenon, since it is interesting and simple. The canonical ensemble partition for any physical system is

$$Z[\beta] = \sum_{\psi} e^{-\beta E_\psi} \tag{11.5}$$

where as usual $\beta = 1/T$, and the sum is over all quantum mechanical states ψ in the theory. The sum can be re-organized as

$$Z[\beta] = \sum_{E} e^{S(E) - \beta E} \tag{11.6}$$

where $e^{S(E)}$ is the number of states with energy E – this defines the entropy as a function of E. Note that if $S(E)$ grows faster than $\beta_* E$ for some β_* in the limit that $E \to \infty$, then this sum diverges and partition function is ill defined!

To understand this phenomenon better, note that the expectation value of the (total) energy is

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z[\beta] \tag{11.7}$$

Let us consider what happens if we have the borderline case $S(E) \to \beta_* E$ in the limit that $E \to \infty$. Then we find

$$Z[\beta] \approx \sum_E^\infty e^{(\beta_* - \beta)E} = \frac{1}{\beta_* - \beta} \tag{11.8}$$

so we find that the expectation value of the energy is

$$\langle E \rangle = \frac{1}{\beta_* - \beta} \tag{11.9}$$

Thus we see that as $\beta \to \beta_*$, the expectation value of the energy diverges. Reversing the logic, it takes an infinite amount of energy to achieve the temperature $T_* = 1/\beta_*$. *This T_* is the Hagedorn temperature, and it is the maximum possible temperature for the system.* Hagedorn behavior is relevant because it is the thermodynamic behavior of a gas of strings. Thus if a stringy regime exists in AdS, then there will be a range of temperatures showing a Hagedorn behavior.

There is an even more extreme behavior, where $S(E)$ grows more rapidly than $\beta_* E$ for any fixed β_*. For example, in $d+1$ dimensional flat spacetime, black holes have an entropy

$$S(R_S) \sim \frac{R_S^{d-1}}{G_{d+1}} \tag{11.10}$$

which translates into

$$S(E) \sim (G_{d+1})^{\frac{1}{2-d}} E^{\frac{d-1}{d-2}} \tag{11.11}$$

This means that $S(E)$ grows with a power of E greater than one. This implies the famous statement that flat space black holes have a specific heat

$$\frac{\partial}{\partial T}\langle E \rangle < 0 \tag{11.12}$$

Increasing the temperature actually decreases the black hole mass, because the black hole entropy dominates the free energy. This is relevant to AdS because this is the behavior of 'small' AdS black holes, with $R_S \ll R_{AdS}$. So these states can also contribute to the partition function in AdS/CFT.

11.2 Thermodynamic Details from AdS Geometry

To make all of this more precise, let us study the Euclidean CFT on $S^1 \times S^{d-1}$, following [21]. Now we will be using the canonical ensemble, so we will be studying states that minimize the free energy, which includes the effects of entropy. Compactifying the Euclidean time direction with period β corresponds to studying the partition function of the theory on S^{d-1} at a temperature $T = 1/\beta$, as usual. Note that the S^{d-1} also has some radius β', and since we are studying a CFT, the physics only depends on the ratio of β/β'. For convenience we can fix $\beta' = 1$, although one can also find a scaling limit [21] where $\beta' \to \infty$, so that we get a Euclidean CFT on $S^1 \times R^{d-1}$.

We can write Euclidean AdS in the coordinates

$$ds^2_{AdS} = \left(\frac{r^2}{R^2_{AdS}} + 1\right) dt^2 + \frac{dr^2}{\frac{r^2}{R^2_{AdS}} + 1} + r^2 d\Omega^2 \tag{11.13}$$

This coordinate system follows immediately from our usual global coordinates by taking $r = R_{AdS} \tan\rho$. We have introduced this particular coordinate system so that we can naturally write the AdS-Schwarzschild solution

$$ds^2_{Sch} = \left(\frac{r^2}{R^2_{AdS}} + 1 - \frac{\omega_d M}{r^{d-2}}\right) dt^2 + \frac{dr^2}{\frac{r^2}{R^2_{AdS}} + 1 - \frac{\omega_d M}{r^{d-2}}} + r^2 d\Omega^2 \tag{11.14}$$

where M is the physical mass of a black hole, and the constant

$$\omega_d = \frac{16\pi G_{d+1}}{(d-1)\mathrm{Vol}(S^{d-1})} \tag{11.15}$$

is written in terms of the $d+1$ dimensionsal Newton constant G_{d+1}. There is a horizon at the largest root r_+ of

$$\frac{r^2}{R^2_{AdS}} + 1 - \frac{\omega_d M}{r^{d-2}} = 0 \tag{11.16}$$

The physical space ends at r_+, since after this point the metric is no longer Euclidean. The metric for the full $S^1 \times S^{d-1}$ will only be smooth if we must avoid conical singularities in t. In the vicinity of r_+, we can write $r = r_+ + \delta r$ and the metric looks like

$$ds^2 \approx \delta r \left(\frac{dr_+^2 + (d-2)R^2_{AdS}}{R^2_{AdS} r_+}\right) dt^2 + \frac{1}{\delta r}\left(\frac{R^2_{AdS} r_+}{dr_+^2 + (d-2)R^2_{AdS}}\right) d(\delta r)^2 + r_+^2 d\Omega^2 \tag{11.17}$$

We can define a new coordinate $x = 2\sqrt{\delta r}$, in which case we see that we have the metric for a cone in x and t. To avoid a conical singularity at $x = 0$, we must take the periodicity in $t \to t + \beta$ with

$$\beta = \frac{4\pi R^2_{AdS} r_+}{dr_+^2 + (d-2)R^2_{AdS}} \tag{11.18}$$

This result means that *the temperature* $1/\beta$ *is fixed in terms of the black hole mass* M. Furthermore, β has a maximum as a function of r_+, so we see that for $\beta > \beta_{min}$, which means

$$T < T_{min} = \frac{1}{R_{AdS}} \frac{\sqrt{d(d-2)}}{2\pi} \tag{11.19}$$

there is no thermodynamically stable black hole. In other words, when $T < T_{min}$ the thermodynamically stable state of the spacetime will be a gas of particles (and perhaps strings and small black holes) in otherwise empty AdS, while for $T > T_{min}$ we can also have the AdS-Schwarzschild spacetime describing a large black hole.

The phase transition between these two states is called the Hawking-Page phase transition. We can compute the transition temperature by studying the difference in Free Energies between AdS and the AdS-Schwarzschild solution. It is computed in [21], and the result is

$$F = \frac{\mathrm{Vol}(S^{d-1})(R_{AdS}^2 r_+^2 - r_+^{d+1})}{4G_{d+1}(dr_+^2 + (d-2)R_{AdS}^2)} \tag{11.20}$$

We see that in the large r_+ limit the free energy of AdS-Schwarzschild dominates. The two free energies are equal when $r_+ = R_{AdS}$, so that we have a phase transition temperature of

$$T_c = \frac{1}{R_{AdS}} \frac{d-1}{2\pi} \tag{11.21}$$

In other words, heating up the AdS/CFT system above this temperature (in the canonical ensemble) will result in an equilibrium state consisting of a large AdS black hole.

12 Partition Functions, Sources, and Holographic RG

12.1 Various Dictionaries

In these notes we have defined CFT operators directly in terms of limits of bulk fields, via

$$\mathcal{O}(t, \Omega) = \lim_{\epsilon \to 0} \frac{\phi(t, \rho(\epsilon; t, \Omega), \Omega)}{\epsilon^\Delta} \tag{12.1}$$

where $\rho(\epsilon; t, \Omega) = \frac{\pi}{2} - \epsilon f(t, \Omega)$ and the function $f(t, \Omega)$ determines the geometry of the boundary on which the CFT lives. This allowed us to compute correlators for CFT operators $\mathcal{O}$ in terms of limits of AdS correlators using AdS Feynman diagrams.

In fact, there are two AdS/CFT 'dictionaries', and our choice has been the less common one in the literature. Our choice has the great advantage that it makes it obvious how to identify the Hilbert spaces of the AdS and CFT theories, and it also seems more immediate and intuitive to me. The moreconventional dictionary has the advantage that it is easier to use when considering finite deformations of the CFT.

The conventional dictionary involves computing the AdS path integral, where the bulk fields $\phi(t, \rho, \Omega) \to \phi_0(t, \Omega)$ for some classical field configuration $\phi_0(t, \Omega)$ on the boundary of AdS. This gives a partition functional

$$Z_{AdS}[\phi_0] = \int_{\phi_0} D\phi \, e^{-S_{AdS}[\phi]} \tag{12.2}$$

Alternatively, we could imagine computing a CFT partition function

$$Z_{CFT}[\phi_0] = \int_{CFT} D\chi e^{-S_{CFT}[\chi] + \int d^d x \phi_0(x) \mathcal{O}(x)} \tag{12.3}$$

where $\mathcal{O}(x)$ is a CFT operator and $\phi_0(x)$ is a classical source for this operator. The claim of the usual AdS/CFT dictionary is that

$$Z_{AdS}[\phi_0] = Z_{CFT}[\phi_0] \tag{12.4}$$

In particular, this means that we can compute CFT correlators by differentiating Z_{AdS} with respect to ϕ_0 and then setting it to zero, giving

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta}{\delta \phi_0(x_1)} \cdots \frac{\delta}{\delta \phi_0(x_n)} Z_{AdS}[\phi_0] \Big|_{\phi_0=0} \tag{12.5}$$

This form of the dictionary is particularly useful because it tells us how to compute the partition function of the CFT for finite values of the sources ϕ_0, which we can then interpret as coupling constants, since in effect, $\phi_0 \mathcal{O}$ has been added to the CFT action.

Let us see why these two dictionaries are equivalent, following [22]. To make the argument, let us use the Poincaré patch for convenience. We will introduce two reference surfaces, one at some $z = \epsilon$ and another at $z = \ell$, with $\epsilon \ll \ell \ll 1$. The bulk path integral can be broken up as

$$\begin{aligned} Z_{AdS}[\phi_0] &= \int \mathcal{D}\phi_{z<\ell}\, e^{-S_{z<\ell}[\phi]} \int \mathcal{D}\tilde{\phi}_{z=\ell} \int \mathcal{D}\phi_{z>\ell}\, e^{-S_{z>\ell}[\phi]} \\ &= \int \mathcal{D}\tilde{\phi} \Psi_{UV}[\phi_0, \tilde{\phi}; \epsilon, \ell] \Psi_{IR}[\tilde{\phi}, \ell] \end{aligned} \tag{12.6}$$

where $\tilde{\phi}(x)$ is the field $\phi(\ell, x)$ at the surface $z = \ell$ (so it only depends on x, not z), and we have written the path integrals over the intervals $[\epsilon, \ell]$ and $[\ell, \infty]$ in terms of a 'UV' and 'IR' wavefunction. Only the UV wavefunction knows about ϕ_0. Our usual dictionary is defined by

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \lim_{\ell \to 0} \ell^{-n\Delta} \int \mathcal{D}\tilde{\phi} \Psi_{IR} \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) \Psi_{UV} \Big|_{\phi_0=0, \epsilon=0} \tag{12.7}$$

The other dictionary based on partition functions will be equivalent if

$$\lim_{\ell \to 0} \int \mathcal{D}\tilde{\phi} \Psi_{IR} \left[\frac{\delta}{\delta \phi_0(x_1)} \cdots \frac{\delta}{\delta \phi_0(x_n)} \Psi_{UV} \right]_{\phi_0=0} \propto \lim_{\ell \to 0} \ell^{-n\Delta} \int \mathcal{D}\tilde{\phi} \Psi_{IR} \tilde{\phi}(x_1) \cdots \tilde{\phi}(x_n) \Psi_{UV} \Big|_{\phi_0=0} \tag{12.8}$$

The 'UV' region plays a simple role – it is only present so that we can differentiate Ψ_{UV} to obtain the field $\tilde{\phi}$ at the intermediate surface at $z = \ell$.

We have previously discussed the fact that AdS interactions become irrelevant as we approach the boundary, so we can assume that Ψ_{UV} is computed purely from the free theory. In this case we can just calculate it directly. It is

$$\Psi_{UV}[\phi_0, \tilde{\phi}] = \int_{\phi(\epsilon)=\phi_0 \epsilon^{d-\Delta}}^{\phi(\ell)=\tilde{\phi}} \mathcal{D}\phi \exp\left[-\frac{1}{2} \int_\epsilon^\ell d^d x dz \sqrt{-g} \left((\nabla\phi)^2 - m^2 \phi^2 \right) \right] \tag{12.9}$$

This is just a Gaussian integral, so we can evaluate it by solving the equations of motion for ϕ with the correct boundary conditions and then inputting the solution back into the action. The equations of motion

$$\nabla^2\phi - m^2\phi = 0 \tag{12.10}$$

will have two solutions, which we can refer to as $G_k^\epsilon(z,x)$ and $G_k^\ell(z,x)$. One can compute these G_k in terms of bessel functions in the case of pure AdS. We choose a linear combination of solutions so that $G_k^\epsilon(\epsilon,x) = 1$ and $G_k^\epsilon(\ell,x) = 0$, and vice versa for $G_k^\ell(z,x)$. The label k is a momentum label for the x_μ coordinates, although in principle it could be any label associated with a complete basis for solutions. Now the solution for $\phi(z,k)$ (we have gone to momentum space for convenience) with the desire boundary conditions is simply

$$\phi(z,k) = \phi_0(k)\epsilon^{d-\Delta}G_k^\epsilon(z) + \tilde{\phi}(k)G_k^\ell(z) \tag{12.11}$$

When we evaluate the action with this solution it vanishes up to a total derivative term, and so the result only involves boundary terms of the form $\phi\nabla_z\phi$ evaluated at ϵ and ℓ. Terms quadratic in ϕ_0 or $\tilde{\phi}$ will not contribute when we differentiate Ψ_{UV} and then set $\phi_0 \to 0$. Thus the relevant terms will be

$$\Psi_{UV}[\phi_0, \tilde{\phi}] \propto \exp\left[-\frac{1}{2}\int d^dk\, \phi_0(-k)\tilde{\phi}(k)\left(\epsilon^{-\Delta}\nabla_z G_k^\ell(\epsilon) - \ell^{-d}\nabla_z G_k^\epsilon(\ell)\right)\right] \tag{12.12}$$

where we note that the z derivatives of the G solutions can be non-vanishing on both ends of the interval. Evaluating the derivatives for the actual solutions gives

$$\Psi_{UV} = \exp\left[-(2\Delta - d)\ell^{-\Delta}\int d^dx\phi_0(x)\tilde{\phi}(x) + \cdots\right] \tag{12.13}$$

where the ellipsis denotes other terms that either vanish more quickly as $\ell \to 0$ or that are quadratic in ϕ_0 or $\tilde{\phi}$. So we see that

$$\left(\frac{1}{\Psi_{UV}}\frac{\delta\Psi_{UV}}{\delta\phi_0}\right)_{\phi_0=0} = \ell^{-\Delta}(2\Delta - d)\tilde{\phi}(x) \tag{12.14}$$

as desired, and the equivalence between the two dictionaries has been proven.

12.2 Fefferman-Graham and the Holographic RG Formalism

12.3 Breaking of Conformal Symmetry

In AdS/CFT, the conformal symmetry of the CFT is realized geometrically as the group of AdS isometries. If we want to use holography to study theories that are not conformally invariant, then we must study quantum field theories in spacetimes with less symmetry than AdS. In order to do minimal violence to the conformal symmetry, we can study Poincaré invariant theories that are approximately conformal in some regime, for example at very high or very low energies. However, we should note that recently AdS/CFT type ideas have been used to attempt to study toy non-relativistic systems in order to make contact with condensed matter physics.

Since we wish to preserve Poincaré symmetry, it is most natural to view AdS in the coordinates

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \eta^{\mu\nu} dx_\mu dx_\nu\right) \tag{12.15}$$

Now we would like to break conformal symmetry. The laziest option is to simply declare, by fiat, that there exist membranes, conventionally called either the 'UV Brane' and the 'IR Brane', that end spacetime at some finite positions $z_{UV} \ll z_{IR}$. In this case the AdS isometries will only be broken by the presence of these membrane, in the same way that a brick wall breaks translation invariance.

Note that the positions z_{UV} and z_{IR} transform under dilatations

$$D = z\partial_z + x^\mu \partial_\mu \tag{12.16}$$

as lengths, so they specify a short and long distance scale, respectively. These obviously correspond to energy scales $\mu_{UV} = 1/z_{IR}$ and $\mu_{IR} = 1/z_{IR}$.

Let us first consider the case when $z_{IR} = \infty$, so that the IR brane is absent and we only have a UV brane. If we like *we can add whatever d dimensional quantum field theory we like by including an action localized to the UV brane.* We can (roughly) interpret the UV Brane as setting a UV cutoff μ_{UV} for the CFT, and the fields localized to the UV brane represent some other theory that couples to the CFT. In particular, when we quantize gravity in this space, the boundary conditions (or UV brane action) for the graviton will be important. If we do not impose Dirichlet boundary conditions then the graviton will have a normalizable zero mode, and so *our theory will include d-dimensional gravity.*

Now let us consider an IR brane at z_{IR} in the absence of a UV Brane. In this case there can be no dynamical gravity, or any other degrees of freedom aside from the CFT. However, what we do have is some kind of conformal symmetry breaking associated with the energy scale μ_{IR}. Since this IR brane comes out of nowhere, we do not have any particular insight into the nature of the conformal symmetry breaking in the CFT. However, what we can say is that the IR brane will fluctuate in position (do to gravitational effects if nothing else), and in the absence of other fields these fluctuations will be a massless goldstone boson mode. This mode is often called the radion or dilaton; it is the goldstone boson of broken conformal symmetry.

We can also put d-dimensional quantum fields directly on the IR brane. These can be very roughly interpreted as composite or emergent degrees of freedom that arise when conformal symmetry is broken. Again, this is rather ad hoc from the CFT point of view.

With both an IR and a UV brane, we have both a UV cutoff for the CFT and conformal symmetry breaking in the IR. In this case the dynamics become a bit more interesting, because we can stabilize the position z_{IR} by giving fields in the 'slice' of AdS appropriate boundary conditions. The idea is that the configuration of the AdS fields will depend on z_{IR}, and therefore the total bulk energy will depend on z_{IR}. Minimizing this energy determines the equilibrium value of z_{IR}. The original implementation of these ideas was called the Goldberger-Wise mechanism. It was interesting because this setup, when applied to weak-scale physics, is called a Randall-Sundrum model, and it has the potential to explain the hierarchy problem.

One can consider more involved and also much more physically plausible models of conformal symmetry breaking where the AdS metric itself is modified to a metric of the form

$$ds^2 = \frac{1}{A(z)^2}\left(dz^2 + \eta^{\mu\nu}dx_\mu dx_\nu\right) \tag{12.17}$$

where $A(z)$ is some function that asymptotes to $A(z) \sim z$ for $z \sim 0$. Such a model represents a theory that is approximately conformal in the UV, where $z \to 0$, but that breaks conformal symmetry in a more complex and interesting way. Such models have been studied in great detail in the literature, both phenomenologically and as precise string theory backgrounds.

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