**How did string theory help Borcherds solve the monstrous moonshine conjecture?**

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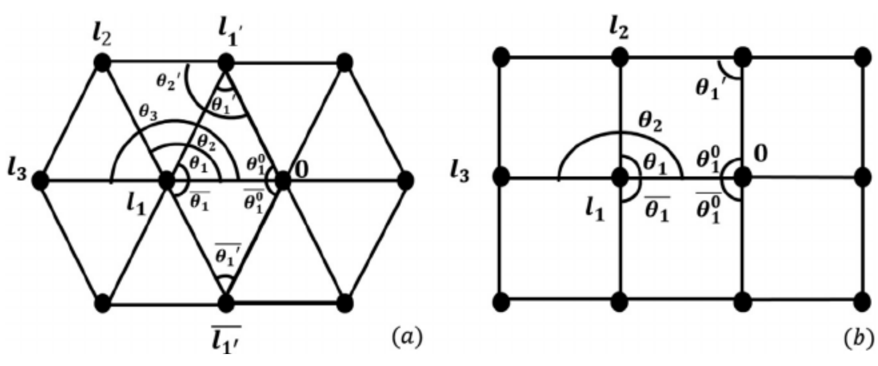
First, let me state a simplified version of the Monstrous moonshine conjecture. The conjecture states that the numbers 196,883 and 196,884 that appear at two seemingly totally different places of mathematics are so close for a rational reason. It says that it isn't just a coincidence.

What are the two places? First, consider the Monster group. It is some group (set) of operations (the "group" means that there is a rule how to compose i.e. "multiply" two operations) analogous to the rotations of a cube or dodecahedron that keep the shape unchanged. Except that the monster group has almost $10^{54}$ elements.

In the classification of all possible finite groups, the monster group is the largest "exceptional case" among the 26 or 27 exceptional groups known as "sporadic groups". All other groups are organized in infinite families that generalize some "groups of operations" that an undergraduate student could understand after some "mild" generalization of what he learned in the linear algebra course. But the sporadic groups are harder, exceptions, and the monster group has the largest number of elements.

You may represent the sporadic group as a finite set of some "rotations" on an N-dimensional space, in analogy with the group of symmetries of the dodecahedron. The smallest faithful space on which the monster group may be defined as a "space of some selected rotations" is 196,883-dimensional. We say that the smallest non-trivial (the 1-dimensional "never-changing" representation is forbidden here) is 196,883-dimensional.

So this number 196,883 appears in the theory of finite groups, some generalization of the science about the symmetries of Platonic solids, operations you may do with Rubik's cube, and similar considerations.



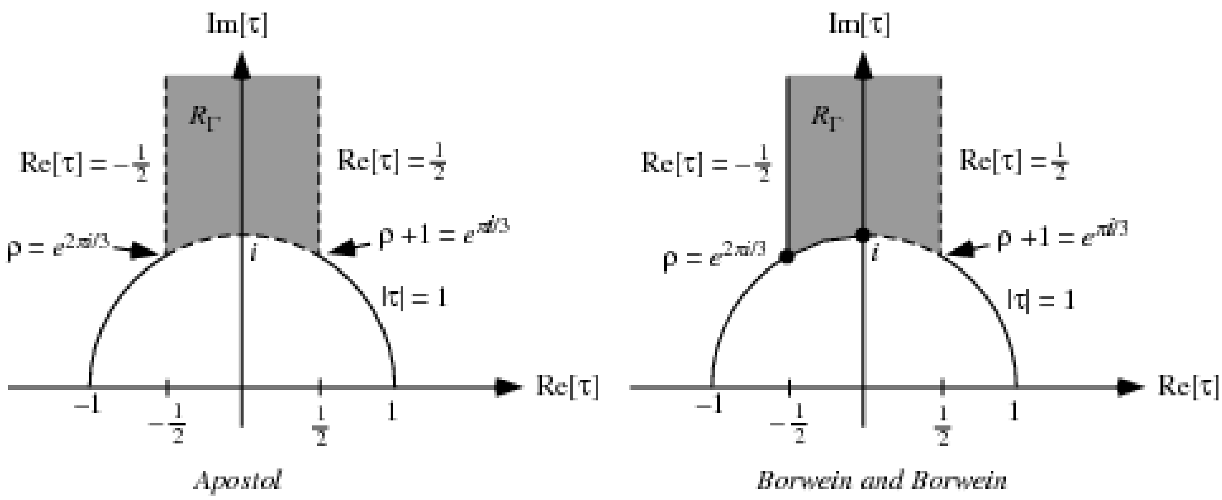

Now, where does 196,884 appear? Consider the space of all possible shapes of 2-dimensional tori or, equivalently, shapes of lattices in a 2-dimensional plane. A lattice is something generalizing ℤ2, the grid of points whose both coordinates are integer-valued. Except that you may take the lattice to be the set

$\{z=m+n\tau,\ m,n\in\mathbb{Z}\}$

Two examples of a two-dimensional lattice are seen above. So the shape of the grid is parameterized by the complex number $\tau\in\mathbb{C}$. Its imaginary part is variable and says that the spacing of the grid may be different in the vertical and horizontal direction: the ratio matters for the shape. Moreover, the real part of $\tau$ may make the grid slanted. For each grid, one may define a 2-torus by an identification, but I don't need to talk about it.

One may see that the values $\tau$ and $-1/\tau$ are equivalent because they just correspond to the exchange of the two basis vectors. Also, changing $\tau$ to $\tau+1$ keeps the lattice unchanged. So the space of inequivalent lattices isn't the whole $\tau$-plane but just a region of the plane, the "fundamental domain". I don't even need to tell you that the two operations generate a group $SL(2,\mathbb{Z})$ known as the modular group.

Because of the $\tau\to\tau+1$ identification, you may take the real part of $\tau$ to be between −1/2−1/2 and +1/2+1/2. And because of the $\tau\to-1/\tau$ identification, you may take $|\tau|>1$. So the fundamental domain is a semi-infinite strip with a round boundary.




The two pictures of the fundamental (shaded) region above only differ by the choice which boundary belongs to it. Not a big deal, I needed a minute to spot the difference.

OK, the left and right boundary are identified, and the circular boundary is identified in a "mirror way", too. The points at $\tau\to i\infty$ are conformally really just one point. So the topology of the fundamental region is equivalent to a two-sphere. For this reason, there must exist a holomorphic function that maps this $\tau$ from the fundamental region to a whole complex plane – which is also equivalent to a two-sphere, once you add the single point anywhere at infinity to complete the "stereographic projection" of the sphere.

The function (mapping the fundamental domain to the whole complex plane) is unique up to some undetermined added $SL(2,\mathbb{C})$ transformations (the Möbius transformation $z\to z'=(az+b)/(cz+d)$ mapping the plane onto the same plane, with four complex parameters) but one may pick one of the most natural choices for the map and it is given by the jinvariant, a specific function $j(\tau)$. Interestingly enough, the function may be expanded for $\tau \rightarrow i\infty$ and the expansion starts as

$\exp(-2\pi i\tau) + 196{,}884 \exp(+2\pi i\tau) + \ldots$

The following terms are proportional to $\exp(4\pi i\tau)$ and higher powers of $\exp(2\pi i\tau) \equiv q$, and their coefficients are increasing integers. But you see that the first nontrivial integer is 196,884, very close to the dimension of the minimum representation of the monster group.

We found almost the same two large integers in two very different branches of mathematics: large sporadic finite groups (group theory) and some modular functions describing conformal maps in the complex plane (theory of complex functions).

Is it a coincidence that these two numbers differ by one? Or is there a deeper explanation that unifies both situations? The answer is, of course, that there exists a deeper explanation.

You may define a mathematical – but we may say physical – system whose symmetry is the monster group; but whose partition function in the physics sense is the $j$-function. What is the physical system? It's the two-dimensional world sheet "conformal" theory describing the propagation of a string – the same string as used in realistic string theory – moving on the most beautiful 24-dimensional torus.

What is the torus? It is a 24-dimensional space whose 24 coordinates are periodically identified, so basically only the 24 fractional parts of the coordinates matter. But the lattice we must use is the Leech lattice. It is the most beautiful 24-dimensional lattice. One of the reasons is that when balls of the right radius are placed at all the lattice sites, they demonstrably define the "densest packing" in 24 dimensions. In 8 dimensions, the analogous lattice with the densest packing is the $E_8$ lattice (also a proven result).

The Leech lattice is a lattice – a stretched and tilted version of $\mathbb{Z}^{24}$, much like the two-dimensional lattices discussed above – which is unique if you impose the following conditions: the squared length of all the lattice sites is an even integer, the lattice is self-dual, so the lattice of all points that have an integer inner products with all the elements of the Leech lattice is the Leech lattice itself, and it has no lattice sites whose squared length is +2 – this minimum possible value "after zero" is accidentally banned. (If you dropped the last condition, and considered all the even self-dual lattices, there would be 24 distinct lattices in 24 dimensions – the Leech lattice would be one of them.)

This lattice by itself has some nice discrete symmetry analogous to the symmetries of the Platonic solids above. The group of automorphisms of the Leech lattice is known as the Conway group Co0 – a smaller of the 26 or 27 sporadic groups (Co1 is just a quotient of Co0 by its 2-element center, and Co2 is another subgroup of Co0). But if you allow strings to propagate on the torus obtained by the identification using this lattice, you will find out that the strings have a bigger symmetry group – an extension of the Conway group which happens to be the monster group itself.

So the string theory for strings propagating on the 24-dimensional "Leech torus" demonstrably has the monster group as its symmetry group. So all excited energy levels of the string must form representations of the monster group. If you count the states at the first excited level, you will see one singlet and one copy of the minimum 196,883-dimensional nontrivial representation of the monster group. In total, there are 196,884 states at the first excited level. This clarifies why the string theory compactification is linked to the "group theoretical" appearance of the number.

On the other hand, the partition sum of this string theory may be shown to be the simple *j*-function. In the expansion of the partition sum, string theory guarantees that the coefficients are related to the number of states at various excited energy levels of the vibrating string. While more complicated "modular" functions could appear as the partition sum of the particular string theory, one may show that due to the special properties of the Leech lattice, a partition sum that is as simple as the *j*-function is the right one. So this establishes the connection of the string theory with the expansion of the *j*-function in calculus.

In total, the string theory is connected to both sides – the group-theoretical and the complex-plane-based "version" of the number 196,883 or 196,884. That means that the version of string theory connects these two numbers with one another, too.

There exist various other versions of string theory, e.g. those using the K3 surfaces, that similarly explain various "cousins" of the monstrous moonshine, e.g. the Umbral moonshine. Also, all the other terms in the expansion of the *j*-function have coefficients that are dimensions of other simple enough representations (the direct sums of several irreps: the monster group has 194 distinct irreps – as you might know, it means that it has 194 conjugacy classes, too).

Borcherds has used all the mathematical features of the string theory above and completed a rigorous proof, while avoiding much of the (motivating and heuristic) physicists' jargon that was mostly used above. So he would talk about the vertex operator algebra (VOA) and not string theory, and so on, but the beef is the same.