Weil conjectures and étale cohomology

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June 2, 2021

– Undergraduate Forum, –
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Riemann zeta function

Definition

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This expression defines a meromorphic function on the half plane $\mathrm{Re}(s) > 1$. It was firstly studied by Euler as a real function. Riemann extended Euler's definition to a complex variable.

Theorem (Euler product formula)

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, *we have*

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

where $\mathbb{P}$ *is the set of prime numbers.*

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Eular product links Riemann zeta function (analysis) with prime numbers (number theory).

In his seminal article "*Über die Anzahl der Primzahlen unter eine gegebene Grösse*" *(On the Number of Primes Less Than a Given Magnitude)* in 1859, Riemann gave a functional equation of zeta function and proved a meromorphic continuation:

Theorem

The function $\Lambda(s) = (s-1)\pi^{-s/2}\Gamma(s/2+1)\zeta(s)$ *has an analytic continuation in* $\mathbb{C}$. *We have* $\Lambda(s) = \Lambda(1-s)$ *for any* $s \in \mathbb{C}$. *Here* $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ *denotes the* Γ*-function.*
In particular, $\zeta(s)$ *has an analytic continuation in* $\mathbb{C}\backslash\{1\}$ *with a single pole at* $s = 1$.

Conjecture (Riemann Hypothesis, 1859)

All non-trivial zeros of $\zeta(s)$ *lies on the line* $\mathrm{Re}(s) = \frac{1}{2}$.

The conjecture is still open.

Theorem (Dirichlet, 1826)

$m, n \in \mathbb{N}, (n, m) = 1$. *Then there are infinitely many prime numbers in* $\{n, n+m, n+2m, n+3m \cdots\}$.

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Theorem (Hadamard and de la Vallée-Poussin, 1896)

$$\lim_{n\to\infty} \frac{\ln(n)\pi(n)}{n} = 1,$$

where $\pi(n) = \sharp\{p \in \mathbb{P} \mid p \leq n\}$.

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- If Riemann conjecture is proved, we will have more accurate estimations for the distribution of prime numbers.

Weil Conjectures

$$\text{spaces} \quad \Longleftrightarrow \quad \text{functions}$$

$$X \quad (\text{compact Hausdroff}) \quad \longmapsto \quad C(X)$$

$$X = \mathrm{Max}(C(X)) \quad \longleftarrow\!\shortmid \quad C(X)$$

In algebraic geometry, we study the space of zeros of polynomials of several variables:

$$X = \{\underline{x} = (x_1, \dots, x_n) | f_1(\underline{x}) = f_2(\underline{x}) = \cdots = f_m(\underline{x}) = 0, f_i \in k[x_1, \cdots, x_n]\}$$

where k is an algebraically closed field.

Definition

Let k be an arbitrary field and $A = k[x_1, \dots, x_n]/I$ a domain. The set of maximal ideals

$$X = \mathrm{Max}(k[x_1, \cdots, x_n]/(f_1, \dots, f_m))$$

is called an *affine algebraic variety* over k.

Definition

Let A be a commutative ring (with a multiplicity identity 1), the set

$$\mathrm{Spec}(A) = \{\text{prime ideals of } A\}$$

is called the *spectrum* of A. The space $\mathrm{Spec}(A)$ is called an *affine scheme*.

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The definition of an affine scheme is more general than that of an affine variety.

A nature topology on $X = \mathrm{Max}(k[x_1, \cdots, x_n]/(f_1, \ldots, f_m))$ (resp. on $\mathrm{Spec}(A)$) has the base

$$\{D_f \mid \mathfrak{m} \in D_f \text{ iff } f \notin \mathfrak{m}\}_{f \in k[x_1, \cdots, x_n]/(f_1, \ldots, f_m)},$$

$$(\text{resp. } \{D_f \mid \mathfrak{p} \in D_f \text{ iff } f \notin \mathfrak{p}\}_{f \in A}).$$

The topology above is called the *Zariski topology*.

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The topology above is called the *Zariski topology*.

- An algebraic variety over k is a glue of finitely many affine varieties over k and separated over k ($\Delta : X \to X \times_k X$ is a closed immersion).
- A scheme is a glue of affine schemes.
- A morphism $f : X \to Y$ between schemes is a glue of $f_{ij} : \mathrm{Spec}(B_i) \to \mathrm{Spec}(A_j)$, which corresponds to a ring homomorphism $f_{ij}^* : A_j \to B_i$. A morphism between schemes is continuous with respect to their Zariski topologies.

Arithmetic:

$$\mathrm{Spec}(\mathbb{Z}) = \{2, 3, 5, 7, \ldots\} \bigcup \mathrm{Spec}(\mathbb{Q})$$

Algebra:

$$\mathbb{A}^1_{\mathbb{F}_q} = \mathrm{Spec}(\mathbb{F}_q[x]) = \{\text{irr. poly. in } \mathbb{F}_q[x]\} \bigcup \mathrm{Spec}(\mathbb{F}_q(x))$$

Geometry:

$$\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x]) = \{\text{complex plane}\} \bigcup \mathrm{Spec}(\mathbb{C}(x))$$

Arithmetic:

$$\zeta(s) = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1}$$
$$= \prod_{p \in |\mathrm{Spec}(\mathbb{Z})|} \frac{1}{1 - (\sharp(\mathbb{Z}/p\mathbb{Z}))^{-s}}$$

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Algebra:

$$\zeta(\mathbb{A}^1_{\mathbb{F}_q}, s) = \prod_{\mathfrak{p} \in |\mathrm{Spec}(\mathbb{F}_q[x])|} \frac{1}{1 - (\sharp(\mathbb{F}_q[x]/\mathfrak{p}))^{-s}}$$

More generally,

$$\zeta(X, s) = \prod_{x \in |X|} \frac{1}{1 - (\sharp(k(x)))^{-s}},$$

where X is an algebraic variety over $\mathbb{F}_q$ ($q = p^m$ for a prime ideal p);
$|X|$ is the set of closed points (maximal ideals) of X;
$k(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ is the residue field of X at x (it is a finite extension of $\mathbb{F}_q$).

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We call $\deg(x) = [k(x) : \mathbb{F}_q]$ the degree of x. We put $T = q^{-s}$.

$$\zeta(X,s) = Z(X,T) = \prod_{x\in|X|} \frac{1}{1-T^{\deg(x)}}$$

We denote by $X(\mathbb{F}_{q^m})$ the set of $\mathbb{F}_{q^m}$ rational points of X (i.e., the set $\mathrm{Hom}_{\mathrm{Spec}(\mathbb{F}_q)}(\mathrm{Spec}(\mathbb{F}_{q^m}), X)$). We have $\sharp(X(\mathbb{F}_{q^m})) = \sum_{\deg(x)|m} \deg(x)$.

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$$\begin{aligned}
Z(X,T) &= \exp(\ln Z(X,T)) \\
&= \exp(\sum_{x\in|X|} -\ln(1-T^{\deg x})) \\
&= \exp(\sum_{x\in|X|}\sum_{n=1}^{\infty} \frac{T^{n\deg(x)}}{n}) \\
&= \exp(\sum_{m=1}^{\infty}(\sum_{\deg(x)|m} \deg(x)\frac{T^m}{m})) \\
&= \exp(\sum_{m=1}^{\infty} \sharp(X(\mathbb{F}_{q^m}))\frac{T^m}{m}) \in \mathbb{Q}[[T]].
\end{aligned}$$

Example

$X = \mathbb{P}^1_{\mathbb{F}_q} = \mathbb{A}^1_{\mathbb{F}_q} \bigcup \{\infty\}$. We have $\mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^m}) = q^m + 1$, for any $m \geq 1$.

$$\begin{aligned} Z(\mathbb{P}^1_{\mathbb{F}_q}, T) &= \exp(\sum_{m=1}^{\infty} \sharp(\mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^m}))\frac{T^m}{m}) \\ &= \exp(\sum_{m=1}^{\infty} (q^m + 1)\frac{T^m}{m}) \\ &= \exp(\sum_{m=1}^{\infty} \frac{T^m}{m}) \exp(\sum_{m=1}^{\infty} \frac{(qT)^m}{m}) \\ &= \frac{1}{(1-T)(1-qT)} \end{aligned}$$

Example

X is a projective curve over $\mathbb{F}_q$ of genus $g > 0$. We have

$$Z(X,T) = \frac{P_X(T)}{(1-T)(1-qT)},$$

where $P_X(T)$ is a polynomial of degree $2g$ in $\mathbb{Z}[T]$.

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Weil proved an analogue of Riemann Hypothesis for curves: any zero of $P_X(q^{-s})$ lies on the line $\mathrm{Re}(s) = 1/2$ (i.e., any root α of $P_X(T) = 0$ satisfies $|\alpha| = q^{-1/2}$).

Conjecture (Weil conjectures)

Let X be a smooth projective variety over $\mathbb{F}_q$ of dimension d. Then, we have

(A) (Analytic Continuation) *$Z(X,T)$ is a rational function.*

(B) (Functional Equation) *$Z(X,T)$ has a functional equation*

$$Z(X, \frac{1}{q^d T}) = \pm q^{d\chi(X)/2} T^{\chi(X)} Z(X,T)$$

where $\chi(X)$ is the Euler-Poincaré characteristic of X.

(C) (Riemann Hypothesis)

$$Z(X,T) = \frac{P_1(T)P_3(T)\cdots P_{2d-1}(T)}{P_0(X)P_2(T)\cdots P_{2d}(T)}$$

where $P_i(T) \in \mathbb{Z}[T]$, $P_0(T) = 1 - T$, $P_{2d}(T) = 1 - q^d T$. Any root of $P_i(T) = 0$ has absolute value $q^{-i/2}$ $(i = 0, 1, \ldots, 2d)$.

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- A milestone achievement is due to Grothendieck *et al.*. They developed the theory of étale cohomology in SGA. Using this cohomology theory, Grothendieck obtained (A), (B) and gave a cohomology description of $P_i(T)$ in (C) (1960s).

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- Grothendieck also proposed a famous conjecture called *standard conjecture* which implies (C). However it is still open now.
- Deligne brought the notion of *weight* to ℓ-adic sheaves and finished the proof of Weil conjectures. He also proved Weil conjectures for L-functions (1973, 1980).

Étale cohomology

Definition

A morphism $f : X \to Y$ between schemes is called *étale* if

- $f : X \to Y$ is flat, i.e., it is a glue of morphisms of affine schemes $f_{ij} : \mathrm{Spec}(B_i) \to \mathrm{Spec}(A_j)$ where B_i are flat A_j-algebras.
- $f : X \to Y$ is *unramified*, i.e., $f^{-1}(y)$ is discrete and the diagonal map $\Delta : X \to X \times_Y X$ is an open immersion.

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Definition (very vague)

An *étale topology* of a scheme Y is a category Y_{et} with all étale morphisms $f : X \to Y$ as its objects and X-morphisms as morphisms between objects. A morphism $g : Z \to Y$ is called an *étale cover* if it is surjective and étale.

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Example

(1) Let K be a field. A morphism from a connected scheme X to $\mathrm{Spec}(K)$ is étale if $X = \mathrm{Spec}(L)$, where L/K is a finite separable extension.

(2) The morphism

$$\mathrm{Spec}(\mathbb{C}[x^{\pm 1}]) \to \mathrm{Spec}(\mathbb{C}[x^{\pm 1}]), \quad a \mapsto a^n$$

is étale for any $n \geq 1$.

(3) Open immersions are étale.

(4) The morphism $\mathrm{F} : \mathbb{A}^1_{\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q}$ induced by

$$\mathbb{F}_q[x] \to \mathbb{F}_q[x], \quad x \mapsto x^q$$

is not étale. (relative Frobenius)

Definition

A *presheaf of abelian groups* on a category $\mathscr{C}$ is a contravariant functor from $\mathscr{C}$ to the category of abelian groups.

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Definition

An *étale sheaf* $\mathscr{F}$ on a scheme X is an étale presheaf on X satisfying the following extra condition:
for any surjective morphism $g : V \to U$ between schemes étale over X, we have an exact sequence of abelian schemes

$$0 \to \mathscr{F}(U) \to \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V).$$

The category of étale sheaves on X is denoted by $\widetilde{X}_{\text{et}}$.

Example

Let K be a field and let $\overline{K}$ be a separable closure of K. Let n be an integer invertible in K. The following two categories are equivalent:

- The category of étale sheaves of finitely generated $\mathbb{Z}/n\mathbb{Z}$-modules on $\mathrm{Spec}(K)$.
- The category of finitely dimensional representations of $\mathrm{Gal}(\overline{K}/K)$ with the coefficient $\mathbb{Z}/n\mathbb{Z}$.

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Example

Let X be a variety over a field K. Let n be an integer invertible in K. The following two categories are equivalent:

- The category of locally constant étale sheaves of finitely generated $\mathbb{Z}/n\mathbb{Z}$-modules on X.
- The category of finitely dimensional representations of $\pi_1^{\mathrm{et}}(X, \bar{x})$ with the coefficient $\mathbb{Z}/n\mathbb{Z}$.

$\pi_1^{\mathrm{et}}(X, \bar{x})$ is called the *étale fundamental group* of X.

Definition

Let X be a scheme and $\mathscr{F}$ an étale sheaf on X. The i-th right derived functor of

$$\Gamma(X,-):\widetilde{X}_{\mathrm{et}}\to \mathrm{Ab},\quad \Gamma(X,\mathscr{F})=\mathscr{F}(X)$$

is called the i-th étale cohomology group. We denote it by $H^i_{\mathrm{et}}(X,-)=R^i\Gamma(X,-)$.

Example

Let C be a connected projective and smooth curve over an algebraically closed field k, let ℓ be a prime number invertible in k and let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ be a constant étale sheaf on X. we have

$$H^0_{\text{et}}(C,\Lambda) \cong \Lambda, \quad H^1_{\text{et}}(C,\Lambda) \cong \Lambda^{2g}, \quad H^2_{\text{et}}(C,\Lambda) \cong \Lambda.$$

Example

Let X be a smooth variety over $\mathbb{C}$. We have

$$H^i_{\text{et}}(X,\mathbb{Z}/n\mathbb{Z}) \cong H^i_{\text{sing}}(X,\mathbb{Z}/n\mathbb{Z}).$$

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$$H^i_{\text{et}}(X, \mathbb{Z}/n\mathbb{Z}) \cong H^i_{\text{sing}}(X, \mathbb{Z}/n\mathbb{Z}).$$

Zariski topology is too far away from singular topology. Let X be a variety over $\mathbb{C}$ of dimension d and $\mathscr{F}$ a Zariski sheaf on X. We have $H^r(X, \mathscr{F}) = 0$ for $r > d$.

Definition

Let X be a variety over a field K and let ℓ be a prime number invertible in K. The group

$$H^i_{\text{et}}(X, \mathbb{Q}_\ell) = \varprojlim_n H^i_{\text{et}}(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is called the *i-th ℓ-adic cohomology* of X.

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All $H^*_{\text{et}}(X, \mathbb{Q}_\ell)$ are finitely dimensional $\mathbb{Q}_\ell$-vector spaces.

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All $H^*_{\text{et}}(X, \mathbb{Q}_\ell)$ are finitely dimensional $\mathbb{Q}_\ell$-vector spaces.

Example

We fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. For any smooth variety over $\mathbb{C}$, we have

$$H^i_{\text{et}}(X, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell \cong H^i_{\text{sing}}(X, \mathbb{C}).$$

Let X be a variety over $\mathbb{F}_q$. We put $\overline{X} = X \times_{\operatorname{Spec}(\mathbb{F}_q)} \operatorname{Spec}(\overline{\mathbb{F}}_q)$. The topological generator of $\operatorname{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ induces a $\overline{\mathbb{F}}_p$-morphism $\mathrm{F} : X \to X$, called *relative Frobenius*. It induces linear maps

$$\mathrm{F}^* : H^i_{\text{et}}(\overline{X}, \mathbb{Q}_\ell) \to H^i_{\text{et}}(\overline{X}, \mathbb{Q}_\ell)$$

Approaches to Weil conjectures

Theorem (Lefschetz fix point theorem, due to Grothendieck)

Let $f : Y \to Y$ be an endomorphism of the projective variety Y over an algebraically closed field K with finitely many fix point. Let ℓ be a prime number invertible in K. Then

$$|Y^f| = \sum_{i=0}^{2\dim Y} (-1)^i \mathrm{tr}(f^* | H^i_{\mathrm{et}}(Y, \mathbb{Q}_\ell)).$$

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Let X be a projective and smooth variety over $\mathbb{F}_q$ of dimension d. Applying Lefschetz fix point theorem, we have

$$\sharp(X(\mathbb{F}_{q^m})) = |\overline{X}^{\mathrm{F}^n}| = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(\mathrm{F}^{*n} | H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell))$$

Corollary (Grothendieck)

Let X be a projective and smooth variety over $\mathbb{F}_q$ of dimension d. Then,

$$Z(X,T) = \exp\left(\sum_{n=1}^{\infty} \sharp(X(\mathbb{F}_{q^n}))\frac{T^n}{n}\right)$$

$$= \prod_{i=0}^{2d} \det(1 - \mathrm{F}^* T | H^i_{\mathrm{et}}(\overline{X}, \mathbb{Q}_\ell))^{(-1)^{i+1}}$$

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It answers (A) and a part of (C) of Weil conjectures.

Theorem (Poincaré duality, due to Grothendieck)

Let Y be a projective and smooth variety over an algebraically closed field K of dimension d. Let ℓ be a prime number invertible in K. Then,

$$H^i_{\text{et}}(Y, \mathbb{Q}_\ell) \times H^{2d-i}_{\text{et}}(Y, \mathbb{Q}_\ell)(d) \xrightarrow{\cup} H^{2d}_{\text{et}}(Y, \mathbb{Q}_\ell)(d) \xrightarrow{\text{tr}} \mathbb{Q}_\ell$$

are perfect parings of $\mathbb{Q}_\ell$-vector spaces.

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Let X be a smooth projective variety over $\mathbb{F}_q$ of dimension d. If $\alpha_1, \ldots, \alpha_s$ are eigenvalues of F^* in $H^i_{\text{et}}(\overline{X}, \mathbb{Q}_\ell)$ iff $q^d/\alpha_1, \ldots, q^d/\alpha_s$ are eigenvalues of F^* in $H^{2d-i}_{\text{et}}(\overline{X}, \mathbb{Q}_\ell)$.

Corollary (Grothendieck)

Let X be a smooth projective variety over $\mathbb{F}_q$ of dimension d. Then, $Z(X,T)$ has a functional equation

$$Z(X, \frac{1}{q^d T}) = \varepsilon(X)(q^d T)^{\chi(X)} Z(X,T)$$

where

$$\chi(X) = \sum_{i=0}^{2d} (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{\text{et}}(\overline{X}, \mathbb{Q}_\ell)$$

is the Euler-Poincaré characteristic of X and

$$\varepsilon(X) = \prod_{i=0}^{2d} \det(-\mathrm{F}^* | H^i_{\text{et}}(\overline{X}, \mathbb{Q}_\ell))^{(-1)^{i+1}}$$

is called the epsilon factor of X.

It answers (B) of Weil conjecture.

Theorem (*"Sur la conjecture de Weil I"*, Deligne 1973)

Let X be a projective and smooth variety over $\mathbb{F}_q$. Then each polynomial $P_i(T) = \det(1 - \mathrm{F}^* T | H^i_{\text{et}}(\overline{X}, \mathbb{Q}_\ell))$ has integral coefficients independent of ℓ ($\ell \neq p$) and each complex root of $P_i(T)$ is an algebraic number of absolute value $q^{-i/2}$ ($i = 0, 1, \dots, 2d$).

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Deligne's theorem fully solved Weil conjectures.

Let X be a variety over $\mathbb{F}_q$ and let $\mathscr{G}$ be an étale $\mathbb{Q}_\ell$-sheaf. For each closed point $x \in X$, the relative Frobenius F_x act on the étale stalk $\mathscr{G}_{\overline{x}}$.

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Definition

Let X be a variety over $\mathbb{F}_q$.

(1) An étale $\mathbb{Q}_\ell$-sheaf $\mathscr{G}$ on X is called *pure of weight* $r \in \mathbb{Q}$ if, for every closed point $x \in X$, all eigenvalues of F_x on $\mathscr{G}|_{\overline{x}}$ are algebraic numbers ($\in \overline{\mathbb{Q}}$) and all their complex conjugations have absolute value $(\sharp k(x))^{r/2}$.

(2) An étale $\mathbb{Q}_\ell$-sheaf $\mathscr{G}$ on X is called *mixed of weight* $\leq r \in \mathbb{Q}$, if it has a finite filtration whose successive quotients are pure of weight $\leq r$.

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The constant sheaf $\mathbb{Q}_\ell$ on X is pure of weight 0.

Theorem (key theorem of Deligne)

Let $f : X \to Y$ be a morphism of varieties over $\mathbb{F}_q$ and let $\mathscr{F}$ be a $\mathbb{Q}_\ell$-sheaf on X. Suppose $\mathscr{F}$ is mixed of weight $\leq r$. Then, for any $i \geq 0$, $Rf^i_!\mathscr{F}$ is mixed of weight $\leq r + i$.

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- Laumon gives a simpler proof of key theorem by Deligne-Fourier transform for étale sheaves.

Further developments

Perverse sheaves (Bernstein-Beilinson-Deligne-Gabber)

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- Intersection cohomology
- Geometric Satake in the geometric Langlands correspondence

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Étale cohomology of schemes and moduli spaces

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Cohomology theories

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Thank you!