Understanding Decoherence without Density Matrix

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1 Introduction

Quantum decoherence is often misunderstood as solving the measurement problem in quantum mechanics. This note aims to clarify the mathematical framework of decoherence and its implications for quantum measurement and quantum computing.

2 The Original Quantum System

Consider a quantum system S with basis states $\{|x\rangle\}$. The most general pure state of this system is:

$$|h\rangle = \sum_x a(x)\,|x\rangle \tag{1}$$

where $a(x)$ are complex amplitudes satisfying $\sum_x |a(x)|^2 = 1$.

3 The Environment/Measurement Device

The environment (or measurement device) E is itself a quantum system with basis states $\{|e(x)\rangle\}$. We make the following assumptions:

1. The environment states are orthonormal:

$$\langle e(x)|e(y)\rangle = \delta_{xy} \tag{2}$$

2. The environment is initially in a superposition:

$$|e\rangle = \sum_x b(x)\,|e(x)\rangle \tag{3}$$

where $|b(x)|^2 = 1$ for all x, ensuring equal probabilities.

4 Environment-Induced Superselection

A crucial assumption in decoherence theory is that the joint system-environment state $|x_1\rangle |e(x_2)\rangle$ can only exist when $x_1 = x_2$. This is not just a mathematical convenience but reflects the physical reality that the environment "measures" the system in a specific basis.

Mathematically:

$$|x_1\rangle |e(x_2)\rangle = 0 \text{ if } x_1 \neq x_2 \tag{4}$$

5 System-Environment Interaction

When the system interacts with the environment, we get:

$$|H\rangle = |h\rangle |e\rangle \to |he\rangle = \sum_x a(x)b(x) |x\rangle |e(x)\rangle \tag{5}$$

Several key observations:

1. The amplitude for each basis state $|x\rangle$ is modified by $b(x)$
2. Since $|b(x)|^2 = 1$, $b(x)$ only affects the phase
3. If the phases of $b(x)$ are random, the phase relationships between different $a(x)$ become inaccessible
4. The probabilities $|a(x)|^2$ remain unchanged

6 Observable Expectation Values

A particularly illuminating way to understand decoherence is through the calculation of expectation values for observables.

6.1 Pure State Case

For a pure state $|h\rangle = \sum_x a(x) |x\rangle$, the expectation value of an observable O is:

$$\langle h| O |h\rangle = \sum_{x,y} a^*(x)a(y) \langle x| O |y\rangle \tag{6}$$

Alternatively, using the density matrix formalism:

$$\langle h| O |h\rangle = \text{tr}(O |h\rangle \langle h|) = \text{tr}(O\rho) \tag{7}$$

where $\rho = |h\rangle \langle h|$ is the density matrix. While mathematically equivalent, the direct calculation using state vectors often provides more direct physical insight.

Note the presence of cross-terms ($x \neq y$) that depend on the phases of the amplitudes $a(x)$. This is characteristic of quantum interference.

6.2 Entangled State Case

For an entangled state $|H\rangle = \sum_x a(x)b(x)\,|x\rangle\,|e(x)\rangle$, the expectation value becomes:

$$\langle H|\,O\,|H\rangle = \sum_{x,y} a^*(x)b^*(x)a(y)b(y)\,\langle x|\,O\,|y\rangle\,\langle e(x)|e(y)\rangle \tag{8}$$

$$= \sum_x |a(x)|^2|b(x)|^2\,\langle x|\,O\,|x\rangle \tag{9}$$

$$= \sum_x |a(x)|^2\,\langle x|\,O\,|x\rangle \tag{10}$$

where we used $\langle e(x)|e(y)\rangle = \delta_{xy}$ and $|b(x)|^2 = 1$.

This calculation reveals several profound insights:

1. Cross-terms vanish due to the orthogonality of environment states
2. Only classical probabilities $|a(x)|^2$ appear, not complex amplitudes
3. Phase information in both $a(x)$ and $b(x)$ becomes inaccessible
4. The Born rule classical probabilities emerge naturally, but the Born rule itself must still be assumed

This direct calculation provides a clearer understanding of decoherence than the density matrix formalism, showing explicitly how entanglement with the environment leads to classical-like behavior while preserving quantum coherence in the full state.

7 Conclusion

Decoherence provides a profound mathematical framework for understanding how quantum systems interact with their environment. The calculation $\langle H|\,O\,|H\rangle = \sum_x |a(x)|^2\,\langle x|\,O\,|x\rangle$ beautifully demonstrates how the orthogonality of environmental states naturally eliminates quantum interference terms from our observations.

However, it's crucial to understand that decoherence does not solve the measurement problem or explain wavefunction collapse. Here's why:

1. The quantum state remains a superposition

   $$|H\rangle = \sum_x a(x)b(x)\,|x\rangle\,|e(x)\rangle$$

   This is still a pure quantum state - we've just moved the superposition to a larger Hilbert space.

2. The Born rule must still be postulated

   - While we naturally get $|a(x)|^2$ terms in our calculations
   - We still need the Born rule to interpret these as probabilities

- The theory doesn't explain why we observe specific outcomes with these probabilities

3. The and/or problem remains

   - Decoherence explains why we can't observe quantum interference
   - But it doesn't explain why we experience a single definite outcome
   - The transition from "and" (superposition) to "or" (definite outcome) remains mysterious

What decoherence does provide is a precise mechanism for how quantum systems appear classical through environmental interactions. The quantum coherence isn't lost - it becomes inaccessible due to entanglement with environmental degrees of freedom. This understanding is crucial for quantum computing, where maintaining coherence requires careful isolation from environmental interactions.

The elegance of decoherence theory lies in showing exactly how quantum behavior becomes hidden from our view, while leaving the deeper mysteries of quantum measurement intact. This reminds us that analyzing the mechanics of measurement is not the same as solving the measurement problem.