Vertex Algebras and String Theory in Monstrous Moonshine

THESIS

submitted in partial fulfillment of the
requirements for the degree of

BACHELOR OF SCIENCE
in
PHYSICS AND MATHEMATICS

Author :	L. M. Grotenhuis
Student ID :	s1868578
Supervisors :	Prof.dr. K.E. Schalm,
	Dr. R.I. van der Veen

Leiden, The Netherlands, June 19, 2020

Vertex Algebras and String Theory in Monstrous Moonshine

L. M. Grotenhuis

Huygens-Kamerlingh Onnes Laboratory, Leiden University
P.O. Box 9500, 2300 RA Leiden, The Netherlands

June 19, 2020

Abstract

Monstrous moonshine describes the unexpected relation between the modular J-function and the largest sporadic simple group known as the Monster. Both objects arise naturally in a conformal vertex algebra $V^\natural$; the J-function as the character and the Monster as the automorphism group. This vertex algebra can be interpreted as the quantum theory of a bosonic string living on a $\mathbb{Z}_2$-orbifold of spacetime compactified by the Leech lattice. We will discuss basic properties of vertex algebras and construct vertex algebras associated to even lattices, which form the main building block of $V^\natural$. We will calculate the characters of these vertex algebras and discuss their modular properties. Subsequently, we will develop the quantum theory of free bosonic open and closed strings, including the toroidal and orbifold compactificaton of the latter. We will conclude with calculating the one-loop partition function of the closed string on the $\mathbb{Z}_2$-orbifold compactified by the Leech lattice, and show it to be equal to the J-function.

Contents

Version of June 19, 2020– Created June 19, 2020 - 11:32

Introduction

The term *monstrous moonshine* popped up for the first time as title of a paper by Conway and Norton in 1979 [CN79]. 'Monstrous' referred to an extraodinary algebraic structure, namely the largest sporadic finite simple group $\mathbb{M}$, also known as *the Monster*. The name was not given lightly, as the order of $\mathbb{M}$ is

$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

which is roughly $8 \cdot 10^{53}$. The sporadic groups are a category in the classification of all finite simple groups. To recall, simple groups are groups with exactly two normal subgroups, namely the trivial group and itself. In a way, finite simple groups form the building blocks of all finite groups, as every finite group G allows a *composition chain*: a finite set of groups $G_1, \ldots, G_n$ satisfying

$$1 = G_1 \subset \cdots \subset G_n = G,$$

where G_i is normal in G_{i+1} and G_{i+1}/G_i is simple for all $i \in \{1\ldots, n-1\}$. The simple groups G_{i+1}/G_i are called composition factors. Even though a finite group might have more than one composition chain, the Jordan-Hölder theorem implies that every composition chain involves the same composition factors, that is, the same simple groups [Lan02]. A natural question is therefore whether one can produce a list of *all* the finite simple groups. It took most of the 20th century, a vast number of mathematicians and a proof of several thousand pages long to arrive at the following classification theorem: *Every finite simple group is isomorphic to either (i) a cyclic group of prime order, (ii) an alternating group A_n with $n \geq 5$, (iii) a group of Lie type, or (iv) one of the 26 sporadic simple groups* [Wil09]. The sporadic groups are the ‘odd ones out’ that do not belong to any of the infinite families described in (i)-(iii), and of these special finite simple groups, the Monster is the largest one.

‘Moonshine’ referred to the unexpected, and at that time mostly speculative, connection between the sporadic groups and a completely different class of objects, namely modular functions. Modular functions are complex functions that are invariant under certain transformations, and play a crucial role in number theory. In their paper, Conway and Norton proposed a deep connection between representations of the Monster group and certain modular functions, the *j-function* in particular. This became known as the *Monstrous Moonshine Conjecture*, and was eventually proven by Borcherd in 1992 [Bor92], building on contributions from many other mathematicians such as Kac, Frenkel, Lepowsky and Meurman. Remarkably, the proof makes use of results from modern physics, namely the no-ghost theorem in string theory. The road to the proof involved an interplay between many special structures from both mathematics and physics, such as unimodular lattices, modular forms, Lie algebras, conformal field theory, vertex algebras and string theory. The term ‘monstrous’ is therefore quite fitting in a second sense, given the wide spectrum of theories that are connected to the conjecture. We do not have a full understanding of moonshine yet, and certainly will not manage to give one in this thesis. Instead, we will focus on the role of vertex algebras in monstrous moonshine and their connection to string theory. First, however, in order to understand the full statement of the conjecture and its connection to string theory, several of the above-mentioned topics require a more in-depth introduction.

Modular functions

Modular functions can be understood by means of complex lattices and tori. A *complex lattice* L is a subgroup of $\mathbb{C}$ of rank 2 that spans $\mathbb{C}$, so we have

$$L = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$$

for certain $\lambda_1, \lambda_2 \in \mathbb{C}^\times$ (where $\mathbb{C}^\times$ denotes the group of units in $\mathbb{C}$, i.e. $\mathbb{C} \setminus \{0\}$) with $\frac{\lambda_1}{\lambda_2} \notin \mathbb{R}$. Given such a lattice, we obtain a *complex torus* by taking the quotient $\mathbb{C}/L$. This quotient describes a compact *Riemann surface*, that is, a connected complex 1-manifold. In simple terms, this means that $\mathbb{C}/L$ locally looks like $\mathbb{C}$. When tori are viewed as a real manifolds (so locally homeomorphic to $\mathbb{R}^2$) there is essentially only *one* torus, as any two real tori can be smoothly deformed into each other. When tori are viewed as complex manifolds, these deformations need to be *holomorphic*, and thus necessarily preserve angles. In this way, different complex lattices can give rise to different complex tori. A morphism ϕ between two complex tori $\mathbb{C}/L$ and $\mathbb{C}/L'$ is by definition a (locally) holomorphic map. One can prove that any such ϕ must be given by $\phi(z+L) = \alpha z + \beta + L'$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha L \subset L'$, and that ϕ is invertible if and only if $\alpha L = L'$ [Shu20]. Isomorphic complex tori are often called *conformally equivalent*, since conformal maps are by definition angle-preserving. As we will see (and as the name suggests), these maps play a fundamental role in 2-dimensional conformal field theory.

We would like to have a simple description of each isomorphism class of complex tori. Let L be a complex lattice with basis vectors λ_1, λ_2 and consider the torus $\mathbb{T} := \mathbb{C}/L$. Without loss of generality, we may assume that $\tau := \frac{\lambda_1}{\lambda_2}$ lies in the upper half-plane $\mathbb{H} := \{c \in \mathbb{C} : \mathrm{Im}(c) > 0\}$, also known as the Poincaré plane. Since scaling of L will leave $\mathbb{T}$ invariant (up to isomorphism), we may exchange L for the lattice L_τ with basis vectors τ and 1. $\mathbb{T}$ is now defined by one complex number, namely τ, which is often called the *modular parameter*. Conversely, every $\tau \in \mathbb{H}$ describes a complex torus $\mathbb{C}/L_\tau$. Now suppose we have $\tau, \tau' \in \mathbb{H}$ such that $\mathbb{C}/L_\tau$ and $\mathbb{C}/L_{\tau'}$ are isomorphic. Then there exists an $\alpha \in \mathbb{C}$ with $L_\tau = \alpha L_{\tau'}$ and thus we have

$$\begin{pmatrix}\tau\\1\end{pmatrix} = M\begin{pmatrix}\alpha\tau'\\\alpha\end{pmatrix} \text{ and } \begin{pmatrix}\alpha\tau'\\\alpha\end{pmatrix} = M'\begin{pmatrix}\tau\\1\end{pmatrix} \text{ for some } M, M' \in \mathrm{Mat}(2,\mathbb{Z}).$$

One easily checks that the matrices M and M' are each other's inverse. Since the coefficients are integer, this implies $\det M = \pm 1$. Writing $M = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, we obtain

$$\tau = \frac{\tau}{1} = \frac{a\alpha\tau' + b\alpha}{c\alpha\tau' + d\alpha} = \frac{a\tau' + b}{c\tau' + d}.$$

By requiring $\frac{a\tau'+b}{c\tau'+d} \in \mathbb{H}$ and using that $\tau' \in \mathbb{H}$, one finds that $ad - bc = \det M = 1$. This implies that M is an element of the *special linear group* $\mathrm{SL}_2(\mathbb{Z})$. We conclude that $\tau, \tau' \in \mathbb{H}$ define the same torus if and only if

$$\tau = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \cdot \tau' := \frac{a\tau' + b}{c\tau' + d} \text{ for some } \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

(Strictly speaking we only showed the 'only if' part; the other implication should be easy to see.) The expression above defines an action of $\mathrm{SL}_2(\mathbb{Z})$

on $\mathbb{H}$. Since for any $M \in \mathrm{SL}_2(\mathbb{Z})$, the matrix -$M$ acts identically on $\mathbb{H}$, it is more natural to consider the action of the quotient group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$ on $\mathbb{H}$. This is known as the *modular group*, which we will denote as Γ. We thus find that the isomorphism classes of complex tori correspond to the orbits of $\mathbb{H}$ under the action of the modular group.

Given the action of Γ on $\mathbb{H}$, it is natural to look for functions on $\mathbb{H}$ that are invariant under this action. Generally, a *modular function* is defined as a meromorphic function f on $\mathbb{H}$ that satisfies $f(\tau) = f(g \cdot \tau)$ for all $g \in \Gamma$ and in addition is 'meromorphic at the cusp $i\infty$'. Before explaining this last criterium, note that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, so any modular function f is periodic as it satisfies $f(\tau) = f(\tau + 1)$. This implies that f has a Fourier expansion $f(\tau) = \sum_{n=-\infty}^{\infty} c_n q^n$ for certain $c_n \in \mathbb{C}$, where we write $q = e^{i2\pi\tau}$. For f to be 'meromorphic at $i\infty$' then means that f, viewed as function of q, has a pole at $q = 0$. In other words, its q-expansion is of the form

$$f(\tau) = \sum_{n=M}^{\infty} c_n e^{i2\pi\tau n} = \sum_{n=M}^{\infty} c_n q^n$$

for some $M \in \mathbb{Z}$ [Mil17].

Modular functions are easy to define, yet apriori quite hard to construct. Therefore they are often expressed in terms of functions that are not Γ-invariant yet do behave 'nicely' under the action of Γ. These functions are the *modular forms*: holomorphic functions f on $\mathbb{H}$ that satisfy $f(g \cdot \tau) = (c\tau + d)^{2k} f(\tau)$ for some $k \in \mathbb{Z}$ and for all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and that in addition are holomorphic at the cusp $i\infty$ (i.e. holomorphic at $q = 0$). The integer k is called the *weight* of the modular form. Examples of modular forms are the *modular discriminant* Δ of weight 12 given by

$$\Delta(\tau) = \eta(\tau)^{24} = q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

where $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ is the *Dedekind eta-function*, and the *lattice theta functions* $\Theta_L(\tau)$ of weight $n/2$ given by

$$\Theta_L(\tau) = \sum_{\lambda \in L} q^{(\lambda,\lambda)/2}$$

for some even unimodular lattice L of rank n [Ser12]. In this context, a lattice L is defined as a free abelian group of finite rank equipped with a bilinear symmetric form $(\cdot,\cdot) : L \times L \to \mathbb{Q}$. Such a lattice is called *even* if $(\lambda, \lambda) \in 2\mathbb{Z}$ for all $\lambda \in L$, and *unimodular* if its Gram determinant equals 1.

One can construct modular functions by taking the quotient of two modular forms of the same weight. An important modular function that can be obtained this way is the *j-function*. It can be defined as

$$j(\tau) = \frac{\Theta_{E_8 \times E_8 \times E_8}(\tau)}{\Delta(\tau)},$$

where E_8 is the unique even unimodular lattice of rank 8 [FLM89]. Although the construction of j takes some effort, it turns out that with j we are actually done: one can show that the set of all modular functions is equal to the field $\mathbb{C}(j)$ generated by j [Apo12]. In other words, every modular function is a rational function of j, and vice versa. Another remarkable property of j is that the coefficients of its Fourier series are integers [MDG15]. The first few coefficients are given below:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

Up to addition by a constant, j is the unique modular function that is holomorphic on $\mathbb{H}$ and whose Fourier coefficients satisfy $c_{-1} = 1$ and $c_{-n} = 0$ for all $n > 1$*. Setting the constant term to zero, we obtain the *normalised j-function*

$$J(\tau) := j(\tau) - 744 = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

Note that nothing in our previous discussion requires the coefficients of modular functions to be integral. One is therefore led to wonder if these large numbers have any special meaning. As we will see in chapter 3, the coefficients of the Dedekind eta-function count the number of partitions of natural numbers. Do the coefficients of the J-function count something as well? The answer, as we now know, is yes: unexpectedly (and for a long time inexplicably) these coefficients turn out to count dimensions of certain 'natural' representations of the Monster.

Moonshine

The existence of the Monster group was predicted in 1973 by Fisher and Griess [FLM89], when the classification project of finite simple groups was

*This follows from the fact that the difference of any two such functions is a cusp form of weight 0, which is known to vanish [Ser12].

in full swing. Although it took another decade before Griess [Gri82] explicitly constructed $\mathbb{M}$ and thereby proved its existence, the conjectured properties of $\mathbb{M}$ were enough for Fisher, Livingstone and Thorne to calculate its character table in 1978. This table gives information about the representations of a finite group. A representation of a finite group G is a homomorphism $\rho : G \to \mathrm{GL}(V)$, where V is a finite-dimensional complex vector space (the standard definition is more general, but we will restrict ourselves to this one). Such a representation is called irreducible if V contains no proper subspaces that are closed under the action of G. Often ρ is left implicit and the representation is imply denoted as V. To each representation, one can assign a *character* $\chi_V : G \to \mathbb{C}$ defined by $g \mapsto \mathrm{Tr}(\rho(g))$. Standard results in representation theory state that every representation is a direct sum of irreducible ones and that characters take constant values on conjugacy classes. Moreover, the number of conjugacy classes of a group equals the number of irreducible representations up to isomorphism [Lan02].

From the character table, one can deduce the dimensions of all irreducible representations. The Monster has 194 irreducible representations and the first few dimensions d_i (in increasing order) of these are

$$d_0 = 1,\ d_1 = 196883,\ d_2 = 21296876,\ d_3 = 842609326,$$

where d_0 corresponds to the trivial representation. At first sight, these large numbers do not seem to have any special meaning. However, in 1978 John Mckay noticed that

$$c_1 = 196884 = 1 + 196883 = d_0 + d_1,$$

where we let c_n denote the n-th coefficient of $J(\tau)$. Soon thereafter, Thompson [Tho79b] used the character table of $\mathbb{M}$ to obtain

$$\begin{aligned} c_2 &= d_0 + d_1 + d_2, \\ c_3 &= 2d_0 + 2d_1 + d_2 + d_3, \end{aligned}$$

and similar expressions for c_4 and c_5. This led to the question whether there exist representations V_n of $\mathbb{M}$ with $\dim V_n = c_n$, such that their direct sum $V^\natural = \bigoplus_{n=-1}^{\infty} V_n$ forms a somehow 'natural' representation of the Monster. Any vector space of the form $\bigoplus_{i\in\mathbb{Z}} W_i$ is called $\mathbb{Z}$-graded, and its *graded dimension* is the series $\sum_{i\in\mathbb{Z}} \dim W_i q^i$. For this 'natural' representation $V^\natural$, the graded dimension is equal to

$$J(\tau) = \sum_{n=-1}^{\infty} c_n q^n = \sum_{n=-1}^{\infty} \dim V_n q^n = \sum_{n=-1}^{\infty} \chi_{V_n}(1) q^n.$$

In [Tho79a], Thompson generalised this by studying for each $g \in \mathbb{M}$ the series

$$J_g(\tau) := \sum_{n=-1}^{\infty} \chi_{V_n}(g) q^n,$$

which appeared to have modular properties as well. These are now called *Thompson series*. As conjugate elements of $\mathbb{M}$ give the same series, there are at most 194 Thompson series.

Thompson's work led Conway and Norton to their monstrous moonshine conjecture given in [CN79]:

Conjecture (Conway & Norton, 1979). *There exists a 'natural' representation $V^\natural = \bigoplus_{n=-1}^{\infty} V_n$ of the Monster group such that*

$$J(\tau) = \sum_{n=-1}^{\infty} \dim V_n q^n$$

and such that for each $g \in \mathbb{M}$ the series $J_g(\tau) := \sum_{n=-1}^{\infty} \chi_{V_n}(g) q^n$ is a normalised generator of a genus zero function field.

A *genus zero function field* is a field of 'modular' functions that are not necessarily Γ-invariant, but invariant under the action of a discrete subgroup $\Gamma' \subset \mathrm{PSL}_2(\mathbb{R})$ for which (some compactification of) the quotient $\mathbb{H}/\Gamma'$ defines a compact Riemann surface of genus zero, that is, a surface homeomorphic to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. One can show that the j-function defines a homeomorphism between $\mathbb{H}/\Gamma \cup \{i\infty\}$ and the Riemann sphere [Apo12]. The set of all modular functions is therefore a genus zero function field, and the J-function is its normalised generator. As $J(\tau) = J_1(\tau)$, this is in line with the conjecture. However, the conjecture makes a much stronger statement: even if $g \in \mathbb{M}$ is not the unit element, the series $J_g(\tau)$ generates a field of functions that are 'modular' with respect to an appropriate subgroup of $\mathrm{SL}_2(\mathbb{R})$. All this modular information is carried by the conjectured representation $V^\natural$ of $\mathbb{M}$, which therefore must have a very rich structure. This structure turned out to be that of a *vertex algebra*: an infinite-dimensional vector space equipped with a multiplication that depends on a formal parameter. As we will see, these algebras are the mathematical equivalent of a conformal field theory, and giving a full definition takes some effort.

The proof of the conjecture makes extensive use of the theory of Lie algebras. A *Lie algebra* is a vector space equipped with a certain bilinear operation $[\cdot,\cdot]$ called the *Lie bracket*. In the 1970s, Kac developed a theory of so-called *affine Kac-Moody* Lie algebras and their representations, and showed that their characters have modular properties. Soon after, representations of affine Lie algebras were constructed in [LW78] using a kind of differential operators, which were recognised as the *vertex operators* used in string theory to describe scattering amplitudes. The vertex operator representation essentially gave a correspondence $L \mapsto V_L$ between lattices and graded vector spaces [FLM89]. The details of this construction will be given in section 3.1. For $L = E_8 \times E_8 \times E_8$ (the lattice we used to define the j-function), the graded dimension of V_L is equal to $j(\tau)$. Another lattice with ties to the Monster is the *Leech lattice* Λ: the unique 24-dimensional even unimodular lattice containing no elements of norm 2. It was used in Griess' first construction of $\mathbb{M}$, and the graded dimension of its associated space V_Λ is equal to $J(\tau) + 24$. The next step in proving the conjecture was to adjust V_Λ to get rid of the constant 24. To this end, Frenkel, Lepowsky and Meurman constructed a new vertex operator representation $L \mapsto V_L'$, called the *twisted* representation. They conjectured that the right space was

$$V^\natural = V_\Lambda^+ \oplus V_\Lambda'^+ ,$$

where the '$+$' indicates an invariant subspace under certain involutions of V_Λ and V_Λ'. They showed that the graded dimension of $V^\natural$ indeed equals $J(\tau)$ and that the Monster group can be realised as the full automorphism group of $V^\natural$ when this space is equipped with a vertex algebra structure [FLM84]. $V^\natural$ is therefore known as the *Monster vertex algebra*. The last piece of the puzzle was to show that $V^\natural$ indeed gave rise to the right Thompson series $J_g(\tau)$, in which Borcherds succeeded in 1992 [Bor92]. The proof involves his self-built theory on *generalized* Kac-Moody Lie algebras and an extension of $V^\natural$ to such an algebra, called the *Monster Lie algebra*. For the construction of the latter, Goddard and Thorn's no-ghost theorem from string theory was used, making the result a truly shared accomplishment of both mathematics and physics.

In this thesis

We have seen that monstrous moonshine interlinks a wide spectrum of special structures and general theories. One of the crucial players (if not the most crucial) are vertex algebras. Vertex algebras are now understood as the mathematical equivalent of a 2-dimensional conformal field theory. Essentially, this is a quantum field theory in a 2-dimensional space that is invariant under transformations that preserve angles. Given this interpretation, it is not surprising that modular functions are connected to vertex algebras. In string theory, strings moving through spacetime trace out a 2-dimensional surface, and the fields on this surface give rise to a conformal field theory. Remarkably, FLM's construction of $V^\natural$ via twisting is now understood as the algebraic counterpart of *orbifolding* in string theory, a concept that was only introduced after the construction of $V^\natural$. More precisely, the vertex algebra $V^\natural$ can be viewed as the quantum theory of a bosonic string living on an orbifold that has been *compactified* (i.e. folded in into a torus) using the Leech lattice [Tui92]. The J-function is then obtained as the *one-loop partition function* of this string. Recall that the dimension of the Leech lattice is 24; this compactification therefore only makes sense if this string moves in a 24-dimensional space. Surprisingly, bosonic string theory is *only* consistent in exactly 24 (free) dimensions, as we will see in chapter 4.

The purpose of this thesis is to give an introductory account of the two different faces of vertex algebras, namely the algebraic one and the string theoretic one, and their connection to monstrous moonshine. To this end, it is naturally divided into two parts: chapters 1-3 address the algebraic aspects and chapters 4-6 focus on string theory.

The first chapter is fully dedicated to the definition of a vertex algebra. It starts with a brief description of conformal field theory, which will motivate our algebraic definitions throughout the chapter. In chapter 2, we encounter our first example, namely the *Heisenberg vertex algebra*. In chapter 3, we will associate vertex algebras to lattices and discuss the modular properties of their characters. Here we will encounter V_Λ and find that its graded dimension equals $J(\tau) + 24$.

After the first three chapters, we move towards string theory. As we do not assume any prior knowledge of string theory, chapter 4 provides an introduction to bosonic strings. We solve the equations of motion for open strings, develop the quantum theory by means of the light cone gauge and obtain the critical dimension. In chapter 5, we generalise our results to closed strings and discuss two kinds of compactifications for closed strings, namely via lattices and via orfibolds. We then have enough equipment to calculate the one-loop partition function of the string theory associated with $V^\natural$, which is done in chapter 6. In this last chapter, we first discuss general string diagrams and how vertex operators were initially introduced to calculate their amplitudes. We then restrict ourselves to the one-loop diagram of closed strings and calculate the partition function of the string theory associated with $V^\natural$. As it should, this function will be equal to the J-function.

Part I

Vertex Algebra Theory

Chapter 1

Defining vertex algebras

As discussed in the introduction, vertex algebras can be viewed as the mathematical description of a 2-dimensional conformal field theory. We will first briefly discuss the main features of a conformal field theory; this discussion is largely based on [Rib14] and [FS03]. Subsequently, we will see how these properties are made precise in the definition of a conformal vertex algebra. The definitions in this chapter are based on the text [FBZ04] by Frenkel and Ben-Zhu. The decomposition theorem comes from [Noz08].

1.1 Conformal field theory as motivation

A *conformal field theory* is a *quantum field theory* whose symmetries include conformal mappings. Of course, this only shifts the question. What is a quantum field theory? In essence, it is a probabilistic theory in which the main objects of study are functions defined on some space M. These functions are called *fields*. A typical choice for M would be 4-dimensional spacetime, and a typical field would be the electromagnetic vector field $\vec{E}$. The fields can be thought of as a description of the (elementary) particles in the space M; in particular, the creation and annihilation of particles is governed by interactions of the fields.* Now, any physical theory should contain quantities that can be measured in experiment. For a quantum

*One can view interacting fields as interfering waves. In this way, the description of particles in terms of fields is a manifestiation of the wave-particle duality in quantum mechanics.

field theory, these are given by the *space of states* and the *correlation functions*. The space of states is a vector space (typically a Hilbert space) which represents all possible states of the physical system the theory describes. The correlation functions realise the probabilistic nature of the theory: they provide the probability amplitude for moving from one state to another.

Typically, states are written as $|v\rangle$ and are interpreted as particle states, that is, a certain particle configuration in M. The fields are represented by linear operators that act on the space of states. One state, usually denoted as $|0\rangle$, represents the *vacuum*, which should be interpreted as the absence of particles. From the vacuum, we can create particle states by letting linear (field) operators act on $|0\rangle$.

Physicists often speak of 'solving' a quantum field theory, which means finding the space of states and calculating the probability amplitudes. The standard way to tackle this, is to identify the symmetries of the system, decide how we want our fields to transform under these symmetries, and then to write down an appropriate *Lagrangian* L for the fields. The correlation functions are then obtained from the Langrangian by means of *path integrals* [Zee03]. We will use the Lagrangian method to obtain the theory of the free bosonic string in chapter 4. We will not discuss path integrals in detail; however, they will be mentioned in chapter 6 when we calculate the one-loop probability amplitude for the closed string.

Symmetry is a central concept in modern physics. The term generally refers to the invariance of a theory under certain transformations, however the precise meaning of this usually depends on the context. For our purposes, we can define a symmetry of a field theory as an invertible mapping of the underlying space M onto itself under which the *action* S, as induced by the Langrangian L, remains invariant.[†] Via composition, these maps naturally form a group, which is known as the *symmetry group*. Often, the group elements can be described by a continuous parameter. For example, let us consider a theory that is invariant under rotations, where M is equal to the complex plane $\mathbb{C}$. Then for each $t \in \mathbb{R}$, we obtain an element $z \mapsto e^{it}z$ of the symmetry group. Due to this continuous parameterisation, a symmetry group often carries the structure of a *Lie group*: a group that is also a smooth real manifold. Unsurprisingly, Lie groups are closely connected to Lie algebras.

Definition 1.1. *(Lie algebra) A Lie algebra is an algebra* $\mathfrak{g}$ *whose product, denoted*

[†]This is known as *spacetime symmetry*. One can also have *internal symmetries*, which are transformations of the *fields* (instead of the underlying space) leaving the action invariant.

by $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$*, is anticommutative and satisfies the Jacobi-identity*

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

for all $x, y, z \in \mathfrak{g}$*. The product map* $[\cdot,\cdot]$ *is referred to as the Lie bracket.*

As we will see, Lie algebras and their representations play a fundamental role in the theory of vertex algebras. For readers unfamiliar with Lie algebras, some basic definitions and results that are relevant for this text can be found in the appendix.

Every Lie group gives rise to a Lie algebra in a canonical manner [FS03]. The vector space of this induced algebra is the tangent space of the unit element of the Lie group. Thus, thinking of the group elements as symmetry maps, the elements of the Lie algebra can informally be viewed as *infinitesimal* symmetry maps, that is, maps that are 'infinitely close' to the identity[‡]. A fundamental result of Lagrangian mechanics is *Noether's theorem*, which states that every infinitesimal symmetry transformation gives rise to a conserved quantity. For example, rotation invariance of a system results in the conservation of angular momentum. In the quantum theory, these conserved quantities become operators whose commutation relations generally correspond to those of the Lie algebra induced by the symmetry group[§]. The important takeaway of this is that the Hilbert space of the quantum theory will necessarily be a *representation* of the Lie algebra that describes its symmetries. For example, any relativistic quantum theory must display a representation of the *Lorentz Lie algebra*, the algebra of infinitesimal Lorentz transformations. This requirement leads to the critical dimension of bosonic string theory, which will be discussed in section 4.9.

For a conformal field theory, the symmetries are *conformal maps*. We will restrict ourselves to two dimensions and take M to be a 2-dimensional Riemannian manifold, that is, a real smooth manifold equipped with a positive definite metric.

[‡]Infinitesimal transformations are a common tool for physicists, who tend to be very comfortable and skilled in dealing with them, whereas mathematicians often feel uneasy by their informal approach. For readers sharing this unease, a mathematically rigorous approach to infinitesimal transformations is given in chapter 2 of [BK89]

[§]I say 'generally', as certainly not all theories follow this pattern precisely. Symmetries can not always be described by Lie groups, and sometimes only an extended version of the Lie algebra is found in the quantum theory. See [FS03] for a thorough introduction on symmetries and Lie algebras.

Definition 1.2. *(Conformal map) A conformal map ϕ between Riemannian manifolds (M, g) and (N, h) is a smooth map $\phi : M \to N$ such that there exists a smooth function $\lambda : M \to \mathbb{R}$ with*

$$\forall x \in M, \forall v, w \in T_x M : h_{\phi(x)}(\phi'(x)v, \phi'(x)w) = \lambda^2(x) \cdot g_x(v, w). \tag{1.1}$$

We thus see that the pull-back metric of a conformal map is (locally) equal to the original metric scaled by a positive real number. This implies that conformal maps preserve *angles*, since the cosine of the angle α between two smooth curves $\beta, \gamma : (0, 1) \to M$ that intersect at some point $x = \beta(t) = \gamma(t) \in M$ is defined by

$$\cos\alpha := \frac{g_x(\beta'(t), \gamma'(t))}{\sqrt{g_x(\beta'(t), \beta'(t)) g_x(\gamma'(t), \gamma'(t))}} = \frac{\langle \beta'(t), \gamma'(t) \rangle_x}{\|\beta'(t)\|_x \, \|\gamma'(t)\|_x}.$$

When we interchange g with the pull-back metric in the equation above, the factor $\lambda^2(x)$ will cancel out.

When M is simply the Euclidean plane $\mathbb{R}^2$, the conformal maps from M to itself can be viewed as complex functions by identifying $\mathbb{R}^2$ with $\mathbb{C}$. One can show that the conformal maps are then *holomorphic* and *antiholomorphic* functions on $\mathbb{C}$ [vK20]. This is not surprising, as holomorphic functions preserve angles and orientation, whereas antiholomorphic functions preserve angels and invert orientation. For a conformal field theory in two dimensions, it is therefore more natural to view M as a Riemann surface, that is, a 1-dimensional *complex* manifold. The symmetry maps are then simply holomorphic and antiholomorphic functions, so these conformal theories can benefit from the elegant and rich theory of complex analysis.

The Lie algebra of infinitesimal holomorphic transformations is the *Witt algebra* $\mathfrak{W}$ [BLT12]. This is a complex algebra generated by an infinite basis $\{l_n : n \in \mathbb{Z}\}$ whose Lie bracket is given by

$$[l_m, l_n] = (m - n) l_{m+n}, \quad \forall n, m \in \mathbb{Z}.$$

We get another copy $\overline{\mathfrak{W}}$ of the Witt algebra for the antiholomorphic transformations, whose generators are conventionally denoted by $\bar{l}_n$. The full Lie algebra of the infinitesimal conformal transformations (for a conformal field theory in two dimensions) is then $\mathfrak{W} \oplus \overline{\mathfrak{W}}$ with $[l_m, \bar{l}_n] = 0$.

The Witt algebra allows for a unique non-trivial *central extension*. Details of this construction can be found in [FLM89], but for our purposes it is sufficient to say that this amounts to adding a non-trivial term to the Lie brackets that commutes with all the generators. The result is the *Virasoro algebra*.

Definition 1.3. *(Virasoro algebra) For $c \in \mathbb{C}^\times$, the Virasoro algebra $\mathfrak{Vir}_c$ with central charge c is the Lie algebra spanned by the elements $\mathbb{1}$ and L_n, $n \in \mathbb{Z}$, with the Lie bracket given by $[\mathbb{1}, L_n] = 0$ and*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\mathbb{1}. \tag{1.2}$$

For any two $c, c' \in \mathbb{C}^\times$, $\mathfrak{Vir}_c$ and $\mathfrak{Vir}_{c'}$ are isomorphic as Lie algebras, so there is indeed only one Virasoro algebra. The factor $\frac{c}{12}$ is a convention from quantum field theory, motivated by the fact that if we let the 'new' element $\mathbb{1}$ act as the identity, representations of the Virasoro algebra usually take this form. The number c is called the *central charge* of the representation.

Interestingly, it is the Virasoro algebra, and not the Witt algebra, that is represented by the space of states V of a 2-dimensional conformal field theory. This is due to the fact that strictly speaking, the Witt algebra should act on the *projective* space $\mathbb{P}(V)$, since the probability amplitude of moving from one state $|v_1\rangle$ to another $|v_2\rangle$ is independent of the scaling of $|v_1\rangle$ and $|v_2\rangle$. When we lift this to an action on V in an appropriate way, we obtain an action of the Virasoro algebra [Sch08].

The space of states will be a representation of the direct sum of two copies of the Virasoro algebra, typically denoted as $\mathfrak{Vir}_c \oplus \overline{\mathfrak{Vir}}_{\bar{c}}$, one for each copy of the Witt algebra. In chapter 5, we will see how this happens for the free bosonic closed string. Often, the space V can be factorised into a representation of $\mathfrak{Vir}_c$ and a representation of $\overline{\mathfrak{Vir}}_{\bar{c}}$, referred to as the *chiral* and *antichiral* part of the theory, respectively. It usually suffices to study the chiral part, as the dynamics of the antichiral part are essentially the same. In this chapter, we will work our way up to the definition of *conformal vertex algebras*, which give an algebraic description of the chiral part of a 2-dimensional conformal field theory.

1.2 Formal power series

Let us start building towards the definition of a vertex algebra. From now until chapter 4, all algebras and vector spaces will be taken over the field $\mathbb{C}$ unless stated otherwise.

The main objects in vertex algebras, namely fields, will be expressed in terms of formal power series.

Definition 1.4. *(Formal power series) Let R be an algebra and let z be a formal variable. We define $R[[z^{\pm 1}]]$ as the set of R-valued formal power series in z, that is, elements of the form $\sum_{n\in\mathbb{Z}} a_n z^n$ with $a_n \in R$.*

More formally, an R-valued power series $\sum_{n\in\mathbb{Z}} a_n z^n$ can be viewed as a function $\mathbb{Z} \to R$. Note that the above definition is easily generalized to any finite number of variables. To illustrate, we take elements of $R[[z^{\pm 1}, w^{\pm 1}]]$ to be of the form $\sum_{n,m\in\mathbb{Z}} a_{n,m} z^n w^m$, or equivalently to be functions $\mathbb{Z} \times \mathbb{Z} \to R$.

Since R is a $\mathbb{C}$-algebra, $R[[z^{\pm 1}]]$ is readily made into a $\mathbb{C}$-vector space with coefficient-wise addition and scalar multiplication. We can identify the following subspaces: $R[z]$ denotes the polynomials in z, $R[[z]]$ the power series with only positive powers in z, and $R((z))$ the power series containing only a finite number of terms with negative powers of z. We can multiply in $R[[z]]$ as follows:

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{m=0}^{\infty} b_m z^m\right) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m z^{n+m} \\ &= \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i b_{k-i}\right) z^k \in R[[z]].\end{aligned}$$

In the same way, we can multiply in $R[z]$ and $R((z))$. If R is a ring, note that this multiplication gives these three subspaces a ring structure as well. We cannot, however, extend this to a well-defined multiplication in $R[[z^{\pm 1}]]$, since in that case the coefficients of the product will become infinite sums of coefficients of the initial terms. These products are therefore in general not well-defined. Meanwhile, we can always multiply formal power series in *different* formal variables z and w: for $\sum_{n\in\mathbb{Z}} a_n z^n \in R[[z^{\pm 1}]]$ and $\sum_{m\in\mathbb{Z}} b_m w^m \in R[[w^{\pm 1}]]$ we can view their product

$$\left(\sum_{n\in\mathbb{Z}} a_n z^n\right)\left(\sum_{m\in\mathbb{Z}} b_m z^m\right) = \sum_{n,m\in\mathbb{Z}} a_n b_m z^n w^m$$

as an element of $R[[z^{\pm 1}, w^{\pm 1}]]$.

One important formal power series in the study of vertex algebras is the formal delta function.

Definition 1.5. *(Delta function) The formal delta function $\delta(z,w)$ in $\mathbb{C}[[z^{\pm 1}, w^{\pm 1}]]$ is defined by*

$$\delta(z,w) = \sum_{n\in\mathbb{Z}} z^{-n-1} w^n = \sum_{n\in\mathbb{Z}} z^n w^{-n-1}.$$

As the name suggests, this power series comprises the defining property of the well-known Dirac delta distribution. For a fixed $w \in \mathbb{C}^\times$, the series $\delta(z,w)$ becomes an element of $\mathbb{C}[[z^{\pm 1}]]$. We can view an element $f(z)$ in $\mathbb{C}[[z^{\pm 1}]]$ as a functional (or distribution) on $\mathbb{C}[z^{-1}, z]$, that maps any meromorphic function $g(z)$ to

$$\mathrm{Res}_z\, f(z)g(z) \in \mathbb{C}.$$

Here Res_z denotes the residue at $z = 0$ just as in complex analysis, so for any series $a(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ we have $\mathrm{Res}_z\, a(z) = a_{-1}$. Now for any series $g(z) = \sum_k g_k z^k$ in $\mathbb{C}[z^{-1}, z]$ we have

$$\begin{aligned}\mathrm{Res}_z\, \delta(z,w)g(z) &= \mathrm{Res}_z \sum_{n\in\mathbb{Z}} \sum_k g_k z^{k+n} w^{-n-1} \\ &= \mathrm{Res}_z \sum_{m\in\mathbb{Z}} \left(\sum_k g_k w^{-m+k-1} \right) z^m = \sum_k g_k w^k = g(w),\end{aligned}$$

and thus the Cauchy Residue Theorem implies

$$\frac{1}{2\pi i} \oint g(z)\delta(z,w)dz = g(w),$$

where we integrate over a closed contour that winds once around zero. This is the complex generalisation of the real Dirac delta distribution characterised by $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$.

For any formal series $a(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, we can define its (partial) derivative with respect to z in the familiar way:

$$\partial_z a(z) = \sum_{n\in\mathbb{Z}} n a_n z^{n-1}.$$

One easily checks that this formal derivative satisfies the Leibniz rule. The partial derivatives of the delta function satisfy a recursive relation which will be useful in proving the decomposition theorem in the next section.

Lemma 1.6. *For $n \geq 1$, we have $(z-w)\frac{1}{n!}\partial_w^n \delta(z,w) = \frac{1}{(n-1)!}\partial_w^{n-1}\delta(z,w)$.*

Proof. We use induction on n. For $n = 1$, we have

$$\begin{aligned}(z-w)\partial_w \delta(z,w) &= (z-w) \sum_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1} \\ &= \sum_{n\in\mathbb{Z}} n z^{-n} w^{n-1} - \sum_{n\in\mathbb{Z}} n z^{-n-1} w^n \\ &= \sum_{n\in\mathbb{Z}} (n+1) z^{-(n+1)} w^n - \sum_{n\in\mathbb{Z}} n z^{-n-1} w^n = \delta(z,w).\end{aligned}$$

Assuming the equality holds for some $n \geq 1$, we find

$$\begin{aligned}
&(z-w)\frac{1}{(n+1)!}\partial_w^{n+1}\delta(z,w) \\
&= \partial_w\left((z-w)\frac{1}{(n+1)!}\partial_w^n\delta(z,w)\right) + \frac{1}{(n+1)!}\partial_w^n\delta(z,w) \\
&= \partial_w\left(\frac{1}{(n+1)}\frac{1}{(n-1)!}\partial_w^{n-1}\delta(z,w)\right) + \frac{1}{(n+1)!}\partial_w^n\delta(z,w) \\
&= \frac{1}{n!}\partial_w^n\delta(z,w).
\end{aligned}$$

□

1.3 Fields and locality

We can now define our *fields* as formal power series with linear maps as coefficients.

Definition 1.7. *(Field) Let V be a vector space. A field is a formal power series $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n}$ in* End $V[[z^{\pm 1}]]$ *satisfying the following condition: for all $v \in V$ there exists an $N \in \mathbb{Z}$ such that $A_n \cdot v = 0$ for all $n \geq N$.*

The name 'field' should not be confused with the algebraic notion of a field, that is, with a commutative division ring. As we always take algebras en vector spaces over $\mathbb{C}$ and will not be concerned with other algebraic fields, no problems should arise here.

Fields can be viewed as operators, as we can let a field $A(z)$ act on our vector space V by $A(z) \cdot v = \sum_{n\in\mathbb{Z}}(A_n \cdot v)z^{-n} \in V[[z^{\pm 1}]]$. Note that the condition for a field then becomes: for all $v \in V$ we have $A(z) \cdot v \in V((z))$. Fields can thus be viewed as linear maps from V to $V((z))$. Moreover, the choice for a minus in the power of z is a mere convention coming from quantum theory, where operators with positive indices are generally viewed as 'annihilation operators'. These will be further discussed in the chapters on string theory.

For two fields $A(z)$ and $B(w)$ in different formal variables, we want to formulate a notion of commutativity. This notion is referred to as *locality*, as it reflects the idea that two fields at different locations do not directly influence each other.

Definition 1.8. *(Locality) Two fields $A(z)$ and $B(w)$ in different formal variables z and w are called local with respect to each other if there exists an $N \in \mathbb{N}$ such that*

$$(z-w)^N [A(z), B(w)] = 0$$

in End $V[[z^{\pm 1}, w^{\pm 1}]]$.

Here the square brackets denote the commutator of $A(z)$ and $B(w)$, which is well-defined in End $V[[z^{\pm 1}, w^{\pm 1}]]$. Note that locality is in fact a weaker form of commutativity, as $(z-w)^N A(z,w) = 0$ does not necessarily imply $A(z,w) = 0$ for a formal power series $A(z,w)$. Indeed, we have $\delta(z,w) \neq 0$ and

$$\begin{aligned}(z-w)\delta(z,w) &= (z-w) \sum_{m \in \mathbb{Z}} z^{-m-1} w^m \\ &= \sum_{m \in \mathbb{Z}} z^{-m} w^m - \sum_{m \in \mathbb{Z}} z^{-(m+1)} w^{m+1} = 0.\end{aligned}$$

When studying the structure of vertex algebras, one is often interested in determining the commutator $[A(z), B(z)]$ of two fields that are local with respect to each other. A nice property of these commutators is that they can be decomposed in a sum over derivatives of the delta function. The result below is sometimes referred to as the *decomposition theorem* [Noz08].

Theorem 1.9. *Let R be an algebra and let $a(z,w) \in R[[z^{\pm 1}, w^{\pm 1}]]$. Suppose there exists an $N \in \mathbb{N}$ such that $(z-w)^N a(z,w) = 0$. Then we have*

$$a(z,w) = \sum_{j=0}^{N-1} \frac{1}{j!} c^j(w) \partial_w^j \delta(z,w)$$

where $c^j(w) = Res_z (z-w)^j a(z,w)$.

Proof. Consider $a(z,w) = \sum_{n,m \in \mathbb{Z}} a_{n,m} z^n w^m$ and $N \in \mathbb{N}$ such that $(z-w)^N a(z,w) = 0$ and define $c^j(w)$ as in the theorem. If $N = 1$, we have $(z-w)a(z,w) = 0$ which implies $a_{n-1,m} = a_{n,m-1}$ and thus $a_{n,m} = a_{-1,n+m+1}$. So we obtain

$$\begin{aligned}a(z,w) &= \sum_{n,m \in \mathbb{Z}} a_{-1,n+m+1} z^n w^m = \sum_{k,n \in \mathbb{Z}} a_{-1,k} z^n w^{k-n-1} \\ &= \sum_{k \in \mathbb{Z}} a_{-1,k} w^k \sum_{n \in \mathbb{Z}} z^n w^{-n-1} = (\mathrm{Res}_z a(z,w))\, \delta(z,w).\end{aligned}$$

Now for general N, we will use the above result to prove by induction that for $1 \leq m \leq N$ we have

$$(z-w)^{N-m}\left(a(z,w)-\sum_{i=1}^{m} c^{N-i}(w)\frac{1}{(N-i)!}\partial_w^{N-i}\delta(z,w)\right)=0. \tag{1.3}$$

The theorem then follows from the case $m = N$. The proof by induction is given by (i) and (ii) below.

(i) We first show that (1.3) holds for $m = 1$. Since $(z-w)^N a(z,w) = 0$ and the theorem holds for $N = 1$, we can apply the theorem to $(z-w)^{N-1}a(z,w)$. We find

$$\begin{aligned}(z-w)^{N-1}a(z,w) &= \left(\mathrm{Res}_z (z-w)^{N-1}a(z,w)\right)\delta(z,w)\\ &= c^{N-1}(w)\delta(z,w).\end{aligned}$$

By applying lemma 1.6 repeatedly, we find

$$\delta(z,w) = (z-w)^{N-1}\frac{1}{(N-1)!}\partial_w^{N-1}\delta(z,w),$$

and thus it follows that (1.3) holds for $m = 1$.

(ii) Suppose (1.3) holds for some $m \in \{1,\ldots,N-1\}$. Again using that the theorem holds for $N = 1$, it follows that

$$\begin{aligned}(z-w)^{N-m-1}&\left(a(z,w)-\sum_{i=1}^{m} c^{N-i}(w)\frac{1}{(N-i)!}\partial_w^{N-i}\delta(z,w)\right)\\ &= \mathrm{Res}_z\Bigg[(z-w)^{N-m-1}\Bigg(a(z,w)\\ &\qquad -\sum_{i=1}^{m} c^{N-i}(w)\frac{1}{(N-i)!}\partial_w^{N-i}\delta(z,w)\Bigg)\Bigg]\delta(z,w).\end{aligned} \tag{1.4}$$

Note that for all $1 \leq i \leq m$ we have

$$\mathrm{Res}_z\left[(z-w)^{N-m-1}\partial_w^{N-i}\delta(z,w)\right] = 0,$$

since we can use lemma 1.6 to get rid of the term $(z-w)^{N-m-1}$ and one easily sees that $\mathrm{Res}_z\partial_w^k\delta(z,w) = 0$ for all $k \geq 0$. It follows that (1.4) equals $c^{N-(m+1)}(w)\delta(z,w)$. If we now apply lemma 1.6 for $N - (m+1)$ times to the delta function again, we find that (1.3) is satisfied by $m+1$.

□

1.4 Definition of a vertex algebra

Now we are finally in the position to define a vertex algebra.

Definition 1.10. *A vertex algebra consists of the following objects:*

- *(Space of states) a vector space V;*
- *(Vacuum state) a vector $|0\rangle \in V$;*
- *(Translation operator) a linear map $T \in End\ V$;*
- *(Vertex operators) for any formal variable z a linear map $Y(\cdot, z) : V \to (End\ V)[[z^{\pm 1}]]$ denoted by $A \mapsto \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$, such that $Y(A,z)$ is a field for all $A \in V$.*

These objects satisfy the following axioms:

(VA1) (Vacuum axiom) We have $Y(|0\rangle, z) = id_V$ and for all $A \in V$ we have $A_n|0\rangle = 0$ for $n \geq 0$ and $A_{-1}|0\rangle = A$.

(VA2) (Translation axiom) We have $T|0\rangle = 0$ and for all $A \in V$ we have $[T, Y(A,z)] = \partial_z Y(A,z)$.

(VA3) (Locality axiom) For all $A, B \in V$, the fields $Y(A,z)$ and $Y(B,w)$ are local with respect to each other.

Again, the choice to take z to the power of $-n-1$ instead of n is a convention inspired by quantum theory. In this way, coefficients with non-negative indices will act as annihilation operators and the (-1)-term becomes the 'creation operator' of the initial state.

In the realm of vertex algebras, we have natural analogues of standard algebraic notions such as subalgebras and homomorphisms.

Definition 1.11. *(Vertex subalgebra) For a vertex algebra $(V, |0\rangle, T, Y)$, a subspace $W \subset V$ is a vertex subalgebra if $|0\rangle \in W$ and if the restricted maps $T|_W$ and $Y(\cdot, z)|_W$ make U into a vertex algebra.*

Definition 1.12. *(Vertex algebra homomorphism) For two vertex algebras $(V_1, |0\rangle_1, T_1, Y_1)$ and $(V_2, |0\rangle_2, T_2, Y_2)$, a linear map $\phi : V_1 \to V_2$ is a vertex algebra homomorphism if $\phi(|0\rangle_1) = |0\rangle_2$ and for all $A, B \in V_1$ we have*

$$\phi(Y_1(A,z)B) = Y_2(\phi(A), z)\phi(B),$$

or equivalently, if for all $n \in \mathbb{Z}$ we have

$$\phi(A_n B) = (\phi(A))_n \phi(B).$$

Following the definition above, we define isomorphisms, endomorphisms and automorphisms of vertex algebras in the standard way. In particular, the automorphism group of a vertex algebra $(V, |0\rangle, T, Y)$ is the group consisting of the bijective vertex algebra homomorphisms from V to itself, with the composition of maps as group operation.

Note that (VA1) implies that the mapping $A \mapsto A_{-1}$ for $A \in V$ is injective, and thus the vertex operator map Y is injective. The next theorem shows that for any $A \in V$, the corresponding vertex operator $Y(A,z)$ is uniquely defined by how it acts on the vacuum. In this way, the map Y represents the *state-field correspondence* in conformal field theory.

Theorem 1.13. *(Goddard* [God89]*)* Let V be a vertex algebra and $F(z)$ a field on V that is local with respect to each vertex operator $Y(B,z)$ with $B \in V$. If there exists an $A \in V$ such that $F(z)|0\rangle = Y(A,z)|0\rangle$, then $F(z) = Y(A,z)$.

Proof. Let $B \in V$. By assumption, for large enough $N \in \mathbb{N}$ we obtain the equalities

$$\begin{aligned}(z-w)^N F(z) Y(B,w)|0\rangle &= (z-w)^N Y(B,w) F(z)|0\rangle \\ &= (z-w)^N Y(B,w) Y(A,z)|0\rangle \\ &= (z-w)^N Y(A,z) Y(B,w)|0\rangle.\end{aligned}$$

Note that $Y(B,w)|0\rangle \in V[[w]]$ due to the vacuum axiom, so we can evaluate the first and last expression in $w = 0$. This gives $z^N F(z) B = z^N Y(A,z) B$ for all $B \in V$, which implies $F(z) = Y(A,z)$. □

The proof of Theorem 1.13 also works for a general formal power series $A(z)$ (so not necessarily a field), if we extend our definition of locality to general power series in the obvious way.

A few comments can be made on the translation operator. It follows from the axioms that the action of T on V is in fact completely determined by the vertex operator map. Indeed, for any $A \in V$, we have

$$\begin{aligned}TA = TA_{-1}|0\rangle &= \left(\sum_{n\in\mathbb{Z}} TA_n|0\rangle z^{-n-1}\right)\Bigg|_{z=0} = TY(A,z)|0\rangle\big|_{z=0} \\ &= [T, Y(A,z)]|0\rangle\big|_{z=0} = \partial_z Y(A,z)|0\rangle\big|_{z=0} = A_{-2}|0\rangle.\end{aligned}$$

We could therefore exclude the transition operator from the definition. However, the convention is to include it, as T serves its own purpose as *generator* of translation in the formal variable z. One can derive from (VA1) and (VA2) that for all $A \in V$ we have $Y(A,z)|0\rangle = e^{zT}A \in V[[z]]$ (where the exponential is defined as $e^{zT} = \sum_{n=0}^{\infty} \frac{1}{n!}T^n z^n$), so it follows that $e^{wT}Y(A,z)|0\rangle = Y(A,z+w)|0\rangle$.

Theorem 1.13 provides us with an efficient way to prove equalities between vertex operators and fields, once we know they act identically on the vacuum. To determine whether this condition is indeed satisfied, the following lemma comes in handy.

Lemma 1.14. *Let W be a vector space and $\phi \in \operatorname{End} W$ a linear map. Then there exists a unique solution in $W[[z]]$ to the differential equation $\partial_z A(z) = \phi A(z)$ with initial condition $A(0) = w_0 \in W$,*

Proof. Writing $A(z) = \sum_{n=0}^{\infty} A_n z^n$, we find $\partial_z A(z) = \sum_{n=0}^{\infty}(n+1)A_{n+1}z^n$. Thus $\partial_z A(z) = \phi A(z)$ implies the recursive relation $(n+1)A_{n+1} = \phi A_n$ for $n \geq 0$. Given an initial condition $A_0 = w_0$, this relation fully determines $A(z)$. $\square$

1.5 Conformal vertex algebras

Our definition of a vertex algebra includes the key ingredients of a quantum field theory: a space of states V and the fields $Y(A,z)$ that act on this space. To describe a *conformal* field theory, we must have an action of the Virasoro algebra on V. In 2-dimensional conformal field theory, the Virasoro element L_{-1} arises as generator of translations and the element L_0 as generator of dilations, which in the radial picture are viewed as time translations [BLT12]. This suggests that L_0 corresponds to the quantum operator of the Hamiltonian. When we quantise the bosonic string, we will indeed find that this is the case. Viewing L_0 as the Hamiltonian, it is natural to write our space of states in terms of eigenstates of L_0, as these are the energy eigenstates. Ideally, we can write V as a direct sum of eigenspaces of L_0. Such a space is called *graded* and linear maps that respect its graded decomposition are called *homogeneous.*

Definition 1.15. *(Graded space) For I an arbitrary set, an I-gradation of a vector space V is a decomposition $V = \bigoplus_{i \in I} V_i$. Given such a decomposition, V is called I-graded and the elements of V_i are called homogeneous elements of degree i.*

Definition 1.16. *(Homogeneous map) A linear map $f : V \to W$ between I-graded vector spaces is called homogeneous if $f(V_i) \subset W_i$ for all $i \in I$. For $I = \mathbb{Z}$, we call f homogeneous of degree $m \in \mathbb{Z}$ if $f(V_n) \subset W_{n+m}$ for all $n \in \mathbb{Z}$.*

For a state in a given conformal field theory, the eigenvalue corresponding to L_0 is called the *conformal dimension* of this state. This conformal dimension is also reflected in the field that corresponds to this state.

Definition 1.17. *(Conformal dimension) Let V be a $\mathbb{Z}$-graded vector space. A field $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n}$ in $\operatorname{End} V[[z^{\pm 1}]]$ is called (homogeneous) of conformal dimension $\Delta \in \mathbb{Z}$ if for all $n \in \mathbb{Z}$, A_n is a homogeneous map of degree $-n + \Delta$.*

We can now define a *conformal vertex algebra* as a graded vertex algebra equipped with a specific action of the Virasoro algebra, such that the graded components correspond the eigenspaces of L_0 and such that L_{-1} acts as the translation operator.

Definition 1.18. *(Graded vertex algebra) A vertex algebra $(V, |0\rangle, T, Y)$ is called graded if V is $\mathbb{N}$-graded, $|0\rangle$ has degree zero, T is homogeneous of degree 1, and for all $A \in V_m$ the field $Y(A, z)$ is of conformal dimension m.*

Definition 1.19. *(Conformal vertex algebra) A graded vertex algebra $(V, |0\rangle, T, Y)$ is called conformal of central charge $c \in \mathbb{C}$ if there exists a conformal vector $\omega \in V_2$, which means that, if we write*

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \tag{1.5}$$

we have $L_{-1} = T$, $L_0 v = nv$ for all $v \in V_n$ and

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{1.6}$$

In other words, the L_n give an action of $\mathfrak{Vir}_c$ on V where L_0 acts as gradation operator and L_{-1} as translation operator.

In the literature, conformal vertex algebras are often referred to as *vertex operator algebras*. This can be confusing, as the more general vertex algebras from definition 1.10 certainly include vertex operators as well. We will therefore not be using this terminology.

We define a *subalgebra* of a conformal vertex algebra V as a vertex subalgebra that contains the conformal vector of V. Similarly, we define a *homomorphism* between conformal vertex algebras V_1 and V_2 with conformal vectors ω_1 and ω_2, respectively, as a vertex algebra homormophism that sends ω_1 to ω_2. Conformal vertex algebra isomorphisms and autmorhpisms are then defined in the obvious way. The monster vertex algebra $V^\natural$ constructed by Frenkel, Meurman and Lepowksy is a conformal vertex algebra, and its automorphism group is precisely the monster group $\mathbb{M}$.

Naturally, all these definitions beg for an example. Unfortunately, easy examples conformal vertex algebras are hard to come by. In the next chapter, we will carefully construct one of the more simple ones, namely the Heisenberg vertex algebra.

Chapter 2

Constructing a vertex algebra

In this chapter, we will encounter our first conformal vertex algebra, namely the *Heisenberg vertex algebra*. This fundamental structure will be the main building block for constructing the lattice vertex algebra in the next chapter. Our construction is based on [FBZ04] and theorem 2.8 can be found in [Noz08].

2.1 The Heisenberg vertex algebra

We will build the space of states using an action of the *Heisenberg Lie algebra*. This Lie algebra is found in the quantum field theories of free bosons. Indeed, we will come across it when study the quantum theory of the free bosonic string in chapter 4.

Definition 2.1. *The Heisenberg Lie algebra $\mathfrak{h}$ is the Lie algebra spanned by the elements $\mathbb{1}$ and b_n, $n \in \mathbb{Z}$, with Lie bracket given by $[\mathbb{1}, b_n] = 0$ and $[b_n, b_m] = n\delta_{n,-m}\mathbb{1}$.*

We now want to construct a 'simplest' representation $(\mathcal{F}, \rho)$ of $\mathfrak{h}$ in which $\mathbb{1}$ acts as the identity. Note that we cannot let all b_n act by zero or let $\mathcal{F}$ be one-dimensional, since in that case we would find

$$\rho(\mathbb{1}) = \rho([b_1, b_{-1}]) = \rho(b_1)\rho(b_{-1}) - \rho(b_{-1})\rho(b_1) = 0. \tag{2.1}$$

Also note that any representation of $\mathfrak{h}$ gives a representation of the universal enveloping algebra $U(\mathfrak{h})$. By the PBW-theorem (see the appendix), we know that $U(\mathfrak{h})$ has a basis

$$\{b_{n_1}b_{n_2}\dots b_{n_m}\mathbb{1}^k\}_{k\in\mathbb{N},n_1\leq n_2\leq\dots\leq n_m}.$$

This motivates us to set $\mathcal{F}=\mathbb{C}[b_{-n}:n>0]$ on which the b_n with negative indices simply act by multiplication, and the b_n with nonnegative indices annihilate the constants in $\mathcal{F}$. The action of the b_n with $n\geq 0$ on an arbitrary polynomial $p\in\mathcal{F}$ is determined by moving b_n in $b_np\in U(\mathfrak{h})$ to the right (that is, by writing b_np in terms of the PBW-basis). For example, the action of b_3 on $b_{-1}b_{-3}^2$ is given by

$$\begin{aligned}
b_3b_{-1}b_{-3}^2 &= b_{-1}b_3b_{-3}^2 = b_{-1}b_{-3}b_3b_{-3}+b_{-1}[b_3,b_{-3}]b_{-3}\\
&= b_{-1}b_{-3}^2b_3+b_{-1}b_{-3}[b_3,b_{-3}]+3b_{-1}b_{-3}\\
&= 0+3b_{-1}b_{-3}+3b_{-1}b_{-3}=6b_{-1}b_{-3}\in\mathcal{F}.
\end{aligned}$$

The resulting representation is referred to as the *Fock representation* of $\mathfrak{h}$ and is explicitly given by:

$$\rho:\mathfrak{h}\to\operatorname{End}\mathcal{F}\quad\text{with}\quad \rho(b_n)\cdot p=\begin{cases} b_np & \text{if } n<0,\\ n\partial_{b_{-n}}p & \text{if } n>0,\\ 0 & \text{if } n=0.\end{cases}\tag{2.2}$$

We now wish to endow $\mathcal{F}$ with a vertex algebra structure. We take the constant polynomial $1\in\mathcal{F}$ as vacuum vector $|0\rangle$ and we define an $\mathbb{N}$-gradation on $\mathcal{F}$ via $\deg|0\rangle=0$ and $\deg b_{k_1}b_{k_2}\dots b_{k_m}=-\sum_{i=1}^m k_i$. As we have seen, the translation operator is completely determined by the vertex operator map Y. By the vacuum axiom, we must have $Y(|0\rangle,z)=\mathrm{id}$. In order to further construct Y, we will start with the field $Y(b_{-1}|0\rangle,z)$ and use this as 'generator' of the other fields. We set

$$Y(b_{-1}|0\rangle,z)=\sum_{n\in\mathbb{Z}}b_nz^{-n-1},\tag{2.3}$$

which is clearly a field and satisfies (VA1). The proposition below shows it also satisfies (VA3).

Proposition 2.2. *The fields $Y(b_{-1}|0\rangle,z)$ and $Y(b_{-1}|0\rangle,w)$ are local with respect to each other.*

Proof. We have

$$\begin{aligned}
[Y(b_{-1}|0\rangle,z),Y(b_{-1}|0\rangle,w)] &= \sum_{n,m}[b_n,b_m]z^{-n-1}w^{-m-1}=\sum_n nz^{-n-1}w^{n-1}\\
&= \partial_w\sum_n z^{-n-1}w^n,
\end{aligned}$$

so with $(z-w)\delta(z,w) = 0$ and the product rule we find

$$\begin{aligned} 0 &= \partial_w \left((z-w)^2 \sum_n z^{-n-1} w^n \right) \\ &= -2(z-w) \sum_n z^{-n-1} w^n + (z-w)^2 [Y(b_{-1}, z), Y(b_{-1}, |0\rangle w)] \\ &= (z-w)^2 [Y(b_{-1}|0\rangle, z), Y(b_{-1}, w)]. \end{aligned}$$

□

For $k > 0$, the requirement $(b_{-k})_{-1}|0\rangle = b_{-k}$ from (VA2) motivates us to set

$$Y(b_{-k}|0\rangle, z) = \frac{1}{(k-1)!} \partial_z^{k-1} Y(b_{-1}|0\rangle, z). \tag{2.4}$$

Note that the right-hand side of (2.4) again defines a field on $\mathcal{F}$. Moreover, the fields $Y(b_{-k}, z)$ and $Y(b_{-m}, z)$ are local with respect to each other for any $k, m > 0$. This follows from the fact that if $(z-w)^N [A(z), B(w)] = 0$ for some $N \in \mathbb{N}$ and two fields $A(z)$ and $B(w)$, we obtain

$$\begin{aligned} &(z-w)^{N+1} [\partial_z A(z), B(w)] \\ &\qquad = (z-w)^{N+1} [\partial_z A(z), B(w)] + N(z-w)^N [A(z), B(w)] \\ &\qquad = (z-w) \partial_z \big((z-w)^N [A(z), B(w)] \big) = 0. \end{aligned}$$

Thus, since $Y(b_{-1}|0\rangle, z)$ is local with respect to itself, we find by induction that $\partial_z^{k-1} Y(b_{-1}|0\rangle, z)$ and $\partial_z^{m-1} Y(b_{-1}|0\rangle, z)$ are local as well.

We are now left to determine the fields corresponding to monomials $b_{k_1} b_{k_2} \ldots b_{k_m} |0\rangle$. Naively, one might set

$$Y(b_{k_1} b_{k_2} |0\rangle, z) = Y(b_{k_1}|0\rangle, z) Y(b_{k_2}|0\rangle, z).$$

However, the product on the right is not always well-defined in $\mathrm{End} V[[z^{\pm 1}]]$. We can fix this by changing the order of the operators $(b_{k_1})_n$ and $(b_{k_2})_n$ in the product $Y(b_{k_1}|0\rangle, z) Y(b_{k_2}|0\rangle, z)$, as to ensure that the annihilation operators will appear on the right and will thereby act first on a given vector. The following definition makes this procedure precise. Interestingly, the normal ordered product is more than just 'a' solution to the problem at hand. In the next section, we will see that it is actually the *only* way to define the vertex operators $Y(b_{k_1} b_{k_2} |0\rangle, z)$.

Definition 2.3. *(Normal ordered product) The normal ordered product of two fields $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$ and $B(z) = \sum_{n\in\mathbb{Z}} B_n z^{-n-1}$ is defined as the formal power series*

$$:A(z)B(w): = \sum_{n\in\mathbb{Z}} \left(\sum_{m<0} A_m B_n z^{-m-1} + \sum_{m\geq 0} B_n A_m z^{-m-1} \right) w^{-n-1} \in End\, V[[z^{\pm 1}, w^{\pm 1}]]. \tag{2.5}$$

A staightforward calculation gives the following lemma, which can be found in [FBZ04].

Lemma 2.4. *For two fields $A(z)$ and $B(z)$, the normal ordered product $:A(z)B(z):$ is again a well-defined field.*

One easily checks that the normal ordered product is a linear operation and satisfies the Leibniz rule, in the sense that

$$\partial_z :A(z)B(z): = :\partial_z A(z)B(z): + :A(z)\partial_z B(z):. \tag{2.6}$$

Moreover, it should be noted that the normal ordered product is generally not commutative nor associative. We therefore define

$$:A(z)B(z)C(z): = :A(z)(:B(z)C(z):):.$$

However, due to the simple Lie bracket of the Heisenberg Lie algebra, it is both commutative and associative when applied to $Y(b_{-1}|0\rangle, z)$ and its derivatives. For now, we can therefore use a much simpler definition: the normal ordered product

$$:\partial_z^{n_1} Y(b_{-1}|0\rangle, z) \cdots \partial_z^{n_p} Y(b_{-1}|0\rangle, z): \tag{2.7}$$

is obtained by replacing every monomial $b_{k_1} \cdots b_{k_m}$ with the monomial $:b_{k_1} \cdots b_{k_m}:$, which in turn is obtained by moving all b_{k_i} with positive indices to the right of those with negative indices.

With the normal ordered product at hand, we can finally define our vertex operator map Y by

$$Y(b_{-k_1} \cdots b_{-k_m}|0\rangle, z) = :Y(b_{-k_1}|0\rangle, z) \cdots Y(b_{-k_m}|0\rangle, z):. \tag{2.8}$$

To prove that (2.8) indeed provides $\mathcal{F}$ with a vertex algebra structure, we need one more lemma, often referred to as *Dong's Lemma*. It states that locality is preserved when taking normal ordered products. The proof can be found in [Kac97].

Lemma 2.5. *(Dong's Lemma) For $A(z)$, $B(z)$ and $C(z)$ pairwise local fields, the fields $:A(z)B(z):$ and $C(z)$ are again local with respect to each other.*

Theorem 2.6. *The $\mathbb{N}$-graded vector space $\mathcal{F}$ with vacuum vector $|0\rangle = 1$ and vertex operator map $Y(\cdot, z)$ given by (2.8) is a graded vertex algebra.*

Proof. From lemma 2.4, we know that $Y(\cdot, z)$ assigns a well-defined field to every vector in $\mathcal{F}$. Moreover, from proposition 2.2, the remarks below (2.4) and Dong's Lemma, it follows that all vertex operators are local with respect to each other. By definition we have $Y(|0\rangle, z) = \mathrm{id}$ and we know $Y(b_{-1}|0\rangle, z)$ satisfies (VA1). Now suppose $Y(A, z)$ satisfies (VA1) for some $A \in \mathcal{F}$ and let $k \in \mathbb{Z}_{>0}$. Then we have

$$Y(b_{-k}A, z) = :Y(b_{-k}|0\rangle, z)Y(A, z): = \frac{1}{(k-1)!} :\partial_z^{k-1} Y(b_{-1}|0\rangle, z)Y(A, z):$$

$$= \frac{1}{(k-1)!} \sum_{n\in\mathbb{Z}} \left(\sum_{m\le -k} \lambda_{m,k} b_m A_n z^{-m-k} + \sum_{m>-k} \lambda_{m,k} A_n b_m z^{-m-k} \right) z^{-n-1}$$

where $\lambda_{m,k}$ denotes $(-m-1)(-m-2)\cdots(-m-k+1)$. Note that $\lambda_{m,k} = 0$ if $-k+1 \le m \le -1$, and we know $b_m|0\rangle = 0$ for $m \ge 0$. By assumption, $A_n|0\rangle = 0$ for all $n \ge 0$, thus we find

$$Y(b_{-k}A, z)|0\rangle = \frac{1}{(k-1)!} \sum_{n<0, m\le -k} \lambda_{m,k} b_m A_n |0\rangle z^{-m-k-n-1} \in V[[z]]. \tag{2.9}$$

Evaluating (2.9) in $z = 0$ gives $(b_{-k}A)_{-1}|0\rangle = \frac{1}{(k-1)!}\lambda_{-k,k} b_{-k} A_{-1}|0\rangle = b_{-k}A$, so (VA1) holds for $Y(b_{-k}A, z)$ as well. The vacuum axiom then follows by induction.

To prove that $\mathcal{F}$ is a vertex algebra, we now only have to check the translation axiom (VA2). We could derive the action of translation operator T from (2.8), since earlier we showed that $TA = A_{-2}|0\rangle$ for all $A \in \mathcal{F}$. However, we will simply give a definition for T and prove that it works. We define T by $T|0\rangle = 0$ and

$$T \cdot b_{k_1} b_{k_2} \cdots b_{k_m}|0\rangle = -\sum_{i=1}^{m} k_i b_{k_1} \cdots b_{k_i - 1} \cdots b_{k_m}|0\rangle. \tag{2.10}$$

From (2.10), one obtains that $[T, b_n] = -nb_{n-1}$ for all $n \in \mathbb{Z}$. It follows that

$$\begin{aligned}
[T, \partial_z^m Y(b_{-1}|0\rangle, z)] &= \sum_{n\in\mathbb{Z}} (-n-1)\cdots(-n-m)[T, b_n] z^{-n-m-1} \\
&= \sum_{n\in\mathbb{Z}} (-n)\cdots(-n-m) b_{n-1} z^{-n-m-1} \\
&= \partial_z^{m+1} Y(b_{-1}|0\rangle, z)
\end{aligned}$$

for $m \geq 0$. Using the Leibniz rule for the normal ordered product in (2.6), if follows by induction that every vertex operator satisfies (VA2). Thus $(\mathcal{F}, |0\rangle, T, Y)$ is a vertex algebra.

Lastly, we show that the vertex algebra is graded. Since $\deg|0\rangle = 0$ by definition and (2.10) shows that $\deg T = 1$, it is left to show that if $A \in \mathcal{F}$ has degree $m \in \mathbb{N}$, then the field $Y(A,z)$ is of conformal dimension m. In other words, we have to show that every operator A_n of $Y(A,z)$ has degree $-n+m-1$. Note that the operators b_n satisfy $\deg b_n = -n$, thus $Y(b_{-1}|0\rangle, z)$ is indeed of conformal dimension 1. Since taking the derivative of a field results in shifting the operator from z^{-n} over to z^{-n-1}, one easily sees that $\partial_z^{k-1} Y(b_{-1}|0\rangle, z)$ has conformal dimension k. A simple calculation shows that the normal ordered product of two fields of conformal dimension m_1 and m_2, respectively, results in a homogeneous field of conformal dimension $m_1 + m_2$. This concludes the proof. □

2.2 Reconstruction theorem

Our construction of the Heisenberg vertex algebra can be generalized as follows.

Theorem 2.7. *(Reconstruction theorem) Let V be a vector space, $|0\rangle \in V \setminus \{0\}$ a vector, and $T \in \operatorname{End} V$ an endomorphism. Suppose we have a countable collection of linearly independent vectors v^m in V, with for each such vector a well-defined field $v^m(z) = \sum_{n\in\mathbb{Z}} v_n^m z^{-n-1}$. If these objects satisfy the vertex algebra axioms as far as a possible, that is*

- *for all m, $v^m(z)$ satisfies (VA1),*
- *T satisfies (VA2) for the given fields,*
- *the fields $v^m(z)$ are local w.r.t. each other,*

and moreover the vectors $v_{j_1}^{m_1} v_{j_2}^{m_2} \dots v_{j_k}^{m_k}|0\rangle$ with $j_i < 0$ span V, then V carries a unique vertex algebra structure such that $Y(v^m, z) = v^m(z)$, which is given by

$$Y(v_{j_1}^{m_1} v_{j_2}^{m_2} \dots v_{j_k}^{m_k}|0\rangle, z) = \frac{:\partial_z^{(-j_1-1)} v^{m_1}(z) \partial_z^{(-j_2-1)} v^{m_2}(z) \dots \partial_z^{(-j_k-1)} v^{m_k}(z):}{(-j_1-1)!(-j_2-1)! \dots (-j_k-1)!}. \tag{2.11}$$

The proof of 2.7 is analogous to that of the Heisenberg vertex algebra (see [Kac97]). The uniqueness of the vertex algebra structure in the reconstruction theorem is a consequence of the following result.

Theorem 2.8. *Let V be a vertex algebra and let $A, B \in V$. Write $A(z) = Y(A,z)$ and $B(z) = Y(B,z)$. Then we have*

$$Y(A_n B, z) = \begin{cases} \frac{1}{(-n-1)!} :(\partial_z^{-n-1} A(z))B(z): & \text{if } n < 0, \\ Res_w (w-z)^n [A(w), B(z)] & \text{if } n \geq 0. \end{cases} \tag{2.12}$$

Proof. We will prove the theorem for $n = -1$. The general case goes analogously (see [Noz08] for a full proof). We will first use lemma 1.14 to show that $Y(A_{-1}B,z)$ and $:A(z)B(z):$ act identically on the vacuum. By the vacuum axiom, we have $Y(A_{-1}B,z)|0\rangle \in V[[z]]$ and

$$:A(z)B(z):|0\rangle = \sum_{m<0} \sum_{k<0} A_k B_m |0\rangle z^{-k-m-2} \in V[[z]]. \tag{2.13}$$

Again using the vacuum axiom, we find

$$Y(A_{-1}B, 0)|0\rangle = A_{-1}B = A_{-1}B_{-1}|0\rangle = :A(z)B(z):|0\rangle\big|_{z=0}. \tag{2.14}$$

Now by the translation axiom,

$$\partial_z Y(A_{-1}B,z)|0\rangle = [T, Y(A,_{-1}B,z)]|0\rangle = TY(A_{-1}B,z)|0\rangle \tag{2.15}$$

and

$$\begin{aligned} \partial_z :A(z)B(z):|0\rangle &= :\partial_z A(z)B(z):|0\rangle + :A(z)\partial_z B(z):|0\rangle & (2.16)\\ &= :[T,A(z)]B(z):|0\rangle + :A(z)[T,B(z)]:|0\rangle & (2.17)\\ &= :(TA(z))B(z):|0\rangle = T:A(z)B(z):|0\rangle. & (2.18) \end{aligned}$$

It follows from lemma 1.14 that $Y(A_{-1}B,z)|0\rangle = :A(z)B(z):|0\rangle$. By Dong's lemma, $:A(z)B(z):$ is local with respect to all vertex operators, thus theorem 1.13 implies that $Y(A_{-1}B,z) = :A(z)B(z):$. □

By repeated application of (2.12) for $n < 0$, we obtain that the general vertex operators in the reconstruction theorem *must* be given by the expression in (2.11), once we set $Y(v^m, z) = v^m(z)$.

Corollary 2.9. *For V a vertex algebra and $A, B \in V$, the following hold:*

(1) $Y(A_{-1}B,z) = :Y(A,z)Y(B,z):$,

(2) $Y(TA,z) = \partial_z Y(A,z)$,

(3) $[Y(A,z),Y(B,w)] = \sum_{n\geq 0} \frac{1}{n!} Y(A_nB,w)\partial_w^n \delta(z,w)$,

(4) $[A_m,B_k] = \sum_{n\geq 0} \binom{m}{n} (A_nB)_{(m+k-n)}$ *for all* $m,k \in \mathbb{Z}$, *where we use a generalisation of the binomial coefficient defined by*

$$\binom{m}{n} = \frac{m(m-1)\ldots(m-n+1)}{n!} \text{ and } \binom{m}{0} = 1.$$

Proof. The first equality follows directly from theorem 2.8. The second does so as well, since earlier we saw that for all $A \in V$ we have $TA = A_{-2}|0\rangle$. The third property follows from theorem 2.8 together with theorem 1.9. The fourth property follows from the third together with the observation that

$$[Y(A,z),Y(B,w)] = \sum_{m,k\in\mathbb{Z}} [A_m,B_k] z^{-m-1} w^{-k-1},$$

and that for all $n \geq 0$ we have

$$\begin{aligned} Y(A_nB,w)\frac{1}{n!}\partial_w^n \delta(z,w) &= Y(A_nB,w) \sum_{m\in\mathbb{Z}} \binom{m}{n} z^{-m-1} w^{m-n} \\ &= \sum_{m,l\in\mathbb{Z}} (A_nB)_l \binom{m}{n} z^{-m-1} w^{m-n-l-1}. \end{aligned}$$

□

Property (4) of the corollary is particularly useful to determine whether a vertex algebra is conformal.

Proposition 2.10. *The Heisenberg vertex algebra* $\mathcal{F}$ *is has a conformal vector* $\omega = \frac{1}{2} b_{-1}^2 |0\rangle$ *with central charge 1.*

Proof. First note that $\omega = \frac{1}{2} b_{-1}^2 |0\rangle$ is indeed a vector of degree 2, as $\deg b_{-1}^2 = \deg b_{-1} + \deg b_{-1} = 2$. Following definition 1.19, let us write $Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. We have

$$\begin{aligned} Y(\omega,z) &= \frac{1}{2} {:}Y(b_{-1}|0\rangle,z) Y(b_{-1}|0\rangle,z){:} = \frac{1}{2} \sum_{n,m\in\mathbb{Z}} {:}b_n b_m{:} z^{-n-m-2} \\ &= \frac{1}{2} \sum_{n,k\in\mathbb{Z}} {:}b_{n-k} b_k{:} z^{-n-2} \end{aligned}$$

and thus $L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} : b_{n-k}b_k :$ for all $n \in \mathbb{Z}$. We first show that L_0 and L_{-1} act as the gradation and translation operator, respectively, and subsequently we check that the L_n satisfy the Virasoro relations.

For any polynomial $p \in \mathcal{F}$, we have

$$L_0 p = \frac{1}{2}\sum_{k\in\mathbb{Z}} :b_{-k}b_k: p = b_0^2 p + \sum_{k>0} b_{-k}b_k p = \sum_{k>0} k b_{-k}\frac{\partial p}{\partial b_{-k}},$$

so for $p = b_{-k_1}^{m_1} b_{-k_2}^{m_2} \dots b_{-k_n}^{m_n}|0\rangle$ we obtain $L_0 p = (\sum_{i=1}^n m_i \cdot k_i)\, p$. Thus L_0 indeed acts as gradation operator. To show that $L_{-1} = T$, it suffices to show that $L_{-1}|0\rangle = T|0\rangle = 0$ and $[L_{-1}, b_n] = [T, b_n] = -nb_{n-1}$ for all $n \in \mathbb{Z}$, as these relations fully determine the action of T on $\mathcal{F}$. Since $L_{-1} = \frac{1}{2}\sum_{k\in\mathbb{Z}} :b_{-k-1}b_k:$, we have $L_{-1}|0\rangle = 0$ and for any $n \in \mathbb{Z}$ we obtain

$$\begin{aligned}[L_{-1}, b_n] &= \frac{1}{2}\sum_{k\in\mathbb{Z}} [:b_{-k-1}b_k:, b_n] = [:b_{n-1}b_{-n}:, b_n] \\ &= :b_{n-1}[b_{-n}, b_n]: = -nb_{n-1} = [T, b_n].\end{aligned}$$

To determine the commutation relations of the L_n, we make use of corollary 2.9. Recall that we need to show that for any $m, k \in \mathbb{Z}$, the commutator $[L_m, L_k]$ satisfies

$$[L_m, L_k] = (m-k)L_{m+k} + \frac{1}{12}(m^3 - m)\delta_{m,-n}. \tag{2.19}$$

Note that $[L_m, L_k] = [\omega_{m+1}, \omega_{k+1}]$, so it follows from property (4) of corollary 2.9 that we have

$$[L_m, L_k] = \sum_{n\geq -1} \binom{m+1}{n+1} (L_n\omega)_{(m+k-n+1)}.$$

For n large enough, the terms in the sum on the right-hand side will be zero. Indeed, we have

$$L_n\omega = \frac{1}{4}\sum_{k\in\mathbb{Z}} :b_{n-k}b_k: b_{-1}^2|0\rangle,$$

and $:b_{n-k}b_k: b_{-1}^2|0\rangle$ equals zero when $k > 1$ and when $n - k > 1$. Thus one only needs to consider the cases where $n \leq 1 + k \leq 2$, i.e. we have

$$[L_m, L_k] = \sum_{n=-1}^{2} \binom{m+1}{n+1} (L_n\omega)_{(m+k-n+1)}.$$

Using our knowledge of L_{-1} and L_0, one quickly obtains that $L_{-1}\omega = T\omega = b_{-2}b_{-1}|0\rangle$ and $L_0 = 2\omega = b_{-1}^2|0\rangle$. A straightforward calculation further gives $L_1\omega = 0$ and $L_2\omega = \frac{1}{2}|0\rangle$, so we are left to determine the operators

$$(b_{-2}b_{-1}|0\rangle)_{(m+k+2)}, \ (b_{-1}^2|0\rangle)_{(m+k+1)} \text{ and } \frac{1}{2}(|0\rangle)_{(m+k-1)}.$$

Since $Y(|0\rangle, z) = \mathrm{id}$, we find $(|0\rangle)_{(m+k-1)} = \delta_{m+k-1,-1} = \delta_{m,-k}$, and thus

$$\binom{m+1}{3}(L_2\omega)_{(m+k-1)} = \frac{(m+1)m(m-1)}{3!}\frac{1}{2}\delta_{m,-k} = \frac{1}{12}\left(m^3 - m\right)\delta_{m,-k}.$$

This gives us the constant term of the Virasoro relation, that is, the second term in (2.19). Furthermore, we have

$$Y(b_{-1}^2|0\rangle, z) = \sum_{n,j\in\mathbb{Z}} {:}b_n b_j{:} z^{-(n+j)-2} = \sum_{n\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}} {:}b_{n-j-1}b_j{:}\right) z^{-n-1}$$

and thus $(b_{-1}^2|0\rangle)_{(m+k+1)} = 2L_{m+k}$. Lastly, we find

$$\begin{aligned} Y(b_{-2}b_{-1}|0\rangle, z) &= {:}Y(b_{-2}|0\rangle, z)Y(b_{-1}|0\rangle, z){:} \\ &= {:}\sum_{n\in\mathbb{Z}} -(n+1)b_n z^{-n-2}\sum_{j\in\mathbb{Z}} b_j z^{-j-1}{:} \\ &= \sum_{n,j\in\mathbb{Z}} -(n+1){:}b_n b_j{:}z^{-(n+j+2)-1} \\ &= \sum_{n,j\in\mathbb{Z}} -(n-j-1){:}b_{n-j-2}b_j{:}z^{-n-1} \end{aligned}$$

and thus

$$(b_{-2}b_{-1}|0\rangle)_{(m+k+2)} = \sum_{j\in\mathbb{Z}} -(m+k-j+1){:}b_{m+k-j}b_j{:}. \tag{2.20}$$

We need a little bit of work to see that the sum on the right-hand side is equal to the operator L_{m+k} multiplied by a constant. Fix $l \in \mathbb{Z}$. If $m+k-l = l$, then the term ${:}b_{m+k-l}b_l{:}$ occurs only once in the sum and the factor in front of it equals $-(m+k+2)/2$. Otherwise, the term ${:}b_{m+k-l}b_l{:}$ occurs twice in the sum, namely once for $j = l$ and once for $m+k-j = l$, i.e. for $j = m+k-l$. Adding the factors in front of these two terms, we obtain

$$-(m+k-l+1) - (m+k-(m+k-l)+1) = -(m+k+2).$$

We can therefore rewrite the sum in (2.20) to obtain

$$(b_{-2}b_{-1}|0\rangle)_{(m+k+2)} = -\frac{(m+k+2)}{2}\sum_{j\in\mathbb{Z}} :b_{m+k-j}b_j: = -(m+k+2)L_{m+k}.$$

Combining our results, we conclude that

$$\begin{aligned}&\binom{m+1}{1}(L_0\omega)_{(m+k+1)} + \binom{m+1}{0}(L_{-1}\omega)_{(m+k+2)} \\ &= (2m-(m+k+2))L_{m+k} = (m-k)L_{m+k},\end{aligned}$$

which equals the first term of the Virasoro relation in (2.19). □

With this theorem, we finally obtained our first conformal vertex algebra! As we saw, checking all the axioms by means of a brute-force calculation is rather tedious, even for a relatively simple conformal vertex algbra such as $\mathcal{F}$. Fortunately, the reconstruction theorem may significantly reduce these calculations, as it will in the next chapter, where we use $\mathcal{F}$ to construct a vertex algebra associated to an even lattice.

Chapter 3

Lattices and modular forms

In this chapter, we will associate conformal vertex algebras to even lattices and discuss the modular properties of their characters. We will encounter V_Λ, the vertex algebra associated to the Leech lattice, and find that its character equals $J(\tau) + 24$. Our construction follows that of [Kac97] and [LL04].

3.1 Lattices

Before constructing the vertex algebra, we need to address some basic terminology concerning lattices. The kind of lattices we will be considering are generally referred to as *rational lattices*.

Definition 3.1. *A (rational) lattice of rank $d \in \mathbb{N}$ is a free Abelian group L of rank d with a $\mathbb{Z}$-bilinear symmetric positive definite form $(\cdot, \cdot) : L \times L \to \mathbb{Q}$. Such a lattice is called integral if $(\cdot, \cdot)$ only takes values in $\mathbb{Z}$, and it is called even if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$.*

Note that any even lattice L is necessarily integral, since for any $\alpha, \beta \in L$ we have

$$(\alpha, \beta) = \frac{1}{2}\left((\alpha + \beta, \alpha + \beta) - (\alpha, \alpha) - (\beta, \beta)\right) \in \mathbb{Z}. \tag{3.1}$$

One might be more familiar with defining a lattice as a discrete subgroup of the Euclidean space $\mathbb{R}^d$. Indeed, any rational lattice L of rank d can always be embedded in $\mathbb{R}^d$ with the standard inner product $\langle \cdot, \cdot \rangle$, such that the restriction of $\langle\cdot,\cdot\rangle$ to L reduces to the lattice bilinear form $(\cdot,\cdot)$. This can be done as follows. Given L, we can define a real vector space $\mathbb{R}\otimes_{\mathbb{Z}} L$, on which $(\cdot,\cdot)$ induces an inner product via $(\lambda_1\otimes\alpha_1,\lambda_2\otimes\alpha_2)=\lambda_1\lambda_2(\alpha_1,\alpha_2)$. One may then use Gram-Schmidt orthogonalisation to obtain a orthonormal basis $(v_1,\ldots,v_d)$ of $\mathbb{R}\otimes_{\mathbb{Z}} L$ with respect to $(\cdot,\cdot)$. The mapping $\phi:\mathbb{R}\otimes_{\mathbb{Z}} L\mapsto\mathbb{R}^d$ defined by $v_i\mapsto e_i$ (where the e_i denote the standard basis vectors of $\mathbb{R}^d$) then forms an isometry and thereby the desired embedding. In this way, we can view any lattice as subset of $\mathbb{R}^d$, as we will often do.

Given a lattice L, any basis $(\alpha_1,\ldots,\alpha_d)$ of L defines a *Gram matrix* $A=(a_{ij})_{i,j=1}^d$ given by $a_{ij}=(\alpha_i,\alpha_j)$. While this matrix is basis-dependent, its determinant is not. This determinant can be interpreted as the squared volume of the quotient $\mathbb{R}^d/L$, that is, the squared volume of a unit cell spanned by a basis of L in $\mathbb{R}^d$. When $\det(A)=1$, the lattice L is called *unimodular*.

The *dual* of a lattice L in $\mathbb{R}^d$ is defined as

$$L^*=\{v\in\mathbb{R}^d:\langle v,\alpha\rangle\in\mathbb{Z}\text{ for all }\alpha\in L\}. \tag{3.2}$$

One can show that L^* is a lattice itself and that its Gram determinant equals $(\det A)^{-1}$, where A denotes a Gram matrix of L. It directly follows from the definition that L is integral if and only if $L\subset L^*$. In fact, it can be shown that L is integral and unimodular if and only if L is *self-dual*, that is, if and only if $L=L^*$ [FLM89]. In this chapter, we will only consider even lattices, and so the notions 'unimodular' and 'self-dual' will be equivalent.

3.2 Vertex algebras associated to a lattice

We will now construct the conformal vertex algebra of an even lattice. Let L be an even lattice of rank $d\in\mathbb{N}$ with bilinear form $(\cdot,\cdot)$. Define the complexification $\mathfrak{g}=L\otimes_{\mathbb{Z}}\mathbb{C}$ and extend $(\cdot,\cdot)$ to a symmetric $\mathbb{C}$-bilinear form on $\mathfrak{g}$ via $(\alpha\otimes z,\beta\otimes w)=zw(\alpha,\beta)$. We now view $\mathfrak{g}$ as a commutative Lie algebra and define $\hat{\mathfrak{g}}$ as the so-called *affinization** of $\mathfrak{g}$ equipped with

*The affinization of a Lie algebra is a general construction that also works for non-commutative Lie algebras equipped with a bilinear form. In the general case, the Lie bracket of $\hat{\mathfrak{g}}$ is given by

$$[a\otimes t^m,b\otimes t^n]_{\hat{\mathfrak{g}}}=[a,b]_{\mathfrak{g}}\otimes t^{m+n}+m(a,b)\delta_{m,-n}\mathbb{1}.$$

$(\cdot,\cdot)$, which is the Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}\mathbb{1} = \left(\bigoplus_{n\in\mathbb{Z}} \mathfrak{g} \otimes t^n \right) \oplus \mathbb{C}\mathbb{1}$$

with Lie bracket

$$[a \otimes t^n, b \otimes t^m] = m(a,b)\delta_{m,-n}\mathbb{1}, \quad [\mathbb{1}, \hat{\mathfrak{g}}] = 0. \tag{3.3}$$

For simplicity, from now on we will write a_n for $a \otimes t^n$.

Note the similarity between (3.3) and the Heisenberg Lie bracket. Just as we did for the Heisenberg Lie algebra, take the space

$$\mathcal{F}_L = \mathbb{C}[a_{-n} : a \in \mathfrak{g}, n > 0]$$

to be the Fock representation of $\hat{\mathfrak{g}}$. That is, $\mathcal{F}_L$ is the $U(\hat{\mathfrak{g}})$-module where $\mathbb{1}$ acts as the identity, the a_{-n} with $n > 0$ act simply by left multiplication and the a_n with $n \geq 0$ act trivially on 1, so we have $a_n \cdot 1 = 0$ (for all $a \in \mathfrak{g}$). $\mathcal{F}_L$ then carries a natural vertex algebra structure with vacuum vector $|0\rangle = 1 \in \mathcal{F}_L$ and vertex operators given by

$$Y(a_{-1}|0\rangle) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \tag{3.4}$$

for $a \in \mathfrak{g}$. Since $\mathcal{F}_L$ is completely generated by the action of the a_{-n} $(n > 0)$ on the vacuum, it follows from the reconstruction theorem that (3.4) determines every vertex operator of $\mathcal{F}_L$. The proof that (3.4) indeed defines a vertex algebra is almost identical to that of the Heisenberg vertex algebra. Moreover, one can show that $\mathcal{F}_L$ has a conformal vector

$$\omega = \frac{1}{2} \sum_{i=1}^{d} a_{-1}^{(i)} a_{-1}^{(i)} |0\rangle \tag{3.5}$$

with central charge d, where $(a^{(1)}, a^{(2)}, \ldots, a^{(d)})$ is an orthonormal basis of $\mathfrak{g}$ with respect to $(\cdot,\cdot)$. The proof is a straightforward generalisation of the proof of theorem 2.10 and can be found in [LL04]. If L is one-dimensional and we fix $b \in \mathfrak{g}$ with $(b,b) = 1$, we see that $\mathcal{F}_L$ is simply the Heisenberg vertex algebra with conformal vector $\frac{1}{2}b_{-1}^2|0\rangle$.

Intuitively, the lattice vertex algebra associated with L will be a direct sum of copies of the vertex algebra $\mathcal{F}_L$, with one copy for each $\alpha \in L$. The space of states will be

$$V_L = \mathcal{F}_L \otimes_{\mathbb{C}} \mathbb{C}[L], \tag{3.6}$$

.

where $\mathbb{C}[L]$ denotes the group algebra of L. To avoid confusing notation, we will view $\mathbb{C}[L]$ as the vector space $\mathrm{span}_{\mathbb{C}}\{e^\alpha : \alpha \in L\}$ with multiplication defined by $e^\alpha e^\beta = e^{\alpha+\beta}$ and $e^0 = 1$. We now extend our representation $\hat{\mathfrak{g}} \to \operatorname{End} \mathcal{F}_L$ to a representation $\hat{\mathfrak{g}} \to \operatorname{End} V_L$ in the following way: as usual we let $\mathbb{1}$ act as the identity on V_L, and for any $a \in \mathfrak{g}$, $p \in \mathcal{F}_L$, $n \in \mathbb{Z} \setminus \{0\}$ and $\alpha \in L$ we set

$$\begin{aligned} a_0 \cdot (p \otimes e^\alpha) &= (a, \alpha)(p \otimes e^\alpha), \\ a_n \cdot (p \otimes e^\alpha) &= a_n p \otimes e^\alpha. \end{aligned}$$

It follows that V_L is a direct sum of irreducible $\hat{\mathfrak{g}}$-modules, each of the form $\mathcal{F}_L \otimes \mathbb{C}e^\alpha$ with $\alpha \in L$. Note that $\mathcal{F}_L \otimes \mathbb{C}e^0$ is isomorphic to $\mathcal{F}_L$ as $\hat{\mathfrak{g}}$-module. By identifying $p \in \mathcal{F}_L$ with $p \otimes e^0 \in V_L$, we can view $\mathcal{F}_L$ as submodule of V_L.

We will now show that V_L carries a natural conformal vertex algebra structure, such that $\mathcal{F}_L$ is a conformal vertex subalgebra. In other words, we want to show that V_L has a unique vertex algebra structure such that the vertex operators corresponding to the states $a_{-1} \otimes e^0$ are given by (3.4) and such that (3.5) defines a conformal vector of V_L. Naturally, we take the vacuum to be $|0\rangle = 1 \otimes e^0 \in V_L$. To simplify notation, from now on we will write $|\alpha\rangle$ for $1 \otimes e^\alpha$. Moreover, for any $a \in \mathfrak{g}$ we define

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}.$$

We will construct the vertex operator map Y. By assumption, we have $Y(a_{-1}|0\rangle) = a(z)$. Since V_L is generated by the action of the a_{-n} $(n > 0)$ on the vectors $|\alpha\rangle$, it follows from the reconstruction theorem that we only have to determine $Y(|\alpha\rangle, z)$ for each $\alpha \in L$. Once again, for brevity, define $f_\alpha(z) = Y(|\alpha\rangle, z)$. Let $(a^{(1)}, a^{(2)}, \ldots, a^{(d)})$ be an orthonormal basis of $\mathfrak{g}$ with respect to $(\cdot, \cdot)$. By assumption, we want

$$\omega = \frac{1}{2} \sum_{i=0}^{d} a_{-1}^{(i)} a_{-1}^{(i)} |0\rangle$$

to be a conformal vector. From

$$Y(\omega, z) = \frac{1}{2} \sum_{i=0}^{d} {:}a^{(i)}(z) a^{(i)}(z){:} = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

we obtain that our gradation operator is given by

$$L_0 = \frac{1}{2} \sum_{i=0}^{d} \sum_{m \in \mathbb{N}} {:}a_m^{(i)} a_{-m}^{(i)}{:} = \sum_{i=0}^{d} \left(\frac{1}{2} a_0^{(i)} a_0^{(i)} + \sum_{m>0} a_{-m}^{(i)} a_m^{(i)} \right) \tag{3.7}$$

and our translation operator is

$$T = L_{-1} = \frac{1}{2}\sum_{i=0}^{d}\sum_{m\in\mathbb{N}} :a_m^{(i)}a_{-1-m}^{(i)}:. \tag{3.8}$$

To find the gradation of the vectors $|\alpha\rangle$ with $\alpha \in L$, note that we have $\alpha = \sum_{i=1}^{d}\lambda_i a^{(i)}$ for certain $\lambda_i \in \mathbb{C}$. We thus obtain

$$\begin{aligned} L_0|\alpha\rangle &= \frac{1}{2}\sum_{i=1}^{d} a_0^{(i)}a_0^{(i)}|\alpha\rangle = \frac{1}{2}\sum_{i=1}^{d}(a^{(i)},\alpha)^2 = \frac{1}{2}\sum_{i=1}^{d}\left(a^{(i)},\sum_{j=1}^{d}\lambda_j a^{(j)}\right)^2|\alpha\rangle \\ &= \frac{1}{2}\sum_{i=1}^{d}\lambda_i^2|\alpha\rangle = \frac{1}{2}(\alpha,\alpha)|\alpha\rangle, \end{aligned}$$

using the fact that the $a^{(i)}$ form an orthonormal basis with respect to $(\cdot,\cdot)$. In a similar way, one deduces from (3.7) that $\deg a_{-n} = n$ for $a \in \mathfrak{g}$, just as before. Since L is even and positive definite, note that this defines an $\mathbb{N}$-gradation on V_L. Moreover, we obtain for $\alpha \in L$ that

$$T|\alpha\rangle = \sum_{i=0}^{d} :a_0^{(i)}a_{-1}^{(i)}:|\alpha\rangle = \sum_{i=0}^{d}(a^{(i)},\alpha)a_{-1}^{(i)}|\alpha\rangle = \alpha_{-1}|\alpha\rangle, \tag{3.9}$$

where the last equality again follows by orthonomality.

From corollary 2.9 we obtain

$$\begin{aligned} [a(w),f_\alpha(z)] &= \sum_{n\geq 0}\frac{1}{n!}Y(a_n|\alpha\rangle,z)\partial_z^n\delta(z,w) \\ &= Y(a_0|\alpha\rangle,z)\delta(z,w) \\ &= (a,\alpha)f_\alpha(z)\sum_{m\in\mathbb{Z}} z^m w^{-m-1}, \end{aligned}$$

which implies

$$[a_n,f_\alpha(z)] = (a,\alpha)f_\alpha(z)z^n \text{ for } n \in N. \tag{3.10}$$

We now define the field

$$X_\alpha(z) = e^{-\alpha}e^{\sum_{n<0}\frac{\alpha_n}{n}z^{-n}}f_\alpha(z)e^{\sum_{n>0}\frac{\alpha_n}{n}z^{-n}} \in \operatorname{End} V_L[[z^{\pm 1}]], \tag{3.11}$$

where e^α denotes the linear operator defined by $e^\alpha \cdot p|\beta\rangle = p|\beta+\alpha\rangle$ for $p \in \mathcal{F}_L$, $\beta \in L$. Although the expression above might appear complicated and abitrary, $X_\alpha(z)$ is defined precisely so to ensure that it commutes with all operators a_n ($a \in \mathfrak{g}$, $n \in \mathbb{Z}$).

Lemma 3.2. *For any $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$ we have* $[a_n, X_\alpha(z)] = 0$.

The proof of this lemma is a straightforward calculation. Let us check that $X_\alpha(z)$ is indeed a well-defined field. Note that the a_n with $n > 0$ are annihilation operators, i.e. given any state in V_L the operator $(a_n)^m$ acts trivially if either n or m is large enough. So applying $e^{\sum_{n>0}\frac{\alpha_n}{n}z^{-n}}$ to any state in V_L gives a polynomial in $V_L[z^{-1}]$. For the left exponential term, note that $\sum_{n<0}\frac{\alpha_n}{n}z^{-n}$ is an element of $z(\mathrm{End}\, V_L)[[z]]$, so for any $m \in \mathbb{N}$ we have

$$\left(\sum_{n<0}\frac{\alpha_n}{n}z^{-n}\right)^m \in z^m(\mathrm{End}\, V_L)[[z]],$$

which implies

$$e^{\sum_{n<0}\frac{\alpha_n}{n}z^{-n}} = \sum_{m=0}^{\infty}\left(\sum_{n<0}\frac{\alpha_n}{n}z^{-n}\right)^m \in (\mathrm{End}\, V_L)[[z]].$$

Now since $f_\alpha(z)$ is a field and since multiplication of series with finitely many negative powers of z is well-defined, it follows that $X_\alpha(z)$ is indeed a field.

From (3.11) and the lemma, it follows that

$$f_\alpha(z) = e^\alpha e^{-\sum_{n<0}\frac{\alpha_n}{n}z^{-n}} e^{-\sum_{n>0}\frac{\alpha_n}{n}z^{-n}} X_\alpha(z)$$

and thus

$$\partial_z f_\alpha(z) = \left(\sum_{n<0}\alpha_n z^{-n-1}\right) f_\alpha(z) + f_\alpha(z)\left(\sum_{n>0}\alpha_n z^{-n-1}\right) + e^\alpha E^{-\alpha(z)}E^{+\alpha(z)}\partial_z X_\alpha(z),$$

where $E^{\pm\alpha(z)}$ denotes $e^{-\sum_{\pm n>0}\frac{\alpha_n}{n}z^{-n}}$. On the other hand, by corollary 2.9(2) and (3.9) we have

$$\begin{aligned}\partial_z f_\alpha(z) &= Y(T|\alpha\rangle, z) = Y(\alpha_{-1}|\alpha\rangle, z) = :\alpha(z)f_\alpha(z): \\ &= :\left(\sum_{n\in\mathbb{Z}}\alpha_n z^{-n-1}\right) f_\alpha(z):.\end{aligned}$$

So we obtain $e^\alpha E^{-\alpha(z)}E^{+\alpha(z)}\partial_z X_\alpha(z) = f_\alpha(z)\alpha_0 z^{-1}$ and thus $\partial_z X_\alpha(z) = \alpha_0 z^{-1}X_\alpha(z)$. If we now define the operator $z^{\alpha_0} \in \mathrm{End}\, V_L[z]$ by

$$z^{\alpha_0}\cdot p|\beta\rangle = z^{(\alpha,\beta)}p|\beta\rangle \text{ for } p \in \mathcal{F}_L,\ \beta \in L,$$

we find $X_\alpha(z) = c_\alpha z^{\alpha_0}$, with $c_\alpha \in \operatorname{End} V_L$ some operator that commutes with every a_n ($a \in \mathfrak{g}$, $n \in \mathbb{Z}$).

Lastly, we check which choices of c_α comply with the vacuum and the locality axiom (we already incorporated the translation axiom indirectly via corollary 2.9(2)). From

$$Y(|\alpha\rangle, z) = f_\alpha(z) = e^\alpha c_\alpha z^{\alpha_0} e^{-\sum_{n<0} \frac{\alpha_n}{n} z^{-n}} e^{-\sum_{n>0} \frac{\alpha_n}{n} z^{-n}} \tag{3.12}$$

we find

$$Y(|\alpha\rangle, z)|0\rangle = e^\alpha c_\alpha (1 + \alpha_{-1} z + O(z^2))|0\rangle,$$

thus the vacuum axiom implies $c_\alpha|0\rangle = |0\rangle$. Moreover, (3.12) implies $Y(|0\rangle, z) = c_0$, so we must have $c_0 = \mathrm{id}$. Now for the locality axiom, first note that for all $a \in \mathfrak{g}$ and $\alpha \in L$ we have

$$\begin{aligned} (z-w)[a(z), f_\alpha(w)] &= (z-w)(a,\alpha) f_\alpha(w)\delta(z,w) \\ &= (a,\alpha) f_\alpha(w)(z-w)\delta(z,w) = 0. \end{aligned}$$

So all we have left to check is that the fields $f_\alpha(z)$ and $f_\beta(w)$ are local for all $\alpha, \beta \in L$. To this end, we will make use of the following lemma, which follows from a simple calculation (see (6.3.46) in [LL04]).

Lemma 3.3. *For $a, b \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$ we have*

$$e^{a_n} e^{b_m} = e^{[a_n, b_m]} e^{b_m} e^{a_n}.$$

In fact, the above equality holds for any two linear operators A and B whose commutator satisfies $[[A,B],A] = [[A,B],B] = 0$. With repeated application of this lemma, we obtain

$$\begin{aligned} f_\alpha(z) f_\beta(w) &= e^\alpha c_\alpha z^{\alpha_0} e^\beta c_\beta w^{\beta_0} E^{-\alpha(z)} \prod_{n>0} e^{-\frac{\alpha_n}{n} z^{-n}} \prod_{n<0} e^{-\frac{\beta_n}{n} w^{-n}} E^{+\beta(w)} \\ &= e^\alpha c_\alpha z^{\alpha_0} e^\beta c_\beta w^{\beta_0} E^{-\alpha(z)} \prod_{n>0} e^{-\frac{1}{n^2}[\alpha_n, \beta_{-n}] z^{-n} w^n} E^{-\beta(w)} E^{+\alpha(z)} E^{+\beta(w)} \\ &= e^\alpha c_\alpha z^{\alpha_0} e^\beta c_\beta w^{\beta_0} e^{-(\alpha,\beta) \sum_{n>0} \frac{1}{n} \left(\frac{w}{z}\right)^n} E^{-\alpha(z)} E^{-\beta(w)} E^{+\alpha(z)} E^{+\beta(w)} \\ &= e^\alpha c_\alpha z^{\alpha_0} e^\beta c_\beta w^{\beta_0} e^{-(\alpha,\beta) \sum_{n>0} \frac{1}{n} \left(\frac{w}{z}\right)^n} S^{\alpha(z), \beta(w)}. \end{aligned}$$

where $S^{\alpha(z),\beta(w)}$ denotes the term

$$e^{-\sum_{n<0} \frac{1}{n}(\alpha_n z^{-n} + \beta_n w^{-n})} e^{-\sum_{n>0} \frac{1}{n}(\alpha_n z^{-n} + \beta_n w^{-n})}.$$

Instead of continuing algebraically, let us use our knowledge of complex analysis for a bit. Since z and w are distinct formal variables, we can view $\frac{w}{z}$ as a new variable x. We know that the power series $-\sum_{n>0}\frac{1}{n}x^n$ corresponds to the logarithm of $1-x$, so we find

$$e^{-(\alpha,\beta)\sum_{n>0}\frac{1}{n}x^n} = (1-x)^{(\alpha,\beta)}.$$

Noting that $z^{\alpha_0}e^\beta = e^\beta z^{\alpha_0} z^{(\alpha,\beta)}$, we conclude that

$$f_\alpha(z)f_\beta(w) = e^\alpha c_\alpha e^\beta c_\beta z^{\alpha_0} w^{\beta_0}(z-w)^{(\alpha,\beta)} S^{\alpha(z),\beta(w)}. \tag{3.13}$$

Since $z^{\alpha_0}w^{\beta_0} = w^{\beta_0}z^{\alpha_0}$ and $S^{\alpha(z),\beta(w)} = S^{\beta(w),\alpha(z)}$, it follows now that

$$(z-w)^N[f_\alpha(z), f_\beta(w)] = (z-w)^N(f_\alpha(z)f_\beta(w) - f_\beta(w)f_\alpha(z)) = 0$$

for some $N \in \mathbb{N}$ if and only if

$$e^\alpha c_\alpha e^\beta c_\beta = (-1)^{(\alpha,\beta)} e^\beta c_\beta e^\alpha c_\alpha. \tag{3.14}$$

Summarising our results, we obtain the following theorem.

Theorem 3.4. *There exists a conformal vertex algebra structure on V_L with vacuum vector $|0\rangle$ and conformal vector ω as defined in (3.5) and such that for each $a \in \mathfrak{g}$ we have $Y(a_{-1}|0\rangle, z) = a(z)$, if and only if for each $\alpha \in L$ there exists an operator c_α that commutes with every a_n (for all $a \in \mathfrak{g}$, $n \in \mathbb{Z}$) such that these operators satisfy $c_\alpha|0\rangle = |0\rangle$, $c_0 = \mathrm{id}$ and (3.14). In this case, the mapping*

$$Y(|\alpha\rangle, z) = e^\alpha c_\alpha z^{\alpha_0} e^{-\sum_{n<0}\frac{\alpha_n}{n}z^{-n}} e^{-\sum_{n>0}\frac{\alpha_n}{n}z^{-n}}. \tag{3.15}$$

for $\alpha \in L$ defines a vertex algebra structure on V_L.

The existence of the required vertex algebra structure on V_L now depends on whether we can find suitable operators c_α. As these operators commute with all the a_n, they are fully determined by their action on the vectors $|\beta\rangle$, $\beta \in L$. Let us restrict ourselves to a simple action, where for all $\alpha, \beta \in L$ we have $c_\alpha|\beta\rangle = c_{\alpha,\beta}|\beta\rangle$ for some complex number $c_{\alpha,\beta} \in \mathbb{C}$. Then corollary 5.5 of [Kac97] provides us with the following pleasing result.

Theorem 3.5. *For every even lattice L there exists a solution for the complex numbers $c_{\alpha,\beta}$ such that $c_{\alpha,\beta} \neq 0$ for all $\alpha, \beta \in L$ and such that the induced operators c_α on V_L satisfy the requirements of theorem 3.4. Moreover, any two solutions give rise to isomorphic vertex algebra structures on V_L.*

This implies that there exists a conformal vertex algebra structure on V_L, and there is a unique[†] and such structure that takes on a particularly simple form. From now on, we will refer to the vertex algebra structure on V_L induced by the solution of theorem 3.5 as *the* vertex algebra V_L.

3.3 Characters and modularity

By construction, L_0 defines an $\mathbb{N}$-gradation on V_L. We can therefore write $V_L = \bigoplus_{n\in\mathbb{N}} V_{L,n}$ where $V_{L,n}$ denotes the eigenspace of L_0 with eigenvalue n. As we will see, these eigenspaces $V_{L,n}$ are finite-dimensional. This allows us to define the *character* of a lattice vertex algebra.

Definition 3.6. *Let V be a conformal vertex algebra with central charge c. If the L_0-gradation $\bigoplus_{n\in N} V_n$ of V is such that $\dim V_n < \infty$ for all n, then we define the character of V as the formal power series*

$$char_q(V) = \sum_{n=0}^{\infty} (\dim V_n) q^{n-\frac{c}{24}}.$$

Note that, apart from the factor $\frac{c}{24}$, this expression coincides with the graded dimensions we discussed in the introduction. Physically, the factor $\frac{c}{24}$ is explained by an energy shift due to the vacuum energy. Mathematically, the factor appears arbitrary; we will see, however, that this factor gives the characters modular properties if we write $q = e^{2\pi i\tau}$ and let τ take values in the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

Recall from the introduction that a modular invariant function f on $\mathbb{H}$ satisfies $f(\tau) = f(\frac{a\tau+b}{c\tau+d})$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$. To check whether this is the case, it is useful to determine generators of Γ.

Theorem 3.7. *The modular group Γ is generated by the elements*

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.16}$$

Proof. First note that the inverse of T is $\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ and thus we have $T^k = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ for all $k \in \mathbb{Z}$. Now let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. To be precise, we take M to

[†] One might be worried about the choice of orthonormal basis we made to define the conformal vector ω. However, for two orthonormal bases $\{a^{(i)}\}$ and $\{b^{(i)}\}$ of $\mathfrak{g}$, the mapping $a^{(i_1)}_{-n_1} \dots a^{(i_m)}_{-n_m}|\alpha\rangle \mapsto b^{(i_1)}_{-n_1} \dots b^{(i_m)}_{-n_m}|\alpha\rangle$ defines an isomorphism between the resulting conformal vertex algebras.

represent the equivalence class of the matrices M and $-M$ in $\mathrm{SL}_2(\mathbb{Z})$, so we may assume $c \geq 0$. We prove the theorem by induction on c. For $c = 0$, it follows from $ad - bc = 1$ that $a = d = \pm 1$, and thus we find

$$M = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix} = \left(\begin{smallmatrix} 1 & \pm b \\ 0 & 1 \end{smallmatrix}\right) = T^{\pm b}.$$

Now let $c \geq 1$. Then we have $d = ck + r$ for some $k, r \in \mathbb{Z}$ with $0 \leq r < c$. We obtain

$$MT^{-k}S = \left(\begin{smallmatrix} a & b \\ c & ck+r \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -k \\ 0 & 1 \end{smallmatrix}\right)S = \left(\begin{smallmatrix} a & b-ak \\ c & r \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b-ak & -a \\ r & -c \end{smallmatrix}\right).$$

Since $0 \leq r < c$, the theorem now follows by induction and the fact that T and S are invertible. □

By theorem 3.7, it suffices to check that $f(\tau) = f(\tau + 1)$ and $f(\tau) = f(-\frac{1}{\tau})$ in order to show that f is invariant under the action of Γ.

Let us determine the character of the simplest vertex algebra, namely the conformal Heisenberg vertex algebra V_H with central charge $c = 1$. Recall that states in V_H are simply linear combinations of polynomials in the b_{-n} with $n > 0$, and that any polynomial $b_{-k_1} \cdots b_{-k_m}|0\rangle$ is an eigenvector of L_0 with eigenvalue $\sum_{i=1}^m k_i$. The eigenspace decomposition with respect to L_0,

$$V_H = \bigoplus_{n \in \mathbb{N}} V_{H,n} \, ,$$

is therefore given by

$$V_{H,n} = \mathrm{span}_{\mathbb{C}}\{b_{-k_1} \ldots b_{-k_m}|0\rangle : k_i < 0, \sum k_i = n\}.$$

The dimension of $V_{H,n}$ is equal to the number of *partitions* of n, i.e. the number of different ways in which we can write n as a sum of positive integers. For example, the number of partitions of the integer 3 is three, as 3 can be written as $1+1+1$, $1+2$ or 3. For any n, its number of partitions is clearly finite, so the character of V_H is indeed well-defined. Now note that any partition of $n \in \mathbb{N}$ defines a sequence $(i_1, i_2, i_3, \ldots)$ in $\mathbb{N}$ such that finitely many i_m are nonzero and $n = \sum_{m=1}^{\infty} i_m m$ (so i_m denotes how often m appears in the partition). Conversely, any such sequence defines a partition of n. Letting $p(n)$ denote the number of partitions of n, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} p(n)q^n &= \sum_{(i_1,i_2,\dots)} q^{\sum_{m=1}^{\infty} m i_m} = \sum_{(i_1,i_2,\dots)} q^{i_1+2i_2+3i_3+\cdots} \\ &= (1+q^1+q^2+\dots)(1+q^2+q^4+\dots)(1+q^3+q^6+\dots)\cdots \\ &= \prod_{m=1}^{\infty}\sum_{i=0}^{\infty} q^{im}.\end{aligned} \tag{3.17}$$

Here $\sum_{(i_1,i_2,\dots)}$ indicates that we are summing over all sequences in $\mathbb{N}$ with finitely many nonzero terms. The character of V_H is thus given by

$$\mathrm{char}_q(V_H) = \sum_{n=0}^{\infty} p(n)q^{n-\frac{1}{24}} = q^{-\frac{1}{24}}\prod_{m=1}^{\infty}\sum_{i=0}^{\infty} q^{im} = q^{-\frac{1}{24}}\prod_{m=1}^{\infty}\frac{1}{1-q^m}. \tag{3.18}$$

For the last equality we used convergence of the geometric series, anticipating we will interpret q as $e^{2\pi i\tau}$ with $\mathrm{Im}(\tau) > 0$.

The series $\eta(\tau) = q^{\frac{1}{24}}\prod_{m=1}^{\infty} 1 - q^m$ with $q = e^{2\pi i\tau}$ defines a holomorphic function on $\mathbb{H}$ with no zeros[‡]. This function is known as the *Dedekind eta function*. It is not modular invariant, but does transform rather 'nicely' under modular transformations [BD20]. Namely, we have

$$\eta(\tau+1) = e^{\frac{\pi}{12}i}\eta(\tau) \text{ and } \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau). \tag{3.19}$$

In terms of this function, we have $\mathrm{char}_q(V_H) = \frac{1}{\eta(\tau)}$.

Let us now determine the character of the lattice vertex algebra V_L, where L still denotes an even lattice of rank d. Recall that V_L is conformal with central charge $c = d$ and has a basis of states of the form

$$a^{(j_1)}_{-k_1}a^{(j_2)}_{-k_2}\dots a^{(j_m)}_{-k_m}|\beta\rangle \text{ with } 1 \le j_i \le d, k_i > 0, \beta \in L, \tag{3.20}$$

which are eigenvectors of L_0 with eigenvalue $\sum_{i=1}^{m} k_i + \frac{1}{2}(\beta,\beta)$. The L_0-eigenspace decomposition $V_L = \bigoplus_{n\in\mathbb{N}} V_{L,n}$ is thus given by

$$V_{L,n} = \mathrm{span}_{\mathbb{C}}\{a^{(j_1)}_{-k_1}a^{(j_2)}_{-k_2}\dots a^{(j_m)}_{-k_m}|\beta\rangle : \sum_{i=1}^{m} k_i + \frac{1}{2}(\beta,\beta) = n\}.$$

[‡]This follows from standard results in complex analysis and the fact that $|q| < 1$ for $\tau \in \mathbb{H}$. See for example see Remark IV.1.7 in [Fre05].

That these eigenspaces are finite-dimensional follows directly from the fact that the set $\{\beta \in L : (\beta, \beta) \leq n\}$ is finite for any $n \in \mathbb{N}$. Indeed, if we embed L in the Euclidean space $\mathbb{R}^d$ with inner product $\langle \cdot, \cdot \rangle$, the set $\{\beta \in L : \langle \beta, \beta \rangle \leq n\}$ forms a discrete topological subspace in the compact closed ball $\{v \in \mathbb{R}^d : \langle v, v \rangle \leq n\}$, and any compact and discrete topological space is finite. We thus find that the character of V_L is well-defined.

Reasoning as before, we obtain

$$\begin{aligned} \mathrm{char}_q(V_L) &= \sum_{n=0}^{\infty} (\dim V_{L,n}) q^{n-\frac{d}{24}} \\ &= q^{-\frac{d}{24}} \sum_{\beta \in L} \sum_{(i_1^1, i_2^1, \ldots)} \cdots \sum_{(i_1^d, i_2^d, \ldots)} q^{\sum_{j=1}^{d} \sum_{m=1}^{\infty} m i_m^j + \frac{1}{2}(\beta,\beta)} \\ &= q^{-\frac{d}{24}} \left(\sum_{(i_1, i_2, \ldots)} q^{\sum_{m=1}^{\infty} m i_m} \right)^d \sum_{\beta \in L} q^{\frac{1}{2}(\beta,\beta)} = \frac{1}{\eta^d} \sum_{\beta \in L} q^{\frac{1}{2}(\beta,\beta)}. \end{aligned} \tag{3.21}$$

In the third expression, the i_m^j have the same role as the i_m from before, only now we have one for each dimension of L. The variable i_m^j should be interpreted as 'the number of times $a_{-m}^{(j)}$ appears in the basis state'.

The series $\Theta_L(\tau) = \sum_{\beta \in L} q^{\frac{1}{2}(\beta,\beta)}$ with $q = e^{2\pi i \tau}$ and $\tau \in \mathbb{H}$ defines a holomorphic function that is known as the *theta function* of L[§]. Since L is even, we clearly have $\Theta_L(\tau + 1) = \Theta_L(\tau)$. If L is also unimodular (or equivalently, self-dual) it can be shown that Θ_L satisfies

$$\Theta_L\left(-\frac{1}{\tau}\right) = (i\tau)^{d/2} \Theta_L(\tau), \tag{3.22}$$

making it a modular form of weight $\frac{d}{2}$ [Ser12]. In chapter 6, we will find that the character $\frac{\Theta_L}{\eta^d}$ of V_L can be interpreted as a *partition function* of a certain string theory. By definition, this partition function needs to be modular invariant. Looking at (3.19) and (3.22), this happens precisely when d is a multiple of 24.

There are 24 even unimodular lattices of rank 24 [Nie73]. Among them is the *Leech lattice* Λ, which is the only one without elements of squared

[§]To see that this series converges for $\tau \in \mathbb{H}$, note that for all $n \in \mathbb{N}$ the number of lattice elements $\beta \in L$ with $\frac{1}{2}(\beta, \beta) = n$ can be bounded by a polynomial expression in n. This is due to the fact these lattice points are contained in the closed ball $\{v \in \mathbb{R}^d : \langle v, v \rangle \leq n\}$, whose volume is of the order $n^{d/2}$.

length 2 [CS13]. Let us determine the character of V_Λ. By our observations above, we know that $\frac{\Theta_\Lambda(\tau)}{\eta(\tau)^{24}}$ is a modular function that is holomorphic on $\mathbb{H}$. Moreover, from

$$\frac{\Theta_\Lambda(\tau)}{\eta(\tau)^{24}} = \sum_{n=0}^{\infty} (\dim V_{\Lambda,n}) q^{n-1} \tag{3.23}$$

we find that the lowest power in its q-expansion is q^{-1} and that the corresponding coefficient is equal to one, as only the vacuum $|0\rangle$ has L_0-eigenvalue zero. Now recall from the introduction that the J-function is the unique modular function with these properties up to an additive constant. We thus obtain

$$\sum_{n=0}^{\infty} (\dim V_{\Lambda,n}) q^{n-1} = J(\tau) + C \tag{3.24}$$

for some $C \in \mathbb{C}$. To determine C, note that the constant term corresponds to the number of basis states with q-level zero, i.e. the number of basis states with L_0-eigenvalue 1. Since the Leech lattice is even and contains no elements of squared length 2, we have $(\beta, \beta) > 2$ for all $\beta \in \Lambda \setminus \{0\}$. The only level zero states are therefore of the form $a^{(j)}_{-1}|0\rangle$. As $d = 24$, we have precisely 24 of these states. We therefore conclude

$$\mathrm{char}_q(V_\Lambda) = J(\tau) + 24. \tag{3.25}$$

Note that any other choice of an even unimodular 24-dimensional lattice would have resulted in a higher constant term, as these necessarily include lattice elements of squared length 2.

The result above inspired the construction of the monster vertex algebra $V^\natural$ out of V_Λ. We will not study the algebraic structure of $V^\natural$; instead, we will develop the string theory that has been associated with $V^\natural$. This will enable us to calculate the character of $V^\natural$ in chapter 6, which we will find to be equal to $J(\tau)$.

Part II

String Theory

Chapter 4

Free bosonic open strings

We have defined of vertex algebras, studied some of their properties and seen two important examples: the Heisenberg vertex algebra and the lattice vertex algebra. Thus far, our discussion has been mainly mathematical. We will now shift our focus to the field of study in which vertex operators where first introduced, namely that of string theory. In this chapter, the basic principles of free bosonic string theory will be discussed. We will solve the equations of motion for open strings, develop the quantum theory by means of the light cone gauge and discuss the critical dimension of spacetime for a Lorentz invariant bosonic string theory. Our discussion is largely based on [Zwi04].

4.1 The setting

We consider a free open string of finite length in $(d+1)$-dimensional spacetime for some $d \in \mathbb{N}$, that is, our string lives in $\mathbb{R}^{1,d}$, which is the vector space $\mathbb{R}^{1+d}$ equipped with the relativistic inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle (v_0, v_1, \ldots v_d), (w_0, w_1, \ldots w_d) \rangle = -v_0 w_0 + v_1 w_1 + \cdots + v_d w_d.$$

As the inner product shows, we take the zero-th coordinate to indicate time. By free we understand that the string is not subjected to external forces, and by open we mean that the string has two endpoints (in contrast to closed strings, which form a loop). Moving through both space and time, our 1-dimensional string carves out a 2-dimensional surface in spacetime, referred to as the *world-sheet*. To describe this surface, we use two parameters τ and σ, the first relating to time evolution of the string and the second to position along the string. Formally, we take the world-sheet to be the image of a C^∞ injective map

$$X : \mathbb{R} \times I \ni (\tau, \sigma) \mapsto X(\tau, \sigma) \in \mathbb{R}^{1,d},$$

with I some closed interval of $\mathbb{R}$. For open strings, we let σ take values in the interval $[0, \pi]$. Moreover, as we want τ to represent time evolution, we would expect $\partial_\tau X(\tau, \sigma)$ to be a timelike vector for all τ and σ. However, as we will see, the string endpoints move at the speed of light, so we allow $\partial_\tau X$ to be lightlike occasionally. Thus we set

$$\langle \partial_\tau X, \partial_\tau X \rangle \leq 0$$

with equality in only a finite number of points. In the same way, we want $\partial_\sigma X$ to be a spacelike, so we require

$$\langle \partial_\sigma X, \partial_\sigma X \rangle \geq 0,$$

again with equality in only a finite number of points. Any map X satisfying the above criteria that has the world-sheet of our string as its image, is called a *parameterisation* of this world-sheet. From now on, we will let X^i denote the map obtained by composition of X with the projection onto the i-th coordinate, so that $X^i(\tau, \sigma)$ corresponds to the ith-coordinate of the spacetime vector $X(\tau, \sigma)$ for given τ and σ.

4.2 The relativistic string action

To determine the equations of motion for a relativistic string, we need to formulate an action that is both parameterisation invariant and Lorentz invariant. A simple action satisfying these criteria is the *Nambu-Goto action*

$$S_{NG} = -\frac{T}{c} \int \int_0^\pi \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}\, d\sigma d\tau.$$

In the above expression, multiplication of vectors refers to the relativistic inner product and we have used the simplifying notation $\dot{X} = \frac{\partial X}{\partial \tau}$ and $X' = \frac{\partial X}{\partial \sigma}$. The constant term T has units of force and can be interpreted as the string tension. Frequently, the constant factor in S_{NG} is written in terms of the *Regge slope* $\alpha' = \frac{1}{2\pi T \hbar c}$. Note that the term under the square root is always nonnegative: $(\dot{X} \cdot X')^2$ is nonnegative as the square of a real number, and $(\dot{X})^2(X')^2$ is nonpositive as we required $(\dot{X})^2 \leq 0$ and $(X')^2 \geq 0$. The string action S_{NG} is a generalisation of the action of a relativistic free point particle, which is proportional to the proper length of its world-line. In the same way, S_{NG} can be viewed as the "proper area" of the world-sheet.

With the action at hand, the equations of motion governing the string between some initial time τ_i and final time τ_f can be deduced from Hamilton's variational principle. To this end, define the canonical momentum densities

$$\mathcal{P}^\tau \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}} \quad \text{and} \quad \mathcal{P}^\sigma \equiv \frac{\partial \mathcal{L}}{\partial X'}, \tag{4.1}$$

where $\mathcal{L}$ denotes the Lagrangian density given by

$$\mathcal{L}(\dot{X}, X') = -\frac{T}{c}\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2}\,. \tag{4.2}$$

Let $\eta : \mathbb{R} \times [0, \pi] \to \mathbb{R}^{1,d}$ denote a variation of our string evolution $X(\tau, \sigma)$ between τ_i and τ_f, meaning that η is a C^∞ map such that such that for all $\sigma \in I$ we have

$$\eta(\tau_i, \sigma) = \eta(\tau_f, \sigma) = 0. \tag{4.3}$$

Hamilton's principle states that the true equations of motion define a stationary point of the action, meaning that infinitesimal variations of X should leave S_{NG} unchanged. Formally, this amounts to the following. For $\epsilon \in \mathbb{R}$ and for fixed X and η, the action

$$S_{NG}(X + \epsilon\eta) = -\frac{T}{c}\int_{\tau_i}^{\tau_f}\int_0^\pi \mathcal{L}(\dot{X} + \epsilon\dot{\eta}, X' + \epsilon\eta')d\sigma d\tau$$

can be viewed as a function $\phi : \mathbb{R} \to \mathbb{R}$ of ϵ. The variational principle then requires that

$$\left.\frac{d\phi}{d\epsilon}\right|_{\epsilon=0} = 0.$$

We obtain

$$\begin{aligned}-\frac{T}{c}\int_{\tau_i}^{\tau_f}\int_0^\pi \left(\frac{\partial \mathcal{L}}{\partial \dot{X}}\frac{\partial \eta}{\partial \tau} + \frac{\partial \mathcal{L}}{\partial X'}\frac{\partial \eta}{\partial \sigma}\right) d\sigma d\tau& \\ = -\frac{T}{c}\int_{\tau_i}^{\tau_f}\int_0^\pi \left(\mathcal{P}^\tau\frac{\partial \eta}{\partial \tau} + \mathcal{P}^\sigma\frac{\partial \eta}{\partial \sigma}\right) d\sigma d\tau = 0&\end{aligned}$$

and thus

$$-\frac{T}{c}\int_{\tau_i}^{\tau_f}\int_0^{\pi}\left(\frac{\partial}{\partial\tau}(\eta\mathcal{P}^\tau)+\frac{\partial}{\partial\sigma}(\eta\mathcal{P}^\sigma)-\eta\left(\frac{\partial\mathcal{P}^\tau}{\partial\tau}+\frac{\partial\mathcal{P}^\sigma}{\partial\sigma}\right)\right)d\sigma d\tau=0$$

Since $[\eta\mathcal{P}^\tau]_{\tau_i}^{\tau_f}=0$ due to (4.3), it follows that

$$-\frac{T}{c}\int_{\tau_i}^{\tau_f}[\eta\mathcal{P}^\sigma]_0^\pi d\tau-\int_{\tau_i}^{\tau_f}\int_0^{\pi}\eta\left(\frac{\partial\mathcal{P}^\tau}{\partial\tau}+\frac{\partial\mathcal{P}^\sigma}{\partial\sigma}\right)d\sigma d\tau=0. \tag{4.4}$$

As the above equality should hold for any variation η, we obtain the equation of motion

$$\frac{\partial\mathcal{P}^\tau}{\partial\tau}+\frac{\partial\mathcal{P}^\sigma}{\partial\sigma}=0, \tag{4.5}$$

and we require that $[\eta\mathcal{P}^\sigma]_0^\pi=0$. The latter imposes boundary conditions on the string endpoints. As we put no constraints on η in the endpoints $\sigma=0$ and $\sigma=\pi$ of the string, we must have

$$\mathcal{P}^\sigma(\tau,0)=\mathcal{P}^\sigma(\tau,\pi)=0 \tag{4.6}$$

for any $\tau\in\mathbb{R}$. The above constraint is known as the Neumann or free endpoint boundary condition.

Remark. As mentioned in the introduction, fields living on the world-sheet of the string give rise to a conformal field theory. Why this happens is seen more clearly from the *Polyakov string action* S_P than from the Nambu-Goto action, since S_P includes a metric $\gamma(\tau,\sigma)$ on the world-sheet. The Polyakov action is given by

$$S_P=\int\int_0^\pi d\tau d\sigma\frac{1}{\sqrt{-\det(\gamma(\tau,\sigma))}}\sum_{i,j=1}^{2}\gamma_{ij}(\tau,\sigma)\langle\partial_iX(\tau,\sigma),\partial_jX(\tau,\sigma)\rangle,$$

where $(\gamma_{ij}(\sigma,\tau))_{i,j=1,2}$ denotes the 2×2-matrix corresponding to the metric $\gamma(\tau,\sigma)$ and $\det(\gamma(\tau,\sigma))$ denotes its determinant. To refrain from confusing notation, we let ∂_1X and ∂_2X denote the partial derivatives of X to the τ-coordinate and σ-coordinate, respectively. One easily sees that S_P is invariant under conformal transformations of the world-sheet, as any scaling of $\gamma(\tau,\sigma)$ by a positive factor will cancel out in the integrand. Since S_P is equivalent to S_{NG} (in the sense that it leads to the same equations of motion), we will restrict ourselves to the latter. Details on the Polyakov action and the proof of the equivalence can be found in section 24.6 of [Zwi04].

4.3 Reparameterisation

Although seemingly simple, the equation of motion is a rather complex equation, as (4.1) and (4.2) imply that

$$\mathcal{P}^\tau = -\frac{T}{c}\frac{(\dot{X}\cdot X')X' - (X')^2\dot{X}}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \text{ and } \mathcal{P}^\sigma = -\frac{T}{c}\frac{(\dot{X}\cdot X')\dot{X} - (\dot{X})^2 X'}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} \tag{4.7}$$

Fortunately, we can use our freedom to choose a parameterisation to simplify these expressions substantially.

Let us consider parameterisations that satisfy

$$n\cdot X(\tau,\sigma) = \lambda\tau \tag{4.8}$$

for some $\lambda \in \mathbb{R}$ and $n \in \mathbb{R}^{1,d}$. This means that for fixed τ_0, the string described by $X(\tau_0,\sigma)$ lies at the intersection of the worldsheet and a hyperplane normal to the vector n. As we want two points on the string described by $X(\tau_0,\sigma)$ to be spacelike separated, we require n to be timelike or lightlike. Note that, given a world-sheet surface and such a vector n, such a parameterisation always exists since every point on the worldsheet lies on a unique hyperplane normal to n.

We define the string momentum as

$$p(\tau) = \int_0^\pi \mathcal{P}^\tau d\sigma \in \mathbb{R}^{1,d}.$$

Using the equation of motion (4.5) and the free endpoint boundary conditions (4.6), we obtain

$$\frac{dp}{d\tau} = \int_0^\pi \frac{\partial\mathcal{P}^\tau}{\partial\tau}d\sigma = -\int_0^\pi \frac{\partial\mathcal{P}^\sigma}{\partial\sigma}d\sigma = [\mathcal{P}^\sigma]_\pi^0 = 0.$$

So, given a parameterisation, $p \in \mathbb{R}^{1,d}$ is a well-defined conserved property of the string. In fact, one can use the equation of motion to show that p is independent of the chosen parametrisation. We can therefore set

$$\lambda = \tilde{\lambda}(n\cdot p)$$

for some $\tilde{\lambda} \in \mathbb{R}$. To determine $\tilde{\lambda}$, let us first compare units in the expression $n\cdot X(\tau,\sigma) = \tilde{\lambda}(n\cdot p)\tau$. Here we take τ to be a dimensionless parameter. As $X(\tau,\sigma)$ has units of length and p of momentum, it follows that $\tilde{\lambda}$ has units of velocity per force. A natural choice would therefore be to take $\tilde{\lambda}$ proportional to $\frac{c}{T}$. To simply notation, from now on we will set $c = \hbar = 1$*. In terms of the Regge slope, we have $\tilde{\lambda} \sim \frac{1}{T} = 2\pi\alpha'$. With hindsight, a convenient choice for $\tilde{\lambda}$ will be $2\alpha'$. The parameterisation condition then becomes

$$n \cdot X(\tau,\sigma) = 2\alpha'(n \cdot p)\tau. \tag{4.9}$$

We want to specify our parameterisation even more to ensure $\dot{X} \cdot X' = 0$. To this end, we take $n \cdot \mathcal{P}^\tau$ to be constant†. From $\int_0^\pi n \cdot \mathcal{P}^\tau d\sigma = n \cdot p$, it follows that this condition is equivalent to

$$n \cdot \mathcal{P}^\tau(\tau,\sigma) = \frac{n \cdot p}{\pi}. \tag{4.10}$$

Now the equation of motion (4.5) implies

$$\frac{\partial}{\partial\tau} n \cdot \mathcal{P}^\tau + \frac{\partial}{\partial\sigma} n \cdot \mathcal{P}^\sigma = \frac{\partial}{\partial\tau}\left(\frac{n \cdot p}{\pi}\right) + \frac{\partial}{\partial\sigma} n \cdot \mathcal{P}^\sigma = \frac{\partial}{\partial\sigma} n \cdot \mathcal{P}^\sigma = 0, \tag{4.11}$$

thus $n \cdot \mathcal{P}^\sigma$ is independent of σ. However, the boundary conditions (4.6) imply that $n \cdot \mathcal{P}^\sigma$ vanishes at the endpoints of the string, thus we find

$$n \cdot \mathcal{P}^\sigma(\tau,\sigma) = 0 \quad \text{for all } \tau \text{ and } \sigma.$$

It now follows from (4.7) and (4.9) that

$$n \cdot \mathcal{P}^\sigma = -\frac{1}{2\pi\alpha'} \frac{(\dot{X} \cdot X')\partial_\tau(n \cdot X) - (\dot{X})^2\partial_\sigma(n \cdot X)}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2}} \tag{4.12}$$

$$= -\frac{1}{2\pi\alpha'} \frac{2\alpha'(n \cdot p)(\dot{X} \cdot X')}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2(X')^2}} = 0 \tag{4.13}$$

and thus we indeed get $\dot{X} \cdot X' = 0$. With this result and (4.10), we find

$$\frac{n \cdot p}{\pi} = n \cdot \mathcal{P}^\tau = \frac{1}{2\pi\alpha'} \frac{(X')^2\partial_\tau(n \cdot X)}{\sqrt{-(\dot{X})^2(X')^2}} = \frac{(n \cdot p)(X')^2}{\pi\sqrt{-(\dot{X})^2(X')^2}} \tag{4.14}$$

*This will not result in loss of information, since knowing the units of an expression in which c and $\hbar$ have been set to 1, enables one to reconstruct its original form involving occurrences of c and $\hbar$.

†Such a parameterisation exists, since after choosing our lines of constant τ on the wordsheet, we can choose a σ-parametrisaion as to ensure that $\mathcal{P}^\tau$ is evenly 'spread out' over the string. For details of this construction, see section 7.1 of [Zwi04].

and thus $-(\dot{X})^2 = (X')^2$. Note that $-(\dot{X})^2 > 0$ and $(X')^2 > 0$ hold almost everywhere, so continuity of X ensures that this result also holds in points where $(\dot{X})^2$ or $(X')^2$ is zero. Our two parameterisation conditions (4.9) and (4.10) thus result in

$$(\dot{X} \pm X')^2 = 0. \tag{4.15}$$

Substituting these results in (4.7) we now obtain

$$\begin{aligned}
\mathcal{P}^\tau &= -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')X' - (X')^2\dot{X}}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} = \frac{1}{2\pi\alpha'}\frac{(X')^2\dot{X}}{\sqrt{(X')^2(X')^2}} = \frac{1}{2\pi\alpha'}\dot{X},\\
\mathcal{P}^\sigma &= -\frac{1}{2\pi\alpha'}\frac{(\dot{X}\cdot X')\dot{X} - (\dot{X})^2 X'}{\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}} = \frac{1}{2\pi\alpha'}\frac{-(X')^2 X'}{\sqrt{(X')^2(X')^2}} = -\frac{1}{2\pi\alpha'}X'.
\end{aligned} \tag{4.16}$$

Our equation of motion thus becomes the familiar wave equation

$$\ddot{X} - X'' = 0, \tag{4.17}$$

and the free endpoint boundary conditions become

$$\left.\frac{\partial X}{\partial \sigma}\right|_{\sigma=0} = \left.\frac{\partial X}{\partial \sigma}\right|_{\sigma=\pi} = 0. \tag{4.18}$$

This surely is a great simplification. We see that the string endpoints are free to move in any direction that is transverse to the string. Note that (4.15) and (4.18) imply that $\dot{X}$ is lightlike at the endpoints, and thus the string endpoints move at the speed of light.

4.4 Solving the wave equation

The general solution to the wave equation (4.17) is of the form $X(\tau,\sigma) = \frac{1}{2}(F(\tau-\sigma) + G(\tau+\sigma))$, for arbitrary C^2 maps $F, G : \mathbb{R} \to \mathbb{R}^{1,d}$. As the boundary condition $\mathcal{P}^\sigma(\tau, 0) = 0$ implies $F'(\tau) = G'(\tau)$, the maps F and G only differ by a constant, so we can redefine F to obtain

$$X(\tau,\sigma) = \frac{1}{2}(F(\tau-\sigma) + F(\tau+\sigma)). \tag{4.19}$$

The other boundary condition $\mathcal{P}^\sigma(\tau,\pi) = 0$ then tells us that $F'(\tau-\pi) = F'(\tau+\pi)$, meaning that F' is periodic with period 2π. So we can write F'

in terms of its complex Fourier series $F'(s) = \sum_{n\in\mathbb{Z}} c_n e^{ins}$ with $c_n \in \mathbb{C}^{d+1}$, where c_n^i and c_{-n}^i are complex conjugates for all $n \in \mathbb{Z}$, $i \in \{0, \dots, d\}$. This gives us

$$F(s) = x_0 + c_0 s - \sum_{n\neq 0} \frac{i}{n} c_n e^{ins}$$

for some integration constant $x_0 \in \mathbb{R}$, which implies

$$\begin{aligned} X(\tau,\sigma) &= x_0 + c_0\tau - \frac{1}{2}\sum_{n\neq 0}\frac{i}{n}c_n e^{in\tau}\left(e^{-in\sigma} + e^{in\sigma}\right) \\ &= x_0 + c_0\tau - \sum_{n\neq 0}\frac{i}{n}c_n e^{in\tau}\cos(n\sigma). \end{aligned}$$

The coefficient c_0 is in fact equal to $2\alpha' p$, since we have

$$p = \int_0^\pi \mathcal{P}^\tau d\sigma = \frac{1}{2\pi\alpha'}\int_0^\pi \dot{X} d\sigma = \frac{1}{2\pi\alpha'}\int_0^\pi c_0 d\sigma = \frac{c_0}{2\alpha'}. \tag{4.20}$$

The third equality follows from the fact that all terms involving a factor $\cos(n\sigma)$ do not contribute to the integral. Lastly, we introduce some conventional notation by defining $a_n := \sqrt{\frac{n}{2\alpha'}}c_{-n}$ and its complex conjugate $a_n^* := \sqrt{\frac{n}{2\alpha'}}c_n$ for all $n > 0$. As $\sqrt{\alpha'}$ has units of length, note that these constants are dimensionless. When we quantise the open string, the a_n and a_n^* will become annihilation and creation operators, respectively. We thus obtain the solution

$$X(\tau,\sigma) = x_0 + 2\alpha' p\tau - i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\left(a_n^* e^{in\tau} - a_n e^{-in\tau}\right)\frac{\cos(n\sigma)}{\sqrt{n}}. \tag{4.21}$$

The last term describes the oscillations in the string; when all a_n are zero, $X(\tau,\sigma)$ simply describes the motion of a free point particle.

4.5 The light-cone gauge

The expression in (4.21) alone is not enough to specify our solutions, as we need the a_n to be such that the parameterisation constraints are satisfied. To find a fully explicit solution to both the equation of motion and the constraints in (4.15), we will now choose a specific normal vector for our parameterisation. We will take the lightlike vector

$$n = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0\right). \tag{4.22}$$

This parameterisation is referred to as the *light cone gauge*. When working in this gauge, it is useful to switch from the standard basis $(e_0, e_1, e_2, \ldots, e_d)$ to the so-called *light-cone basis* $(e_+, e_-, e_2, \ldots, e_d)$ of $\mathbb{R}^{1,d}$, where

$$e_+ = \frac{1}{\sqrt{2}}(e_0 + e_1) \text{ and } e_- = \frac{1}{\sqrt{2}}(e_0 - e_1).$$

Note that the e_+ and e_- are lightlike vectors. For $d = 1$, they carve out a 'cone' in 2-dimensional spacetime in which the world-lines of physical objects lie. If we take the plus-coordinate to represent light-cone time, one can show that the minus-coordinate of the spacetime momentum p should be interpreted as the light-cone energy. The relativistic inner product in light-cone coordinates becomes

$$\langle (v_+, v_-, \ldots v_d), (w_+, w_-, \ldots w_d) \rangle = -v_+ w_- - v_- w_+ + \sum_{j=2}^{d} v_j w_j. \tag{4.23}$$

In this new basis, we will refer to the first two coordinates as light-cone coordinates and the other $d-1$ as *transverse* coordinates.

In the light-cone gauge, we get $n \cdot X = X^+$ and $n \cdot p = p^+$, so the parameterisation conditions (4.9) and (4.10) become

$$X^+(\tau, \sigma) = 2\alpha' p^+ \tau \text{ and } p^+ = \pi \mathcal{P}^{\tau +}. \tag{4.24}$$

Using (4.23), our parameterisation constraints become

$$(\partial_\tau X \pm \partial_\sigma X)^2 = -2(\partial_\tau X^+ \pm \partial_\sigma X^+)(\partial_\tau X^- \pm \partial_\sigma X^-) + (\partial_\tau \vec{X} \pm \partial_\sigma \vec{X})^2 = 0$$

where $\vec{X} = (X^2, \ldots, X^d)$. Since we may assume $p^+ \neq 0$[‡], we obtain from (4.24) that

$$(\partial_\tau X^- \pm \partial_\sigma X^-) = \frac{1}{4\alpha' p^+}(\partial_\tau \vec{X} \pm \partial_\sigma \vec{X})^2 \tag{4.25}$$

Here we see why the light-cone gauge was useful. The dynamics of $X^-(\tau, \sigma)$ are completely determined by the dynamics of the transverse

[‡]If not, we can always rotate our coordinate system every so slightly.

dimensions $\vec{X}(\tau,\sigma)$, up to the constant x_0^-. The string dynamics are thus fully determined by a choice of the constants x_0^-, p^+ and of those occurring in $X^j(\tau,\sigma)$ for all $2 \leq j \leq d$.

For brevity, let us once again define new complex vectors $\alpha_n \in \mathbb{C}^{d+1}$ by

$$\alpha_0 = \sqrt{2\alpha'}p, \quad \alpha_n = \sqrt{n}a_n, \quad \text{and} \quad \alpha_{-n} = \sqrt{n}a_n^* \quad \text{for } n \in \mathbb{N}_{>0}. \tag{4.26}$$

One can use (4.25) to deduce that the minus oscillators satisfy

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{2p^+}\sum_{k\in\mathbb{Z}}\sum_{j=2}^{d}\alpha_{n-k}^j\alpha_k^j.$$

The world-sheet parameterisation is then given by the following three expansions:

$$X^+(\tau,\sigma) = 2\alpha' p^+\tau, \tag{4.27}$$

$$X^j(\tau,\sigma) = x_0^j + \sqrt{2\alpha'}\alpha_0^j\tau - i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^j e^{-in\tau}\cos(n\sigma), \tag{4.28}$$

for $2 \leq j \leq d$, and lastly

$$X^-(\tau,\sigma) = x_0^- + \frac{1}{p^+}L_0\tau + \frac{1}{p^+}\sum_{n\neq 0}\frac{1}{n}L_n e^{-in\tau}\cos(n\sigma), \tag{4.29}$$

where L_n denotes $\frac{1}{2}\sum_{k\in\mathbb{Z}}\sum_{j=2}^{d}\alpha_{n-k}^j\alpha_k^j$. As the notation suggests, we will see that these L_n satisfy the Virasoro relations in the quantum theory.

4.6 Quantisation

To quantise the open string, we will use our knowledge of the quantum harmonic oscillator. First, let us consider a simple action that encodes the dynamics of the transverse coordinates $\vec{X}(\tau,\sigma)$, namely

$$S = \frac{1}{4\pi\alpha'}\int d\tau\int_0^{\pi} d\sigma(\partial_\tau\vec{X})^2 - (\partial_\sigma\vec{X})^2 = \int d\tau\int_0^{\pi} d\sigma\mathcal{L}. \tag{4.30}$$

The canonical momenta $\frac{\partial\mathcal{L}}{\partial(\partial_\tau\vec{X})} = \frac{1}{2\pi\alpha'}\partial_\tau\vec{X}$ and $\frac{\partial\mathcal{L}}{\partial(\partial_\sigma\vec{X})} = \frac{1}{2\pi\alpha'}\partial_\sigma\vec{X}$ agree with the expressions we found for $\mathcal{P}^\tau$ and $\mathcal{P}^\sigma$ in (4.16). Using the variational principle, one can show that S also induces the right equation of motion and boundary conditions.

For the rest of this chapter, let j and k denote integers in $\{2, 3, \ldots, d\}$. Let us write $\vec{X}(\tau, \sigma)$ in terms of new $(d-1)$-dimensional variables $q_n(\tau)$, such that

$$X^j(\tau, \sigma) = q_0^j(\tau) + 2\sqrt{\alpha'} \sum_{n=1}^{\infty} q_n^j(\tau) \frac{\cos(n\sigma)}{\sqrt{n}}. \tag{4.31}$$

As we have seen in section 4.4, this is the most general solution to the equation of motion that satisfies the boundary conditions. The action then becomes

$$S = \int d\tau \Bigg[\underbrace{\frac{1}{4\alpha'}(\dot{q}_0(\tau))^2}_{:=L_0} + \sum_{n=1}^{\infty} \underbrace{\frac{1}{2n}(\dot{q}_n(\tau))^2 - \frac{n}{2}(q_n(\tau))^2}_{:=L_n} \Bigg]. \tag{4.32}$$

Note that multiplication of the $(d-1)$-dimensional vectors q_n simply reduces to the standard inner product on $\mathbb{R}^{d-1}$, as we are only considering the transverse coordinates. For every $n \geq 1$, the term L_n can be recognised as the Lagrangian of a harmonic oscillator with frequency n, and the term L_0 as the Lagrangian of a free particle. We therefore declare the q_n^j and their conjugate momenta $p_n^j = \frac{\partial L_n}{\partial \dot{q}_n^j}$ to be time-independent Schrödinger operators with the canonical commutation relations

$$[q_n^j, p_m^k] = i\delta_{jk}\delta_{nm}. \tag{4.33}$$

For every $n \geq 1$, we get the ladder operators

$$a_n^j = \frac{1}{\sqrt{2}}(p_n^j - iq_n^j) \quad \text{and} \quad a_n^{j\dagger} = \frac{1}{\sqrt{2}}(p_n^j + iq_n^j)$$

with the expected commutation relations $[a_n^j, a_m^{k\dagger}] = \delta_{jk}\delta_{nm}$. As the notation suggests, we will find that these operators correspond to the oscillator modes in (4.21).

To find the Heisenberg operators, we need an expression for the Hamiltonian. For each Lagrangian L_n, the corresponding Hamiltonian is given by $H_n = \frac{\partial L_n}{\partial \dot{q}_n}\dot{q}_n - L_n$. So we obtain

$$H_n = \begin{cases} \frac{1}{4\alpha'}\dot{q}_0^2 = \alpha' p_0^2 & \text{if } n = 0, \\ \frac{1}{2n}\dot{q}_n^2 + \frac{n}{2}q_n^2 = \frac{n}{2}(p_n^2 + q_n^2) = n\left(a_n^\dagger a_n + \frac{1}{2}(d-1)\right) & \text{if } n \geq 1. \end{cases} \tag{4.34}$$

The full Hamiltonian is then given by $H = \sum_{n=0}^{\infty} H_n$. Now for any time-independent Schrödinger operator Q, let $Q(\tau)$ denote its corresponding Heisenberg operator, so we have $Q(\tau) = e^{iH\tau} Q e^{-iH\tau}$. The Heisenberg evolution of $Q(\tau)$ is then given by

$$\frac{dQ(\tau)}{d\tau} = i[H(\tau), Q(\tau)]. \tag{4.35}$$

Note that commutation relations are preserved if one switches from Schrödinger operators to Heisenberg operators, i.e. if $[Q_1, Q_2] = Q_3$ for some Schrödinger operators Q_i then we have $[Q_1(\tau), Q_2(\tau)] = Q_3(\tau)$.

From the commutation relations of the ladder operators, we find that $[H, a_n^j] = [H_n, a_n^j] = n\left[a_n^{j\dagger} a_n^j, a_n^j\right] = -n a_n^j$. This implies

$$\frac{da_n^j(\tau)}{d\tau} = i[H(\tau), a_n^j(\tau)] = -in a_n^j(\tau), \tag{4.36}$$

so we obtain $a_n^j(\tau) = e^{-in\tau} a_n^j(0) = e^{-in\tau} a_n^j$. Similarly, we get $a_n^{j\dagger}(\tau) = e^{in\tau} a_n^{j\dagger}$, so we find

$$q_n^j(\tau) = \frac{i}{\sqrt{2}}(a_n^j e^{-in\tau} - a_n^{j\dagger} e^{in\tau}) \quad \text{for } n \geq 1. \tag{4.37}$$

We still have to find an expression for $q_0^j(\tau)$. Since $[H, p_0^j] = 0$, it follows that $p_0^j(\tau)$ is independent of τ and thus $p_0^j(\tau) = p_0^j(0) = p_0^j$. From (4.33) we obtain $[H, q_0^j] = \alpha'[(p_0^j)^2, q_0^j] = -2\alpha' i p_0^j$, so we have

$$\frac{dq_0^j(\tau)}{d\tau} = i[H(\tau), q_0^j(\tau)] = 2\alpha' p_0^j(\tau) = 2\alpha' p_0^j, \tag{4.38}$$

which implies

$$q_0^j(\tau) = x_0^j + 2\alpha' p_0^j \tau, \tag{4.39}$$

where x_0^j is some τ-independent operator that satisfies $[x_0^j, p_0^k] = i\delta_{jk}$, since we need $[q_0^j, p_0^k] = i\delta_{jk}$.

Substituting our results in (4.31), we obtain

$$X^j(\tau, \sigma) = x_0^j + 2\alpha' p_0^j \tau + i\sqrt{2\alpha'} \sum_{n=1}^{\infty} \left(a_n^j e^{-in\tau} - a_n^{j\dagger} e^{in\tau}\right) \frac{\cos(n\sigma)}{\sqrt{n}}. \tag{4.40}$$

Comparing this to our result in (4.21), we see that the operator p_0^j corresponds to the conserved momentum p^j of the string, so from now on we will simply write p^j instead of p_0^j. Looking at our expressions for $X^+(\tau,\sigma)$ and $X^-(\tau,\sigma)$ at the end of section 4.5, we see that in order to complete the quantisation of the light cone string, we still need x_0^- and p^+ to become Schrödinger operators. The canonical commutation relation is $[x_0^-, p^+] = -i$, where the minus stems from the form of the relativistic inner product in light cone coordinates (see (4.23)).

All the zero and oscillator modes of the string have now become Schrödinger operators. Before moving on to the space of states on which these operators act, let us revisit the Hamiltonian. From (4.34), we obtain that the full Hamiltonian is given by

$$\begin{aligned} H = \sum_{n=0}^{\infty} H_n &= \alpha' p^2 + \sum_{n=1}^{\infty} n\left(a_n^\dagger a_n + \frac{1}{2}(d-1)\right) \\ &= \alpha' p^2 + \sum_{n=1}^{\infty} n a_n^\dagger a_n + \frac{1}{2}(d-1)\sum_{n=1}^{\infty} n. \end{aligned} \tag{4.41}$$

The diverging term stems from the nonzero commutator of $a_n^\dagger$ and a_n. Due to this term, our Hamiltonian is *not* well-defined as is. We will solve this problem in section 4.8.

4.7 Space of states

In the last section we obtained the Schrödinger operators for the quantum theory of the open string. However, we have been talking about these operators without specifying a vector space for them to act on. We will now construct this space of states. Within this space the a_n^j and $a_n^{j\dagger}$ will act as lowering and raising operators, also referred to as *annihilation* and *creation* operators, respectively. As we know from the quantum harmonic oscillator, a creation operator $a_n^{j\dagger}$ raises the energy of a state, whereas a annihilation operators lowers it. For our string, the operator pairs (x_0^j, p^j) and (x_0^-, p^+) correspond to the zero modes of the string, whereas the a_n^j and $a_n^{j\dagger}$ correspond to the oscillation modes. Our ground states will therefore be eigenstates of a maximal commuting subset of the operators x_0^j, p^j, x_0^-, and p^+. As we would like our ground states to be eigenstates of energy, we will take eigenstates of the light-cone and transverse momenta operators p^+ and p^j. Thus, for any $c_{p^+} \in \mathbb{R}$ and $c_{\vec{p}} \in \mathbb{R}^{d-1}$ we define a ground state

$$|c_{p^+}, c_{\vec{p}}\rangle \quad \text{with} \quad \begin{aligned} p^+|c_{p^+}, c_{\vec{p}}\rangle &= c_{p^+}|c_{p^+}, c_{\vec{p}}\rangle, \\ p^j|c_{p^+}, c_{\vec{p}}\rangle &= c^j_{\vec{p}}|c_{p^+}, c_{\vec{p}}\rangle. \end{aligned}$$

The energy of these ground states cannot be lowered by the annihilation operators, thus we set

$$a^j_n|c_{p^+}, c_{\vec{p}}\rangle = 0 \text{ for all } n \geq 1.$$

We can create new states by letting the creation operators $a^{j\dagger}_n$ act upon the ground states. The basis of our state space is therefore given by vectors of the form

$$a^{j_1\dagger}_{n_1} a^{j_2\dagger}_{n_2} \ldots a^{j_k\dagger}_{n_k} |c_{p^+}, c_{\vec{p}}\rangle \tag{4.42}$$

with $n_i \geq 1$ and $2 \leq j_i \leq d$ for all $i \in \{1, 2, \ldots, k\}$. The vector space V_{open} that is generated by these basis vectors is the state space of the open string. The action of the operators a^j_n on V_{open} follows from their commutation relation with the $a^{j\dagger}_n$.

The construction of V_{open} should remind the reader of the Fock space representation of the Heisenberg Lie algebra. To see the resemblance more clearly, let us switch to the notation α^j_n as we did at the end of section 4.5. We define operators

$$\alpha^j_0 = \sqrt{2\alpha'} p^j, \quad \alpha^j_n = \sqrt{n} a^j_n, \quad \text{and} \quad \alpha^j_{-n} = \sqrt{n} a^{j\dagger}_n \quad \text{for } n \in \mathbb{N}, \tag{4.43}$$

so the vectors

$$\alpha^{j_1}_{n_1} \alpha^{j_2}_{n_2} \ldots \alpha^{j_k}_{n_k} |c_{p^+}, c_{\vec{p}}\rangle \ \text{ with } \ n_i < 0, \ c_{p^+} \in \mathbb{R}, \ c_{\vec{p}} \in \mathbb{R}^{d-1} \tag{4.44}$$

form a basis of V_{open}. Since $[a^j_n, a^{k\dagger}_m] = \delta_{jk}\delta_{nm}$ and $[a^j_n, a^k_m] = [p^j, a^j_n] = [p^j, a^{j\dagger}_n] = 0$ for all $n, m \geq 1$, it follows that

$$[\alpha^j_n, \alpha^k_m] = n\delta_{jk}\delta_{n,-m} \quad \text{for all } n, m \in \mathbb{Z}. \tag{4.45}$$

We see that for every $j \in \{2, \ldots, d\}$, we get a copy of the Heisenberg Lie algebra! If we let $\mathfrak{g} = \bigoplus_{j=2}^{d+1} \mathfrak{h}$ denote the Lie algebra obtained by taking the direct sum of $d-1$ copies of the Heisenberg Lie algebra $\mathfrak{h}$, we can view V_{open} as a direct sum

$$V_{open} = \bigoplus_{c_{p^+}, c_{\vec{p}}} \mathfrak{g}|c_{p^+}, c_{\vec{p}}\rangle \tag{4.46}$$

where each subspace $\mathfrak{g}|c_{p^+}, c_{\vec{p}}\rangle$ is a Fock representation of $\mathfrak{g}$ in which α_0^j acts as multiplication by $\sqrt{2\alpha'}c_{\vec{p}}^j$. Note that this is quite similiar to the state space of the lattice vertex algebra, except that we are summing over $\mathbb{R}^{d-1}$ instead of over a lattice. This is due to the fact that the momentum spectrum of our string is not discrete but continuous. When we compactify the closed string in chapter 5, we will find that the transverse momenta become quantised.

4.8 Virasoro operators

Recall the expression

$$X^-(\tau, \sigma) = x_0^- + \frac{1}{p^+} L_0 \tau + i \frac{1}{p^+} \sum_{n \neq 0} \frac{1}{n} L_n e^{-in\tau} \cos(n\sigma) \tag{4.47}$$

with $L_n = \sum_{k\in\mathbb{Z}} \sum_{j=2}^{d} \alpha_{n-k}^j \alpha_k^j$ from section 4.5. For brevity, let us write

$$\alpha_{n-k}\alpha_k = \sum_{j=2}^{d} \alpha_{n-k}^j \alpha_k^j.$$

In deriving (4.47), the α_n^j where assumed to be complex numbers and thus to commute with each other. In the quantum theory, the α_n become operators and we have to worry about their ordering in the sums $\frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_{n-k}\alpha_k$. To ensure that the L_n are well-defined operators, we use the normal ordered product, just as we did in chapter 2 for vertex operators.

For all $n \in \mathbb{Z}$, we define

$$\tilde{L}_n = :L_n: = \frac{1}{2} \sum_{k\in\mathbb{Z}} \sum_{j=2}^{d} :\alpha_{n-k}^j \alpha_k^j:. \tag{4.48}$$

We know that the α_n^j generate the Heisenberg Lie algebra, i.e. we have $[\alpha_n^j, \alpha_m^k] = n\delta_{jk}\delta_{n,-m}$ for all $n, m \in \mathbb{Z}$. Noting the similarity between the expression for the $\tilde{L}_n$ in (4.48) and the conformal vector of the lattice vertex algebra in equation (3.5) of section 3.2, it follows directly from our discussion below (3.5) that the $\tilde{L}_n$ satisfy the commutation relations of the Virasoro algebra with central charge $d-1$. Note that for $n \neq 0$, we have $\tilde{L}_n = L_n$ since in that case the operators α_{n-k} and α_k commute with each other for all $k \in \mathbb{Z}$. However, this is not the case for $n = 0$. We have

$$L_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_{-k}\alpha_k = \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{k=1}^{\infty} \alpha_{-k}\alpha_k + \frac{1}{2}\sum_{k=1}^{\infty} \alpha_k\alpha_{-k} \tag{4.49}$$

$$= \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \alpha_{-k}\alpha_k + \frac{1}{2}(d-1)\sum_{k=1}^{\infty} k = \tilde{L}_0 + \frac{1}{2}(d-1)\sum_{k=1}^{\infty} k, \tag{4.50}$$

where we used that $\alpha_k^j\alpha_{-k}^j = \alpha_k^j\alpha_{-k}^j + [\alpha_k^j, \alpha_{-k}^j] = \alpha_k^j\alpha_{-k}^j + k$. Since the last sum diverges, L_0 is not a well-defined operator as is. We therefore redefine L_0 by setting

$$L_0 := \tilde{L}_0 = \frac{1}{2}\alpha_0^2 + \sum_{k=1}^{\infty} \alpha_{-k}\alpha_k. \tag{4.51}$$

Now we have $L_n = \tilde{L}_n$ for all n, thus we conclude from our results in section 3.2 that

$$\left[L_m, L_n\right] = (m-n)L_{m+n} + \frac{d-1}{12}(m^3 - m)\delta_{m,-n} \quad \text{for all } n, m \in \mathbb{Z}, \tag{4.52}$$

which shows that V_{open} is a representation of the Virasoro algebra with central charge $d-1$.

Recall from section 4.5 that $2\alpha' p^- = \sqrt{2\alpha'}\alpha_0^- = \frac{1}{p^+}L_0$. To make up for simply omitting the term $\frac{1}{2}(d-1)\sum_{k=1}^{\infty} k$ from the operator L_0, we set

$$2\alpha' p^- := \frac{1}{p^+}(L_0 - a) \tag{4.53}$$

for some constant a. Also recall that the diverging term $\frac{1}{2}(d-1)\sum_{k=1}^{\infty} k$ appeared in the Hamiltonian given in (4.41). Substituting the constant a, we obtain

$$\begin{aligned} H &= \alpha' p^2 + \sum_{n=1}^{\infty} n a_n^\dagger a_n - a \\ &= \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n - a \\ &= L_0 - a = 2\alpha' p^- p^+. \end{aligned} \tag{4.54}$$

We now have a well-defined Hamiltonian and well-defined Virasoro operators. To complete the quantum theory, we have to determine the value of the constant a, which is called the *normal ordering constant*.

4.8.1 The normal ordering constant

Before we determine the value of a properly in the next section, let us give a naive interpretation of the diverging sum $\sum_{k=1}^{\infty} k$. To this end, we consider the complex Riemann zeta function

$$\zeta(z) = \sum_{k=1}^{\infty} k^{-z}, \quad \mathrm{Re}(z) > 1. \tag{4.55}$$

This function allows for an analytic continuation on the whole complex plane (except for a simple pole at $z = 1$). Given this continuation, we have $\zeta(-1) = -\frac{1}{12}$. One is therefore tempted to suggest the value $-\frac{1}{12}$ for our diverging sum $\sum_{k=1}^{\infty} k$. We then obtain

$$a = -\frac{1}{2}(d-1)\sum_{k=1}^{\infty} k \text{ ``=''} \ \frac{1}{24}(d-1). \tag{4.56}$$

Although this argument might appear terribly unconvincing, it actually does provide the right answer! In the next section, we will use Lorentz invariance to obtain $d = 25$ and $a = 1$, which agrees with our naive result in (4.56).

Let us consider the effect of setting a equal to 1. The Hamiltonian of the open string now becomes

$$H = L_0 - 1. \tag{4.57}$$

The effect of this shift is most prominent when we consider the *mass* of the string states. Classically, the squared mass of the string in light-cone coordinates is given by $M^2 = -p^2 = 2p^+p^- - \sum_{j=2}^{25} p^j p^j$. In the quantum theory, M^2 becomes an operator. Recalling that $L_0 - a = 2\alpha' p^+ p^-$ and $\sqrt{2\alpha'}p^j = \alpha_0^j$, we obtain

$$M^2 = \frac{1}{\alpha'}(L_0 - a) - \frac{1}{2\alpha'}\sum_{j=2}^{25}\alpha_0^j\alpha_0^j = \frac{1}{\alpha'}\left(-a + \sum_{j=2}^{25}\sum_{n=1}^{\infty}\alpha_{-n}^j\alpha_n^j\right). \tag{4.58}$$

We see that $a = 1$ results in a shift in the mass spectrum. In particular, it ensures that we have massless states aside from the ground states. These new massless states can be recognized as photons [Zwi04].

4.9 Lorentz invariance and the critical dimension

As mentioned in section 1.1, any Lorentz invariant quantum theory must contain *Lorentz generators*: operators that generate the Lorentz transformations and whose commutation relations are those of the Lorentz lie algebra. These operators correspond to the conserved Lorentz charges of our string, which can be found by Noether's theorem. In the string context, this theorem states the following: for any (infinitesimal) transformation $X \mapsto X + \epsilon Y$ that leaves the Lagrangian $\mathcal{L}$ invariant up to first order in ϵ, we obtain *conserved currents*

$$j^\alpha := \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X)} \cdot Y = \mathcal{P}^\alpha \cdot Y, \text{ for } \alpha \in \{\tau, \sigma\},$$

that satisfy

$$\partial_\tau j^\tau + \partial_\sigma j^\sigma = 0.$$

Here Y denotes some smooth map from the parameter space to $\mathbb{R}^{1,d}$ and ϵ is an (infinitesimal) constant. The currents give rise to a conserved *charge* $Q(\tau) = \int_0^\pi j^\tau d\sigma$ that satisfies

$$\frac{dQ}{d\tau} = \int_0^\pi \partial_\tau j^\tau d\sigma = -\int_0^\pi \partial_\sigma j^\sigma d\sigma = -[\mathcal{P}^\sigma \cdot Y]_0^\pi = 0,$$

due to the boundary conditions of the open string.

An (infinitesimal) Lorentz transformation is a transformation

$$X \mapsto X + \epsilon A X,$$

for some linear map $A : \mathbb{R}^{1,d} \to \mathbb{R}^{1,d}$ and (infinitesimal) constant ϵ, that leaves the relativistic dot product unchanged up to first order in ϵ, i.e. we have

$$\langle X + \epsilon A X, X + \epsilon A X \rangle = \langle X, X \rangle + \mathcal{O}(\epsilon^2). \tag{4.59}$$

Clearly, any such transformation leaves the Lagrangian $\mathcal{L}$ in (4.2) invariant (up to first order in ϵ) and thus results in a conserved charge by Noether's theorem. To satisfy (4.59), we must have $\langle X, AX \rangle = 0$. In light-cone coordinates, this means that A is given by a matrix of the form $\eta_{lc} B$, where $B \in \mathrm{Mat}_{d+1}(\mathbb{R})$ denotes an antisymmetric matrix and η_{lc} is the *light-cone* *metric* given by

$$\eta_{lc} = \begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{4.60}$$

The set of antisymmetric matrices is generated by the matrices

$$B_{(\mu\nu)} = \mathbb{1}_{(\mu\nu)} - \mathbb{1}_{(\nu\mu)} \quad \text{with } \mu, \nu \in \{+, -, 2, \dots, d\},$$

where $\mathbb{1}_{(\mu\nu)}$ denotes the matrix with a 1 at the $\mu\nu$-entry and a 0 elsewhere. So, for every $\mu, \nu \in \{+, -, 2, \dots, d\}$ we obtain a conserved current

$$\begin{aligned} j^\tau_{\mu\nu} &= \langle \mathcal{P}^\tau, \eta_{lc} B_{(\mu\nu)} X \rangle = \langle \mathcal{P}^\tau, \eta_{lc}(\mathbb{1}_{(\mu\nu)} - \mathbb{1}_{(\nu\mu)})X \rangle \\ &= \frac{1}{2\pi\alpha'} \left(\langle \dot{X}, \eta_{lc}\mathbb{1}_{(\mu\nu)} X \rangle - \langle \dot{X}, \eta_{lc}\mathbb{1}_{(\nu\mu)} X \rangle \right) \\ &= \frac{1}{2\pi\alpha'} \left(\dot{X}^\mu X^\nu - \dot{X}^\nu X^\mu \right), \end{aligned}$$

which gives rise to a conserved charge. As we are free to choose the scaling of these charges, we can define them as

$$Q_{\mu\nu} = -\int_0^\pi j^\tau_{\mu\nu} d\sigma = \frac{1}{2\pi\alpha'} \int_0^\pi \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu \right) d\sigma. \tag{4.61}$$

Note that the antisymmetry of the Lorentz transformation is reflected in the charges, as we have $Q_{\mu\nu} = -Q_{\nu\mu}$. In particular, we have $Q_{\mu\mu} = 0$.

Recall that the general solution for the components of X is

$$X^\mu(\tau, \sigma) = x_0^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma),$$

which implies $\dot{X}^\mu(\tau, \sigma) = 2\alpha' p^\mu + \sqrt{2\alpha'} \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)$. Since the conserved charges are independent of τ, we only have to worry about the τ-dependent terms when we substitute these expressions in (4.61). From

$$X^\mu \dot{X}^\nu = 2\alpha' x_0^\mu p^\nu + i2\alpha' \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \alpha_{-n}^\nu \cos^2(n\sigma) + (\tau\text{-dependent terms}),$$

we obtain that the Lorentz charges are given by

$$Q_{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right). \tag{4.62}$$

The above is a *classical* result. We now have to determine the quantum operators corresponding to the Lorentz charges. These operators should generate the Lorentz transformations and their commutation relations should be those of the Lorentz Lie algebra. In the light-cone gauge, this Lie algebra takes the following form.

Definition 4.1. *The Lorentz Lie algebra $\mathfrak{L}$ in $(d+1)$ light-cone coordinates is the Lie algebra spanned by the elements $M_{\mu\nu}$ with $\mu, \nu \in \{+, -, 2, \ldots, d\}$, whose Lie bracket is given by*

$$[M_{\mu\nu}, M_{\rho\kappa}] = i\eta_{\mu\rho} M_{\nu\kappa} - i\eta_{\nu\rho} M_{\mu\kappa} + i\eta_{\mu\kappa} M_{\rho\nu} - i\eta_{\nu\kappa} M_{\rho\mu}, \tag{4.63}$$

where $\eta = \eta_{lc}$.

To obtain the quantum operators corresponding to the classical charges $Q_{\mu\nu}$ given in (4.62), one simply replaces the zero and oscillation modes x_0^μ, p^μ and α_n^μ by their respective Schrödinger operators. One should check, however, whether the resulting expression is a well-defined Hermitian operator. In (4.62), the α_n^μ with positive index already appear on the right, so the expression is normal ordered. Moreover, since $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$, the term $i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right)$ is Hermitian. However, the term $x_0^\mu p^\nu - x_0^\nu p^\mu$ is not Hermitian in general. For example, for $\mu = -$ and $\nu = j \in \{2, \ldots, d\}$, the term $x_0^\nu p^\mu$ is not Hermitian as $[x_0^j, p^-] \neq 0$. We therefore redefine Q_{-j} as

$$Q_{-j} = x_0^- p^j - \frac{1}{2} \left(x_0^j p^- - p^- x_0^j \right) - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right). \tag{4.64}$$

We redefine Q_{j-}, Q_{-+} and Q_{+-} in similar ways. Note that by doing so we still have $Q_{\mu\nu} = -Q_{\nu\mu}$ in general.

The operators we have constructed must satisfy the commutation relations in (4.63). Since the oscillators α_n^- are given by

$$\sqrt{2\alpha'} \alpha_n^- = \begin{cases} \frac{1}{p^+} L_n & \text{if } n \neq 0, \\ \frac{1}{p^+} (L_0 - a) & \text{if } n = 0, \end{cases} \tag{4.65}$$

the most complicated charges are those of the form Q_{-j}. Since $Q_{--} = 0$, we must demand that

$$[Q_{-j}, Q_{-k}] = 0 \quad \text{for } j, k \in \{2, \ldots, d\}. \tag{4.66}$$

in order to satisfy the commutation relation in (4.63). The commutator $[Q_{-j}, Q_{-k}]$ can be calculated explicitly, since we know the commutators of all the zero and oscillator modes occurring in this expression. The calculation is too long and tedious to be given here; the important steps in this calculation can be found in [Wei11] on page 44. The result is

$$[Q_{-j}, Q_{-k}] = \frac{1}{\alpha' p^+ p^+} \sum_{n=1}^{\infty} C_n \left(\alpha^j_{-n} \alpha^k_n - \alpha^k_{-n} \alpha^j_n \right)$$

with

$$C_n = n \left(\frac{25 - d}{24} \right) + \frac{1}{n} \left(\frac{25 - d}{24} + (1 - a) \right). \tag{4.67}$$

It then follows from (4.66) that C_n must be zero for all $n \geq 1$, and thus we obtain

$$\begin{aligned} d &= 25, \\ a &= 1. \end{aligned} \tag{4.68}$$

This is a remarkable result. The requirement that our string theory is Lorentz invariant turns out to be so constraining that it fixes the dimension of spacetime! Moreover, this dimension is *much* larger than the four spacetime dimensions we are familiar with. One might hope that this is a mere artefact of bosonic string theory, which is not a realistic physical theory anyway as it does not include fermions. In superstring theory, however, Lorentz invariance requires a 10-dimensional spacetime. One way to 'deal' with these unobserved extra dimensions is via compactifications, which will be discussed in the next chapter.

Chapter 5

Closed strings and their compactifications

Thus far we only looked at *open* strings, i.e. strings whose endpoints are not joined together. However, any bosonic string theory must include *closed* strings, as open strings can generally close up on themselves. In this chapter, we generalise our former analysis to closed strings. Furthermore, we discuss the toroidal and orbifold compactification of these strings. The results in this chapter are largely based on [Zwi04] and [Sch14].

5.1 Parametrisation

In contrast to open strings, the world-sheet of closed strings traces out a tube in spacetime. We therefore alter our analysis of the open string by simply imposing a periodicity constraint on the world-sheet parameterisation X of our string. This condition reads

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma) \text{ for all } \tau, \sigma \in \mathbb{R}. \tag{5.1}$$

We thus obtain a full description of the string by letting σ take values in $[0, 2\pi]$ (recall that for the open string we let σ take values between 0 and π). The closed string satisfies the same equation of motion as the open string, which is given in (4.5). In contrast to the open string, we do not have to impose boundary conditions due to the simple fact that closed strings do not have boundaries (looking at equation (4.4), we indeed see that the first term vanishes by the periodicity constraint). Analogously to the open string, we can choose a reparameterisation such that

$$
\begin{aligned}
n \cdot X(\tau,\sigma) &= \alpha'(n\cdot p)\tau,\\
n\cdot p &= 2\pi n \cdot \mathcal{P}^{\tau}(\tau,\sigma),\\
n\cdot \mathcal{P}^{\sigma}(\tau,\sigma) &= 0,
\end{aligned}
$$

for some spacelike or lightlike vector $n \in \mathbb{R}^{1,d}$. The first two constraints are the closed string analogues of (4.9) and (4.10). The third can be imposed by choosing suitable points for $\sigma = 0$ on the lines of constant τ. Together they imply $(\dot{X} \pm X')^2 = 0$ (see the derivation of (4.15)), which again gives us the simple expressions

$$\mathcal{P}^{\tau} = \frac{1}{2\pi\alpha'}\dot{X} \quad \text{and} \quad \mathcal{P}^{\sigma} = -\frac{1}{2\pi\alpha'}X', \tag{5.2}$$

and so the equation of motion again becomes the familiar wave equation

$$\ddot{X} - X'' = 0. \tag{5.3}$$

5.2 Solving the wave equation and quantisation

We solve the wave equation as we did for the open string, but this time we have to implement the constraint in (5.1). We know the general solution is of the form $X(\tau,\sigma) = X_L(\tau+\sigma) + X_R(\tau-\sigma)$ for arbitrary C^2 maps $X_L, X_R : \mathbb{R} \mapsto \mathbb{R}^{1,d}$. The notation X_L and X_R refers to the left-moving and right-moving waves, respectively. By the periodicity of X, we obtain

$$X_L(\tau+\sigma+2\pi) + X_R(\tau-\sigma-2\pi) = X_L(\tau+\sigma) + X_R(\tau-\sigma), \tag{5.4}$$

and thus

$$X_L(u+2\pi) - X_L(u) = X_R(v) - X_R(v-2\pi), \tag{5.5}$$

if we write $u = \tau+\sigma$ and $v = \tau-\sigma$. As u and v are independent variables, differentiating with respect to u makes the right-hand side vanish, whereas differentiating with respect to v makes the left-hand side vanish. Thus we find that the derivatives of X_L and X_R are periodic functions, since we obtain

$$X_L'(u) = X_L'(u+2\pi) \quad \text{and} \quad X_R'(v) = X_R'(v+2\pi). \tag{5.6}$$

We can therefore write X_L' and X_R' in terms of their complex Fourier series. In contrast to the open string, we now obtain *two* sets of oscillation modes, which we will write as α_n and $\bar{\alpha}_n$:

$$X_L'(u) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n e^{-inu} \quad \text{and} \quad X_R'(v) = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \bar{\alpha}_n e^{-inv} \tag{5.7}$$

where for all $n \in \mathbb{Z}$, we have $\alpha_n = \alpha_{-n}^*$ and $\bar{\alpha}_n = \bar{\alpha}_{-n}^*$ *. We call the modes α_n and $\bar{\alpha}_n$ the left-moving and right-moving oscillators, respectively. The factor $\sqrt{\frac{\alpha'}{2}}$ in front is merely for convenience. Integrating gives us

$$X_L(u) = x_0^L + \sqrt{\frac{\alpha'}{2}}\alpha_0 u + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n e^{-inu},$$

where $x_0^L \in \mathbb{R}$ is some integration constant, and we get a similar expression for X_R. Substituting these expressions in (5.5) then gives

$$\alpha_0 = \bar{\alpha}_0. \tag{5.8}$$

The full solution is therefore given by

$$X(\tau,\sigma) = \left(x_0^L + x_0^R\right) + \sqrt{2\alpha'}\alpha_0\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma}\right). \tag{5.9}$$

Just as we showed for the open string, the zero mode α_0 corresponds to the momentum

$$p = \int_0^{2\pi} \mathcal{P}^\tau d\sigma = \frac{1}{2\pi\alpha'}\int_0^{2\pi} \dot{X} d\sigma = \frac{1}{2\pi\alpha'}\int_0^{2\pi} \sqrt{2\alpha'}\alpha_0 d\sigma = \sqrt{\frac{2}{\alpha'}}\alpha_0.$$

We thus arrive at two sets of oscillation modes, with one zero mode for the momentum. Looking at (5.9), we see that we are only interested in the sum of x_0^L and x_0^R, so we define the coordinate zero mode to be $x_0 = x_0^L + x_0^R$. We then arrive at the final expression

$$X(\tau,\sigma) = x_0 + \alpha' p\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma}\right). \tag{5.10}$$

*Here the asterisks denote complex conjugation, whereas the bars do not.

For the quantisation of the closed string, we move to light-cone coordinates and proceed as with the open string. The derivation is similar, only now we end up with two sets of independent oscillator operators: the α_n^j and the $\bar{\alpha}_n^j$. The Hamiltonian is given by

$$H = \alpha' p^- p^+,$$

and the full set of Schrödinger operators consists of the $x_0^j, p^j, x_0^-, p^+, \alpha_n^j$ and $\bar{\alpha}_n^j$ with canonical commutation relations

$$\begin{aligned}
[x_0^j, p^k] &= i\delta_{jk}, \\
[x_0^-, p^+] &= -i, \\
[\alpha_n^j, \alpha_m^k] &= n\delta_{n,-m}\delta_{jk}, \\
[\bar{\alpha}_n^j, \bar{\alpha}_m^k] &= n\delta_{n,-m}\delta_{jk},
\end{aligned}$$

for $n, m \in \mathbb{Z}$ and $j, k \in \{2, 3, \ldots, d\}$. Any other pair than those mentioned above commutes with each other, so in particular we have $[\alpha_n^j, \bar{\alpha}_m^k] = 0$.

5.3 Space of states and Virasoro operators

As we have obtained two sets of creation and annihilation operators, the space state of the closed string will have basis vectors of the form

$$\alpha_{n_1}^{j_1} \ldots \alpha_{n_m}^{j_m} \bar{\alpha}_{\bar{n}_1}^{k_1} \ldots \bar{\alpha}_{\bar{n}_\ell}^{k_\ell} |c_{p^+}, c_{\vec{p}}\rangle \text{ with } n_i, \bar{n}_i < 0,\ c_{p^+} \in \mathbb{R},\ c_{\vec{p}} \in \mathbb{R}^{d-1}. \tag{5.11}$$

Here $|c_{p^+}, c_{\vec{p}}\rangle$ denotes a ground state momentum eigenvector, as it did in the previous chapter for the open string. Note that the ordering of barred and unbarred creation operators in (5.11) does not matter, as they all commute with each other. However, we do have to keep in mind that the α_n and $\bar{\alpha}_n$ are not completely independent, as we found $\alpha_0 = \bar{\alpha}_0$ in (5.8). The effect of this constraint becomes clear when we consider the Virasoro operators of the closed string. We have two sets of Virasoro operators, defined by

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \sum_{j=2}^{d} :\alpha_{n-k}^j \alpha_k^j: \text{ and } \bar{L}_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \sum_{j=2}^{d} :\bar{\alpha}_{n-k}^j \bar{\alpha}_k^j: \text{ for } n \in \mathbb{Z}. \tag{5.12}$$

Note that normal ordering is only necessary for $n = 0$. As for the open string, the oscillator modes α_n^- and $\bar{\alpha}_n^-$ of the X^- coordinate of the closed string can be expressed in terms of the L_n and $\bar{L}_n$, namely

$$\sqrt{2\alpha'}\alpha_n^- = \frac{2}{p^+}L_n \quad \text{and} \quad \sqrt{2\alpha'}\bar{\alpha}_n^- = \frac{2}{p^+}\bar{L}_n \tag{5.13}$$

for $n \neq 0$. For $n = 0$, we need to correct for the normal ordering by setting

$$\sqrt{2\alpha'}\alpha_0^- = \frac{2}{p^+}(L_0 - a) \quad \text{and} \quad \sqrt{2\alpha'}\bar{\alpha}_0^- = \frac{2}{p^+}(\bar{L}_0 - \bar{a}) \tag{5.14}$$

for certain constants a and $\bar{a}$. As for the open string, the diverging term for both L_0 and $\bar{L}_0$ is $\frac{1}{2}(d-1)\sum_{k=1}^{\infty} k$, and so we expect $a = \bar{a} = \frac{1}{24}(d-1)$. Indeed, it can be shown that Lorentz invariance requires both $a = \bar{a} = 1$ and $d = 25$. The Hamiltonian of the closed string then becomes

$$H = \alpha' p^+ p^- = L_0 + \bar{L}_0 - 2.$$

We recognise this as the sum of two open string Hamiltonians: one for the left-moving and one for the right-moving oscillations.

The constraint $\bar{\alpha}_0 = \alpha_0$ implies that in particular we have $\bar{\alpha}_0^- = \alpha_0^-$, so it follows from equation (5.13) that $\bar{L}_0$ and L_0 should act identically on the state space of the closed string. Note that L_0 and $\bar{L}_0$ act as gradation operators: the first with respect to the left-moving creation operators and the second with respect to the right-moving creation operators. The state space of the closed string must therefore be generated by vectors of the form

$$\alpha_{n_1}^{j_1} \dots \alpha_{n_m}^{j_m} \bar{\alpha}_{\bar{n}_1}^{k_1} \dots \bar{\alpha}_{\bar{n}_\ell}^{k_\ell} |c_{p^+}, c_{\vec{p}}\rangle \quad \text{with} \quad n_i, \bar{n}_i < 0, \ c_{p^+} \in \mathbb{R}, \ c_{\vec{p}} \in \mathbb{R}^{24}, \tag{5.15}$$

that in addition satisfy

$$\sum_{i=1}^{m} n_i = \sum_{i=1}^{l} \bar{n}_i. \tag{5.16}$$

The constraint above is also known as the *level-matching condition*, as it requires the total degree (or *level*) in right and left moving oscillators to be equal.

5.3.1 World-sheet momentum

We have just seen that the operator $L_0 - \bar{L}_0$ should act as annihilator on the whole state space of the closed string. In 2-dimensional conformal field theory, the operator $L_0 - \bar{L}_0$ arises as generator of rotations on the punctured complex plane $\mathbb{C}^\times$. Recall from the remark at the end of section 4.2 that we have a conformal field theory on the world-sheet. For the closed string, the parameter space underlying this world-sheet can be viewed as a cylinder. We can map this cylinder to $\mathbb{C}^\times$ via

$$(\tau, \sigma) \mapsto e^{\tau + i\sigma}. \tag{5.17}$$

The rotations on $\mathbb{C}^\times$ therefore correspond to σ-translations in our context. We will now check that $L_0 - \bar{L}_0$ indeed generates σ-translations along the string.

First, note that the mode expansion

$$X(\tau, \sigma) = x_0 + \sqrt{2\alpha'}\alpha_0 \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left(\alpha_n e^{-in\sigma} + \bar{\alpha}_n e^{in\sigma}\right) \tag{5.18}$$

implies

$$\dot{X} + X' = \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n e^{-in(\tau+\sigma)}. \tag{5.19}$$

Given the commutation relations $[\alpha_n^j, \alpha_m^k] = n\delta_{n,-m}\delta_{jk}$ and $[\alpha_0^j, x_0^k] = \sqrt{\frac{\alpha'}{2}}[p^k, x_0^k] = -i\sqrt{\frac{\alpha'}{2}}\delta_{jk}$, a straightforward calculation gives

$$[L_0, \alpha_n^j] = -n\alpha_n^j \qquad \text{and} \qquad [L_0, x_0^j] = -i\sqrt{\frac{\alpha'}{2}}\alpha_0^j. \tag{5.20}$$

Combining these results, we find

$$\begin{aligned} [L_0, X^j(\tau,\sigma)] &= [L_0, x_0^j] + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} e^{-in\sigma}[L_0, \alpha_n^j] \\ &= -i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} e^{-in(\tau+\sigma)}\alpha_n^j = -\frac{i}{2}(\dot{X} + X'). \end{aligned} \tag{5.21}$$

Here we used in the first equality that each left-moving mode commutes with all right-moving modes. Similar calculations give

$$[\bar{L}_0, X^j(\tau,\sigma)] = -\frac{i}{2}(\dot{X} - X'). \tag{5.22}$$

From (5.21) and (5.22), we deduce

$$[L_0 + \bar{L}_0, X^j(\tau,\sigma)] = -i\partial_\tau X^j(\tau,\sigma) \text{ and}$$
$$[L_0 - \bar{L}_0, X^j(\tau,\sigma)] = -i\partial_\sigma X^j(\tau,\sigma),$$

The first commutator shows that $L_0 + \bar{L}_0$ generates τ-translations. Of course, this was to be expected, as the Hamiltonian is the generator of time translations and we have $[H, X^j] = [L_0 + \bar{L}_0 - 2, X^j] = [L_0 + \bar{L}_0, X^j]$. The second commutator shows that $L_0 - \bar{L}_0$ generates σ-translations. The operator $L_0 - \bar{L}_0$ can therefore be interpreted as the *world-sheet momentum*. Standard notation for this momentum is

$$P := L_0 - \bar{L}_0. \tag{5.23}$$

Note that this is *not* the same as the spacetime momentum p of our string. On the closed string state space, P must vanish, which means that our string states are invariant under σ-translations. This reflects the fact that the string dynamics are independent of our choice of the $\sigma = 0$ point on the closed string.

5.4 Toroidal compactification

Both bosonic string theory and superstring theory predict the existence of 10 or more spacetime dimensions. Clearly, we only observe four of these. The question arises where these extra dimensions are hiding. One way to explain this, is to assume that extra dimensions have been 'curled up' around themselves. To obtain the string theory corresponding to the monster vertex algebra, we will have to curl up all 24 transverse dimensions. More precisely, we will let the closed string move freely on a 24-dimensional torus. First, however, let us consider a closed string in a space where one dimension has been rolled up into a circle with radius $R \in \mathbb{R}$. For example, we can compactify the 25-th dimension by identifying

$$v^{25} \sim v^{25} + 2\pi R \quad \text{for } v \in \mathbb{R}^{1,25}. \tag{5.24}$$

Formally, we take the quotient

$$S^1 := \mathbb{R}/2\pi R\mathbb{Z} \tag{5.25}$$

where $2\pi R\mathbb{Z}$ denotes the additive subgroup $\{2\pi Rn : n \in \mathbb{Z}\}$ of $\mathbb{R}$, and let our string move in the space

$$\mathbb{R}^{1,24} \times S^1 \tag{5.26}$$

instead of in $\mathbb{R}^{1,25}$. The periodicity condition of the closed string given in (5.1) can then be weakened to

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + m2\pi R \quad \text{for some } m \in \mathbb{Z} \tag{5.27}$$

for the compactified dimension. Here we still view X as a map to $\mathbb{R}^{1,25}$; the condition in (5.27) ensures that the string becomes closed *after* compactification. The integer m indicates how many times the string will be wrapped around the circle S^1 when moving from $\mathbb{R}^{1,25}$ to $\mathbb{R}^{1,24} \times S^1$. Compactifications of this form, where the resulting compact space is a circle or a torus (in the case of multiple compactified dimensions), are referred to as *toriodal compactifications*.

5.4.1 One transverse coordinate

The compactification described above can be done for any of the string coordinates. However, since we want to be able the use the light-cone gauge, we will only compactify the transverse coordinates. Let $k \in \{2, 3, \ldots, 25\}$ and suppose that

$$X^k(\tau, \sigma + 2\pi) = X^k(\tau, \sigma) + m2\pi R \quad \text{for some fixed } m \in \mathbb{Z}. \tag{5.28}$$

For the other coordinates, we assume the standard periodicity condition of the closed string. Now once again, we solve the wave equation. Note that the solution will only differ from the 'regular' closed string in the k-th coordinate. For this coordinate, equation (5.5) becomes

$$X_L^k(u + 2\pi) - X_L^k(u) = X_R^k(v) - X_R^k(v - 2\pi) + m2\pi R, \tag{5.29}$$

where we write $u = \tau + \sigma$ and $v = \tau - \sigma$. As the new term $m2\pi R$ does not affect the derivative of the right-hand side, our earlier analysis still applies. We therefore find

$$X_L^k(u) = x_0^{Lk} + \sqrt{\frac{\alpha'}{2}}\alpha_0^k u + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^k e^{-inu} \quad \text{and} \tag{5.30}$$

$$X_R^k(v) = x_0^{Rk} + \sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^k v + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \bar{\alpha}_n^k e^{-inv}, \tag{5.31}$$

where α_n^k and $\bar{\alpha}_n^k$ denote the complex Fourier coefficients of $X_L'^k(u)$ and $X_R'^k(v)$, respectively. Due to the extra term in (5.29), we generally do *not* have $\alpha_0^k = \bar{\alpha}_0^k$. Instead, (5.29) implies $2\pi\sqrt{\frac{\alpha'}{2}}\alpha_0^k = 2\pi\sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^k + m2\pi R$ and thus

$$\alpha_0^k - \bar{\alpha}_0^k = m\sqrt{\frac{2}{\alpha'}}R. \tag{5.32}$$

The momentum p^k is related to the sum of $\bar{\alpha}_0^k$ and α_0^k, since we have

$$\begin{aligned}\sqrt{2\alpha'}p^k &= \sqrt{2\alpha'}\int_0^{2\pi}\mathcal{P}^{\tau k}d\sigma = \frac{\sqrt{2\alpha'}}{2\pi\alpha'}\int_0^{2\pi}\dot{X}^k d\sigma \\ &= \frac{\sqrt{2\alpha'}}{2\pi\alpha'}\int_0^{2\pi}\left(\sqrt{\frac{\alpha'}{2}}\bar{\alpha}_0^k + \sqrt{\frac{\alpha'}{2}}\alpha_0^k\right)d\sigma = \bar{\alpha}_0^k + \alpha_0^k.\end{aligned} \tag{5.33}$$

The full solution of the compactified coordinate is thus given by

$$X^k(\tau,\sigma) = x_0^k + \alpha' p^k\tau + mR\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\alpha_n^k e^{-in\sigma} + \bar{\alpha}_n^k e^{in\sigma}\right) \tag{5.34}$$

with $x_0^k = x_0^{Rk} + x_0^{Lk}$. The difference with the 'regular' closed string is the extra zero mode mR, which is proportional to the winding number. This zero mode can be interpreted as a different kind of momentum: just as p is the momentum of the string with respect to τ, $\frac{mR}{\alpha'}$ reflects the 'internal' momentum with respect to σ. This internal energy is due to the stretching of the string by wrapping it m times around the circle. The usual notation for this momentum is

$$w := \frac{mR}{\alpha'}$$

and is commonly referred to as the 'winding number'.

Since we only compactified a transverse coordinate, we can still use the light-cone gauge. Quantisation leads to the expected commutation relations

$$[x_0^k, p^k] = i, \quad [\alpha_n^k, \alpha_m^k] = n\delta_{n,-m}, \quad [\bar{\alpha}_n^k, \bar{\alpha}_m^k] = n\delta_{n,-m}, \quad [\bar{\alpha}_n^k, \alpha_m^k] = 0.$$

Since equation (5.32) implies $w = \frac{1}{\sqrt{2\alpha'}}(\alpha_0 - \bar{\alpha}_0)$, the winding number w also becomes an operator. This operator commutes with all the others, as it should, since for a single closed string it is merely a constant. However, if we are to allow for strings with different winding numbers in our state space, we should view w as an operator whose eigenvalues are the possible winding numbers, i.e. values from the set $\frac{R}{\alpha'}\mathbb{Z}$. The winding number w is thus a quantized observable.

As a result of restricting the k-th dimension from $\mathbb{R}$ to S^1, the momentum p^k that is conjugate to x_0^k also becomes quantised. To see this, consider the operator e^{iap^k}, where $a \in \mathbb{R}$. By standard results in quantum mechanics, we know that this operator should translate a state an amount of a along the k-th direction. However, since the k-th dimension is compactified, a translation of a should have the same result as a translation of $a + 2\pi R$. This implies that the operator $e^{i2\pi R p^k}$ should act as the identity operator, which in turn requires that p^k has eigenvalues of the form

$$c_{p^k} = \frac{n}{R}, \quad n \in \mathbb{Z}. \tag{5.35}$$

We can now construct the state space of a closed string with one compactified dimension. Note that, in comparison to the regular closed string, we lost states due to the quantisation of p^k. However, we also gained new states, namely those corresponding to different winding numbers. Our ground states are eigenstates of the operators p^+, p^k, w and p^j for $j \in \{2, 3, \ldots, 25\} \setminus \{k\}$, which we write as

$$|c_{p^+}, c_{\vec{p}}, n, m\rangle \quad \text{with} \quad c_{p^+} \in \mathbb{R},\ c_{\vec{p}} \in \mathbb{R}^{23}, n, m \in \mathbb{Z}. \tag{5.36}$$

Here n corresponds to a p^k-eigenvalue of $\frac{n}{R}$, whereas m corresponds to a w-eigenvalue of $\frac{mR}{\alpha'}$. General basis vectors are then obtained by acting upon these ground states with creation operators, so they are of the form

$$\alpha_{n_1}^{j_1} \ldots \alpha_{n_s}^{j_s} \bar{\alpha}_{\bar{n}_1}^{k_1} \ldots \bar{\alpha}_{\bar{n}_\ell}^{k_\ell} |c_{p^+}, c_{\vec{p}}, n, m\rangle \quad \text{with} \quad n_i, \bar{n}_i < 0. \tag{5.37}$$

Not every state of the above form is in our state space, as we still need to consider the constraint $\bar{\alpha}_0^k - \alpha_0^k = \sqrt{2\alpha'}w$ for the compactified dimension and the constraint $\bar{\alpha}_0^\mu = \alpha_0^\mu$ for the other dimensions. Again, the effect of these constraints becomes clear via the Virasoro operators. Since $k \neq 1, 2$ we still have $\bar{\alpha}_0^- = \alpha_0^-$, which implies that $\bar{L}_0$ and L_0 should act identically on the state space. From

$$L_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \sum_{j=2}^{25} :\alpha_{-n}^j \alpha_n^j: = \frac{1}{2}\alpha_0^k \alpha_0^k + \frac{1}{2} \sum_{\substack{j=2 \\ j \neq k}}^{25} \alpha_0^j \alpha_0^j + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{j=2}^{25} \alpha_{-n}^j \alpha_n^j \tag{5.38}$$

and a similar expression for $\bar{L}_0$, we obtain

$$L_0 - \bar{L}_0 = \frac{1}{2}(\alpha_0^k\alpha_0^k - \bar{\alpha}_0^k\bar{\alpha}_0^k) + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=2}^{25}\alpha_{-n}^j\alpha_n^j - \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=2}^{25}\bar{\alpha}_{-n}^j\bar{\alpha}_n^j.$$

Note that $\frac{1}{2}(\alpha_0^k\alpha_0^k - \bar{\alpha}_0^k\bar{\alpha}_0^k) = \frac{1}{2}(\alpha_0^k + \bar{\alpha}_0^k)(\alpha_0^k - \bar{\alpha}_0^k) = \alpha' p^k w$, so to ensure that $L_0 - \bar{L}_0$ annihilates every state in our state space, the difference between the degree in right-moving and left-moving creation operators must be equal to the eigenvalue of the operator $\alpha' p^k w$. To be precise, the basis vectors in (5.37) must satisfy

$$\alpha'\frac{n}{R}\frac{mR}{\alpha'} - \sum_{i=1}^{s} n_i + \sum_{i=1}^{l}\bar{n}_i = nm - \sum_{i=1}^{s} n_i + \sum_{i=1}^{l}\bar{n}_i = 0. \tag{5.39}$$

5.4.2 The closed string on a torus

We can easily extend the toroidal compactification to *all* transverse directions instead of just one. In the example above, we compactified along the k-th dimension. Note, however, that we could have chosen any transverse direction, i.e. we could have set $\vec{X} = \vec{X} + 2\pi R\vec{m}$ for some transverse vector $\vec{m}$. Compactifying all transverse directions thus comes down to choosing 24 linearly independent transverse vectors along which we will roll up our space. These vectors span a full rank lattice L in $\mathbb{R}^{24}$ (which need not be rational in the sense of definition 3.1). The periodicity condition for a closed string then becomes

$$\vec{X}(\tau, \sigma + 2\pi) = \vec{X}(\tau, \sigma) + 2\pi R\beta \;\text{ for some fixed } \beta \in L. \tag{5.40}$$

As before, $\vec{X}$ here denotes the vector $(X^2, \ldots, X^{25})$. This compactification reduces the transverse dimensions to the 24-dimensional torus that is obtained by taking the quotiënt of $\mathbb{R}^{24}$ by $2\pi RL$. To obtain to Monster vertex algebra, one must take L to be the Leech lattice.

The quantum theory of the closed string on the torus is almost immediate from our analysis in the last section, as we have $X^j = X^j + 2\pi R\beta^j$ for each transverse dimension j. We thus obtain two sets of oscillators α_n^j and $\bar{\alpha}_n^j$, which satisfy

$$\alpha_0^j - \bar{\alpha}_0^j = \sqrt{\frac{2}{\alpha'}}R\beta^j \qquad \text{and} \qquad \alpha_0^j + \bar{\alpha}_0^j = \sqrt{2\alpha'}p^j. \tag{5.41}$$

Instead of defining a winding number for each transverse direction j, it is more informative to study the *left-moving and right-moving transverse momentum*, which we define as

$$\vec{p}_L = (\alpha_0^2, \dots, \alpha_0^{25}) \qquad \text{and} \qquad \vec{p}_R = (\bar{\alpha}_0^2, \dots, \bar{\alpha}_0^{25}). \tag{5.42}$$

In order for $e^{-i\vec{a}\cdot\vec{p}}$ to be a well-defined operator, we need $\frac{\lambda\cdot\vec{p}}{R}$ to be integral for any $\lambda \in L$. This means that the allowed momenta $\frac{1}{R}\vec{p}$ lie precisely in the *dual lattice* L^* of L in $\mathbb{R}^{24}$, which is defined as

$$L^* = \{x \in \mathbb{R}^{24} : x \cdot \lambda \in \mathbb{Z} \text{ for all } \lambda \in L\}. \tag{5.43}$$

We thus obtain from (5.41) and (5.42) that the left and right momenta satisfy

$$\vec{p}_L = \frac{1}{2}\left(\sqrt{2\alpha'}\vec{p} + \sqrt{\frac{2}{\alpha'}}R\beta\right) = \sqrt{\alpha'}\left(\frac{1}{\sqrt{2}}\vec{p} + \frac{1}{\alpha'\sqrt{2}}R\beta\right) \tag{5.44}$$

$$\vec{p}_R = \frac{1}{2}\left(\sqrt{2\alpha'}\vec{p} - \sqrt{\frac{2}{\alpha'}}R\beta\right) = \sqrt{\alpha'}\left(\frac{1}{\sqrt{2}}\vec{p} - \frac{1}{\alpha'\sqrt{2}}R\beta\right), \tag{5.45}$$

with $\frac{1}{R}\vec{p} \in L^*$ and $\beta \in L$.

To obtain the state space of the lattice vertex algebra, we will have to set $R = \sqrt{2}$ and $\alpha' = 1$. The result above then simplifies to

$$\vec{p}_L = \frac{1}{R}\vec{p} + \beta \qquad \text{and} \qquad \vec{p}_R = \frac{1}{R}\vec{p} - \beta, \tag{5.46}$$

This result is particularly nice if L is *self-dual*, i.e. if $L = L^*$, since then $\vec{p}_L$ and $\vec{p}_R$ are lattice elements themselves. Indeed, this will be the case when L is the Leech lattice. We can then take our ground states to be eigenvectors of p^+, p_L^j and p_R^j, and write them as

$$|c_{p^+}, \beta_L, \beta_R\rangle \;\text{ with }\; c_{p^+} \in \mathbb{R},\; \beta_L, \beta_R \in L. \tag{5.47}$$

The action of p_L^j is then given by $p_L^j|c_{p^+}, \beta_L, \beta_R\rangle = \beta_L^j|c_{p^+}, \beta_L, \beta_R\rangle$ (and similarly for p_R^j) and the general basis vectors are of the form

$$\alpha_{n_1}^{j_1} \dots \alpha_{n_s}^{j_s}\bar{\alpha}_{\bar{n}_1}^{k_1} \dots \bar{\alpha}_{\bar{n}_\ell}^{k_\ell}|c_{p^+}, \beta_L, \beta_R\rangle \;\text{ with }\; n_i, \bar{n}_i < 0. \tag{5.48}$$

As before, these states must be annihilated by $L_0 - \bar{L}_0$. From

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n = \frac{1}{2}p_L^2 + \sum_{n=1}^{\infty}\alpha_{-n}\alpha_n, \tag{5.49}$$

and a similar expression for $\bar{L}_0$, we find that the states in (5.48) must satisfy

$$\frac{1}{2}\left(\beta_L^2 - \beta_R^2\right) - \sum_{i=1}^{s} n_i + \sum_{i=1}^{l} \bar{n}_i = 0. \tag{5.50}$$

The above result implies that $\frac{1}{2}\left(\beta_L^2 - \beta_R^2\right)$ must be integer, so we need L to be an even lattice. Again, the Leech lattice satisfies this requirement.

5.5 Orbifold compactification

A different way of changing the boundary conditions and obtaining new states is via orbifolding. Formally, an oribfold is a generalization of a manifold, which locally looks like quotients of open subsets of $\mathbb{R}^n$ by finite groups. For our purposes, it suffices to think of an orbifold as an identification of points in spacetime which includes fixed points. To obtain the Monster vertex algebra, we will have to orbifold the 24-dimensional torus whose underlying lattice is the Leech lattice. First, however, let us consider a simpler construction. We reduce the 24 transverse dimensions of the uncompactified closed string to the orbifold $\mathbb{R}^{24}/\mathbb{Z}_2$ by identifying

$$\vec{v} \sim -\vec{v} \quad \text{for } v \in \mathbb{R}^{1,25}. \tag{5.51}$$

Here, as before, we have $\vec{v} = (v^2, \ldots, v^{25})$. More formally, we let $\mathbb{Z}_2 = \{\pm 1\}$ act on $\mathbb{R}^{24}$ by $\vec{v} \mapsto \pm\vec{v}$, and let our string move in the space

$$\mathbb{R}^{1,1} \times \mathbb{R}^{24}/\mathbb{Z}_2. \tag{5.52}$$

Note that the identification has one fixed point in $\mathbb{R}^{24}$, namely zero. The quantum theory of this orbifold consists of two kinds of states: superpositions of closed strings in $\mathbb{R}^{1,25}$ that are invariant under the orbifold identification, and states corresponding to open strings in $\mathbb{R}^{1,25}$ that become closed under the orbifold identification. The former are known as the *untwisted states* and the latter as the *twisted states*.

5.5.1 Untwisted states

To find the untwisted states, we consider the state space of the (uncompactified) closed string and remove those states that not invariant under the orbifold identification. Recall this space is generated by the vectors

$$\alpha_{n_1}^{j_1} \ldots \alpha_{n_m}^{j_m} \bar{\alpha}_{\bar{n}_1}^{k_1} \ldots \bar{\alpha}_{\bar{n}_\ell}^{k_\ell} |c_{p^+}, c_{\vec{p}}\rangle \text{ with } n_i, \bar{n}_i < 0,\ c_{p^+} \in \mathbb{R},\ c_{\vec{p}} \in \mathbb{R}^{24}, \tag{5.53}$$

that satisfy $\sum n_i = \sum \bar{n}_i$. We now define an operator U on this space which implements the orbifold. As the mapping $\vec{v} \mapsto -\vec{v}$ is an involution, we want $UU = \mathrm{id}$ and thus $U = U^{-1}$. For each transverse direction, we set

$$UX^j(\tau,\sigma)U = -X^j(\tau,\sigma). \tag{5.54}$$

As the above holds for all τ and σ, it follows from (5.9) that we have

$$Ux_0^jU = -x_0^j, \quad U\alpha_n^jU = -\alpha_n^j, \quad \text{and} \quad U\bar{\alpha}_n^jU = -\bar{\alpha}_n^j. \tag{5.55}$$

We only orbifold the transverse dimensions, so we need U to leave X^+ and X^- invariant. In particular, we expect U to have no effect on the light-cone momentum p^+ and zero mode x_0^-. We therefore define

$$Up^+U = p^+ \qquad \text{and} \qquad Ux_0^-U = x_0^-. \tag{5.56}$$

These equations imply $UX^\pm U = X^\pm$, since $X^+ = \alpha' p^+ \tau$ and since X^- is quadratic in the transverse oscillators α_n^j. Note that the Hamiltonian $L_0 + \bar{L}_0 - 2$ is quadratic in transverse oscillators as well and thus commutes with U. This means that the U-invariant sector is well-defined, as any quantum state that is U-invariant at some point in time will remain U-invariant.

To find the U-invariant states, we still have to determine the action of U on the ground states. If a state has no transverse momentum, we expect U to have no effect, so we set

$$U|c_{p^+},0\rangle = |c_{p^+},0\rangle. \tag{5.57}$$

Since the transverse momenta p^j are conjugate to the position zero modes x_0^j, the operator $e^{iax_0^j}$ should 'translate' the transverse momentum p^j by a. We thus obtain from $Ux_0^jU = -x_0^j$ that

$$U|c_{p^+},c_{\vec{p}}\rangle = Ue^{ic_{\vec{p}}\cdot\vec{x}_0}|c_{p^+},0\rangle = (Ue^{ic_{\vec{p}}\cdot\vec{x}_0}U)U|c_{p^+},0\rangle \tag{5.58}$$

$$= e^{-ic_{\vec{p}}\cdot\vec{x}_0}|c_{p^+},0\rangle = |c_{p^+},-c_{\vec{p}}\rangle. \tag{5.59}$$

We can now determine the U-invariant states. Consider the states of form

$$|c_{p^+},c_{\vec{p}}\rangle + |c_{p^+},-c_{\vec{p}}\rangle, \quad \text{and} \tag{5.60}$$

$$|c_{p^+},c_{\vec{p}}\rangle - |c_{p^+},-c_{\vec{p}}\rangle, \tag{5.61}$$

The first are U-invariant and the second have U-eigenvalue -1. The basis states of the untwisted space are therefore obtained by acting with an even number of creation operators on (5.60) or with an odd number on (5.61) (in a way that respects the constraint $\sum n_i = \sum \bar{n}_i$).

5.5.2 Twisted states

The untwisted states are obtained as a U-invariant subspace of the parent theory. The states in this subspace are necessarily superpositions, and can be interpreted as two closed strings that transform into each other under the orbifold identification. The twisted states, on the other hand, correspond to open strings in the parent theory whose endpoints join together under the orbifold identification. These strings satisfy the boundary condition

$$X^j(\tau, \sigma + 2\pi) = -X^j(\tau, \sigma) \tag{5.62}$$

for the transverse dimensions j. For the other two dimensions, we still have the closed string periodicity condition $X^{\pm}(\tau, \sigma + 2\pi) = X^{\pm}(\tau, \sigma)$. Note that, as before, we still view X as a map to $\mathbb{R}^{1,25}$. The condition in (5.62) ensures that the string becomes closed *after* the orbifold identification.

Let us solve the wave equation for the transverse dimensions. With the new boundary condition, equation (5.5) becomes

$$X_L^j(u + 2\pi) + X_L^j(u) = -X_R^j(v) - X_R^j(v - 2\pi), \tag{5.63}$$

so differentiating with respect to the independent variables u and v gives

$$\partial_u X_L^j(u) = -\partial_u X_L^j(u + 2\pi) \text{ and } \partial_v X_R^j(v) = -\partial_v X_R^j(v + 2\pi). \tag{5.64}$$

This implies that $\partial_u X_L^j$ and $\partial_v X_R^j$ are periodic functions with a period of 4π instead of 2π. Their Fourier expansions are of the form

$$\partial_u X^j{}_L(u) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in 2\mathbb{Z}+1} \alpha^j_{\frac{n}{2}} e^{-i\frac{n}{2}u} \text{ and } \partial_v X^j{}_R(v) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in 2\mathbb{Z}+1} \bar{\alpha}^j_{\frac{n}{2}} e^{-i\frac{n}{2}v},$$

so integrating gives

$$X_L^j(u) = x_0^{Lj} + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in 2\mathbb{Z}+1} \frac{2}{n} \alpha^j_{\frac{n}{2}} e^{-i\frac{n}{2}u}, \tag{5.65}$$

and similar expression for $X_R^j(v)$. It follows from (5.63) that $x_0^{Lj} = -x_0^{Rj}$, so the full solution is given by

$$X^j(\tau, \sigma) = i\sqrt{2\alpha'} \sum_{n \in 2\mathbb{Z}+1} \frac{e^{-i\frac{n}{2}\tau}}{n} \left(\alpha^j_{\frac{n}{2}} e^{-i\frac{n}{2}\sigma} + \bar{\alpha}^j_{\frac{n}{2}} e^{i\frac{n}{2}\sigma} \right). \tag{5.66}$$

In contrast to the regular closed string, there is no zero coordinate or zero momentum mode for the orbifolded dimensions. This is a natural consequence of the constraint in (5.62). If we fix τ_0 and consider the 24-dimensional string described by $\vec{X}(\tau_0, \sigma)$ in $\mathbb{R}^{24}$, the constraint in (5.62) implies that this string is located symmetrically around the fixed point $\vec{0}$ of the orbifold. The average (transverse) position is therefore equal to this fixed point. As time flows, the string may oscillate around this point, but it will not move away from it.

Quantisation of the orbifolded string leads to the expected commutation relations

$$
\begin{aligned}
[\alpha^k_n, \alpha^k_m] &= n\delta_{n,-m}, \\
[\bar{\alpha}^k_n, \bar{\alpha}^k_m] &= n\delta_{n,-m}, \\
[\bar{\alpha}^k_n, \alpha^k_m] &= 0
\end{aligned}
$$

for $n, m \in \mathbb{Z} + \frac{1}{2}$. The ground states for the twisted states are simply eigenvectors of p^+, since we have no transverse momentum operators. The general basis states are thus given by

$$
\alpha^{j_1}_{n_1} \ldots \alpha^{j_m}_{n_m} \bar{\alpha}^{k_1}_{\bar{n}_1} \ldots \bar{\alpha}^{k_\ell}_{\bar{n}_\ell} |c_{p^+}\rangle \text{ with } n_i, \bar{n}_i < 0, \; c_{p^+} \in \mathbb{R}. \tag{5.67}
$$

Since we have no zero modes, we do not have any constraints on the left-moving and right-moving energy levels $\sum n_i$ and $\sum \bar{n}_i$.

To finalise the orbifold theory, let us consider the Virasoro operators. The shift to half-integer indices of the α^j_n does not affect the form of the L_n and $\bar{L}_n$ in the expansion of $X^-(\tau, \sigma)$, so we have

$$
L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}+\frac{1}{2}} \sum_j \alpha^j_{n-k} \alpha^j_k \quad \text{for } n \in \mathbb{Z}, \tag{5.68}
$$

and a similar expression for the $\bar{L}_n$. However, the half-integer indices *do* affect the normal ordering constant for $n = 0$. We have

$$
\frac{1}{2} \sum_{k \in \mathbb{Z}+\frac{1}{2}} \alpha^j_{-k} \alpha^j_k = \sum_{k \in \mathbb{N}+\frac{1}{2}} \alpha^j_{-k} \alpha^j_k + \frac{1}{2} \sum_{k \in \mathbb{N}+\frac{1}{2}} [\alpha^j_k, \alpha^j_{-k}] \tag{5.69}
$$

$$
= \sum_{k \in \mathbb{N}+\frac{1}{2}} \alpha^j_{-k} \alpha^j_k + \frac{1}{2} \sum_{k \in \mathbb{N}+\frac{1}{2}} k \tag{5.70}
$$

The diverging term in both L_0 and $\bar{L}_0$ is therefore $24 \cdot \frac{1}{2} \sum_{k \in \mathbb{N}+\frac{1}{2}} k$. Let us once again give a naive interpretation of this infinite sum by using the 'result' from section 4.8.1

$$\sum_{k \in \mathbb{N}} k \text{ “=” } -\frac{1}{12}. \tag{5.71}$$

From

$$\sum_{k \in \mathbb{N}} k = \sum_{k \in \mathbb{N}_{\text{even}}} k + \sum_{k \in \mathbb{N}_{\text{odd}}} k = 2 \sum_{k \in \mathbb{N}} k + \sum_{k \in \mathbb{N}_{\text{odd}}} k, \tag{5.72}$$

we suggest that the value of the diverging term should be

$$24 \cdot \frac{1}{2} \sum_{k \in \mathbb{N}+\frac{1}{2}} k = 24 \cdot \frac{1}{4} \sum_{k \in \mathbb{N}_{\text{odd}}} k = -6 \cdot \sum_{k \in \mathbb{N}} k \text{ “=” } \frac{1}{2}. \tag{5.73}$$

Again, one might be unconvinced by this procedure. It turns out, however, that this value does give rise to a consistent string theory.

Following our strategies from before, one might be tempted to redefine L_0 as $\frac{1}{2} \sum_{k \in \mathbb{Z}+\frac{1}{2}} \sum_j : \alpha^j_{-k} \alpha^j_k :$ and to adjust for the normal ordering constant by setting $H = (L_0 + \frac{1}{2}) + (\bar{L}_0 + \frac{1}{2})$. However, due the to half-integer indices, the operators L_n will then *not* satisfy the Virasoro relations. As shown in theorem 1.9.6 of [FLM89], we have to redefine

$$L_0 := \frac{1}{2} \sum_{k \in \mathbb{Z}+\frac{1}{2}} \sum_j : \alpha^j_{-k} \alpha^j_k : + \frac{3}{2} \tag{5.74}$$

(and similarly for $\bar{L}_0$) in order to obtain the proper commutation relations. The 'effective' normal ordering constant is therefore still equal to -1. So, with this definiton of L_0 and $\bar{L}_0$, the Hamiltonian remains

$$H = (L_0 - 1) + (\bar{L}_0 - 1). \tag{5.75}$$

In the next chapter, we will use this Hamiltonian and the expression for L_0 to calculate the partition function of an orbifold theory. We will find that this function is modular invariant, which is a necessary condition for any consistent string theory.

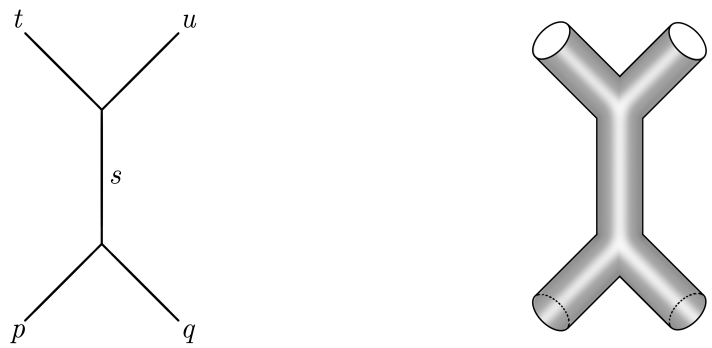
Chapter 6

String interactions

In this chapter, we first sketch how vertex operators are used to calculate the probability amplitudes corresponding to diagrams of interacting strings. We then restrict ourselves to the one-loop diagram of closed strings and calculate the partition function of the string theory associated with the monster vertex algebra $V^\natural$. As it should, this partition function will be equal to the J-function. The definition of the one-loop probability amplitude and calculation of the partition function of the uncompactified string are based on [Sch14] and [BLT12]. The derivation of the partition function of $V^\natural$ follows [Tui92] and [DGH88].

6.1 String diagrams and vertex operators

In the last two chapters, we determined the quantum states of free open and closed strings. As explained in section 1.1, one of the main goals in quantum field theory is to determine the probability amplitudes of state transitions. In the particle picture, these transitions are scattering processes which are often depicted by a *Feynman diagram*. Figure 6.1a shows such a diagram, in which two particles p and q annihilate to produce a new particle s, which in turn decays into two particles t and u. We thus consider the time axis to run upwards and the space axis to run from left to right. The lines can be interpreted as particle trajectories and the vertices as interactions. To determine the probability amplitude corresponding to this transition, one should sum over the amplitudes of all possible Feynman diagrams with incoming particles p, q and outgoing particles t, u. To calculate the amplitude of such a diagram, one has to perform a path integral over the particle trajectories and the interaction vertices. Each vertex will contribute a multiplicative factor to the amplitude of the diagram; this factor depends on the theory one is dealing with.



(a) ***(b)***

Figure 6.1: *Figure (a) shows a Feynman diagram with two incoming particles p, q, an intermediate particle s and two outgoing particles t, u. Figure (b) shows the closed string version of this interaction.*

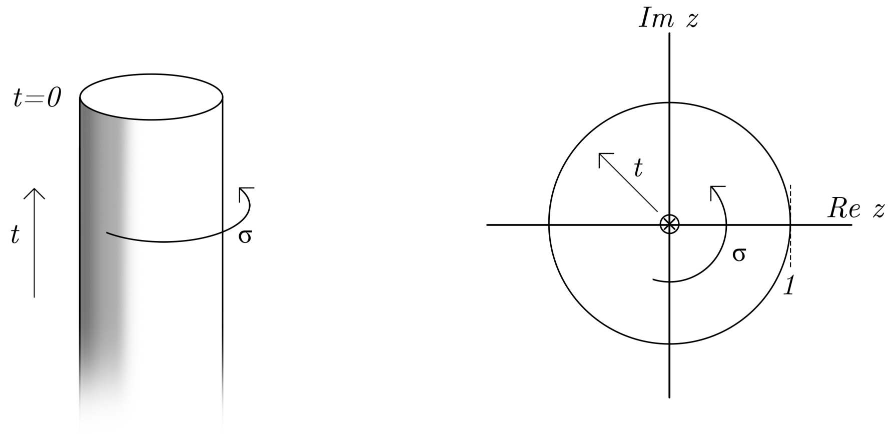
In the string picture, the annihilation and decay of particles becomes the merging and splitting of strings. Figure 6.1b shows the closed string analogue of the particle diagram in 6.1a. The most important difference with the particle picture is that we do not have a well-defined location in spacetime for the interaction vertices anymore. Locally, any part of the string diagram simply looks like the world-sheet of a free string. The theory of interacting strings is therefore fully determined by the free theory.

To determine the amplitude of a string diagram such as the one given in 6.1b, one has to perform a path integral over all world-sheets of this form. One way to make this feasible is to use conformal invariance* to transform the world-sheet into a compact surface. Each incoming or outgoing closed string forms a semi-infinite cylinder as shown in figure 6.2a. If we parametrise this cylinder by a complex coordinate $w = t + i\sigma$ with $-\infty \leq t \leq 0$ and $0 \leq \sigma \leq 2\pi$, it can be mapped conformally to the punc-

*Recall from the remark in section 4.2 that the Polyakov string action is invariant under conformal transformations of the world-sheet.

tured unit disk in $\mathbb{C}^\times$ via the mapping

$$w \mapsto z = e^w. \tag{6.1}$$



(a) ***(b)***

Figure 6.2: *Figure (a) shows the parametrised semi-infinite cylinder, which describes an incoming or outgoing closed string. Figure (b) shows the result of the mapping $t + i\sigma \mapsto e^{t+i\sigma}$, whose image is the punctured complex unit disk.*

In this way, we can transform the worldsheet into a sphere with a finite number of punctures, as shown in figure 6.3. Each of these punctures corresponds to an incoming or outgoing state. Via the state-field correspondence of conformal field theory, such a state $|A\rangle$ corresponds to a local field operator $V_A(z, \bar{z})$ that satisfies

$$|A\rangle = \lim_{z\to 0} V_A(z, \bar{z})|0\rangle, \tag{6.2}$$

where $|0\rangle$ denotes the vacuum and $\bar{z}$ denotes the complex conjugate of z. Since $z = e^{t+i\sigma}$, the limit $z \to 0$ is equivalent to $t \to -\infty$. The state $|A\rangle$ can thus be interpreted as the initial state in the 'infinite past' (for outgoing states, it makes more sense to parametrise the cylinder with $0 \leq t \leq \infty$, but the result is the same). The operators $V_A(z, \bar{z})$ are called *vertex operators*, as they form the string analogue of the interaction vertices of the particle diagrams. Indeed, we recognise equation (6.2) as the vacuum axiom of the vertex operators $Y(A, z)$ we defined in chapter 1. Generally, the $V_A(z, \bar{z})$ are also taken to satisfy the translation and locality axiom.[†]

[†]This is, of course, what inspired the mathematical definition of a vertex algebra in terms of the three axioms in the first place.

For a string state $|k\rangle$ of momentum k, the corresponding vertex operator is given by

$$V_k(z,\bar{z}) = :e^{ikX(z,\bar{z})}: \tag{6.3}$$

where

$$X(z,\bar{z}) = x_0 - i\alpha' p \ln|z|^2 + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n z^{-m} + \bar{\alpha}\bar{z}^{-n}\right). \tag{6.4}$$

Note that the above expression can be obtained from our general solution for the closed string given in (5.10) by setting $z = e^{i(\tau+\sigma)}$ and $\bar{z} = e^{i(\tau-\sigma)}$, and assuming τ to take imaginary values[‡]. In particular, we obtain

$$X_L(z) = x_0^L - i\alpha' p \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n z^{-m}. \tag{6.5}$$

and thus

$$:e^{ikX_L(z)}: = e^{ikx_0^L} z^{\alpha' kp} e^{-k\sum_{n<0}\frac{\alpha_n}{n}z^{-n}} e^{-k\sum_{n>0}\frac{\alpha_n}{n}z^{-n}}. \tag{6.6}$$

We recognise this expression as the lattice vertex operators $Y(|\alpha\rangle, z)$ given in (3.15). In this way, our mathematical definitions in chapter 1 describe the chiral (i.e. left-moving) vertex operators in string theory.[§]

With the vertex operators of the incoming and outgoing states at hand, the probability amplitude can be obtained via a path integral over all the compactified world-sheets, in which each vertex operator contributes a multiplicative factor to the integrand. Simply put, we fill up the punctures and make up for this by inserting appropriate vertex operators. Details on this procedure can be found in [GSW12].

In the next section, we will calculate the probability amplitude of the *one-loop* diagram of the closed string. This diagram is simply a torus with no

[‡]Note that this reduces to our earlier definition of z if we set $\tau = -it$. This identification is called a *Wick rotation*, and it transforms the Minkowski metric into the Euclidean metric. The variable t is therefore called *Euclidean time*. This change to Euclidean time is necessary to ensure that the path integral is well-defined.

[§]Recall that the reconstruction theorem from chapter 2 implies that the vertex operators $Y(|\alpha\rangle, z)$ determine the vertex operators of the whole state space. Assuming that the 'string' vertex operators $V_A(z,\bar{z})$ satisfy the vertex algebra axioms as well, we obtain that not just the pure momentum vertex operators (as given by (6.3)) but also the vertex operators of the oscillator states correspond to the 'mathematical' lattice vertex operators.

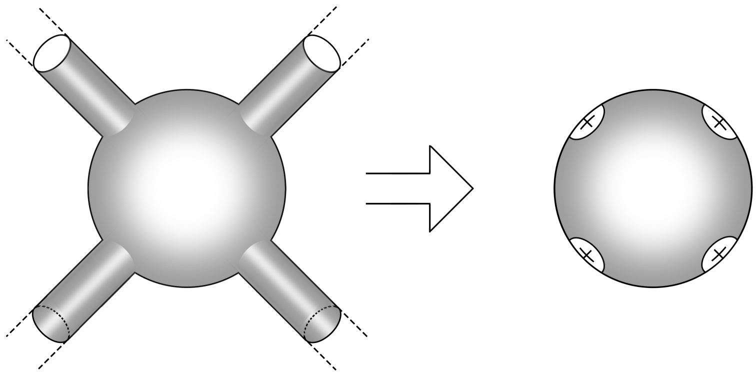


Figure 6.3: *Conformal transformation of a world-sheet with extending incoming and outgoing strings to a compact sphere with a finite number of punctures.*

extending tubes, that is, no incoming or outgoing states. The calculation of the probability amplitude can therefore be performed without the use of vertex operators. Of course, this diagram does not describe any interaction process; in fact, it does not even describe a proper world-sheet according to our definitions in section 4.1. Nevertheless, its amplitude is of interest since it can be interpreted as a contribution to the vacuum energy of spacetime, and thereby as a contribution to the cosmological constant [Pol98].

6.2 Amplitude of the one-loop diagram

The one-loop diagram of the closed string is a torus. It is important to note that this is *not* the kind of torus we obtained via toroidal compactification. In toroidal compactification, we let the string move in a *spacetime* whose spatial part is shaped like a torus. Here, we are considering the case that the world-sheet 'happens' to form a torus. So, essentially, it is not the spacetime but the *parameter space* of σ and τ that is shaped like a torus. By defining a complex coordinate $z = \tau + i\sigma$, we can view this torus in parameter space as a *complex torus* as defined in the introduction. There we showed that such a torus can be described by a modular parameter in $\mathbb{H}$. Unfortunately, the standard notation for this parameter is τ, so from now on let us denote the parameters of our world-sheet by σ_0 and σ_1 instead of τ and σ, respectively, so that τ will be available to denote the modular parameter of our complex torus. In the introduction, we showed that two complex tori with modular parameters $\tau, \tau' \in \mathbb{H}$ are equivalent if and only if $\tau = \frac{a\tau'+b}{c\tau'+d}$ for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Naturally, we expect the probability amplitude of our diagram to be independent of our choice between equivalent modular parameters. In other words, our results should be modular invariant.

The probability amplitude $\mathcal{A}(\tau) \in \mathbb{C}$ corresponding to the torus with modular parameter τ is calculated via a path integral over all world-sheet fields $X(\sigma_0, \sigma_1)$, which looks something like

$$\mathcal{A}(\tau) = \int \mathcal{D}[X] e^{-S_E(X,\tau)}. \tag{6.7}$$

Here S_E denotes the *Euclidean* Polyakov string action, which can be obtained by interchanging the time coordinate σ_0 with an imaginary time coordinate $-i\sigma_E$, where $\sigma_E \in \mathbb{R}$ denotes the Euclidean time. Furthermore, the dependence of S_E on τ reflects the dependence of the Polyakov action on a world-sheet metric.

We will not explain the precise meaning of the expression in (6.7) nor how to solve it, but simply state its solution as given in [Sch14]:

$$\int \mathcal{D}[X] e^{-S_E(X,\tau)} = \mathrm{Tr}_V(e^{-2\pi \,\mathrm{Im}(\tau) H} e^{2\pi i \,\mathrm{Re}(\tau) P}), \tag{6.8}$$

where V denotes the space of states corresponding to the free dynamics of the string, that is, the dynamics in the 24 transverse dimensions. Furthermore, H denotes the Hamiltonian and P denotes the world-sheet momentum operator. Recall that H and P are the generators of time and space translations, respectively, and that we defined the imaginary axis as the time direction and the real axis as the space direction. The operator $e^{-2\pi \,\mathrm{Im}(\tau) H} e^{2\pi i \,\mathrm{Re}(\tau) P}$ can therefore be interpreted as inducing a translation of $\mathrm{Im}\,\tau$ along the imaginary axis and a translation of $\mathrm{Re}\,\tau$ along the real axis. Together, this simply amounts to a translation by the complex number τ, so we have made precisely one loop around the torus.¶

As shown in the last chapter, for the closed string these operators are given by $H = L_0 + \bar{L}_0 - 2$ and $P = L_0 - \bar{L}_0$. We therefore obtain

$$\mathcal{A}(\tau) = \mathrm{Tr}_V \left(e^{2\pi i \tau (L_0 - 1)} e^{-2\pi i \bar{\tau} (\bar{L}_0 - 1)} \right), \tag{6.9}$$

¶The fact that the time translation operator $e^{-2\pi \,\mathrm{Re}(\tau) H}$ misses the complex factor i is due the conversion to Euclidean time. Moreover, the factor 2π is necessary as we defined the torus as having a period of 1 instead of 2π.

where $\bar{\tau}$ denotes the complex conjugate of τ.

The expression in (6.9) is called a *partition function*, since (as we will see in the next section) it counts the number of eigenstates per energy level. In the easiest cases, the partition function factorises into a the left-moving and right-moving part. When this occurs, we have

$$\mathcal{A}(\tau) = \mathrm{Tr}_{V_L}\left(e^{2\pi i\tau(L_0-1)}\right)\mathrm{Tr}_{V_R}\left(e^{-2\pi i\bar{\tau}(\bar{L}_0-1)}\right) =: Z_L(\tau)Z_R(-\bar{\tau}), \quad (6.10)$$

where V_L and V_R denote the state spaces of the left-moving and right-moving dynamics, respectively. Since the quantum theory of the closed string is perfectly symmetric in the left- and right-moving oscillators, we have $Z_L = Z_R$ as complex functions. To calculate $\mathcal{A}(\tau)$, it therefore often suffices to study the left-moving (also called *chiral* or *meromorphic*) part of the string theory.

It is not difficult to see that this factorisation happens for the uncompactified closed string, once we have integrated over the continuous spectrum of the transverse momenta. For the compactified string the situation is more complicated. However, it turns out that the partition function still factorises if we consider a toroidal compactification by an even and unimodular (or equivalently, self-dual) lattice [DGH88]. So, in particular, it factorises when we compactify by Leech lattice Λ. In section 6.4, we will calculate the left-moving part of the partition function for a closed string on a $\mathbb{Z}_2$-orbifold of the compactified space $\mathbb{R}^{24}/\Lambda$.

6.3 The partition function

In the last section, we saw that calculating the probability amplitude of the one-loop diagram of a closed string comes down to determining the so-called partition function

$$Z_V(\tau) = \mathrm{Tr}_V\left(e^{2\pi i\tau(L_0-1)}\right). \quad (6.11)$$

where V denotes the state space corresponding to the left-moving dynamics of the string. However, what does the expression $\mathrm{Tr}_V(e^{2\pi i\tau(L_0-1)})$ mean precisely? The operator $\mathrm{e}^{2\pi i\tau(L_0-1)}$ acts on an infinite dimensional vector space, which means that the trace of this operator is generally not well-defined. However, let us write $q = e^{2\pi i\tau}$ and treat q as a formal variable. If we can decompose V in L_0-eigenspaces V_n, it makes sense to interpret

$\mathrm{Tr}_V(q^{L_0-1})$ as the formal power series

$$\sum_n (\dim V_n) q^{n-1}, \tag{6.12}$$

where we sum over the eigenvalues of L_0. We thus see that one-loop partition functions are essentially vertex algebra characters! Of course, (6.12) only defines a proper power series if we have countably many V_n and if their dimensions are finite. Moreover, even this is the case, we still have to check whether this series actually defines a complex function on $\mathbb{H}$. Fortunately, most of the state spaces that we are concerned with satisfy these properties. For the uncompactified closed string, however, we first have to integrate over the transverse momenta, as these do not have countably many eigenvalues.

6.3.1 The uncompactified closed string

For the uncompactified closed string, the state space is generated by the basis vectors

$$\alpha^{j_1}_{n_1} \dots \alpha^{j_m}_{n_m} \bar{\alpha}^{k_1}_{\bar{n}_1} \dots \bar{\alpha}^{k_\ell}_{\bar{n}_\ell} |p\rangle \text{ with } n_i, \bar{n}_i < 0,\ p \in \mathbb{R}^{24},\ \sum_i n_i = \sum_i \bar{n}_i. \tag{6.13}$$

Note that the states do not carry a light-cone momentum p^+, as we are only considering the free dynamics in the transverse directions. Moreover, note that we have a continuous spectrum of zero mode momenta p. The only sensible way to interpret the trace of an operator with a continuous spectrum of eigenvalues is by integrating over these values. As the momenta p are independent of the oscillators, they contribute a multiplicative factor to the amplitude $\mathcal{A}(\tau)$ given by

$$\prod_{j=2}^{25} \int_{-\infty}^{\infty} e^{-2\pi(\mathrm{Im}\,\tau)\frac{1}{2}\alpha'(p^j)^2} dp^j = \left(\sqrt{(\alpha' \,\mathrm{Im}\,\tau)^{-1}}\right)^{24} = (\alpha' \,\mathrm{Im}\,\tau)^{-12}.$$

To calculate the contribution of the left-moving oscillators, we determine $\mathrm{Tr}_V\left(e^{2\pi i \tau(L_0-1)}\right)$ where V is the space generated by the left-moving states

$$\alpha^{j_1}_{-n_1} \dots \alpha^{j_m}_{-n_m} |0\rangle \text{ with } n_i > 0, \tag{6.14}$$

which do not have any zero mode momentum. This space reminds us of the state space of the Heisenberg vertex algebra, with 24 copies of

Heisenberg Lie operators. The L_0-eigenspace decomposition is given by $V = \bigoplus_{n\in\mathbb{N}} V_n$ with

$$V_n = \operatorname{span}\{\alpha^{j_1}_{-n_1} \dots \alpha^{j_m}_{-n_m}|0\rangle : \sum_i n_i = n\}.$$

Looking at our calculation in (3.21), we can immediately conclude that

$$\operatorname{Tr}_V \left(e^{2\pi i\tau(L_0-1)}\right) = \sum_{n=0}^{\infty} (\dim V_n) q^{n-1} = \frac{1}{\eta(\tau)^{24}}.$$

Including the contribution of the right-moving oscillators and the continuous momenta, we obtain

$$\mathcal{A}(\tau) \propto (\operatorname{Im}\tau)^{-12}\eta(\tau)^{-24}\eta(-\bar{\tau})^{-24} = \left(\sqrt{\operatorname{Im}\tau}|\eta(\tau)|^2\right)^{-24}.$$

Here we used that $\eta(\tau) = e^{\frac{2\pi i\tau}{24}} \prod_{n=1}^{\infty}(1-e^{2\pi i\tau n})$ is the complex conjugate of $\eta(-\bar{\tau})$. Recall from chapter 3 that the modular group is generated by the elements

$$T : \tau \mapsto \tau + 1 \qquad \text{and} \qquad S : \tau \mapsto -\frac{1}{\tau}.$$

Since η transforms as $\eta(\tau+1) = e^{\frac{\pi}{12}i}\eta(\tau)$ and $\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)$, it follows that $|\eta|^2$ transforms as

$$|\eta(\tau+1)|^2 = |\eta(\tau)|^2 \qquad \text{and} \qquad |\eta(-1/\tau)|^2 = |\tau||\eta(\tau)|^2. \tag{6.15}$$

Noting that $\operatorname{Im}(-1/\tau) = \operatorname{Im}(-\bar{\tau}/|\tau|^2) = \operatorname{Im}(\tau)/|\tau|^2$, we find that $\mathcal{A}(\tau)$ is indeed modular invariant.

6.4 Partition function of the Monster

Let us now calculate the one-loop partition function of the string theory that has been associated with the Monster vertex algebra. We start by determining the partition function of the closed string on the torus $\mathbb{R}^{24}/\Lambda$, where Λ denotes the Leech lattice. It follows from our analysis in section 5.4 that the space of the left-moving states is generated by the vectors

$$\alpha^{j_1}_{-k_1} \dots \alpha^{j_m}_{-k_m}|\beta\rangle \text{ with } k_i > 0,\ \beta \in \Lambda. \tag{6.16}$$

This is precisely the space of states of the lattice vertex algebra V_Λ! The L_0-eigenspace decomposition $V_\Lambda = \bigoplus_{n\in\mathbb{N}} V_{\Lambda,n}$ is thus given by

$$V_{\Lambda,n} = \text{span}_{\mathbb{C}} \left\{ \alpha^{j_1}_{k_1} \dots \alpha^{j_m}_{k_m} |\beta\rangle : \sum_{i=1}^{m} k_i + \frac{1}{2}\beta^2 = n \right\}.$$

Recall from chapter 3 that these eigenspaces are finite-dimensional, so the partition function is indeed well-defined. In fact, it is equal to the character of V_Λ, which we already calculated. We thus obtain

$$Z_{V_\Lambda}(\tau) = \text{Tr}_{V_\Lambda}\left(q^{(L_0-1)}\right) = \frac{\Theta_\Lambda(\tau)}{\eta(\tau)^{24}} = J(\tau) + 24. \tag{6.17}$$

Recall that the constant term 24 corresponds to the 24 level zero states of the form $\alpha^j_{-1}|0\rangle$. We will now implement an orbifold to remove the level zero states. As the orbifold theory consists of an untwisted and a twisted sector, the full partition function will be the sum of the partition functions corresponding to the two sectors.

6.4.1 Untwisted sector

Let us first determine the partition function of the untwisted states, which are the states in V_Λ that are invariant under the orbifold involution. Following our analysis in section 5.5, we introduce a linear operator U with $UU = \text{id}$, which acts on V_Λ via

$$U\alpha^j_n U = -\alpha^j_n \text{ for all } n \in \mathbb{Z}, \tag{6.18}$$

$$U|\beta\rangle = |-\beta\rangle. \tag{6.19}$$

Note that the above holds for every $2 \leq j \leq 25$, i.e. for all transverse directions. As before, U commutes with L_0. We can now decompose each of the eigenspaces $V_{\Lambda,n}$ into two eigenspaces of U with eigenvalues $+1$ and -1. We obtain $V_{\Lambda,n} = V^+_{\Lambda,n} \oplus V^-_{\Lambda,n}$ with

$$V^+_{\Lambda,n} = \text{span}_{\mathbb{C}}\left(\{\alpha^{j_1}_{n_1}\dots\alpha^{j_{2m}}_{n_{2m}}(|\beta\rangle + |-\beta\rangle)\} \cup \{\alpha^{j_1}_{n_1}\dots\alpha^{j_{2m+1}}_{n_{2m+1}}(|\beta\rangle - |-\beta\rangle)\}\right),$$
$$V^-_{\Lambda,n} = \text{span}_{\mathbb{C}}\left(\{\alpha^{j_1}_{n_1}\dots\alpha^{j_{2m}}_{n_{2m}}(|\beta\rangle - |-\beta\rangle)\} \cup \{\alpha^{j_1}_{n_1}\dots\alpha^{j_{2m+1}}_{n_{2m+1}}(|\beta\rangle + |-\beta\rangle)\}\right).$$

The untwisted state space V^+_Λ is equal to the direct sum $\bigoplus_{n\in\mathbb{N}} V^+_{\Lambda,n}$ of the U-invariant eigenspaces. To calculate the partition function, we can use the following trick:

$$\mathrm{Tr}_{V_\Lambda^+}\left(q^{(L_0-1)}\right) = \mathrm{Tr}_{V_\Lambda}\left(\frac{1+U}{2}q^{(L_0-1)}\right). \tag{6.20}$$

The operator $\frac{1+U}{2}$ is called the *projector*. The 'trick' works due to the decomposition of the L_0-eigenspaces $V_{\Lambda,n}$ into U-eigenspaces of eigenvalue ± 1: when we sum over the eigenvalues of the operator $\frac{1+U}{2}q^{(L_0-1)}$ on V_Λ, only the contributions of the U-invariant states will be left. We thus find

$$\begin{aligned}
Z_{V_\Lambda^+}(\tau) &= \frac{1}{2}\mathrm{Tr}_{V_\Lambda}\left(q^{(L_0-1)}\right) + \frac{1}{2}\mathrm{Tr}_{V_\Lambda}\left(Uq^{(L_0-1)}\right) \\
&= \frac{1}{2}Z_{V_\Lambda}(\tau) + \frac{1}{2}\mathrm{Tr}_{V_\Lambda}\left(Uq^{(L_0-1)}\right).
\end{aligned}$$

To calculate the last term, consider the decomposition of V_Λ into $\bigoplus_{n\in\mathbb{N}}\left(V_{\Lambda,n}^+ \oplus V_{\Lambda,n}^-\right)$. Note that the spaces $V_{\Lambda,n}^+$ and $V_{\Lambda,n}^-$ are eigenspaces of the operator $Uq^{(L_0-1)}$ with eigenvalues $+q^{n-1}$ and $-q^{n-1}$, respectively. Now let $\beta \in \Lambda \setminus \{0\}$. If $\alpha_{n_1}^{j_1}\dots\alpha_{n_{2m}}^{j_{2m}}(|\beta\rangle + |-\beta\rangle)$ is a basis state of $V_{\Lambda,n}^+$, then $\alpha_{n_1}^{j_1}\dots\alpha_{n_{2m}}^{j_{2m}}(|\beta\rangle - |-\beta\rangle)$ is a basis state of $V_{\Lambda,n}^-$ and their contributions to the trace $\mathrm{Tr}_{V_\Lambda}\left(Uq^{(L_0-1)}\right)$ will cancel out. The same occurs for the basis states of the form $\alpha_{n_1}^{j_1}\dots\alpha_{n_{2m+1}}^{j_{2m+1}}(|\beta\rangle - |-\beta\rangle)$. We thus find that *only* the states with $\beta = 0$ will contribute to the trace, i.e. we have

$$\mathrm{Tr}_{V_\Lambda}\left(Uq^{(L_0-1)}\right) = \mathrm{Tr}_{V_\Lambda^0}\left(Uq^{(L_0-1)}\right)$$

where V_Λ^0 is generated by the states $\alpha_{-k_1}^{j_1}\dots\alpha_{-k_m}^{j_m}|0\rangle$ with $k_i > 0$. Note that the $Uq^{(L_0-1)}$-eigenvalue of such a state is $(-1)^m q^{\sum k_i}$. This trace can be determined in the same way we determined the vertex algebra characters in section 3.3 (in particular, see (3.17) and (3.21)). We obtain

$$\begin{aligned}
\mathrm{Tr}_{V^0_\Lambda}\left(Uq^{(L_0-1)}\right) &= q^{-1}\left(\sum_{(i_1,i_2,\dots)}(-1)^{\sum_{m=1}^\infty i_m}q^{\sum_{m=1}^\infty m i_m}\right)^{24}\\
&= q^{-1}\left(\left(\sum_{i_1=0}^\infty(-1)^{i_1}q^{i_1}\right)\left(\sum_{i_2=0}^\infty(-1)^{i_2}q^{2i_2}\right)\dots\right)^{24}\\
&= q^{-1}\left(\prod_{k=1}^\infty\sum_{i_k=0}^\infty(-q^k)^{i_k}\right)^{24}\\
&= q^{-1}\left(\prod_{k=1}^\infty\frac{1}{1+q^k}\right)^{24}.
\end{aligned}$$

By standard results from complex analysis[‖], one can show that $q^{-1}\left(\prod_{k=1}^\infty\frac{1}{1+q^k}\right)^{24}$ defines a holomorphic function on $\mathbb{H}$, since $|q|<1$ for $\tau\in\mathbb{H}$. Moreover, it has a single pole at the cusp $\tau=i\infty$, that is, a single pole at $q=0$ when viewed as a function of q. The residue at $q=0$ is equal to 1, just as that of the J-function.

We can express our result in terms of the *theta function* θ_2 (not to be confused with the lattice theta functions Θ_L). This function can be defined as $\theta_2(\tau)=\eta(\tau)2q^{\frac{1}{12}}\prod_{k=1}^\infty(1+q^k)^2$, so we have

$$\mathrm{Tr}_{V^0_\Lambda}\left(Uq^{(L_0-1)}\right)=\left(2\frac{\eta(\tau)}{\theta_2(\tau)}\right)^{12}.$$

We will soon encounter two other theta functions, θ_3 and θ_4, which satisfy

$$\theta_3(\tau)=\eta(\tau)q^{-\frac{1}{24}}\prod_{k=0}^\infty(1+q^{k+\frac12})^2 \quad\text{and}\quad \theta_4(\tau)=\eta(\tau)q^{-\frac{1}{24}}\prod_{k=0}^\infty(1-q^{k+\frac12})^2.$$

Interestingly, these functions are not modular invariant but transform into each under the actions of the modular group. Their transformations can be found in [BLT12] and are given by

$$\theta_2(\tau+1)=e^{\frac{\pi i}{4}}\theta_2(\tau) \quad\text{and}\quad \theta_2(-\frac{1}{\tau})=\sqrt{-i\tau}\theta_4(\tau), \tag{6.21}$$

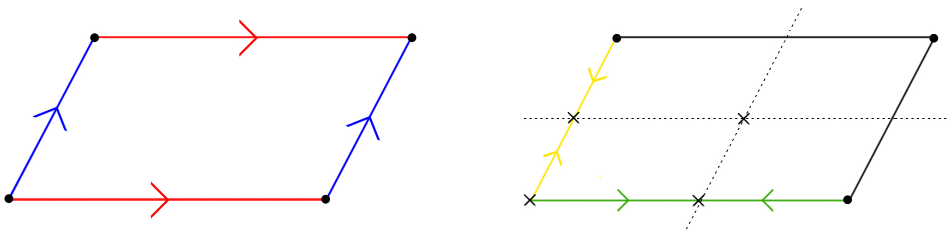
$$\theta_3(\tau+1)=\theta_4(\tau) \quad\text{and}\quad \theta_3(-\frac{1}{\tau})=\sqrt{-i\tau}\theta_3(\tau), \tag{6.22}$$

$$\theta_4(\tau+1)=\theta_3(\tau) \quad\text{and}\quad \theta_4(-\frac{1}{\tau})=\sqrt{-i\tau}\theta_2(\tau). \tag{6.23}$$

‖For example, see Remark IV.1.7 in [Fre05].

6.4.2 Twisted sector

Let us now determine the partition function of the twisted sector. Recall that the twisted strings lack zero modes, as they are 'stuck' around the fixed points of the orbifold. In section 5.5, we orbifolded uncompactified dimensions and therefore had only one fixed point. When we orbifold the torus, we get 2^{24} fixed points. To see how this happens, let us consider 2 dimensions instead of 24. In figure 6.4a, we see a schematic picture of a two-dimensional torus. In figure 6.4b, we see the effect of the orbifold identification. For every basis vector of the underlying lattice, we get a fixed point half-way through. In this way, we obtain r 'basis' fixed points, where r denotes the rank of our lattice. The total set of fixed points is then obtained by taking $\mathbb{Z}$-linear combinations of these points. Since adding the same point twice does not give us anything new, to total number of fixed points is 2^r. In the present case, we thus have 2^{24} fixed points, and each of these gives rise to a different vacuum state of the quantum theory.



(a) ***(b)***

Figure 6.4: *Figure (a) shows a schematic picture of the 2-dimensional torus. The blue (respectively red) lines are to be identified with each other. The black dots are lattice points; one can view the left blue and lower red line as basis vectors of the lattice. In figure (b), we see the effect of the orbifold identification. The $\times$-marks indicate the fixed points.*

We thus obtain the state space V_{tw} generated by the basis

$$\alpha^{j_1}_{-k_1} \dots \alpha^{j_m}_{-k_m} |s\rangle \text{ with } k_i \in \mathbb{N} + \frac{1}{2},\ s \in \{1, \dots, 2^{24}\}. \tag{6.24}$$

By (5.74), we find that the L_0-eigenspaces are given by

$$V_{tw,n} = \text{span}_{\mathbb{C}}\{\alpha^{j_1}_{-k_1} \dots \alpha^{j_m}_{-k_m} |s\rangle : \sum k_i + \frac{3}{2} = n\} \quad \text{with } n \in \frac{1}{2}\mathbb{N}$$

and the involution operator U acts on V_{tw} via $U\alpha^j_n U = -\alpha^j_n$ and $U|s\rangle =$

$-|s\rangle$**. We thus find that the U-eigenspace decomposition of the $V_{tw,n}$ is given by

$$
\begin{aligned}
V_{tw,n} &= V^+_{tw,n} \oplus V^-_{tw,n}, \\
V^+_{tw,n} &= \operatorname{span}_{\mathbb{C}}\{\alpha^{j_1}_{-k_1} \dots \alpha^{j_{2m+1}}_{-k_{2m+1}}|s\rangle : \sum k_i + \frac{3}{2} = n\}, \\
V^-_{tw,n} &= \operatorname{span}_{\mathbb{C}}\{\alpha^{j_1}_{-k_1} \dots \alpha^{j_{2m}}_{-k_{2m}}|s\rangle : \sum k_i + \frac{3}{2} = n\}.
\end{aligned}
$$

The true twisted states are those that are invariant under U. The twisted space state is thus given by $V^+_{tw} = \bigoplus_n V^+_{tw,n}$.

We could now calculate the partition function $Z_{V^+_{tw}}(\tau)$ just as we did for the untwisted sector. However, this will not give us the J-function. The crux lies in the fact that the construction of the Monster vertex algebra of Frenkel, Lepowsky and Meurman involves a twisting method that is now understood as an *asymmetric orbifold* in string theory. Unfortunately, asymmetric orbifolds are beyond the scope of this thesis. The main idea, however, is that the orbifold only acts on the left-moving states of the theory. More information on asymmetric orbifolds can be found in [DGH88] and [NSV87]. For our purposes, the important result is that only 2^{12} of the fixed-point vacuum states are employed. We will therefore let s take values from 1 to 2^{12} instead of 2^{24}.

Let us now determine $Z_{V^+_{tw}}(\tau) = \operatorname{Tr}_{V^+_{tw}}\left(q^{(L_0-1)}\right)$. We proceed as before, to find

$$
\begin{aligned}
\operatorname{Tr}_{V^+_{tw}}\left(q^{(L_0-1)}\right) &= \operatorname{Tr}_{V_{tw}}\left(\frac{1+U}{2} q^{(L_{0}-1)}\right) \\
&= \frac{1}{2}\operatorname{Tr}_{V_{tw}}\left(q^{(L_0-1)}\right) + \frac{1}{2}\operatorname{Tr}_{V_{tw}}\left(U q^{(L_0-1)}\right).
\end{aligned}
\tag{6.25}
$$

We will calculate the last two terms individually. The calculations are similar to what we did before, only now we have half-integer eigenvalues. For

**This action of U on the fixed point vacua $|s\rangle$ might surprise the reader. As explained in [Tui95], one has a choice to let U act as multiplication by $+1$ or -1 on the states $|s\rangle$. Only the latter option, however, results in the J-function as partition function.

the first term, we have

$$\begin{aligned}
\mathrm{Tr}_{V_{tw}}\left(q^{(L_0-1)}\right) &= q^{-1}q^{\frac{3}{2}}2^{12}\left(\sum_{(i_1,i_2,\dots)} q^{\sum_{m=1}^{\infty}(m-\frac{1}{2})i_m}\right)^{24}\\
&= q^{\frac{1}{2}}2^{12}\left(\prod_{k=1}^{\infty}\sum_{i_k=0}^{\infty}(q^{k-\frac{1}{2}})^{i_k}\right)^{24}\\
&= q^{\frac{1}{2}}2^{12}\left(\prod_{k=0}^{\infty}\frac{1}{1-q^{k+\frac{1}{2}}}\right)^{24} = \left(2\frac{\eta(\tau)}{\theta_4(\tau)}\right)^{12}.
\end{aligned}$$

Here we encounter another theta function. The fact that it pops up in the same form as θ_2 is no coincidence. As we expect the full partition function to be modular invariant, we could have anticipated that the twisted partition function involves the functions θ_3 and θ_4 by considering the transformations in (6.21). Indeed, we find that the last trace is equal to

$$\begin{aligned}
\mathrm{Tr}_{V_{tw}}\left(Uq^{(L_0-1)}\right) &= q^{\frac{1}{2}}\left(-2^{12}\right)\left(\sum_{(i_1,i_2,\dots)}(-1)^{\sum_{m=1}^{\infty}i_m}q^{\sum_{m=1}^{\infty}(m-\frac{1}{2})i_m}\right)^{24}\\
&= -q^{\frac{1}{2}}2^{12}\left(\prod_{k=0}^{\infty}\frac{1}{1+q^{k+\frac{1}{2}}}\right)^{24} = -\left(2\frac{\eta(\tau)}{\theta_3(\tau)}\right)^{12}.
\end{aligned}$$

Here the minus stems from the fact that $U|s\rangle = -|s\rangle$ for each of the 2^{12} fixed points s. The two traces $(2\eta(\tau)/\theta_4(\tau))^{12}$ and $(2\eta(\tau)/\theta_3(\tau))^{12}$ again define holomorphic functions on the Poincaré plane. In contrast to the expression $(2\eta(\tau)/\theta_2(\tau))^{12}$ we found before, the traces here are also holomorphic at the cusp $\tau = i\infty$.

6.4.3 The J-function

The space of states of the Monster vertex algebra $V^\natural$ is precisely the state space of the orbifold string theory we just constructed. We thus have

$$V^\natural = V_\Lambda^+ \oplus V_{tw}^+ \tag{6.26}$$

as $\mathfrak{Vir}_{24}$-modules, that is, as vector spaces with an action of the Virasoro operators L_n with central charge 24. To realise $\mathbb{M}$ as automorphism group of $V^\natural$, one has to endow $V^\natural$ with the vertex algebra structure induced by the orbifold theory. It should come as no surprise that this construction is rather involved; the details and the proof that the automorphism group is $\mathbb{M}$ are given in the book [FLM89] by Frenkel, Meurman and Lepowsky. Nevertheless, we do have enough information at this point to calculate the character of $V^\natural$, which is equal to the partition function of the orbifold theory. We will now show that this function is equal to the J-function, as stated in the monstrous moonshine conjecture.

Collecting our results from the previous sections, we obtain

$$\begin{aligned} Z_{V^\natural}(\tau) &= Z_{V_\Lambda^+}(\tau) + Z_{V_{tw}^+}(\tau) \\ &= \frac{1}{2} Z_{V_\Lambda}(\tau) + \frac{1}{2}\operatorname{Tr}_{V_\Lambda}\left(U q^{(L_0-1)}\right) + \frac{1}{2}\operatorname{Tr}_{V_{tw}}\left(q^{(L_0-1)}\right) + \frac{1}{2}\operatorname{Tr}_{V_{tw}}\left(U q^{(L_0-1)}\right) \\ &= \frac{1}{2}\left(J(\tau) + 24 + \left(2\frac{\eta(\tau)}{\theta_2(\tau)}\right)^{12} + \left(2\frac{\eta(\tau)}{\theta_4(\tau)}\right)^{12} - \left(2\frac{\eta(\tau)}{\theta_3(\tau)}\right)^{12}\right). \end{aligned} \tag{6.27}$$

Using the transformations of the eta-function and theta functions given in (3.19) and (6.21), we find that

$$\begin{aligned} &\frac{\eta}{\theta_2}(\tau+1) = e^{-\frac{2\pi i}{12}}\frac{\eta}{\theta_2}(\tau), && \frac{\eta}{\theta_4}(\tau+1) = e^{-\frac{\pi i}{12}}\frac{\eta}{\theta_3}(\tau), && \frac{\eta}{\theta_3}(\tau+1) = e^{-\frac{\pi i}{12}}\frac{\eta}{\theta_4}(\tau), \\ &\frac{\eta}{\theta_2}(-1/\tau) = \frac{\eta}{\theta_4}(\tau), && \frac{\eta}{\theta_4}(-1/\tau) = \frac{\eta}{\theta_2}(\tau), && \frac{\eta}{\theta_3}(-1/\tau) = \frac{\eta}{\theta_4}(\tau). \end{aligned}$$

Together with the fact that $J(\tau)$ is modular invariant, this implies that the whole partition function is modular invariant. Moreover, we know that it is holomorphic on $\mathbb{H}$ and has a simple pole at $\tau = i\infty$. This simple pole stems from the terms $J(\tau)$ and $(2\eta(\tau)/\theta_2(\tau))^{12}$ in (6.27). Since both have a unit residue at $q = 0$, we find that the residue of the whole partition function is also equal to 1 (due to the factor $\frac{1}{2}$ at the front). Recall that the J-function is the *unique* modular function that is holomorphic on $\mathbb{H}$ and has a simple pole with unit residue at the cusp, up to an additive constant. We thus obtain that $Z_{V^\natural}(\tau)$ is equal to $J(\tau)$ plus some constant C.

We could determine the constant mathematically from the expression in (6.27). However, it is more informative to think about the quantum states that are counted by this constant. Recall that the constant represents the number of level zero states, that is, the states with eigenvalue zero for the operator $L_0 - 1$. However, due to the extra term $\frac{3}{2}$ in the twisted L_0-operator, such twisted states do not exist! The terms $(2\eta(\tau)/\theta_4(\tau))^{12}$ and

$(2\eta(\tau)/\theta_3(\tau))^{12}$ therefore do not contribute to the constant C. So, we only have to consider the contribution of the term $(2\eta(\tau)/\theta_2(\tau))^{12}$, which is equal to the trace $\mathrm{Tr}_{V_\Lambda}\left(Uq^{(L_0-1)}\right)$. As we noted before, the level zero states of V_Λ are precisely the 24 states of the form $\alpha^j_{-1}|0\rangle$. Each of these states has U-eigenvalue -1. The constant term in $\mathrm{Tr}_{V_\Lambda}\left(Uq^{(L_0-1)}\right)$ is therefore equal to -24, which is canceled out by the constant term of $Z_\Lambda(\tau)$. We have hereby proven the desired result:

$$Z_{V^\natural}(\tau) = J(\tau).$$

Conclusion

Although the monstrous moonshine conjecture is a purely mathematical statement, we have seen that all of the key objects involved find their common ground in a specific string theory. String theory therefore provides a conceptual explanation for the connection between the Monster and the J-function. As shown in this thesis, this explanation holds that the J-function is precisely the partition function of an orbifold theory whose state space is equal to that of the monster vertex algebra $V^\natural$. To be more precise, this orbifold theory describes a closed bosonic string living on a $\mathbb{Z}_2$-orbifold compactified by the 24-dimensional Leech lattice. One could say that this string theoretic interpretation justifies calling $V^\natural$ a 'natural' representation of the Monster, as it provides a physical model in which this representation arises. We have, however, only touched upon the state space of $V^\natural$; as a follow-up research, it would be interesting to see whether the full vertex algebra structure of $V^\natural$, and thereby its automorphism group $\mathbb{M}$, can be obtained from the given orbifold theory. Moreover, moonshine does not end with the Monster. Ever since Conway and Norton's paper was published, several other 'moonshines' have been discovered, such as the *generalised moonshine* involving sporadic subquotients of the Monster [Car12] and the *umbral moonshine* involving the sporadic Mathieu groups [DGO15]. The physical interpretation of the latter in terms of superstring theories on so-called *K3-surfaces* is still being developed. The interplay between theoretical physics and mathematics that was inspired by monstrous moonshine is therefore far from played out. With this thesis, we hope to have convinced the reader of the fruitfulness of this collaboration and to leave them excited to see its future results.

Bibliography

[Apo12] T. M. Apostol. *Modular functions and Dirichlet series in number theory*, volume 41. Springer Science & Business Media, 2012.

[BD20] P. Bruin and S. Dahmen. Modular forms, 2020. Available at http://www.few.vu.nl/~sdn249/modularforms16/notes.pdf.

[BK89] G.W. Bluman and S. Kumei. *Symmetries and Differential Equations*. Springer New York, 1989.

[BLT12] R. Blumenhagen, D. Lüst, and S. Theisen. Introduction to conformal field theory. In *Basic Concepts of String Theory*, pages 63–106. Springer, 2012.

[Bor92] R. E. Borcherds. Monstrous moonshine and monstrous lie-superalgebras. *Inventiones Mathematicae*, 109(2):405–444, 1992.

[Car12] S. Carnahan. Generalized moonshine iv: Monstrous lie algebras, 2012. Available at https://arxiv.org/abs/1208.6254.

[CN79] J. H. Conway and S. P. Norton. Monstrous moonshine. *Bulletin of the London Mathematical Society*, 11(3):308–339, 1979.

[CS13] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290. Springer Science & Business Media, 2013.

[DGH88] L. Dixon, P. Ginsparg, and J. Harvey. Beauty and the beast: Superconformal symmetry in a monster module. *Communications in Mathematical Physics*, 119(2):221–241, 1988.

[DGO15] J. F. R. Duncan, M. J. Griffin, and K. Ono. Proof of the umbral moonshine conjecture. *Research in the Mathematical Sciences*, 2(1):26, 2015.

[FBZ04] E. Frenkel and D. Ben-Zvi. *Vertex Algebras and Algebraic Curves: Second Edition*. American Mathematical Society, 2nd edition, 2004.

[FLM84] I. B. Frenkel, J. Lepowsky, and A. Meurman. A natural representation of the fischer-griess monster with the modular function j as character. *Proceedings of the National Academy of Sciences of the United States of America*, 81(10):3256–3260, 1984.

[FLM89] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex operator algebras and the Monster*. Academic press, 1989.

[Fre05] E. Freitag. *Complex analysis*. Springer, 2005.

[FS03] J. Fuchs and C. Schweigert. *Symmetries, Lie algebras and representations: A graduate course for physicists*. Cambridge University Press, 2003.

[God89] P. Goddard. *Meromorphic Conformal Field Theory*. Mathematical Sciences Research Institute, 1989.

[Gri82] R. Griess. The friendly giant. *Inventiones mathematicae*, 69(1):1–102, 1982.

[GSW12] M. B. Green, J. H. Schwarz, and E. Witten. *Superstring Theory: Volume 1, Introduction*. Cambridge University Press, 2012.

[Hum72] J. E. Humphreys. *Introduction to Lie algebras and representation theory*. Springer-Verlag, 1972.

[Kac97] V. G. Kac. *Vertex algebras for beginners*. American Mathematical Society, 1997.

[Lan02] S. Lang. *Algebra*. Springer New York, rev. third edition, 2002.

[LL04] J. Lepowsky and H. Li. *Introduction to vertex operator algebras and their representations*. Birkhäuser, 2004.

[LW78] J. Lepowsky and R. Wilson. Construction of the affine lie algebra $a_1(1)$. *Communications in Mathematical Physics*, 62(1):43–53, 1978.

[MDG15] M. R. Murty, M. Dewar, and H. Graves. *Problems in the Theory of Modular Forms*. Springer, 2015.

[Mil17] J. S. Milne. Modular functions and modular forms (v1.31), 2017. Available at https://www.jmilne.org/math/CourseNotes/MF.pdf.

[Nie73] H.V. Niemeier. Definite quadratische formen der dimension 24 und diskriminante 1. *Journal of Number Theory*, 5(2):142–178, 1973.

[Noz08] C. Nozaradan. Introduction to vertex algebras, 2008. Available at https://arxiv.org/abs/0809.1380.

[NSV87] K.S. Narain, M.H. Sarmadi, and C. Vafa. Asymmetric orbifolds. *Nuclear Physics, Section B*, 288(C):551–577, 1987.

[Pol98] J. Polchinski. *String theory: Volume 1, an introduction to the bosonic string*. Cambridge university press, 1998.

[Rib14] S. Ribault. *Conformal field theory on the plane*. CEA Saclay, Institut de Physique Théorique, 2014.

[Sch08] M. Schottenloher. *A mathematical introduction to conformal field theory*. Springer, 2nd ed edition, 2008.

[Sch14] A.N. Schellekens. Introduction to String Theory, 2014. Available at https://www.nikhef.nl/~t58/StringLectures2014.pdf.

[Ser12] J. P. Serre. *A course in arithmetic*. Springer Science & Business Media, 2012.

[Shu20] J. Shurman. Complex tori, 2020. Available at https://people.reed.edu/~jerry/311/tori.pdf.

[Tho79a] J. G. Thompson. Finite groups and modular functions. *Bulletin of the London Mathematical Society*, 11(3):347–351, 1979.

[Tho79b] J. G. Thompson. Some numerology between the fischer-griess monster and the elliptic modular function. *Bulletin of the London Mathematical Society*, 11(3):352–353, 1979.

[Tui92] M. P. Tuite. Monstrous moonshine from orbifolds. *Communications in Mathematical Physics*, 146(2):277–309, 1992.

[Tui95] M. P. Tuite. On the relationship between monstrous moonshine and the uniqueness of the moonshine module. *Communications in Mathematical Physics*, 166(3):495–532, 1995.

[vK20] B.W. van Kootwijk. *String theory and conformal field theory*. Leiden University, The Netherlands, 2020.

[Wei11] T. Weigand. *Introduction to String Theory*. Institut für Theoretische Physik, Universität Heidelberg, Germany, 2011.

[Wil09] R. A. Wilson. *The Finite Simple Groups*. Springer London, 2009.

[Zee03] A. Zee. *Quantum Field Theory in a Nutshell*. Princeton Univ. Press, 2003.

[Zwi04] B. Zwiebach. *A first course in string theory*. Cambridge University Press, 2004.

Appendix

We list some standard definitions and results concerning Lie algebras that are used in this text.

Definition. *(Algebra over a field) Let K be a field. A K-algebra is a vector space A over K together with a K-bilinear operation $\phi : A \times A \to A$. We usually write $\phi(a, b) = ab$.*

If ϕ is associative and A contains a unit element 1 satisfying $\phi(1, a) = \phi(a, 1) = a$ for all $a \in A$, then ϕ provides A with a ring structure. In this text, all algebras and vector spaces will be taken over the field $\mathbb{C}$ unless stated otherwise.

Definition. *(Lie algebra) A Lie algebra is an algebra $\mathfrak{g}$ whose product, denoted by $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, is anticommutative and satisfies the Jacobi-identity*

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

for all $x, y, z \in \mathfrak{g}$. The product map $[\cdot, \cdot]$ is referred to as the Lie bracket.

Note that any associative algebra A becomes a Lie algebra with the commutator as Lie bracket, so $[a, b] = ab - ba$ for all $a, b \in A$.

Definition. *(Lie algebra ideal) A subspace $\mathfrak{i}$ of a Lie algebra $\mathfrak{g}$ is called an ideal if we have $[i, x] \in \mathfrak{i}$ for all $i \in \mathfrak{i}$ and $x \in \mathfrak{g}$. Given such an ideal, the quotient vector space $\mathfrak{g}/\mathfrak{i}$ becomes a Lie algebra via*

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i} \quad \text{for } x, y \in \mathfrak{g},$$

and is therefore called the quotient Lie algebra.

Definition. *(Universal enveloping algebra) Let $\mathfrak{g}$ be a Lie algebra with bracket $[\cdot,\cdot]$. For each $n \in \mathbb{N}$, let $\mathfrak{g}^{\otimes n}$ be the vector space $\mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ where $\mathfrak{g}$ appears n times in the tensor product. Here we set $\mathfrak{g}^{\otimes 0} = \mathbb{C}$. Now define the tensor algebra $T(\mathfrak{g})$ as the associative algebra given by*

$$T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

and

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \cdot (y_1 \otimes y_2 \otimes \cdots \otimes y_m) \\ = x_1 \otimes \cdots \otimes x_k \otimes y_1 \otimes \cdots \otimes y_m \in \mathfrak{g}^{\otimes k+m}.$$

Then $T(\mathfrak{g})$ is a Lie algebra with the commutator as bracket. Let $\mathfrak{i}$ be the ideal of $T(\mathfrak{g})$ generated by $\{x \otimes y - y \otimes x - [x,y] : x, y \in \mathfrak{g}\}$. The universal enveloping algebra of $\mathfrak{g}$ is then defined as the quotient Lie algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/\mathfrak{i}.$$

Theorem. *(Poincaré-Birkhoff-Witt) Let $\mathfrak{g}$ be a Lie algebra. For any basis $\{x_i : i \in I\}$ of $\mathfrak{g}$ where I is a totally ordered set with relation $<$, we get a basis*

$$\{x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_k}^{n_k} : i_1, \ldots, i_k \in I,\ i_1 < i_2 < \cdots < i_k \text{ and } n_1, \ldots, n_k \in \mathbb{N}\}$$

of the universal enveloping algebra $U(\mathfrak{g})$ [Hum72].

Definition. *(Lie algebra representation) Let $\mathfrak{g}$ be a Lie algebra. A Lie algebra representation is a vector space V with a linear map $\rho : \mathfrak{g} \to \mathrm{End}\, V$ such that for all $x, y \in \mathfrak{g}$ we have*

$$\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x).$$

In other words, ρ is a Lie algebra homomorphism if we view $\mathrm{End}\, V$ as a Lie algebra with the commutator as Lie bracket.