# Introduction to Schemes

Geir Ellingsrud and John Christian Ottem

June 18, 2025

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`edit?usp=sharing`



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# Preface



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# Notation

$\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of positive integers.

$\subset$ means 'is subset of', i.e., the same thing as $\subseteq$.

If $X$ and $Y$ are two sets, then $X - Y$ denotes the set of elements in $X$ which are not in $Y$.

All rings are commutative with 1.

Ring maps are required to send 1 to 1.

The zero ring is not an integral domain (and hence not a field).

For a ring $A$, we write $A_{\mathfrak{p}}$ and $A_f$ for the localizations in the multiplicative sets $S = A - \mathfrak{p}$ and $S = \{1, f, f^2, \dots\}$ respectively. Thus, $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid p \nmid b\}$ and $\mathbb{Z}_p = \mathbb{Z}[\frac{1}{p}]$.

The fraction field of an integral domain $A$ is denoted by $K(A)$.

A 'map' is a morphism in the relevant category, e.g., a 'map of rings' is ring homomorphism.

We will occasionally write $A = B$ if there is a *canonical* isomorphism $A \simeq B$. So for instance, $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$.

If $\mathcal{C}$ is a category, we denote its *opposite category* by $\mathcal{C}^{\text{op}}$. This category has the same objects as $\mathcal{C}$, but for any pair of objects, the morphisms from $X$ to $Y$ in $\mathcal{C}^{\text{op}}$ correspond to the morphisms from $Y$ to $X$ in $\mathcal{C}$.



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# 1

# Algebraic sets

In this chapter, we discuss algebraic sets and affine varieties, which will serve as the main motivating examples in the theory of schemes. We will confine ourself to the basic definitions, examine a few key examples, and establish notation for later chapters. As we progress through the book, we will develop the theory of varieties in greater depth.

## 1.1 Algebraic sets

Algebraic sets and affine varieties are defined over a fixed ground field $k$. In this chapter, we will assume that $k$ is algebraically closed. It is useful to keep some specific fields in mind, e.g. the field of complex numbers $\mathbb{C}$, the field of algebraic numbers $\bar{\mathbb{Q}}$, or perhaps the algebraic closure of a finite field, $\bar{\mathbb{F}}_p$.

We will denote the set $k^n$ by $\mathbb{A}^n(k)$, and refer to it as the *affine $n$-space*. The change in notation is meant to underline that there is more to $\mathbb{A}^n(k)$ than just the set of its elements; it will soon be equipped with a topology, and ultimately, it will be a scheme. The elements of $\mathbb{A}^n(k)$ will be called *points*.

**Definition 1.1.** For a subset $S$ of $k[x_1, \dots, x_n]$, we define its *zero set* as

$$Z(S) = \{\, P \in \mathbb{A}^n(k) \mid f(P) = 0 \text{ for all } f \in S \,\}.$$

An *algebraic set* is a subset of $\mathbb{A}^n(k)$ of this form.

If $f_1, \dots, f_r \in S$, then every polynomial of the form $\sum_{i=1}^r b_i f_i$, with the $b_i$'s being polynomials, also vanishes at the points of $Z(S)$. This means that the zero set of the ideal $\mathfrak{a}$ generated by the elements of $S$ is the same as $Z(S)$; that is, $Z(S) = Z(\mathfrak{a})$. We will therefore almost exclusively work with ideals and tacitly replace a set of polynomials by the ideal they generate. Hilbert's basis theorem (Theorem A.21) tells us that any ideal in $k[x_1, \dots, x_n]$ is finitely generated, so an algebraic subset is always described as the set of common zeros of *finitely* many polynomials.

The more constraints imposed, the smaller the solution set will be, so if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals with $\mathfrak{a} \subset \mathfrak{b}$, one has $Z(\mathfrak{b}) \subset Z(\mathfrak{a})$. The assignment $\mathfrak{a} \mapsto Z(\mathfrak{a})$ therefore defines an inclusion-reversing map from the partially ordered set of ideals in $k[x_1, \dots, x_n]$ to the partially ordered set of subsets of $\mathbb{A}^n(k)$.

Under this correspondence, different ideals can define the same algebraic set. For instance, the ideals $(x)$ and $(x^2)$ in $k[x]$, both have the origin in $\mathbb{A}^1(k)$ as their zero set. To deal with



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this ambiguity, we recall the *radical* $\sqrt{\mathfrak{a}}$ of $\mathfrak{a}$, which is defined by

$$\sqrt{\mathfrak{a}} = \{\, f \mid f^r \in \mathfrak{a} \text{ for some } r \in \mathbb{N} \,\}.$$

For a point $P \in \mathbb{A}^n(k)$ and a polynomial $f$, we have $f^r(P) = 0$ for some $r > 0$ if and only if $f(P) = 0$, so see that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$. Two ideals with the same radical therefore have the same zero sets. In fact, Hilbert's Nullstellensatz, which we will state below, tells us that the converse holds as well.

Here are a few basic properties of the zero sets of ideals:

**Proposition 1.2.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals and $\{\mathfrak{a}_i\}_{i\in I}$ a collection of ideals in the polynomial ring $k[x_1, \ldots, x_n]$. Then:

(i) If $\mathfrak{a} \subset \mathfrak{b}$, then $Z(\mathfrak{b}) \subset Z(\mathfrak{a})$
(ii) $Z(\sum_{i\in I} \mathfrak{a}_i) = \bigcap_{i\in I} Z(\mathfrak{a}_i)$
(iii) $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$
(iv) $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$.

*Proof* We have already proved (i) and (iv). For item (ii): $P \in \bigcap_{i\in I} Z(\mathfrak{a}_i)$ if and only if $f(P) = 0$ for all $f \in \mathfrak{a}_i$ and all $i \in I$. This is equivalent to $f(P) = 0$ for all $f \in \bigcup_{i\in I} \mathfrak{a}_i$. Since $\sum_{i\in I} \mathfrak{a}_i$ is the ideal generated by $\bigcup_{i\in I} \mathfrak{a}_i$, so (ii) follows.

For (iii), we have $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ is contained in both $\mathfrak{a}$ and $\mathfrak{b}$. This shows that $Z(\mathfrak{a}) \cup Z(\mathfrak{b}) \subset Z(\mathfrak{a} \cap \mathfrak{b}) \subset Z(\mathfrak{a}\mathfrak{b})$. Conversely, if $P \notin Z(\mathfrak{a}) \cup Z(\mathfrak{b})$, then there exist $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ so that $f(P) \neq 0$ and $g(P) \neq 0$. But then $(fg)(P) \neq 0$, and hence $P \notin Z(\mathfrak{a}\mathfrak{b})$. □

The identities (ii) and (iii) tell us that finite unions and arbitrary intersections of algebraic sets are again algebraic. Furthermore, as $\mathbb{A}^n(k) = Z(0)$ and $\varnothing = Z(1)$, the algebraic sets constitute the closed sets of a topology on the affine space $\mathbb{A}^n(k)$. It is called the *Zariski topology*.

If $X \subset \mathbb{A}^n(k)$ is any subset, we get an induced Zariski topology on $X$ by declaring that the closed sets of $X$ are of the form $X \cap W$, where $W$ is a closed set in $\mathbb{A}^n(k)$.

### *Examples*

**Example 1.3.** The affine space $\mathbb{A}^1(k)$ is called the *affine line*. Every proper nonzero ideal in $k[t]$ is principal, generated by some polynomial $f$. Since $k$ is algebraically closed, $f$ factors as a product $(t - a_1) \cdots (t - a_r)$, and its zero set $Z(f)$ is exactly the finite set of roots $\{a_1, ..., a_r\}$. Conversely, any finite set of points arises this way. Hence, the closed subsets of $\mathbb{A}^1(k)$ are precisely the finite sets and the whole space.

This means that the Zariski topology on $\mathbb{A}^1(\mathbb{C})$ behaves very differently from the usual topology on $\mathbb{C}$. For instance, $\mathbb{A}^1(\mathbb{C})$ is not Hausdorff - any two non-empty open sets must intersect. △

**Example 1.4** (Plane conics)**.** Plane conics are classical examples of algebraic sets. They are defined by a single quadratic equation in $\mathbb{A}^2(k)$. Three familiar examples include (i) the circle $x^2 + y^2 - 1 = 0$; (ii) the parabola $y - x^2 = 0$; and (iii) the hyperbola $xy - 1 = 0$.

Note also the conic $x^2 + y^2 + 1 = 0$, which has no solutions when say $k = \mathbb{R}$. However,

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over $\mathbb{C}$, it becomes equivalent to $xy - 1 = 0$, by the change of variables $u = x + iy$, $v = -x + iy$. In fact, when $k$ is algebraically closed of characteristic $\neq 2$, any conic can be transformed via a linear change of coordinates to one of the types $x^2 + y^2 + 1 = 0$, $y + x^2 = 0$, $x^2 = 0$, $xy = 0$ and $x^2 + 1 = 0$. △

**Example 1.5.** A more interesting example is the so-called *Clebsch cubic surface*; the algebraic set in $\mathbb{A}^3(\mathbb{C})$ defined by the equation

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3.$$

The real points of the surface, i.e. the points in $\mathbb{A}^3(\mathbb{R})$ satisfying the equation, is depicted below. This surface contains 27 lines, all defined over the real numbers. △

**Figure 1.1** The Clebsch cubic surface

## 1.2 Hilbert's Nullstellensatz

For a subset $V \subset \mathbb{A}^n(k)$, we can consider the set of all polynomials which vanish at every point of $V$:

$$I(V) = \{\, f \in k[x_1, \ldots, x_n] \mid f(P) = 0 \text{ for all } P \in V \,\}. \tag{1.1}$$

Note that $I(V)$ is naturally a radical ideal of $k[x_1, \ldots, x_n]$. Moreover, if $W \subset V$, then $I(V) \subset I(W)$.

We thereby have two inclusion-reversing mappings betwen the sets of radical ideals $\mathfrak{a}$ and the sets of closed sets of $\mathbb{A}^n(k)$: $\mathfrak{a} \mapsto Z(\mathfrak{a})$ and $X \mapsto I(X)$. The precise relationship between them is explained by Hilbert's Nullstellensatz:

**Theorem 1.6 (Hilbert's Nullstellensatz).** Let $k$ be an algebraically closed field. Then the map

$$\big\{\text{radical ideals } \mathfrak{a} \subset k[x_1, \ldots, x_n]\big\} \longrightarrow \{\text{closed sets } X \subset \mathbb{A}^n(k)\} \tag{1.2}$$

defined by $\mathfrak{a} \mapsto Z(\mathfrak{a})$ is a bijection, with inverse defined by $X \mapsto I(X)$.

To prove the theorem, we have to show that $Z(I(X)) = X$ for an algebraic set $X$ and $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for any ideal $\mathfrak{a}$. Three of the four required inclusions are straightforward, and hold for any field $k$:

(i) $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$: If $f \in \sqrt{\mathfrak{a}}$, then $f^r \in \mathfrak{a}$ for some positive integer $r$. Therefore, $f^r$ vanishes on $Z(\mathfrak{a})$ and hence $f$ also vanishes on $Z(\mathfrak{a})$. Hence $f \in I(Z(\mathfrak{a}))$.

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(ii) $X \subset Z(I(X))$: If $P \in X$, then by definition of $I(X)$, $f(P) = 0$ for all $f \in I(X)$. Therefore, $P \in Z(I(X))$.

(iii) $Z(I(X)) \subset X$: Write $X = Z(\mathfrak{a})$ for an ideal $\mathfrak{a}$. Then by (i), we have $I(Z(\mathfrak{a})) \supset \sqrt{\mathfrak{a}} \supset \mathfrak{a}$. Taking zero loci, $Z(I(Z(\mathfrak{a}))) \subset Z(\mathfrak{a})$, or equivalently, $Z(I(X)) \subset X$.

Thus the real content of the Nullstellsatz is that if $k$ is algebraically closed, then the inclusion $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$ is an equality. This means that $\sqrt{\mathfrak{a}}$ is exactly the set of polynomial functions which vanish on the closed set $Z(\mathfrak{a})$.

We will deduce Theorem 1.6 from the following algebraic result, which is a consequence of Noether's Normalization Lemma. (See Theorem A.38 for a proof)

**Theorem 1.7.** Let $k$ be a field and let $K$ be a finitely generated $k$-algebra. If $K$ is also a field, then $K$ is a finite field extension of $k$.

In particular, if $k$ is algebraically closed, then there are no non-trivial algebraic field extensions, and so $K = k$.

As a first consequence, we have the following result, which tells us that $Z(\mathfrak{a})$ is non-empty whenever $\mathfrak{a}$ is a proper ideal:

**Theorem 1.8 (Weak Nullstellensatz).** Let $k$ be an algebraically closed field.

(i) Every maximal ideal in $k[x_1, \ldots, x_n]$ is of the form

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$$

for some point $(a_1, \ldots, a_n)$ in $\mathbb{A}^n(k)$.

(ii) For an ideal $\mathfrak{a}$, the zero set $Z(\mathfrak{a})$ is empty if and only if $\mathfrak{a} = (1)$.

*Proof* (i): By Theorem 1.7, the field $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite extension of $k$, hence equal to $k$ because $k$ is assumed to be algebraically closed. For each $i = 1, \ldots, n$, let $a_i \in k$ denote the image of $x_i$ under the quotient map $\pi\colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/\mathfrak{m} = k$. Then clearly all the polynomials $x_i - a_i$ belong to the kernel $\mathfrak{m}$ of $\pi$. Since $(x_1 - a_1, \ldots, x_n - a_n)$ is already a maximal ideal, it must be equal to $\mathfrak{m}$.

(ii): If $\mathfrak{a} = (1)$, then clearly $Z(\mathfrak{a}) = \varnothing$. Conversely, if $\mathfrak{a}$ is a proper ideal, there is a maximal ideal $\mathfrak{m}$ in $k[x_1, \ldots, x_n]$ containing it. By (i), $\mathfrak{m}$ has the form $(x_1 - a_1, \ldots, x_n - a_n)$, and consequently $(a_1, \ldots, a_n) \in Z(\mathfrak{a})$, and so $Z(\mathfrak{a})$ is not empty. □

*Proof of Hilbert's Nullstellensatz* To prove the theorem, it is sufficient to prove that for any ideal $\mathfrak{a} \subset k[x_1, \ldots, x_n]$, we have

$$I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}.$$

Let $f \in I(Z(\mathfrak{a}))$ and consider the $k$-algebra obtained by inverting $f$ and modding out by $\mathfrak{a}$:

$$A = k[x_1, \ldots, x_n, t]/(\mathfrak{a} + (tf - 1)).$$

If $A \neq 0$, then there is a maximal ideal $\mathfrak{m} \subset A$. By the Weak Nullstellensatz, $\mathfrak{m}$ corresponds to a point $(a_1, \ldots, a_n, b) \in \mathbb{A}^{n+1}(k)$ such that $(a_1, \ldots, a_n) \in Z(\mathfrak{a})$ and $b \cdot f(a_1, \ldots, a_n) - 1 = 0$. In particular, $f(a_1, \ldots, a_n) \neq 0$, which contradicts the hypothesis $f \in I(Z(\mathfrak{a}))$.

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We deduce that $A = 0$. But note that $A$ is isomorphic to the localization $k[x_1, \dots, x_n]/\mathfrak{a}$ in the multiplcative set $S = \{1, f, f^2, \dots\}$. Therefore, if $A = 0$, then there is some $n \in \mathbb{N}$, so that $f^n \in \mathfrak{a}$, and hence $f \in \sqrt{\mathfrak{a}}$. □

**Example 1.9.** It is important to note that Hilbert's Nullstellensatz only holds when the ground field is algebraically closed. For instance, consider the ideal $\mathfrak{a} = (x^2 + y^2 + 1)$ in $\mathbb{R}[x, y]$, corresponding to the conic with no real points from Example 1.4. To vanish at the empty set is an empty condition, so $I(Z(\mathfrak{a})) = (1)$, and hence the inclusion $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$ is strict. △

**Example 1.10.** If $k = \mathbb{C}$, there is a quick way to prove Theorem 1.7 as follows. Suppose that $K = \mathbb{C}[x_1, \dots, x_n]/\mathfrak{m}$ is a finitely generated $\mathbb{C}$-algebra which is also a field. If there exists an element $t \in K$ which is not algebraic over $\mathbb{C}$, the fractions $\frac{1}{t-a} \in K$ for $a \in \mathbb{C}$ form an uncountable set of elements in $K$ which are linearly independent over $\mathbb{C}$. This is however not possible, because $K$ has a *countable* basis as a $\mathbb{C}$-vector space, as it is generated by the monomials $x_1^{e_1} \cdots x_n^{e_n}$.

See also Exercise 1.8.24 for an alternative proof of Theorem 1.7. △

For an algebraic set $X \subset \mathbb{A}^n(k)$, we define the *coordinate ring* of $X$ as the quotient

$$A(X) = k[x_1, \dots, x_n]/I(X).$$

where $I(X)$ is the ideal of polynomials that vanish on $X$.

The elements of $A(X)$ can be viewed as polynomial functions on $X$: any polynomial in $k[x_1, \dots, x_n]$ defines a function $X \to k$ by restriction, and two polynomials $f$ and $g$ define the same function on $X$ precisely when their difference $f - g$ vanishes on $X$, that is, $f - g \in I(X)$.

## 1.3 The structure of algebraic sets

A consequence of the Hilbert's Nullstellensatz, the geometric properties of algebraic sets can be translated into algebraic properties of the ideals of the polynomial ring $k[x_1, \dots, x_n]$. To formulate these geometric properties, we recall a few notions from point set topology.

A topological space $X$ is defined to be *irreducible* if it is nonempty and cannot be written as the union of two proper closed subsets; that is, if $X = Z \cup Z'$ with $Z$ and $Z'$ closed, then either $Z = X$ or $Z' = X$.

**Proposition 1.11.** An algebraic set $X \subset \mathbb{A}^n(k)$ is irreducible if and only if the ideal $I(X)$ is prime, i.e., $A(X)$ is an integral domain.

*Proof* Suppose $X = X_1 \cup X_2$ with $X_1$ and $X_2$ proper closed subsets. Then $I(X_1) \supset I(X)$ and $I(X_2) \supset I(X)$ are strict inclusions, so we may pick $f_i \in I(X_i) - I(X)$ for $i = 1, 2$. Then $f_1 f_2 \in I(X)$, and hence $I(X)$ is not prime.

Conversely, suppose $I(X)$ is not prime, so there exist $f_1, f_2$ so that $f_i \notin I(X)$ for $i = 1, 2$, but $f_1 f_2 \in I(X)$. Then $X_1 = X \cap Z(f_1)$ and $X_2 = X \cap Z(f_2)$ are closed proper subsets and $X_1 \cup X_2 = X \cap Z(f_1 f_2) = X$, so $X$ is not irreducible. □

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The following is a preliminary definition of an affine variety. We will revisit the definition in Chapter **??** after we have introduced schemes.

**Definition 1.12.** An *affine variety* is an irreducible algebraic set in $\mathbb{A}^n(k)$.

**Example 1.13.** The affine space $\mathbb{A}^n(k)$ is irreducible, as $k[x_1, \ldots, x_n]$ is an integral domain. △

**Example 1.14.** The closed set $Z(xyz) \subset \mathbb{A}^3(k)$ is not irreducible, as it can be written as a union of three planes $Z(x) \cup Z(y) \cup Z(z)$. Likewise, $Z(xz, yz) = Z(x, y) \cup Z(z) \subset \mathbb{A}^3(k)$ is not irreducible, being the union of a plane and a line. △

We say that a topological space $X$ is *Noetherian* if all descending chains of closed subsets

$$X_0 \supset X_1 \supset X_2 \supset \cdots$$

in $X$ eventually stabilize, that is, there is an integer $N$ such that $X_i = X_{i+1}$ for all $i \geqslant N$.

**Proposition 1.15.**
(i) Every algebraic set $X$ is a Noetherian topologial space.
(ii) Every Noetherian topological space has a unique decomposition

$$X = X_1 \cup \cdots \cup X_r \tag{1.3}$$

where $X_i \subset X$ is a closed irreducible subset and $X_i \not\subset X_j$ for all $i \neq j$.

*Proof* (i): If $X_0 \supset X_1 \supset X_2 \supset \cdots$ is a chain of closed subsets, then $I(X_0) \subset I(X_1) \subset I(X_2) \subset \cdots$ is a chain of ideals in $k[x_0, \ldots, x_n]$, which must stabilize because the polynomial ring is Noetherian. Since $X_i = Z(I(X_i))$, we see that the chain of closed subsets also stabilizes.

(ii): Suppose for a contradiction that $X$ does not admit such a decomposition. Then $X$ cannot be irreducible, so we may write $X = X_1 \cup X_1'$ where $X_1, X_1'$ are proper closed subsets. As $X = X_1 \cup X_1'$, the statement in (ii) must therefore fail for one of $X_1$ or $X_2'$. Suppose that it fails for $X_1$. Then again $X_1$ cannot be irreducible, so we write $X_1 = X_2 \cup X_2'$. Continuing in this way, we obtain an infinite chain of closed subsets $X \supset X_1 \supset X_2 \supset \ldots$, where each inclusion is strict, contradicting (i). Removing any redundant subsets (where $X_i \subset X_j$) gives the desired decomposition (1.3)

As to uniqueness, suppose $X = X_1 \cup \cdots \cup X_r$ and $Y = Y_1 \cup \cdots \cup Y_s$ are two such decompositions. For each $i$, we have

$$X_i = \bigcup_{j=1}^{s} (X_i \cap Y_j).$$

Since $X_i$ is irreducible, $X_i = X_i \cap Y_j$ for some $j$, meaning $X_i \subset Y_j$. Likewise, $Y_j \subset X_k$ for some $k$. Thus, $X_i \subset X_k$, and by the minimality condition ($X_i \not\subset X_\ell$ unless $i = \ell$), we must have $i = k$ and hence $X_i = Y_j$. Similarly, every $Y_j$ must equal some $X_i$. Therefore, the two decompositions are the same up to reordering. □

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The irreducible sets appearing in the decomposition (1.3) are called the *irreducible components* of $X$. They are exactly the closed irreducible subsets which are maximal with respect to inclusion.

**Example 1.16.** Consider the algebraic set $X = Z(I)$ in $\mathbb{A}^3(k)$, where $I$ is the ideal

$$I = (xz - y^2, x^2 - y).$$

Let us find the irreducible components of $X$. Let $P = (a, b, c) \in X$ be a point. Then the second equation implies that $b = a^2$. Plugging this into the first equation, we get $ac - a^4 = 0$, which implies that either $a = 0$ or $c = a^3$. Hence $P$ lies in one of the irreducible subsets $X_1 = Z(x, y)$ or $X_2 = Z(y - x^2, z - x^3)$. Conversely, a point in $X_1$ clearly lies in $X$, and if $P = (a, b, c) \in X_2$, it holds that $b = a^2$ and $c = a^3$ so that $ac - b^2 = a^4 - a^4 = 0$, and $P$ lies in $X$. Hence we find that

$$X = Z(x, y) \cup Z(y - x^2, z - x^3).$$

In geometric terms, $X$ is the union of the $z$-axis and the 'twisted cubic curve' (see Example 1.33). △

**Example 1.17.** Consider the algebraic set $Z(I) \subset \mathbb{A}^2(k)$, where $I$ is the ideal

$$I = (y - x^2, x^2 + (y - 1)^2 - 1).$$

Over the real numbers, we recognise the points of $Z(I)$ as the intersection points of the parabola $y = x^2$ and the circle of radius 1 with centre in $(0, 1)$. To find these intersection points, one can either argue as in the previous example, or directly write the ideal as an intersection:

$$\begin{aligned} I &= (y - x^2, x^2 + (x^2 - 1)^2 - 1) \\ &= (y - x^2, x^2(x - 1)(x + 1)) \\ &= (y - x^2, x^2) \cap (y - x^2, x - 1) \cap (y - x^2, x + 1) \\ &= (y, x^2) \cap (y - 1, x - 1) \cap (y - 1, x + 1). \end{aligned} \tag{1.4}$$

This shows that $Z(I)$ consists of the three points $(0, 0)$, $(1, 1)$, and $(-1, 1)$. In some sense, the decomposition of ideals (1.4) contains more refined information than just the components of the zero set. We will revisit this example in when we discuss *non-reduced schemes* in Section 6.1. △

## 1.4 Dimension

One of the advantages of the correspondence an algebraic set $X$ and its coordinate ring $A(X)$, is the ability to define and study invariants of $X$. As a first example of this, let us define the *dimension* of an algebraic set. For the coordinate ring $A(X)$, there is a well-established notion of the Krull dimension, which is defined as the supremum of the lengths of chains of prime ideals. Using the correspondence between prime ideals and irreducible closed subsets as a guide, we make the following similar definition, which works for any topological space.

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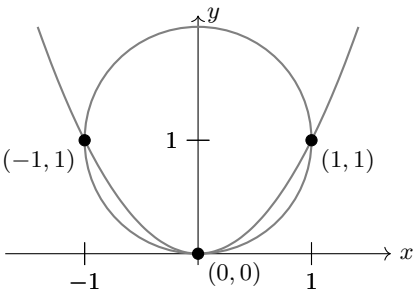


**Figure 1.2** The curves $y = x^2$ and $x^2 + (y-1)^2 = 1$.

**Definition 1.18** (Dimension)**.** Let $X$ be a topological space. The dimension of $X$ is the supremum of the lengths $n$ of chains

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

of distinct irreducible closed subsets of $X$.

This definition is formulated for any topological space, but it mostly makes sense for spaces with 'Zariski-like' topologies such as the ones found in algebraic geometry. For instance, $\mathbb{R}^n$ with the standard topology would be zero-dimensional according to the above definition, as the only irreducible closed subsets are the singleton sets.

The following proposition follows almost immediately:

**Proposition 1.19.** Let $X \subset \mathbb{A}^n(k)$ be an algebraic set. Then

$$\dim X = \dim A(X).$$

*Proof* If $Z_0 \subset Z_1 \subset \cdots \subset Z_r$ is a chain of distinct irreducible subsets in $X$, then $I(Z_0) \supset I(Z_1) \supset \cdots \supset I(Z_r)$ is a chain of distinct prime ideals in $k[x_1, \ldots, x_n]$ containing $I(X)$. These chains correspond exactly to the chains of distinct prime ideals in the ring $A(X) = k[x_1, \ldots, x_n]/I(X)$. As the Krull dimension is defined as the suprema of these lengths, we get the claim. □

**Example 1.20.** The affine $n$-space $\mathbb{A}^n(k)$ has dimension $n$. This follows because the Krull dimension of the polynomial ring $k[x_1, \ldots, x_n]$ is equal to $n$. While this may seem intuitive, it is in fact not so easy to prove. The lower bound, $\dim \mathbb{A}^n_k(k) \geqslant n$, is straightforward, because the following chain of linear spaces

$$Z(x_1, \ldots, x_n) \subset Z(x_2, \ldots, x_n) \subset \cdots \subset Z(x_n)$$

has length $n$ (and each linear space is irreducible). However, the opposite inequality $\dim \mathbb{A}^n(k) \leqslant n$ requires more work (see Proposition A.32). △

**Example 1.21** (Hypersurfaces)**.** Let $f \in k[x_1, \ldots, x_n]$ be a non-constant polynomial, and let $X = Z(f) \subset \mathbb{A}^n(k)$ be the zero set of $f$. Then $A(X) = k[x_1, \ldots, x_n]/(f)$ which has



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Krull dimension $n-1$. This is a consequence of Krull's Principal Ideal Theorem (Theorem A.30). It follows that $\dim(X) = n-1$. △

**Example 1.22.** The algebraic sets in $\mathbb{A}^2(k)$ can be classified according to their dimension:
Dimension 0: $X = \{p_1, \ldots, p_r\}$ is a finite set of points.
Dimension 1: $X = Z(f)$, where $f \in k[x,y]$ is a non-constant polynomial.
Dimension 2: $X = \mathbb{A}^2(k)$.
In contrast, the classification of closed subsets of $\mathbb{A}^3(k)$ is much more complex, as there is no simple description of the ideals in $k[x,y,z]$ defining closed subsets of dimension 1. △

We will study the notion of dimension in more detail in Chapter **??**.

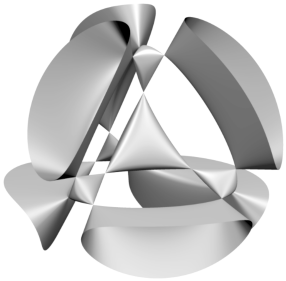

**Figure 1.3** A Kummer surface

## 1.5 Regular and rational functions

The coordinate ring $A(X)$ of an affine variety $X$ is an integral domain, so it has a fraction field, which we will denote by $K(X)$. The field $K(X)$ is called the *function field* or *field of rational functions* on $X$, and its elements are called *rational functions*.

A rational function is by definition an equivalence class of fractions $a/b$ where $a, b \in A(X)$ and $b \neq 0$, where two fractions $a/b$ and $c/d$ represent the same element if $ad = bc$ in $A(X)$. If $A(X)$ is a UFD, a rational function $f$ has a unique minimal representation as a fraction $a/b$ where $a, b \in A(X)$. In general however, there may be different expressions.

While rational functions are formally defined as fractions of polynomials, we may interpret them as functions over open sets. A rational function $f \in K(X)$ is said to be *regular* at a point $P \in X$ if it can be expressed as a fraction $f = a/b$ with $a, b \in A(X)$ and $b(P) \neq 0$. If this holds, we define the *value* $f(P)$ of $f$ at $P$ to be $a(P)/b(P) \in k$. One checks that this is independent of the choice of representatives $a/b$. If $f$ is regular at every point in some open set $U$, then $f$ defines a function

$$f\colon U \longrightarrow k. \tag{1.5}$$

For instance, an element $a/b \in K(X)$ will define a function $D(b) \to k$ over the open set $D(b) = \{\, P \in X \mid b(P) \neq 0 \,\}$. It may however also be regular on a larger open set as well, as illustrated below:

**Example 1.23.** Consider the affine variety $X \subset \mathbb{A}^4(k)$ defined by the equation $xw - yz = 0$.

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In the function field $K(X)$, we have an equality $x/y = z/w$, and the corresponding rational function is therefore regular over the open set $D(y) \cup D(w)$, which is strictly larger than both $D(y)$ and $D(w)$ (for instance, it contains both $(0,1,0,0)$ and $(0,0,0,1)$). △

Rational functions are uniquely determined by their values over open sets: if two rational functions $f, g \in K(X)$ are regular on some open set $V$ and $f(P) = g(P)$ for all $P \in V$, then $f = g$ in $K(X)$. To see this, write $f = a/b$ and $g = c/d$ in $K(X)$. Shrinking $V$ if necessary, we may assume that $a(P)/b(P) = c(P)/d(P)$ for every $P \in V$. Then the polynomial function $ad - bc$ vanishes on a dense subset of $X$, hence on all of $X$. But then $ad - bc = 0$ in $A(X)$ as well, and hence $a/b = c/d$ in $K(X)$.

If $P \in X$ is a point, the set of rational functions which are regular at $P$ forms a subring of $K(X)$ called the *local ring* of $X$ at $P$:

$$\mathcal{O}_{X,P} = \{\, f \in K(X) \mid f \text{ is regular at } P \,\}. \tag{1.6}$$

The ring $\mathcal{O}_{X,P}$ is indeed a local ring with maximal ideal

$$\mathfrak{m}_P = \{\, f \in \mathcal{O}_{X,P} \mid f(P) = 0 \,\}.$$

In fact, if $P = (a_1, \ldots, a_n) \in X$, and we write $\mathfrak{m}_P = (x_1 - a_1, \ldots, x_n - a_n)$ for the maximal ideal in $A(X)$, then

$$\mathcal{O}_{X,P} = A(X)_{\mathfrak{m}_P}.$$

To see this, note that as $A(X)$ is an integral domain, the localization $A(X)_{\mathfrak{m}_P}$ is naturally a subring of $K(X)$. Then an element $f \in K(X)$ lies in $A(X)_{\mathfrak{m}_P}$ if and only if $f$ can be written as $a/b$ with $b \notin \mathfrak{m}_P$, which is equivalent to $f$ being regular at $P$.

If $U \subset X$ is an open set, we set

$$\begin{aligned} \mathcal{O}_X(U) &= \{\, f \in K(X) \mid f \text{ is regular at every point in } U \,\} \\ &= \bigcap_{P \in U} \mathcal{O}_{X,P}. \end{aligned} \tag{1.7}$$

Note that each $\mathcal{O}_X(U)$ is a sub $k$-algebra of $K(X)$. Moreover, if $V \subset U$, then any $f \in K(X)$ which is regular on $U$ is also regular at $V$, so there is an inclusion $\mathcal{O}_X(U) \subset \mathcal{O}_X(V)$.

**Example 1.24.** Let $U = \mathbb{A}^1(k) - \{0\}$ be the complement of the origin in $\mathbb{A}^1$. Then $K(\mathbb{A}^1(k)) = k(x)$, and a rational function $f$ is regular in $U$ if and only if it is of the form $f = a(x)/x^n$ for some polynomial $a(x)$. This means that $\mathcal{O}_{\mathbb{A}^1(k)}(U) = k[x, x^{-1}]$. △

This example generalizes as follows:

**Proposition 1.25.** Let $X$ be an affine variety.

(i) If a rational function $g \in K(X)$ is regular at every point of $X$, then $g$ is a polynomial function. In other words,

$$\mathcal{O}_X(X) = A(X).$$

(ii) More generally, if $f \in A(X)$ and $D(f) = \{\, P \in X \mid f(P) \neq 0 \,\}$, then

$$\mathcal{O}_X(D(f)) = A(X)_f.$$

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*Proof* (i) follows from (ii) by taking $f = 1$.

(ii): Clearly $A(X)_f \subset \mathcal{O}_X(D(f))$. Conversely, given $g \in K(X)$, define the ideal $\mathfrak{a}_g = \{\, b \in A(X) \mid bg \in A(X) \,\}$. This ideal has the property that $g$ is regular at $P$ if and only if $P \notin Z(\mathfrak{a}_g)$. To see this, note that $P \notin Z(\mathfrak{a}_g)$ if and only if some $b \in \mathfrak{a}_g$ does not vanish at $P$, which in turn is equivalent to $g$ being of the form $g = a/b$ for some $b$ with $b(P) \neq 0$. Therefore, if $g$ is regular on all of $D(f)$, it follows that $Z(\mathfrak{a}_g) \subset Z(f)$, and the Nullstellensatz implies that $f^n \in \mathfrak{a}_g$ for some $n > 0$. But then $f^n g \in A(X)$, which shows that $g$ is an element of $A(X)_f$. □

The above proposition is a sort of 'local-to-global principle': being regular is a local condition, which has to be verified near every point, but the conclusion is that a rational function which is regular at every point can be represented globally by a polynomial function.

**Example 1.26.** Consider the complement of the origin in $\mathbb{A}^2(k)$,

$$U = \mathbb{A}^2(k) - \{(0,0)\}$$

We claim that every rational function $f \in \mathcal{O}_{\mathbb{A}^2(k)}(U)$ is the restriction of a polynomial function, that is, $\mathcal{O}_{\mathbb{A}^2(k)}(U) = k[x,y]$. To see this, identify the function field of $\mathbb{A}^2(k)$, with $k(x,y)$, so that $f$ can be expressed as a quotient of two polnomials without common factors $f = a/b$. As $k[x,y]$ is a unique factorization domain, this expression is unique. This means that $f$ cannot be regular at the points in $Z(b)$.

If $b$ is a constant, then $f$ is a polynomial, so the conclusion follows. If $b$ is not constant, then $Z(b)$ will contain infinitely many points (e.g., by Example 1.21). In particular, $f$ will fail to be regular in other points other than the origin $\{(0,0)\}$, contradicting $f \in \mathcal{O}_X(U)$. △

## 1.6 Morphisms of affine varieties

Let $X$ and $Y$ be two affine varieties. For a continuous map $f\colon X \to Y$ and a regular function $g \in \mathcal{O}_Y(V)$, the *pullback of $g$ by $f$* is defined as the composition

$$f^\sharp(g) = g \circ f,$$

which is a regular function on the open set $f^{-1}(V)$.

**Definition 1.27.** A *morphism of affine varieties* is a continuous map $f\colon X \to Y$ which pulls back regular functions back to regular functions, that is, $f^\sharp(g) \in \mathcal{O}_X(f^{-1}V)$ for every open set $V \subset Y$ and every $g \in \mathcal{O}_Y(V)$.

If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms, the composition $g \circ f\colon X \to Z$ is a morphism. Indeed, this map is continuous, and it pulls back regular functions because

$$(g \circ f)^\sharp(h) = f^\sharp(g^\sharp(h)).$$

A morphism of affine varieties $f\colon X \to Y$ is said to be an *isomorphism* if it has an inverse map that is also a morphism.

Given $n$ regular functions $f_1, \ldots, f_n \in A(X)$, we can define a morphism

$$\begin{aligned} f\colon X &\longrightarrow \mathbb{A}^n(k) \\ p &\mapsto (f_1(P), \ldots, f_n(P)). \end{aligned} \tag{1.8}$$

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To verify that $f$ is continuous, we need to check that the preimage of any closed set in $\mathbb{A}^n(k)$ is closed in $X$: if $W = Z(g_1, \ldots, g_r) \subset \mathbb{A}^n(k)$, then

$$\begin{aligned} f^{-1}(W) &= \{\, P \in X \mid g_i(f_1(P), \ldots, f_n(P)) = 0 \text{ for all } i = 1, \ldots, r \,\} \\ &= Z(f^\sharp(g_1), \ldots, f^\sharp(g_r)). \end{aligned}$$

Next, we check that $f$ pulls back regular functions to regular functions. Let $g \in k(y_1, \ldots, y_n)$ be a rational function on $\mathbb{A}^n(k)$ and assume $g$ is regular on an open set $V \subset \mathbb{A}^n(k)$. Let $P \in f^{-1}(V)$ and $Q = f(P)$. Locally around $Q$, we may write $g = a/b$ where $a, b$ are polynomials in $y_1, \ldots, y_n$. Then in a neighborhood of $P \in f^{-1}(V)$, we have

$$f^\sharp(g)(x) = \frac{a(f_1(x), \ldots, f_n(x))}{b(f_1(x), \ldots, f_n(x))}.$$

After expanding, this can be written as a quotient of polynomials where the denominator does not vanish at $P$. Hence $f^\sharp(g) \in \mathcal{O}_X(f^{-1}V)$ and $f$ is a morphism.

**Proposition 1.28.** Let $X$ be an affine variety. Then every morphism $f\colon X \to \mathbb{A}^n(k)$ is of the form

$$f\colon X \longrightarrow \mathbb{A}^n(k); \quad f(P) = (f_1(P), \ldots, f_n(P)) \tag{1.9}$$

where $f_i \in A(X)$ for $i = 1, \ldots, n$.

*Proof* Let $f\colon X \to \mathbb{A}^n(k)$ be a morphism. The coordinate functions $y_1, \ldots, y_n$ are regular functions on $\mathbb{A}^n(k)$, so their pullbacks $f^\sharp(y_i)$ are regular on $X$. The morphism defined by them, $X \to \mathbb{A}^n(k)$, coincides with $f$, as both are defined by the same set of regular functions. □

**Theorem 1.29 (The main theorem of affine varieties).** Let $X$ and $Y$ be affine varieties. Then there is a one-to-one correspondence between morphisms $f\colon X \to Y$ and maps of $k$-algebras $\phi\colon A(Y) \to A(X)$.

*Proof* The correspondence is defined as follows: if $f\colon X \to Y$ is a morphism, then the pullback $f^\sharp\colon A(Y) \to A(X)$ is a map of $k$-algebras.

To define the inverse, let $\phi\colon A(Y) \to A(X)$ be a map of $k$-algebras. Suppose that $Y \subset \mathbb{A}^n(k)$ and let $y_1, \ldots, y_n$ be the coordinate functions on $\mathbb{A}^n(k)$. Now set $f_i = \phi(y_i)$ for $i = 1, \ldots, n$. The $f_1, \ldots, f_n$ define a morphism

$$f\colon X \longrightarrow \mathbb{A}^n(k).$$

For any polynomial $h \in k[y_1, \ldots, y_n]$, we have

$$f^\sharp(h)(x) = h(f(x)) = h(f_1(x), \ldots, f_n(x)) = \phi(h(y_1, \ldots, y_n))(x) \tag{1.10}$$

where the last equality holds because both sides of the equation are maps of $k$-algebras in $h$ and take the same values $f_1, \ldots, f_n$ on the generators $y_1, \ldots, y_n$. This shows that $f^\sharp(h) = \phi(h) = 0$ for every $h \in I(Y)$, as $h$ is zero in $A(Y)$. Therefore, the image of $f$ is contained in $Y = Z(I(Y))$.

Finally, (1.10) shows that $f^\sharp = \phi$, so the correspondence is one-to-one. □

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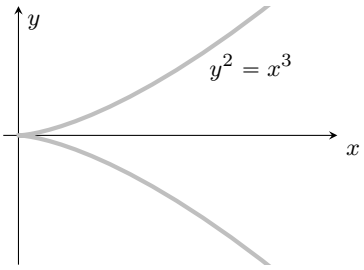


**Figure 1.4** The cuspidal cubic curve

**Corollary 1.30.** Two affine varieties $X$ and $Y$ are isomorphic if and only if the $k$-algebras $A(X)$ and $A(Y)$ are isomorphic.

If $X \subset \mathbb{A}^m(k)$ and $Y \subset \mathbb{A}^n(k)$ are affine varieties, the theorem implies that any morphism $f\colon X \to Y$ is the restriction of a morphism of the form

$$\mathbb{A}^m(k) \longrightarrow \mathbb{A}^n(k); \quad P \mapsto (h_1(P), \dots, h_n(P)) \tag{1.11}$$

where $h_1, \dots, h_n \in k[x_1, \dots, x_m]$ are polynomials. We could have taken this as the definition of morphisms of affine varieties. However, as we will see, the above definition generalizes well to general varieties.

### *Examples*

**Example 1.31.** Any linear map $f\colon \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ is a morphism. Indeed, the components $h_i$ of $f$ are linear polynomials $h_i(x) = \sum_j a_{ij}x_j$. △

**Example 1.32** (The cuspidal cubic curve)**.** Consider the algebraic set $X = Z(y^2 - x^3)$ in $\mathbb{A}^2(k)$. The affine coordinate ring of $X$ is equal to

$$A(X) = k[x, y]/(y^2 - x^3).$$

This is an integral domain because the polynomial $y^2 - x^3$ is irreducible.

Consider the morphism $f\colon \mathbb{A}^1(k) \to \mathbb{A}^2(k)$ defined by $t \mapsto (t^2, t^3)$. The image of $f$ is contained in $X \subset \mathbb{A}^2(k)$, and, in fact, $f$ is a bijection between $\mathbb{A}^1(k)$ and $X$. Indeed, observe that $f(t) = (0, 0)$ only for $t = 0$. Moreover, for points in $X - \{(0, 0)\}$, the assignment $t = y/x$ defines the inverse. However, $f$ is not an isomorphism. As $f^\sharp(x) = t^2$ and $f^\sharp(y) = t^3$, the induced ring map

$$f^\sharp\colon k[x, y]/(y^2 - x^3) \longrightarrow k[t]$$

has image $k[t^2, t^3]$ and so is not surjective.

This means that in order to check that a morphism is an isomorphism, it is not enough to check that it is injective and surjective – also the pullback map $f^\sharp$ has to be an isomorphism. △

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**Example 1.33** (The Twisted Cubic)**.** Consider the morphism

$$\begin{aligned} f\colon \mathbb{A}^1(k) &\longrightarrow \mathbb{A}^3(k) \\ t &\longmapsto (t, t^2, t^3). \end{aligned}$$

If $x, y, z$ are coordinates on $\mathbb{A}^3(k)$, the image of $f$ is the irreducible algebraic set $X = Z(y - x^2, z - x^3)$. $X$ is called the *twisted cubic curve*. The word 'twisted' refers to the fact that $X$ is not contained in a linear plane in $\mathbb{A}^3(k)$.

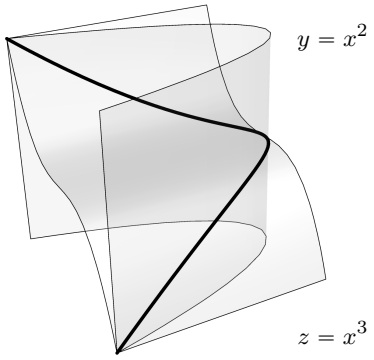


**Figure 1.5** The twisted cubic curve

The corresponding map of $k$-algebras is defined by

$$\phi\colon k[x, y, z] \longrightarrow k[t]$$

mapping $x \mapsto t$, $y \mapsto t^2$, and $z \mapsto t^3$. It is not hard to see that the kernel of $\phi$ is exactly the ideal $\mathfrak{a} = (y - x^2, z - x^3)$. Clearly, $\mathfrak{a} \subset \operatorname{Ker}\phi$. Conversely, if $g \in k[x, y, z]$ is any polynomial, then using $y = x^2$ and $z = x^3$ to remove powers of $y$ and $z$, we can write

$$g(x, y, z) = g(x, x^2, x^3) \mod \mathfrak{a}.$$

If $g \in \operatorname{Ker}\phi$, then $g(x, x^2, x^3)$ is the zero polynomial, and so $g \in \mathfrak{a}$.

As $\phi$ is clearly surjective, we see that

$$A(X) = k[x, y, z]/\mathfrak{a} \simeq k[t].$$

This also implies that $f$ induces an isomorphism $X \simeq \mathbb{A}^1(k)$. △

**Example 1.34.** Let $X = Z(xy - 1) \subset \mathbb{A}^2(k)$ and consider the first projection

$$f\colon X \longrightarrow \mathbb{A}^1(k); \qquad (x, y) \mapsto x$$

The map $f$ is a bijection onto the open set $U = \mathbb{A}^1(k) - \{0\}$. The inverse is given by $g(x) = (x, x^{-1})$ (note that $x^{-1}$ is indeed a regular function on $\mathbb{A}^1(k) - \{0\}$). On the algebraic side, this reflects the fact that $\mathcal{O}_{\mathbb{A}^1(k)}(U) = k[x, x^{-1}] \simeq k[x, y]/(xy - 1)$. △

**Example 1.35.** Consider the morphism $f\colon \mathbb{A}^2(k) \to \mathbb{A}^2(k)$ given by $(x, y) \mapsto (x, xy)$. Then the image of $f$ is equal to $D(x) \cup \{(0, 0)\}$. This shows that the image of an algebraic set need not even be an open set in an algebraic set. △

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**Example 1.36** (Products)**.** If $X \subset \mathbb{A}^m(k)$ and $Y \subset \mathbb{A}^n(k)$ are algebraic sets, we can define the product $X \times Y \subset \mathbb{A}^{m+n}(k)$ as the subset

$$X \times Y = \{\, (a_1, \ldots, a_n, b_1, \ldots, b_n) \mid (a_1, \ldots, a_n) \in X, (b_1, \ldots, b_n) \in Y \,\}.$$

This is indeed an algebraic set: if $X = Z(f_1, \ldots, f_r)$ and $Y = Z(g_1, \ldots, g_s)$, then $X \times Y = Z(f_1, \ldots, f_r, g_1, \ldots, g_s)$. Moreover, the projection maps $\pi_1 \colon \mathbb{A}^{m+n}(k) \to \mathbb{A}^m(k)$ and $\pi_2 \colon \mathbb{A}^{m+n}(k) \to \mathbb{A}^n(k)$ are morphisms. △

## 1.7 Conclusion

The main theme of this chapter is that the geometric properties of an affine variety $X \subset \mathbb{A}^n(k)$ is reflected in the algebraic properties of the coordinate ring

$$A(X) = k[x_1, \ldots, x_n]/I(X).$$

For instance, by the Nullstellensatz the mappings $V \mapsto I(V)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$ give mutually inverse inclusion-reversing bijections between the objects in the following table:

| ALGEBRA | GEOMETRY |
|---|---|
| radical ideals of $A(X)$ | closed subsets of $X$ |
| prime ideals of $A(X)$ | closed irreducible subsets of $X$ |
| maximal ideals of $A(X)$ | points of $X$ |

Since $I(X)$ is a radical ideal, the $k$-algebra $A(X)$ is always a reduced ring, that is, there are no nilpotents. Moreover, $A(X)$ is an integral domain if and only if $X$ is irreducible (Proposition 1.11).

Conversely, every reduced finitely generated $k$-algebra $A$ is isomorphic to $A(X)$ for some algebraic set $X$. Indeed, choose a presentation $A = k[x_1, \ldots, x_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$; then $X = Z(\mathfrak{a})$ is an algebraic set in $\mathbb{A}^m(k)$ with $A(X) = A$. Likewise, any finitely generated $k$-algebra which is also an integral domain is of the form $A(X)$ for some affine variety $X$.

In category theory terms, the affine varieties form a category, $\mathsf{AffVar}$, where the objects are affine varieties and the arrows betwen them are morphisms of affine varieties. By Theorem 1.29, if $X$ and $Y$ are two affine varieties, the assignment $f \mapsto f^\sharp$ defines a bijection

$$\mathrm{Hom}_{\mathsf{AffVar}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{Alg}/\mathsf{k}}(A(Y), A(X)). \tag{1.12}$$

In sum, this means that the functor $X \mapsto A(X)$ defines an equivalence of categories between the category of affine varieties and the opposite category of finitely generated $k$-algebras without zerodivisors (that is, the category where the objects are $k$-algebras, but where all arrows are formally reversed).

## 1.8 Exercises

**Exercise 1.8.1.** Show that $X = \{\, (x, y) \mid y - \cos(x) = 0 \,\}$ is not an algebraic set in $\mathbb{A}^2(\mathbb{C})$. What is the closure of $X$ in the Zariski topology?

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**Exercise 1.8.2.** Show that the algebraic set $Z(y^2 - x^3 - 1) \subset \mathbb{A}^2(k)$ is irreducible.

**Exercise 1.8.3.** Let $f\colon \mathbb{A}^1(k) \to \mathbb{A}^1(k)$ be an isomorphism. Show that $f$ is given by a linear polynomial $ax + b$, where $a, b \in k$ and $a \neq 0$.

**Exercise 1.8.4.** Show that the product $X \times Y$ of two affine varieties $X$ and $Y$ satisfies the following undersal property: for any affine variety $V$, with two morphisms $f_1\colon V \to X$ and $f_2\colon V \to Y$, there is a unique morphism $f\colon V \to X \times Y$ so that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

**Exercise 1.8.5.** Consider the algebraic set in $\mathbb{A}^3(k)$ given by $Y = Z(x^3 + y^3 + z^3 - 3xyz)$. Decide whether $Y$ is irreducible and find its dimension.

**Exercise 1.8.6.** Let $X = Z(y^2 - x^3 + 1, z - x^2) \subset \mathbb{A}^3(k)$. Show that $X$ is isomorphic to the cubic curve $C = Z(v^2 - u^3 + 1) \subset \mathbb{A}^2(k)$.

**Exercise 1.8.7.** Find the irreducible compoents of the algebraic sets below.

a) $Z(x^2y, y^2x) \subset \mathbb{A}^2(k)$.
b) $Z(x^2 - y^2) \subset \mathbb{A}^2(k)$.
c) $Z(x - yz, xz - y^2) \subset \mathbb{A}^3(k)$.
d) $Z(x^2 + y^2, x^2 - z^2 - 1) \subset \mathbb{A}^3(k)$.
e) $Z(x^2 - y, xz - y^2, x^3 - xz) \subset \mathbb{A}^3(k)$.

**Exercise 1.8.8.** Show that the Zariski topology on $\mathbb{A}^2(k)$ is not the product topology on $\mathbb{A}^2(k) = \mathbb{A}^1(k) \times \mathbb{A}^1(k)$.

**Exercise 1.8.9.** Consider the algebraic set in $\mathbb{A}^2(k)$ given by

$$X = Z(x^2 - y^3, x^2 + y^2 - 1)$$

a) Show that $X$ consists of 6 points.
b) If $\pi\colon \mathbb{A}^2(k) \to \mathbb{A}^1(k)$ denotes the projection onto the $x$-axis, find the polynomial describing the closed set $Y = \pi(X)$ in $\mathbb{A}^1(k)$.

**Exercise 1.8.10** (The diagonal)**.** Let $X$ be an affine variety and consider the map

$$\begin{aligned} \Delta\colon X &\longrightarrow X \times X \\ x &\longmapsto (x, x) \end{aligned}$$

a) Show that $\Delta$ is a morphism of affine varieties.
b) Let $X = \mathbb{A}^n(k)$, and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be coordinates on $\mathbb{A}^n(k) \times \mathbb{A}^n(k)$. Show that the ideal defining the image of $\Delta$ is given by

$$\mathfrak{a} = (x_1 - y_1, \ldots, x_n - y_n)$$

c) In general, for an affine variety $X$, show that the image of $\Delta$ is closed in $X \times X$, and that $\Delta$ gives an isomorphism $X \to \Delta(X)$. HINT: Use the two projections $X \times X \to X$.

**Exercise 1.8.11.** Prove that the coordinate ring of the affine curve $X = Z(y - x^5)$ is isomorphic to $k[t]$.

**Exercise 1.8.12.** With notation as in Example 1.23:

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a) Verify that $xw - yz$ is an irreducible polynomial.
b) Show that the rational function $x/y$ is not regular in any open set containing the locus where $y = w = 0$.
c) Show that there is no single distinguished open set so that $D(b)$ contains all the points where $x/y$ is regular.

**Exercise 1.8.13.** Let $x_0, \ldots, x_n$ be coordintes on the affine $(n+1)$-space $\mathbb{A}^{n+1}(k)$ and let $f = f(x_1, \ldots, x_n)$ be a polynomial in $x_1, \ldots, x_n$.
a) Show that the algebraic set $X = Z(x_0 - f)$ is isomorphic to $\mathbb{A}^n(k)$.
b) For which $f$'s is the algebraic set $X = Z(x_0^2 - f)$ irreducible?
c) Find a bijection between the open set $\mathbb{A}^n(k) - Z(f)$ in $\mathbb{A}^n(k)$ and the algebraic set $Z(x_0 f - 1)$ in $\mathbb{A}^{n+1}(k)$.

**Exercise 1.8.14.**
a) Let $\phi\colon A \to B$ be a ring map. Show that $\phi^{-1}(\mathfrak{p})$ is a prime ideal if $\mathfrak{p} \subset B$ is prime.
b) Assume further that $A$ and $B$ are finitely generated $k$-algebras, and $k$ is algebraically closed. Show that $\phi^{-1}(\mathfrak{m})$ is a maximal ideal if $\mathfrak{m} \subset B$ is one. HINT: Use the Nullstellensatz to see that $A/(\phi^{-1}\mathfrak{m}) = k$.

**Exercise 1.8.15.** Let $A$ be a ring and let $\mathfrak{a}$ be an ideal
a) Show that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p}\supset\mathfrak{a}} \mathfrak{p}$, where the intersection is taken over all the prime ideals containing $\mathfrak{a}$. HINT: If $f \notin \sqrt{\mathfrak{a}}$ the ideal $\mathfrak{a}A_f$ is a proper ideal in the localization $A_f$, hence contained in a maximal ideal.
b) Using the Nullstellensatz, show that if $A = k[x_1, \ldots, x_n]$ over an algebraically closed field $k$, we have $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{m}\supset\mathfrak{a}} \mathfrak{m}$ where the intersection is over all the *maximal* ideals containing $\mathfrak{a}$.

**Exercise 1.8.16.** Let $X = Z(f)$ and $Y = Z(g)$ be two algebraic sets in $\mathbb{A}^2(k)$ with $X$ irreducible. Show that either $X \cap Y$ is a finite set, or $X \subset Y$.

**Exercise 1.8.17.** Show that the image of the map

$$\begin{aligned} \phi : \mathbb{A}^1(k) &\longrightarrow \mathbb{A}^3(k) \\ t &\longmapsto (t^2, t^3, t^6) \end{aligned}$$

is given by $V(x^3 - y^2, z - x^3)$. Show that $\phi$ is bijective. Is $\phi$ an isomorphism of affine varieties?

**Exercise 1.8.18.** Let $X = Z(y^2 - x^3 - x) \subset \mathbb{A}^2(k)$. Show that $y/x$ and $(x^2+1)/y$ define the same rational function $f$ on $X$. Show that $f$ is regular on $U = X - \{(0,0)\}$. Show that there are no polynomials $a, b \in k[x, y]$ so that $f$ can be represented as $a/b$ on all of $U$.

**Exercise 1.8.19.** Let $X$ be a topological space.
a) Show a subset $Z \subset X$ is irreducible if and only if its closure $\overline{Z}$ is irreducible;
b) Show that $X$ is irreducible if and only if every non-empty open subset is dense.
c) Show that $X$ is irreducible if and only if every pair of nonempty open sets $U, V \subset X$ intersect.

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d) Let $X_0 \subset X$ be an irreducible component of $X$ and $U \subset X$ be an open subset such that $U \cap X_0$ is nonempty. Show that $X_0 \cap U$ is an irreducible component of $U$.
e) If $f\colon X \to Y$ is a continuous map, show that $f(X)$ is irreducible if $X$ is.

**Exercise 1.8.20.** Let $X \subset \mathbb{A}^4(k)$ be the affine variety defined by the equation $xw - yz = 0$.
a) Compute $\mathcal{O}_X(U)$ for the open set $U = D(x) \cup D(y)$.
b) Compute the local ring $\mathcal{O}_{X,P}$ where $P = (0,0,0,0)$.

**Exercise 1.8.21.** Show that any open set in $\mathbb{A}^n(k)$ can be written as a finite union of distinguished open sets.

**Exercise 1.8.22.** Show that the open set $\mathbb{A}^2 - \{(0,0)\}$ is not isomorphic to an affine variety.

**Exercise 1.8.23.** Show that any irreducible conic in $\mathbb{A}^2(k)$ is isomorphic to either $Z(y - x^2)$ or $Z(xy - 1)$.

**Exercise 1.8.24** (Alternative proof of Hilbert's Nullstellensatz)**.** Let $k$ be a field.
a) Show that $k[x]$ contains infinitely many irreducible polynomials. HINT: Imitate Eucclid's proof of the infinitude of prime numbers.
b) Show that $k(x_1, \ldots, x_n)$ is not finitely generated as a $k$-algebra.
c) Suppose $k \subset K \subset L$ are field extensions and that $L$ is both a finitely generated $k$-algebra and a $k$-vector space of finite dimension. Show that $K$ is also finitely generated over $k$.
d) Prove Theorem 1.7. HINT: Let $K = k[y_1, \ldots, y_n]$. If $y_n$ is transcendental over $k$, write $K = F[y_1, \ldots y_{n-1}]$ where $F = k[y_n]$. Then argue by induction on $n$.

**Exercise 1.8.25** (Varieties defined by matrices)**.** Identify the space of $n \times n$ matrices with entries in $k$ with affine $n^2$-space $\mathbb{A}^{n^2}(k)$.
a) Show that the set of symmetric matrices, i.e. matrices such that $A^T = A$, is an algebraic set in $\mathbb{A}^{n^2}(k)$.
b) Show that the set $\mathrm{GL}_n(k)$ of invertible matrices is an open set in $\mathbb{A}^{n^2}(k)$.
c) Show that the set $\mathrm{SL}_n(k)$ of matrices with determinant 1 is an algebraic set in $\mathbb{A}^{n^2}(k)$. Is it irreducible?
d) Show that the sets $\mathrm{O}(n)$ (resp. $\mathrm{SO}(n)$) consisting of the matrices $A$ such that $A^T A = I_n$ (resp. $A^T A = I_n$ and $\det(A) = 1$), where $I_n$ is the $n \times n$ identity matrix. Show that $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are closed subsets of $\mathbb{A}^{n^2}(k)$. Are they irreducible?

**Exercise 1.8.26.** Identify $\mathbb{A}^{nm}(k)$ with the space of $m \times n$-matrices over the field $k$. For a given integer $r$, show that the set of matrices of rank less than $r$ is an algebraic set. Is it irreducible?

**Exercise 1.8.27.** Let $S \subset \mathbb{A}^{n\times n}(\mathbb{C})$ denote the set of diagonalizable matrices. Determine the Zariski closure of $S$.

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**Exercise 1.8.28** (Nilpotent matrices)**.** As in Exercise 1.8.25 identify the space of $2 \times 2$-matrices with $\mathbb{A}^4(k)$ via the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (a, b, c, d)$. Show that the two ideals in $k[a, b, c, d]$

$$\mathfrak{a} = (a^2 + bc, ab + bd, ac + cd, bc + d^2) \text{ and } \mathfrak{b} = (ad - bc, a + d)$$

satisfy $\mathfrak{a} \neq \mathfrak{b}$ but $\sqrt{\mathfrak{a}} = \mathfrak{b}$. Furthermore, show that

$$Z(\mathfrak{a}) = Z(\mathfrak{b}) = \{M \mid M \text{ is nilpotent}\} \subset \mathbb{A}^4(k).$$

.

**Exercise 1.8.29.** Show that if $X$ is a Hausdorff space, then the only irreducible subsets are the one-point sets $\{x\}$. Deduce that if $X$ is an irreducible space with at least 2 points, then $X$ is not Hausdorff.

**Exercise 1.8.30.** Show that a topological space is Noetherian if and only if every non-empty collection of closed subsets has a minimal element. Equivalently, every collection of open subsets has a maximal element.

**Exercise 1.8.31** (The Frobenius morphism)**.** Let $k$ be an algebraically closed field of positive characteristic $p$.

a) Show that the map $\phi\colon k[t] \to k[t]$ defined by $t \mapsto t^p$, is a map of $k$-algebras.
b) Describe the corresponding morphism $F\colon \mathbb{A}^1(k) \to \mathbb{A}^1(k)$. Show that $F$ is bijective, but not an isomorphism.

**Exercise 1.8.32.** Let $X \subset \mathbb{A}^3(k)$ denote the twisted cubic curve of Example 1.33 and let $p\colon \mathbb{A}^3(k) \to \mathbb{A}^2(k)$ denote the projection map $(x, y, z) \mapsto (x, y)$. Describe the closed set $p(X)$ in $\mathbb{A}^2(k)$.

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# Part I

## Schemes



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# 2

# The spectrum of a ring

In this chapter we make the first step towards the notion of a scheme, by defining the spectrum of a ring. The spectrum of a ring $A$, denoted by $\operatorname{Spec} A$, is a topological space with a topology whose closed sets are defined by the ideals of $A$, in direct analogy with the Zariski topology on affine varieties.

To motivate the construction, let us assume for a moment that $A = A(X)$ is the coordinate ring of an affine variety $X \subset \mathbb{A}^n(k)$. By Hilbert's Nullstellensatz, there is a one-to-one correspondence between the points $P = (a_1, \dots, a_n)$ of $X$ and maximal ideals $\mathfrak{m}_P = (x_1 - a_1, \dots, x_n - a_n)$ in $A$. This allows us to identify $X$ with the set of the maximal ideals of $A$. Note that a point $P \in X$ lies in $Z(\mathfrak{a})$ precisely when $\mathfrak{a} \subset \mathfrak{m}_x$. Therefore, under this identification, the closed sets $Z(\mathfrak{a})$ now take the form $\{\, \mathfrak{m} \mid \mathfrak{a} \subset \mathfrak{m} \,\}$ of maximal ideals containing $\mathfrak{a}$. This shows that the ring $A$ determines both the underlying set $X$ and its Zariski topology. Furthermore, morphisms between affine varieties correspond to ring maps between their coordnate rings.

The rings appearing in this setting are rather special: they are integral domains and finitely generated $k$-algebras. There is also the assumption that $k$ is algebraically closed, which is essential in order to have the above correspondence between points and maximal ideals.

There is a natural way of generalizing this to arbitrary rings, which involves considering all prime ideals, rather than just the maximal ideals. Given a ring $A$, the spectrum $\operatorname{Spec} A$ of $A$ is simply the set of prime ideals of $A$. This set is then equipped with a topology, called the Zariski topology, whose closed sets are the sets of the form $V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{a} \subset \mathfrak{p} \,\}$ where $\mathfrak{a}$ is any ideal in $A$.

The idea of replacing maximal ideals by prime ideals lies at the heart of scheme theory. From a functorial perspective, this is a natural choice, because inverse images of prime ideals under ring maps are prime ideals, and hence a ring map $A \to B$ induces a map $\operatorname{Spec} B \to \operatorname{Spec} A$. If $X$ and $Y$ are affine varieties, the induced map $A(Y) \to A(X)$, in fact, pulls maximal ideals back to maximal ideals, but this is generally not true for arbitrary ring maps (a simple example is the inclusion $\mathbb{Z} \to \mathbb{Q}$).

## 2.1 The spectrum of a ring

Let $A$ be a ring. As always, $A$ is commutative with 1.



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**Definition 2.1.** The *spectrum* of $A$, denoted $\operatorname{Spec} A$, is the set of all prime ideals in $A$:

$$\operatorname{Spec} A = \{\, \mathfrak{p} \mid \mathfrak{p} \subset A \text{ is a prime ideal} \,\}.$$

The set $\operatorname{Spec} A$ has a Zariski topology similar to the Zariski topology as defined in Chapter 1. The definitions are very similar; the closed sets are those of the form

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset \mathfrak{a} \,\}. \tag{2.1}$$

where $\mathfrak{a} \subset A$ is an ideal. Note that $\mathfrak{a} \subset \mathfrak{b}$ implies $V(\mathfrak{a}) \supset V(\mathfrak{b})$.

To draw a closer analogy to the zero sets $Z(\mathfrak{a})$ from Chapter 1, we can think of the sets (2.1) as the 'vanishing set' of a set of functions on $\operatorname{Spec} A$. To make this precise, we introduce the concept of a residue field.

**Definition 2.2.** For a point $\mathfrak{p} \in \operatorname{Spec} A$, we define the *residue field* $\kappa(\mathfrak{p})$ to be the field of fractions of the integral domain $A/\mathfrak{p}$.

The field $\kappa(\mathfrak{p})$ is canonically isomorphic to $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, where $A_{\mathfrak{p}}$ is the localization of $A$ at $\mathfrak{p}$, which is a local ring with maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

Given $f \in A$, we denote by $f(\mathfrak{p}) \in \kappa(\mathfrak{p})$ the image of $f$ modulo $\mathfrak{p}$. In this way, we can view $f$ as a function on $\operatorname{Spec} A$. Note however, that unlike the case of the coordinate ring of an affine variety, the values of $f$ do not lie in a fixed field $k$, but rather in many different fields $\kappa(\mathfrak{p})$, each associated with a different prime ideal $\mathfrak{p}$.

With this notation, it makes sense to talk about the *vanishing set* of $f$; that is,

$$V(f) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) = 0 \text{ in } \kappa(\mathfrak{p}) \,\}.$$

Note that $f(\mathfrak{p}) = 0$ if and only if $f \in \mathfrak{p}$. Therefore, this set is exactly the vanshing set associated to the principal ideal $\mathfrak{a} = (f)$ as defined in (2.1). More generally, if $\mathfrak{a} \subset A$ is an ideal, the set (2.1) can be written as

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) = 0 \text{ for all } f \in \mathfrak{a} \,\}. \tag{2.2}$$

The advantage of expressing $V(\mathfrak{a})$ in this way comes from the fact that all the identities for the $Z(\mathfrak{a})$ remain valid for $V(\mathfrak{a})$, and the proofs are the same.

The next lemma is analogous to Proposition 1.2. It tells us that the vanishing sets indeed form the closed sets of a topology on $\operatorname{Spec} A$.

**Lemma 2.3.** The vanishing sets in $\operatorname{Spec} A$ satisfy the following properties.

(i) (Arbitrary intersections): For any collection of ideals $\{\mathfrak{a}_i\}_{i\in I}$ in $A$,

$$\bigcap_{i\in I} V(\mathfrak{a}_i) = V\big(\sum_{i\in I} \mathfrak{a}_i\big). \tag{2.3}$$

(ii) (Finite unions): For two ideals $\mathfrak{a}$ and $\mathfrak{b}$,

$$V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a} \cdot \mathfrak{b}). \tag{2.4}$$

Additionally, $V(A) = \varnothing$ and $V(0) = \operatorname{Spec} A$.

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*Proof* (i): A prime ideal $\mathfrak{p}$ lies in $\bigcap_{i\in I} V(\mathfrak{a}_i)$ if and only if it contains every $\mathfrak{a}_i$, which in turn happens if and only if $\mathfrak{p}$ contains the sum $\sum_{i\in I} \mathfrak{a}_i$.

(ii): Note that since $\mathfrak{a}\cdot\mathfrak{b} \subset \mathfrak{a}\cap\mathfrak{b}$, it follows that $V(\mathfrak{a}\cap\mathfrak{b}) \subset V(\mathfrak{a}\cdot\mathfrak{b})$. Similarly, we get the inclusion $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subset V(\mathfrak{a}\cap\mathfrak{b})$. To prove (2.4), it suffices to prove $V(\mathfrak{a}\cdot\mathfrak{b}) \subset V(\mathfrak{a}) \cup V(\mathfrak{b})$. If $P \notin V(\mathfrak{a}) \cup V(\mathfrak{b})$, there exist $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ so that $f(P) \neq 0$ and $g(P) \neq 0$, but then $(fg)(P) = f(P)g(P) \neq 0$ in $\kappa(P)$ and hence $P \notin V(\mathfrak{a}\cdot\mathfrak{b})$. This shows that $V(\mathfrak{a}\cdot\mathfrak{b}) \subset V(\mathfrak{a}) \cup V(\mathfrak{a})$.

Finally, prime ideals are by definition proper ideals, so $V(A) = \varnothing$. The zero ideal $(0)$ is contained in every ideal, so $V(0) = \operatorname{Spec} A$. □

The next lemma is about inclusions between the closed sets of $\operatorname{Spec} A$. We recognize them as analogues of some of the statements about algebraic sets in Proposition 1.2.

**Lemma 2.4.** For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $A$, the following hold:
(i) $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$
(ii) $V(\mathfrak{a}) \subset V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$
(iii) $V(\mathfrak{a}) = \varnothing$ if and only if $\mathfrak{a} = A$
(iv) $V(\mathfrak{a}) = \operatorname{Spec} A$ if and only if $\mathfrak{a} \subset \sqrt{(0)}$.

*Proof* (i): For a prime ideal $\mathfrak{p}$, the condition $\mathfrak{p} \supset \mathfrak{a}$ is equivalent to $\mathfrak{p} \supset \sqrt{\mathfrak{a}}$, since prime ideals are radical. Hence $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

(ii): We recall the following identity for the radical of an ideal (see page 418):

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p}\supset\mathfrak{a}} = \bigcap_{\mathfrak{p}\in V(\mathfrak{a})} \mathfrak{p}. \tag{2.5}$$

If $V(\mathfrak{a}) \subset V(\mathfrak{b})$, then the intersection defining $\sqrt{\mathfrak{b}}$ is taken over a larger set than $\sqrt{\mathfrak{a}}$, giving $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$. Conversely, if $\sqrt{\mathfrak{b}} \subset \sqrt{\mathfrak{a}}$, then by (i),

$$V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}}) \subset V(\sqrt{\mathfrak{b}}) = V(\mathfrak{b}).$$

(iii): If $\mathfrak{a} \neq (1)$, then it is contained in a maximal ideal. Hence $V(\mathfrak{a}) = V(1) = \varnothing$ if and only if $\mathfrak{a} = (1)$. (iv) follows from (ii), as $V(\mathfrak{a}) = V(0)$ if and only if $\sqrt{\mathfrak{a}} = \sqrt{(0)}$, which happens if and only if $\mathfrak{a} \subset \sqrt{(0)}$. □

As in the definition (1.1), we define the ideal of a subset $S \subset \operatorname{Spec} A$ as

$$I(S) = \{\, f \in A \mid f(\mathfrak{p}) = 0 \text{ for all } \mathfrak{p} \in S \,\} = \bigcap_{\mathfrak{p}\in S} \mathfrak{p}. \tag{2.6}$$

Note that $I(S)$ is a radical ideal.

**Proposition 2.5.** For any subset $S \subset \operatorname{Spec} A$, we have

$$V(I(S)) = \overline{S}. \tag{2.7}$$

In particular, the closure $\overline{\{\mathfrak{p}\}}$ of the one-point set $\{\mathfrak{p}\}$ is equal to $V(\mathfrak{p})$.

*Proof* First of all, $S \subset V(I(S))$, because if $\mathfrak{p} \in S$ then clearly $\mathfrak{p} \supset \bigcap_{\mathfrak{p}\in S} \mathfrak{p} = I(S)$ and

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so $\mathfrak{p} \in V(I(S))$. It remains to show that $V(I(S))$ is the smallest closed set containing $S$. If $V(\mathfrak{b})$ is any closed subset containing $S$, then for any $\mathfrak{p} \in S$, we have $\mathfrak{p} \in V(\mathfrak{b})$ and hence $\mathfrak{p} \supset \mathfrak{b}$. Taking the intersection over all $\mathfrak{p}$'s in $S$, we see that $I(S) \supset \mathfrak{b}$. Applying $V(-)$, we obtain $V(I(S)) \subset V(\mathfrak{b})$, as required. □

As a consequence, we get the following analogue of the Nullstellensatz:

**Corollary 2.6.** Let $A$ be a ring. Then the map $\mathfrak{a} \mapsto V(\mathfrak{a})$ defines a bijection

$$\big\{\text{radical ideals } \mathfrak{a} \subset A\big\} \longrightarrow \{\text{closed sets } W \subset \operatorname{Spec} A\}, \tag{2.8}$$

with inverse defined by $W \mapsto I(W)$.

*Proof* If $\mathfrak{a}$ is any ideal, then $I(V(\mathfrak{a})) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \sqrt{\mathfrak{a}}$ by the formula (2.5). Hence $I(V(-))$ is the identity map on radical ideals $\mathfrak{a}$. Conversely, if $W$ is a closed set, then $V(I(W)) = \overline{W} = W$ by Proposition 2.5, so also $V(I(-))$ is the identity. □

**Corollary 2.7.** A point in $\mathfrak{p} \in \operatorname{Spec} A$ is closed if and only if $\mathfrak{p}$ is a maximal ideal.

*Proof* $V(\mathfrak{p}) = \{\mathfrak{p}\}$ says that $\mathfrak{p}$ is the only prime ideal which contains $\mathfrak{p}$, which precisely means that $\mathfrak{p}$ is maximal. □

Corollary 2.7 shows that the Zariski topology on $\operatorname{Spec} A$ behaves rather differently from not only the Euclidean topology on manifolds, but also the usual Zariski topology on affine varieties. Indeed, the spectrum of a general ring typically contains many non-closed points corresponding to the prime ideals which are not maximal.

**Definition 2.8** (Generic points)**.** A point $x$ in a closed subset $Z$ of a topological space $X$ is called a *generic point* for $Z$ if $\overline{\{x\}} = Z$.

In our context, the point $\mathfrak{p} \in \operatorname{Spec} A$ is the generic point of the closed set $V(\mathfrak{p})$. The point $\mathfrak{p}$ is in fact the only generic point of $V(\mathfrak{p})$, because if $V(\mathfrak{p}) = V(\mathfrak{q})$, then Lemma 2.4 implies that both $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{q} \subset \mathfrak{p}$.

### *First examples*

**Example 2.9** (Fields)**.** If $K$ is a field, the prime spectrum $\operatorname{Spec} K$ has only one point, which corresponds to the only prime ideal in $K$, the zero ideal. △

**Example 2.10** (The integers)**.** In the ring of integers $\mathbb{Z}$, there are two types of prime ideals: the zero ideal $(0)$ and the maximal ideals $(p)$, one for each prime number $P$.

As $\mathbb{Z}$ is a principal ideal domain, any ideal is of the form $(n)$ for some integer $n$. It follows that the closed subsets are of the form $V(n) = V(p_1) \cup \cdots \cup V(p_r)$ where the $p_i$ are the prime factors of $n$. In other words, the closed sets are either finite sets of closed points or the whole space. Dually, the non-empty open sets are the complements of finite sets of closed points. This means that $\operatorname{Spec} \mathbb{Z}$ is not Hausdorff, as every nonempty open set contains $(0)$.

The residue field at a closed point $(p)$ is equal to $\kappa(p) = \mathbb{Z}/p = \mathbb{F}_p$, whereas the residue

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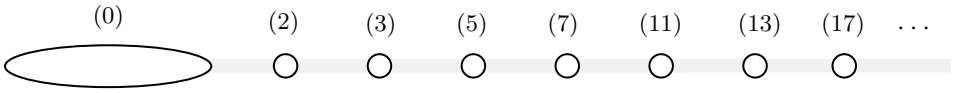


**Figure 2.1** $\operatorname{Spec}\mathbb{Z}$

field at $(0)$ is equal to $\mathbb{Q}$. Each element $f$ of the ring $\mathbb{Z}$ gives rise to a function on $\operatorname{Spec}\mathbb{Z}$ with values in the various residue fields. For instance, the integer $f = 17$ takes the values $f((0)) = 17$, $f((2)) = \bar{1}$, $f((3)) = \bar{2}$, $f((5)) = \bar{2}$, $f((7)) = \bar{3}, \ldots$, in the fields $\mathbb{Q}, \mathbb{F}_2$, $\mathbb{F}_3, \mathbb{F}_5, \mathbb{F}_7, \ldots$, respectively, where the bar indicates the class modulo the relevant prime. △

**Example 2.11** (A polynomial ring)**.** Consider the spectrum of the polynomial ring in one variable, $\mathbb{C}[t]$. Since $\mathbb{C}[t]$ is a principal ideal domain, every ideal is of the form $(f(t))$ for some polynomial $f(t)$. If $f(t)$ is not a constant, we may factor it into a product of terms of the form $t - a_i$, so that

$$V(f(t)) = V(t - a_1) \cup \cdots \cup V(t - a_n).$$

Therefore, the closed sets are either the empty set; the whole space; or a finite set of closed points. As in the previous example, $\operatorname{Spec}\mathbb{C}[t]$ is not Hausdorff.

At the prime $\mathfrak{p} = (t - a)$, the residue field $\kappa(\mathfrak{p})$ is equal to $\mathbb{C}[t]/(t - a) \simeq \mathbb{C}$. Under this isomorphism, a polynomial $f \in \mathbb{C}[t]$ maps to its value $f(a) \in \mathbb{C}$. Hence the 'value' $f(\mathfrak{p}) \in \kappa(\mathfrak{p})$ is identified with the usual evaluation of the polynomial at $t = a$. △

**Example 2.12.** If $A$ is an integral domain, the zero ideal $(0)$ is prime, and as $V(0) = \operatorname{Spec} A$, it is the generic point of all of $\operatorname{Spec} A$. This explains the 'fat' points in the pictures in Examples 2.10 and 2.14 – the closures of these points are the whole space. △

**Example 2.13.** The ring $A = \mathbb{C}[t]/(t^2)$ is not a field, but has only one prime ideal, namely the ideal $(t)$. Note that the ideal $(0)$ is not prime as $t^2 = 0$, but $t \notin (0)$.

The ring $A = \mathbb{C}[t]/(t(t-1))$ has a spectrum which consists of two points. By the Chinese Remainder Theorem,

$$A \simeq \mathbb{C}[t]/t \times \mathbb{C}[t]/(t-1) \simeq \mathbb{C} \times \mathbb{C},$$

which has exactly two prime ideals, namely $0 \times \mathbb{C}$ and $\mathbb{C} \times 0$. △

**Example 2.14** (Discrete valuation rings)**.** Consider a discrete valuation ring $A$, such as $k[t]_{(t)}$ or $\mathbb{Z}_{(p)}$. (See Appendix A for background on discrete valuation rings). The ring $A$ has exactly two prime ideals, the maximal ideal $\mathfrak{m}$ and the zero ideal $(0)$. Therefore, $\operatorname{Spec} A$ consists of just two points: $\operatorname{Spec} A = \{s, \xi\}$ with $s$ corresponding to the maximal ideal $\mathfrak{m}$ and $\xi$ to $(0)$. The closed sets are $\varnothing, \{s\}$ and $\{s, \xi\}$, while the open sets are $\varnothing$, $\operatorname{Spec} A$ and $\{\xi\}$. In particular, $\xi$ is an open point. Once again $\operatorname{Spec} A$ is not Hausdorff, as there are no open sets that separate $s$ and $\xi$. △

## 2.2 Affine spaces

The most important examples of prime spectra are the affine spaces.

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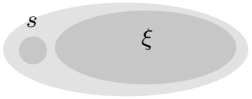


**Figure 2.2** The spectrum of a DVR

**Definition 2.15.** For each non-negative integer $n$, we define the *affine $n$-space* as

$$\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n].$$

More generally, for a ring $R$, we define the *affine $n$-space over $R$* by

$$\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \ldots, x_n].$$

When $R = k$ is a field, each $n$-tuple $(a_1, \ldots, a_n) \in k^n$ determines a maximal ideal $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ in the polynomial ring $A = k[x_1, \ldots, x_n]$, and hence a point of $\mathbb{A}^n_k = \operatorname{Spec} A$ with residue field

$$A/\mathfrak{m} = k[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \simeq k.$$

Conversely, every maximal ideal $\mathfrak{m}$ in $A$ with residue field $A/\mathfrak{m}$ isomorphic to $k$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$: the $a_i$ are uniquely determined as the images of $x_i$ via the quotient map $A \to A/\mathfrak{m} = k$. Therefore, there is a bijection between $k^n$ and the points in $\operatorname{Spec} A$ with residue field isomorphic to $k$.

For these 'traditional points', we will frequently switch between writing the $n$-tuple $(a_1, \ldots, a_n) \in k^n$ and the maximal ideal $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ for the corresponding point in $\mathbb{A}^n_k$.

If $k$ is algebraically closed, then Hilbert's Nullstellensatz tells us that every maximal ideal is of the form $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$, so every closed point of $\mathbb{A}^n_k$ will be of the form $(a_1, \ldots, a_n)$. This means that $\mathbb{A}^n(k)$, as introduced in Chapter 1, forms a subset of the affine $n$-space $\mathbb{A}^n_k$, and the Zariski topology on $\mathbb{A}^n(k)$ is simply the topology induced from $\mathbb{A}^n_k$.

In general, the affine space $\mathbb{A}^n_k$ also contains many points which are not of the 'traditional' type. There can be maximal ideals with residue fields not isomorphic to $k$, and prime ideals which are not maximal (the zero ideal for instance). Hence, $\mathbb{A}^n_k$ is strictly larger than the previously defined $\mathbb{A}^n(k)$. The differences between $\mathbb{A}^n_k$ and $\mathbb{A}^n(k)$ become even more apparent if $k$ is not algebraically closed.

**Example 2.16** (The affine line)**.** The prime spectrum $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ is called the *affine line over $k$*. The polynomial ring $k[t]$ is a principal ideal domain, so the prime ideals are either of the form $(f(t))$ where $f(t)$ is an irreducible polynomial, or the zero ideal $(0)$. In the first case, the ideals are automatically maximal. Therefore, $\mathbb{A}^1_k$ has two types of points: the closed points and the generic point $\eta$.

When $k$ is algebraically closed, the maximal ideals are all of the form $(t - a)$ for $a \in k$, and their residue fields are isomorphic to $k$.

When $k$ is not algebraically closed, there will be other closed points in $\operatorname{Spec} k[t]$ besides the ones of the form $(t - a)$. An interesting example is when $k = \mathbb{R}$. By the Fundamental Theorem of Algebra, a non-zero prime ideal $\mathfrak{p}$ of $\mathbb{R}[t]$ is of the form $\mathfrak{p} = (f(t))$ where $f(t)$

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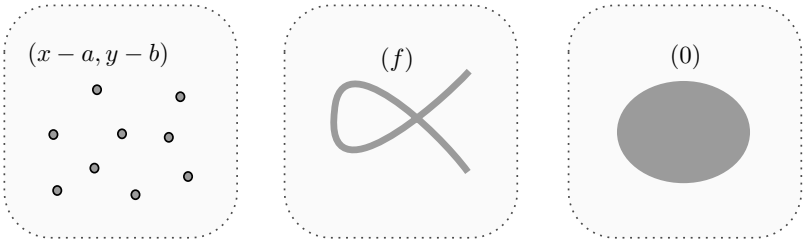
is either linear, that is, $f(t) = t - a$ for some $a \in \mathbb{R}$; or $f$ is quadratic with two conjugate complex non-real roots, that is, $f(t) = (t-a)(t-\bar{a})$ with $a \in \mathbb{C}$ but $a \notin \mathbb{R}$. The closed points in $\operatorname{Spec} \mathbb{R}[t]$ may therefore be identified with the set of pairs $\{a, \bar{a}\}$ with $a \in \mathbb{C}$. For instance, the maximal ideal $(t^2+1)$ corresponds to the pair $\{i, -i\}$. The residue field at this point is isomorphic to $\mathbb{R}[t]/(t^2+1) \simeq \mathbb{C}$.

In general, if a maximal ideal $\mathfrak{m}$ in $k[t]$ is generated by the irreducible polynomial $f(t)$, the residue field at the corresponding point in $\mathbb{A}^1_k$ is the extension of $k$ obtained by adjoining a root of $f$.

Affine spaces over non-algebraically closed fields can be quite complicated. Even the affine line $\mathbb{A}^1_{\mathbb{Q}}$ over $\mathbb{Q}$ is quite mysterious, as the monic irreducible polynomials in $\mathbb{Q}[t]$ have a very intricate structure. △

**Example 2.17** (The affine plane)**.** When $k$ is algebraically closed, one can show that the prime ideals of $k[x,y]$ come in three types: the maximal ideals, which are all of the form $(x-a, y-b)$ for $a, b \in k$; the prime ideals of the form $\mathfrak{p} = (f)$, where $f \in k[x,y]$ is an irreducible polynomial; and the zero ideal $(0)$ (see also Example 1.22).

Note that the point $(f)$ belongs to the closed set $V(f)$. In addition to this point, $V(f)$ contains the maximal ideals $(x-a, y-b)$ which contain $(f)$, or equivalently, $f(a,b) = 0$. Hence the points of $V(f)$ correspond to the points of the 'plane curve' defined by the equation $f(x,y) = 0$. △




## 2.3 Distinguished open sets

The Zariski topology of $\operatorname{Spec} A$ has a convenient basis consisting of particularly simple open sets. For an element $f \in A$, we define the *distinguished open set*, $D(f)$ to be the complement of the closed set $V(f)$, that is,

$$D(f) = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \,\} = \operatorname{Spec} A - V(f).$$

These open sets will turn out to be very important in computations involving the spectrum of a ring.

**Lemma 2.18.** For a ring $A$ and elements $f, g \in A$, we have
(i) $D(f) \cap D(g) = D(fg)$
(ii) $D(g) \subset D(f)$ if and only if some power of $g$ is a multiple of $f$, i.e., we may write $g^r = af$ for some $a \in A$, $r \in \mathbb{N}$.



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*Proof* (i): If $\mathfrak{p}$ is a prime ideal, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$ hold if and only if $fg \notin \mathfrak{p}$.

(ii): The inclusion $D(g) \subset D(f)$ holds if and only if $V(f) \subset V(g)$, and by Lemma 2.4 on page 27, this is equivalent to $(g) \subset \sqrt{(f)}$, i.e., $g^n \in (f)$ for some $n \in \mathbb{N}$. □

**Lemma 2.19.**
(i) The distinguished open sets $\{D(f)\}_{f\in A}$ form a basis for the topology of $\operatorname{Spec} A$.
(ii) A collection $\{D(f_i)\}_{i\in I}$ is an open cover of $\operatorname{Spec} A$ if and only if the $f_i$'s generate the unit ideal. Equivalently, there are finitely many indices $i_1, \dots, i_r \in I$ and $a_1, \dots, a_r \in A$ such that

$$1 = a_1 f_{i_1} + \cdots + a_r f_{i_r}. \tag{2.9}$$

*Proof* (i): We need to show that every open subset $U$ of $\operatorname{Spec} A$ can be written as the union of distinguished open sets. By definition of the Zariski topology, the complement of $U$ is of the form $U^c = V(\mathfrak{a})$ with $\mathfrak{a} \subset A$ an ideal. If we choose a (possibly infinite) set of generators $\{f_i\}_{i\in I}$ for $\mathfrak{a}$, then we have

$$U = V(\mathfrak{a})^c = V\big(\sum_{i\in I}(f_i)\big)^c = \big(\bigcap_{i\in I} V(f_i)\big)^c = \bigcup_{i\in I} D(f_i). \tag{2.10}$$

(ii): By the identity (2.10) with $U = \operatorname{Spec} A$, the collection of open sets $\{D(f_i)\}_{i\in I}$ covers $\operatorname{Spec} A$ if and only if $V(\sum_{i\in I}(f_i)) = \varnothing$. By Lemma 2.4, this in turn happens if and only if $\sum_{i\in I}(f_i) = (1)$, or in other words, the $f_i$ generate the unit ideal. But this happens if and only if 1 can be expressed as a combination of finitely many of the $f_i$'s. □

In algebraic geometry, a topological space is said to be *quasi-compact* if every open covering has a finite subcover. (This terminology is a bit unfortunate, as spaces with this property are usually called 'compact', but the jargon has become standard.)

**Corollary 2.20.** $\operatorname{Spec} A$ is quasi-compact.

*Proof* Let $\{U_i\}_{i\in I}$ be any open cover of $\operatorname{Spec} A$. Then each $U_i$ can be covered by distinguished open sets $D(f_{ij})$ with $f_{ij} \in A$. Since the $D(f_{ij})$'s cover $\operatorname{Spec} A$, finitely many of them will suffice, by Lemma 2.19. As each $U_i$ contains $D(f_{ij})$, we can cover $\operatorname{Spec} A$ with finitely many $U_i$ as well. □

**Example 2.21.** In the affine line $\mathbb{A}^1_k$ over a field, every closed set is of the form $V(f)$ for some polynomial $f$, so every open set is a distinguished open set $D(f)$.

In $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, the set $U = \mathbb{A}^2_k - V(x, y)$ is open, but not of the form $D(f)$. Still, we have $U = D(x) \cup D(y)$. △

**Example 2.22** (The circle)**.** Consider the spectrum $X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$. The maximal ideal $\mathfrak{m} = (x, y - 1)$ defines the point $(0, 1)$ on $X$. Even though the ideal $\mathfrak{m}$ is not a principal ideal (Exercise 2.7.10), the complement $X - \{\mathfrak{m}\}$ *is* a distinguished open set. In fact, it is equal to $D(y - 1)$ because modulo the relation $x^2 = 1 - y^2$, we have

$$\mathfrak{m}^2 = (x^2, x(y - 1), (y - 1)^2) = (y - 1). \tag{2.11}$$

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△

## 2.4 Irreducibility and connectedness

Recall from Chapter 1 that a topological space $X$ is *irreducible* if it cannot be written as the union of two proper closed subsets. From Proposition 1.11, we know that the coordinate ring of an affine variety is an integral domain, and simple examples (like Example 2.28 below) indicate that reducibility of $\operatorname{Spec} A$ is closely linked to zerodivisors in $A$. In general, one has the following:

**Proposition 2.23.** Let $A$ be a ring.

(i) A closed subset $Z \subset \operatorname{Spec} A$ is irreducible if and only if $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$.

(ii) The space $\operatorname{Spec} A$ itself is irreducible if and only if $A$ has just one minimal prime ideal, or equivalently, the nilradical $\sqrt{(0)}$ is prime.

*Proof* (i): By Proposition 2.5 we have $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, and this is irreducible, being the closure of a singleton. For the opposite implication, let $V(\mathfrak{a}) \subset \operatorname{Spec} A$ be a closed subset. Recall that $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}$. If $\sqrt{\mathfrak{a}}$ is not prime, there must be more than one prime involved in the intersection. We may divide them into two different groups, thereby representing $\sqrt{\mathfrak{a}}$ as an intersection $\sqrt{\mathfrak{a}} = \mathfrak{b} \cap \mathfrak{b}'$ where $\mathfrak{b}$ and $\mathfrak{b}'$ are the intersections of the primes in the two groups, and hence are different radical ideals. From this, we get that $V(\mathfrak{a}) = V(\mathfrak{b}) \cup V(\mathfrak{b}')$, and $V(\mathfrak{a})$ is not irreducible.

(ii): The space $\operatorname{Spec} A$ is irreducible if and only if it is of the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ by part (i). Since $\operatorname{Spec} A = V((0))$, this is equivalent to requiring that $\sqrt{(0)}$ is a prime ideal.

The nilradical is the intersection of all prime ideals $\sqrt{(0)} = \bigcap_{\mathfrak{p}} \mathfrak{p}$, and the intersection can be reduced to just the minimal prime ideals. If $A$ has only one minimal prime ideal $\mathfrak{q}$, then $\sqrt{(0)} = \mathfrak{q}$ is prime. Conversely, if $\mathfrak{q}_1, \mathfrak{q}_2$ are distinct minimal primes, then $\sqrt{(0)} \subset \mathfrak{q}_1 \cap \mathfrak{q}_2$, and $\sqrt{(0)}$ could not be prime (as it would be properly contained in both $\mathfrak{q}_1$ and $\mathfrak{q}_2$). □

A consequence of the proposition is that $\operatorname{Spec} A$ is irreducible whenever $A$ is an integral domain, as in that case $(0)$ is a minimal prime ideal. However, $\operatorname{Spec} A$ may certainly be irreducible for other rings as well. For example, the ring $A = \mathbb{C}[t]/(t^2)$ is not an integral domain, and yet has only one prime ideal, namely the principal ideal $(t)$. By part (ii) tells us that this example is typical for such rings: every zerodivisor in the ring is nilpotent. In the spirit of the analogy with functions, there are non-zero functions which vanish everywhere: the element $t$ is nonzero in $A$, but it becomes zero in the residue field $\mathbb{C}[t]/(t) \simeq \mathbb{C}$.

**Example 2.24.** The spectrum $\operatorname{Spec} \mathbb{C}[x, y]/(y^2 + x^5 + 1)$ is irreducible, as the polynomial $y^2 + x^5 + 1$ is irreducible, and hence generates a prime ideal. △

**Example 2.25.** The spectrum $\operatorname{Spec} \mathbb{C}[x, y, z, w]/(x^2, xy, y^2, xw - yz)$ is irreducible. Indeed, the nilradical is given by $\sqrt{(0)} = (x, y)$, and this is a prime ideal. △

Recall that a topological space is *connected* if it cannot be written as a disjoint union of

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two proper open subsets. All of the examples we have seen until now, with the exception of Example 2.13, are connected.

**Example 2.26** (A disconnected spectrum)**.** Suppose that $A = A_1 \times A_2$ is the direct product of two non-trivial rings $A_1$ and $A_2$. In $A$ we have the two *orthogonal idempotents* $e_1 = (1, 0)$ and $e_2 = (0, 1)$; they satisfy the relations $e_1^2 = e_1$, $e_1 e_2 = 0$, $e_2^2 = e_2$ and $e_1 + e_2 = 1$.

The spectrum $\operatorname{Spec} A$ decomposes as the disjoint union $\operatorname{Spec} A = V(e_1) \cup V(e_2)$ of the two closed sets $V(e_i)$. Indeed, since $e_1 + e_2 = 1$, it holds that $V(e_1) \cap V(e_2) = V(e_1, e_2) = \varnothing$. Since $e_1 e_2 = 0$, either $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$ for each prime $\mathfrak{p} \in \operatorname{Spec} A$, so the $V(e_i)$'s cover $\operatorname{Spec} A$. △

In fact, there is a converse to this example.

**Proposition 2.27.** A spectrum $\operatorname{Spec} A$ is disconnected if and only if $A$ is isomorphic to a direct product $A = A_1 \times A_2$ of non-trivial rings $A_1$ and $A_2$.

While it would certainly be possible to give a direct proof of this proposition at the present stage, we will wait until the next chapter to do so. There, we will see a much more conceptual proof using the structure sheaf (see Example 3.26 on page 57). For reduced rings however, the argument is straightforward (see Exercise 2.7.5).

**Example 2.28.** Note that any irreducible space is also connected. The converse does not hold: the spectrum $X = \operatorname{Spec} k[x, y]/(xy)$ is connected but not irreducible. The coordinate functions $x$ and $y$ are zerodivisors in the ring $k[x, y]/(xy)$, and their zero-sets $V(x)$ and $V(y)$ show that $X$ has two components. Since these two components intersect at the origin, $X$ is connected. △

## 2.5 Maps between prime spectra

Let $A$ and $B$ be two rings and let $\phi\colon A \to B$ be a ring map between them. The inverse image $\phi^{-1}(\mathfrak{p})$ of a prime ideal $\mathfrak{p} \subset B$ is a prime ideal: that $ab \in \phi^{-1}(\mathfrak{p})$ means that $\phi(ab) = \phi(a)\phi(b) \in \mathfrak{p}$, so at least one of $\phi(a)$ or $\phi(b)$ has to lie in $\mathfrak{p}$. This means that there is an induced map of spectra

$$\begin{aligned} \operatorname{Spec}(\phi)\colon \operatorname{Spec} B &\longrightarrow \operatorname{Spec} A \\ \mathfrak{p} &\mapsto \phi^{-1}(\mathfrak{p}). \end{aligned} \tag{2.12}$$

This map is continuous in the Zariski topology, because preimages of closed sets are closed; this follows from item (i) in the next proposition.

**Proposition 2.29.** $\phi\colon A \to B$ be a ring map and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced map of spectra. Then

(i) $f^{-1}V(\mathfrak{a}) = V(\phi(\mathfrak{a})B)$ for each ideal $\mathfrak{a} \subset A$.
(ii) $f^{-1}D(g) = D(\phi(g))$ for each $g \in A$.
(iii) $\overline{f(V(\mathfrak{b}))} = V(\phi^{-1}(\mathfrak{b}))$ for each ideal $\mathfrak{b}$ of $B$.

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*Proof* (i): Let $\mathfrak{a} \subset A$ be an ideal. Then we have the following equalities:

$$f^{-1}V(\mathfrak{a}) = \{\, \mathfrak{p} \subset B \mid \mathfrak{a} \subset \phi^{-1}(\mathfrak{p}) \,\} = \{\, \mathfrak{p} \subset B \mid \phi(\mathfrak{a}) \subset \mathfrak{p} \,\} = V(\phi(\mathfrak{a})B).$$

(ii): For each element $g \in A$, we have

$$f^{-1}D(g) = \{\, \mathfrak{p} \subset B \mid g \notin \phi^{-1}(\mathfrak{p}) \,\} = \{\, \mathfrak{p} \subset B \mid \phi(g) \notin \mathfrak{p} \,\} = D(\phi(g)).$$

(iii): By Corollary 2.5, the closure $\overline{f(V(\mathfrak{b}))}$ equals $V(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal

$$\mathfrak{a} = \bigcap_{\mathfrak{p} \in f(V(\mathfrak{b}))} \mathfrak{p} = \bigcap_{\mathfrak{b} \subset \mathfrak{q}} \phi^{-1}(\mathfrak{q}).$$

Here the second equality holds because $\mathfrak{p} \in f(V(\mathfrak{b}))$ implies that $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$ for some $\mathfrak{q}$ with $\mathfrak{b} \subset \mathfrak{q}$. So we get that

$$\mathfrak{a} = \bigcap_{\mathfrak{b} \subset \mathfrak{q}} \phi^{-1}(\mathfrak{q}) = \phi^{-1}\Big(\bigcap_{\mathfrak{b} \subset \mathfrak{q}} \mathfrak{q}\Big) = \phi^{-1}(\sqrt{\mathfrak{b}}) = \sqrt{\phi^{-1}(\mathfrak{b})}.$$

Hence $V(\mathfrak{a}) = V(\phi^{-1}(\mathfrak{b}))$, which gives the desired identity. □

The following propositions gives prototypical examples of induced maps on spectra.

**Proposition 2.30.** For an ideal $\mathfrak{a} \subset A$, the quotient map $A \to A/\mathfrak{a}$ induces a homeomorphism

$$f\colon \operatorname{Spec}(A/\mathfrak{a}A) \xrightarrow{\simeq} V(\mathfrak{a}) \subset \operatorname{Spec} A$$

*Proof* If $\phi\colon A \to A/\mathfrak{a}$ denotes the quotient map, the map $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ gives an inclusion preserving one-to-one correspondence between the set of prime ideals in $A/\mathfrak{a}$ and the set of prime ideals in $A$ containing $\mathfrak{a}$, with inverse given by $\mathfrak{q} \mapsto \mathfrak{q}/\mathfrak{a}$. This shows that $f$ is a continuous bijection onto the closed subset $V(\mathfrak{a})$. To show that $f$ is a homeomorphism, it suffices to show that it is closed, and this follows from the equalities

$$f(V(\mathfrak{b}/\mathfrak{a})) = \big\{\mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{b}/\mathfrak{a} \subset \mathfrak{p}/\mathfrak{a} \in \operatorname{Spec}(A/\mathfrak{a})\big\} = V(\mathfrak{b}).$$ □

By the proposition, if the ring map $\phi\colon A \to B$ is surjective, then $\operatorname{Spec} B$ maps homeomorphically onto the closed subset $V(\operatorname{Ker}\phi) \subset \operatorname{Spec} A$.

In the special case where $\mathfrak{a} = \sqrt{(0)}$ is the nilradical of $A$, we have $V(\mathfrak{a}) = \operatorname{Spec} A$, and the proposition implies the following:

**Corollary 2.31.** Let $A$ be a ring and let $A_{\mathrm{red}} = A/\sqrt{(0)}$ be the reduction of $A$. Then the map $A \to A_{\mathrm{red}}$ induces a homeomorphism

$$\operatorname{Spec}(A_{\mathrm{red}}) \simeq \operatorname{Spec}(A).$$

We next consider the maps of spectra induced by localization maps. The next result implies in particular that distinguished open sets $D(f)$ can themselves be viewed as prime spectra. This fact will be important later on.

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**Proposition 2.32.**

(i) For any $f \in A$, there is a canonical homeomorphism

$$\iota\colon \operatorname{Spec}(A_f) \xrightarrow{\simeq} D(f) \subset \operatorname{Spec} A.$$

(ii) More generally, if $S \subset A$ is a multiplicative subset, the localization map $\ell\colon A \to S^{-1}A$ induces a homeomorphism

$$\iota\colon \operatorname{Spec}(S^{-1}A) \xrightarrow{\simeq} D \subset \operatorname{Spec} A,$$

where $D = \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \cap S = \varnothing \,\}$.

*Proof* It suffices to prove the statement (ii). Let $\mathfrak{q} \subset S^{-1}A$ be a prime ideal. As $\mathfrak{q}$ does not contain 1, we see that $\ell^{-1}(\mathfrak{q})$ does not contain any elements of $S$. In particular, $\ell^{-1}(\mathfrak{q}) \in D$. Conversely, if $\mathfrak{p} \subset A$ is a prime ideal with $\mathfrak{p} \cap S = \varnothing$, that is $\mathfrak{p} \in D$, then $\mathfrak{p}S^{-1}A$ is a prime ideal in $S^{-1}A$, and we have $\mathfrak{p} = \ell^{-1}(\mathfrak{p}S^{-1}A)$. From this it follows that $\iota$ is continuous, injective, with image $D$. To show that $\iota$ is a homeomorphism, it is enough to show that it is closed. This follows from the following equalities:

$$\begin{aligned}\iota(V(\mathfrak{a})) &= \{\, \ell^{-1}(\mathfrak{q}) \mid \mathfrak{q} \supset \mathfrak{a} \,\} \\ &= \{\, \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supset \ell^{-1}(\mathfrak{a}) \text{ and } \mathfrak{p} \cap S = \varnothing \,\} \\ &= V(\ell^{-1}(\mathfrak{a})) \cap D.\end{aligned}$$

□

For two distinguished open sets with $D(g) \subset D(f)$, we may write $g^r = cf$ for some $c \in A$ and $r \in \mathbb{N}$ (statement (ii) of Lemma 2.18). This relation implies that $f$ becomes invertible in $A_g$. This means that the localization map $A \to A_g$ factors uniquely as $A \to A_f \to A_g$. Explicitly, the right-most map is given by

$$A_f \longrightarrow A_g; \quad a/f^n \mapsto ac^n/g^{rn}. \tag{2.13}$$

This ring map identifies $D(g)$ with the distinguished open set $D(g/1)$ in $\operatorname{Spec}(A_f)$, where $g/1$ denotes the image of $g$ in $A_f$.

Note also that $D(g) = D(g^r) = D(cf)$. This shows that any distinguished open set contained in $D(f)$ is of the form $D(fg)$ for some $g \in A$.

### *Examples*

**Example 2.33** (Reduction modulo a prime $p$)**.** The reduction mod $p$-map $\mathbb{Z} \to \mathbb{F}_p$ induces a map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$. The one and only point in $\operatorname{Spec} \mathbb{F}_p$ is sent to the point in $\operatorname{Spec} \mathbb{Z}$ corresponding to the maximal ideal $(p)$.

Likewise, the inclusion $\mathbb{Z} \subset \mathbb{Q}$ induces a map $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$. It sends the unique point in $\operatorname{Spec} \mathbb{Q}$ to the generic point $\eta$ of $\operatorname{Spec} \mathbb{Z}$. △

**Example 2.34** (The twisted cubic curve)**.** Let $k$ be a field. The ring map $\phi\colon k[x, y, z] \to k[t]$ given by $x \mapsto t, y \mapsto t^2, z \mapsto t^3$ correspondes to a map of prime spectra

$$f\colon \mathbb{A}^1_k \longrightarrow \mathbb{A}^3_k.$$

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As $\phi$ is surjective, $f$ is a homeomorphism onto the closed set defined by $\mathfrak{a} = \operatorname{Ker}\phi = (y - x^2, z - x^3)$. This is the scheme version of the twisted cubic curve from Example 1.33. △

**Example 2.35.** Let $k$ be a field. The natural inclusion $\phi\colon k[x] \to k[x,y]/(xy-1)$ induces a morphism

$$\operatorname{Spec} k[x,y]/(xy-1) \longrightarrow \mathbb{A}^1_k = \operatorname{Spec} k[x].$$

On the level of closed points, when $k$ is algebraically closed, this maps $(a, a^{-1})$ to $a$. Since $k[x,y]/(xy-1)$ is an integral domain, $\operatorname{Spec} k[x,y]/(xy-1)$ has a unique generic point $\eta$, and this is mapped to the generic point of $\mathbb{A}^1_k$. Note that $\operatorname{Spec} k[x,y]/(xy-1) \simeq D(x) \subset \mathbb{A}^1_k$ via this morphism. In particular, the image is dense, but not closed in $\mathbb{A}^1_k$. △

## 2.6 Fibers

To understand a map of spectra $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$, it is often useful to understand the *fibers* of $f$, that is, the inverse images of points. If $f$ is induced by a ring map $\phi\colon A \to B$ and $y \in \operatorname{Spec} A$ is a point corresponding to $\mathfrak{p}$ in $A$, the fiber $f^{-1}(y)$ consists of the primes $\mathfrak{q}$ in $B$ such that $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$.

If $y \in \operatorname{Spec} A$ is a closed point, so that $\mathfrak{p}$ is a maximal ideal, then $\{y\} = \overline{\{y\}} = V(\mathfrak{p})$, so Proposition 2.29 implies that $f^{-1}V(\mathfrak{p}) = V(\mathfrak{p}B)$. In particular, the fiber $f^{-1}(y)$ is a closed set, homeomorphic to $\operatorname{Spec}(B/\mathfrak{p}B)$.

If $y$ is not closed, the fiber $f^{-1}(y)$ may or may not be closed in $\operatorname{Spec} B$. While we still have $f^{-1}V(\mathfrak{p}) = V(\mathfrak{p}B)$, this closed set may contain other primes than the ones mapping to $y$. For instance, in the sitation when $\mathfrak{p} = (0)$ is prime in $A$, then $V(\mathfrak{p}B) = \operatorname{Spec} B$.

To describe the fiber, we consider the localization $B_{\mathfrak{p}} = S^{-1}B$ in the multiplicative set

$$S = \phi(A - \mathfrak{p}).$$

The idea is that the 'extra primes' in $V(\mathfrak{p}B)$ which do not map to $\mathfrak{p}$ will localize to the unit ideal in $B_{\mathfrak{p}}$.

Consider the composition

$$\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \longrightarrow \operatorname{Spec}(B_{\mathfrak{p}}) \longrightarrow \operatorname{Spec}(B), \tag{2.14}$$

which is induced by the localization map $B \to B_{\mathfrak{p}}$ and the quotient map $B_{\mathfrak{p}} \to B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$.

**Proposition 2.36.** Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a map between prime spectra and let $\mathfrak{p} \in \operatorname{Spec} A$. Then the composition (2.14) induces a homeomorphism

$$f^{-1}(\mathfrak{p}) \simeq \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}).$$

If $\mathfrak{p}$ is a closed point, then $f^{-1}(\mathfrak{p})$ is homeomorphic to $\operatorname{Spec}(B/\mathfrak{p}B)$.

*Proof* The first map is a homeomorphism onto the subset $V(\mathfrak{p}B_{\mathfrak{p}})$ (Proposition 2.29), and the second is a homeomorphism onto the set of primes $\mathfrak{q} \in \operatorname{Spec} B$ disjoint from $S = \phi(A - \mathfrak{a})$ (Proposition 2.32). Therefore the composition is a homeomorphism onto the set of prime ideals $\mathfrak{q} \in \operatorname{Spec} B$ such that $\mathfrak{q} \cap S = \varnothing$ and $\mathfrak{q}B_{\mathfrak{p}} \supset \phi(\mathfrak{p})B_{\mathfrak{p}}$. The second

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equality implies that $\mathfrak{q} \supset \phi(\mathfrak{p})$ and hence $\phi^{-1}(\mathfrak{q}) \supset \mathfrak{p}$. The first equality then implies that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$ (if $\phi^{-1}(\mathfrak{q}) \supset \mathfrak{p}$ were strict, then elements in $\phi^{-1}(\mathfrak{q}) - \mathfrak{p}$ would map to elements of $\mathfrak{q} \cap S$). Therefore, the image of (2.14) consists of precisely those prime ideals $\mathfrak{q}$ in $B$ so that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, that is, $f(\mathfrak{q}) = \mathfrak{p}$. □

In computations involving fibers, we will frequently use the fact that localization commutes with taking quotients, i.e., $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} \simeq S^{-1}(B/\mathfrak{p})$.

**Example 2.37.** Consider the map

$$f\colon \operatorname{Spec} \mathbb{C}[x,y] \longrightarrow \operatorname{Spec} \mathbb{C}[z],$$

induced by the ring map $\mathbb{C}[z] \to \mathbb{C}[x,y$ defined by $z \mapsto xy$. Let us compute the fibers $f^{-1}(\mathfrak{p})$ over the maximal ideals $\mathfrak{p} = (z-a)$. Note that

$$B/\mathfrak{p}B = \mathbb{C}[x,y]/(xy-a).$$

There are two cases. If $a \neq 0$, then $xy - a$ is an irreducible polynomial, and then $\operatorname{Spec}(B/\mathfrak{p}B)$ is irreducible. This is intuitive, as it corresponds to the hyperbola $V(xy-a)$ in $\mathbb{A}^2_{\mathbb{C}}$. If $a = 0$, we are left with $\operatorname{Spec} \mathbb{C}[x,y]/(xy)$, which is not irreducible. It has two components, the two coordinate axes $V(x)$ and $V(y)$.

Let us also consider the fiber over the generic point $\eta$ of $\operatorname{Spec} \mathbb{C}[z]$, which corresponds to $\mathfrak{p} = (0)$. In this case, the ring $(B/\mathfrak{p}B)_{\mathfrak{p}}$ is the localization of $B$ with respect to the multiplicative set $S = \mathbb{C}[z] - \{0\}$; that is, the ring

$$\mathbb{C}(z)[x,y]/(xy-z).$$

This is again an integral domain, so the fiber $f^{-1}(\eta)$ is irreducible. This fiber may be regarded as a hyperbola in the affine plane $\mathbb{A}^2_{\mathbb{C}(z)}$ over the field $\mathbb{C}(z)$. △

**Example 2.38** (Double covers)**.** Let $k$ be a field of characteristic different from 2 and let $g \in k[x_1, \ldots, x_n]$ be a non-zero polynomial. Consider the spectrum

$$X = \operatorname{Spec} k[x_1, \ldots, x_n, y]/(y^2 - g).$$

The inclusion $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/(y^2 - g)$ induces a map

$$f\colon X \longrightarrow \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n].$$

We call $X$ the *double cover* of $\mathbb{A}^n_k$ associated to $g$.

The name 'double cover' comes from the fibers of $f$. For a point $a = (a_1, \ldots, a_n) \in k^n$, let $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$ be the corresponding maximal ideal. Then the fiber $f^{-1}(\mathfrak{m})$ is the spectrum of the ring

$$\frac{k[x_1, \ldots, x_n, y]}{(y^2 - g, x_1 - a_1, \ldots, x_n - a_n)} \simeq \frac{k[y]}{(y^2 - g(a))}.$$

There are three cases, depending on the value $b = g(a)$:

(i) If $b \neq 0$ and $b$ is a square in $k$, say $b = c^2$, then the polynomial $y^2 - b$ factors as $(y-c)(y+c)$. By the Chinese Remainder Theorem, the fiber is

$$\operatorname{Spec}\Big(\frac{k[y]}{(y-c)} \times \frac{k[y]}{(y+c)}\Big),$$

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which is the disjoint union of two copies of $\operatorname{Spec} k$.

(ii) If $b \neq 0$, but $b$ is not a square in $k$, then the fiber equals $\operatorname{Spec} k(\sqrt{b})$, where $k(\sqrt{b})$ is a quadratic field extension of $k$. This fiber is a single point, but with 'multiplicity 2' (in the sense that the degree of the field extension $k \subset k(\sqrt{b})$ is 2).

(iii) If $b = 0$, the fiber is then to $\operatorname{Spec} k[y]/(y^2)$, which is single point. Again there is a 'multiplicity 2', accounted for by the presence of nilpotent elements in the ring ($k[y]/(y^2)$ has dimension 2 as a $k$-vector space).

△

**Example 2.39** (The Möbius strip)**.** Let $A = \mathbb{R}[x,y]/(x^2+y^2-1)$ and consider the $A$-algebra $B = A[u,v]/(vx-uy)$. This induces a map

$$f\colon \operatorname{Spec} A[u,v]/(vx-uy) \longrightarrow \operatorname{Spec} A.$$

Let us compute the fibers of $f$. As $x^2+y^2=1$ in $A$, $x$ and $y$ generate the unit ideal, and so $\operatorname{Spec} A$ is covered by the two affine subsets $D(x)$ and $D(y)$. If $\mathfrak{p} \in D(x)$, then $x$ is invertible in $A_{\mathfrak{p}}$, and so we compute

$$B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = (A_{\mathfrak{p}}/\mathfrak{p})[u,v]/(v-x^{-1}uy) \simeq \kappa(\mathfrak{p})[u],$$

and the fiber is homeomorphic to $\mathbb{A}^1_{\kappa(\mathfrak{p})}$. A similar argument works when $\mathfrak{p} \in D(y)$. Hence all fibers are isomorphic to affine lines. △

**Example 2.40.** The inclusion $\mathbb{R}[t] \subset \mathbb{C}[t]$ induces a map from the affine line over $\mathbb{C}$ to the affine line over $\mathbb{R}$:

$$\pi\colon \operatorname{Spec} \mathbb{C}[t] \longrightarrow \operatorname{Spec} \mathbb{R}[t].$$

By Example 2.16 there are three cases to consider for the fiber $\pi^{-1}(y)$ of a point $y \in \mathbb{A}^1_{\mathbb{R}}$.

(i) The point $y$ corresponds to the maximal ideal $(t-a)$ with $a \in \mathbb{R}$. Then the fiber is given by

$$\pi^{-1}(y) = \operatorname{Spec}\left(\mathbb{C}[t]/(t-a)\right) \simeq \operatorname{Spec} \mathbb{C},$$

and the fiber is a single closed point with residue field $\mathbb{C}$.

(ii) The point $y$ is a closed point corresponding to $\mathfrak{p} = (f(t))$ where $f \in \mathbb{R}[t]$ has two conjugate complex roots $a, \bar{a}$. Then

$$\pi^{-1}(y) = \operatorname{Spec} \mathbb{C}[t]/(f(t)) \simeq \operatorname{Spec}\left(\frac{\mathbb{C}[t]}{(t-a)} \times \frac{\mathbb{C}[t]}{(t-\bar{a})}\right).$$

Hence the fiber consists of two closed points, both with residue field $\mathbb{C}$.

(iii) The point $y$ equals the generic point $\eta$. Then $\pi^{-1}(\eta)$ is the spectrum of the localization $S^{-1}\mathbb{C}[t] = \mathbb{C}(t)$ where $S = \mathbb{R}[t] - (0)$. Therefore, $\pi^{-1}(\eta)$ consists of a single point with residue field $\mathbb{C}(t)$.

The Galois group $G = \operatorname{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2$ acts on the fibers in this example. More precisely, consider the conjugation map $\mathbb{C}[t] \to \mathbb{C}[t]$ on the polynomial ring $\mathbb{C}[t]$ given by conjugating the coefficients of the polynomials; that is, sending a polynomial $\sum_i a_i t^i$ to $\sum_i \bar{a}_i t^i$. This defines an automorphism

$$\iota\colon \operatorname{Spec} \mathbb{C}[t] \longrightarrow \operatorname{Spec} \mathbb{C}[t].$$

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Note that the subring $\mathbb{R}[t] \subset \mathbb{C}[t]$ is fixed by the conjugation map, so the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C}[t] & \xrightarrow{\quad\iota\quad} & \operatorname{Spec}\mathbb{C}[t] \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \operatorname{Spec}\mathbb{R}[t] & \end{array}$$

Therefore, $G = \langle \mathrm{id}, \iota \rangle \simeq \mathbb{Z}/2$ acts by automorphisms on the fibers of $\pi$, and $\operatorname{Spec}\mathbb{R}[t]$ can be viewed as the quotient space of $\operatorname{Spec}\mathbb{C}[t]$ by $G$; i.e. the space of $G$-orbits. Indeed, by Example 2.16, the closed points of $\operatorname{Spec}\mathbb{R}[t]$ correspond exactly to the orbits of $G$ and the generic point of $\operatorname{Spec}\mathbb{C}[t]$ is invariant and corresponds to the generic point of $\operatorname{Spec}\mathbb{R}[t]$. △

**Example 2.41** (The Gaussian integers)**.** The inclusion $\mathbb{Z} \subset \mathbb{Z}[i]$ induces a morphism

$$f\colon \operatorname{Spec}\mathbb{Z}[i] \longrightarrow \operatorname{Spec}\mathbb{Z}.$$

We will study $\operatorname{Spec}\mathbb{Z}[i]$ by studying the fibers of this map. If $p \in \mathbb{Z}$ is a prime, the fiber over $(p) \in \operatorname{Spec}\mathbb{Z}$ is given by $V((p)\mathbb{Z}[i])$. This is homeomorphic to the spectrum

$$\operatorname{Spec}\left(\mathbb{Z}[i]/p\right) \simeq \operatorname{Spec}\left(\frac{\mathbb{F}_p[x]}{x^2+1}\right).$$

The ring $\mathbb{F}_p[x]/(x^2+1)$ is not an integral domain precisely when the equation $x^2+1=0$ has a solution modulo $p$. This happens[1] if and only if $p=2$ or $p \equiv 1 \mod 4$. By Fermat's Two Square Theorem, this is equivalent to $p = a^2+b^2$ being a sum of two squares, or in other words, that $p=(a+ib)(a-ib)$ factors in $\mathbb{Z}[i]$. This means that there are three cases:

(i) $p=2$. Then $(2)\mathbb{Z}[i] = (1+i)^2\mathbb{Z}[i]$. Therefore, $f^{-1}((2))$ consists of the single point corresponding to the prime ideal $(1+i)$.

(ii) $p \equiv 1 \pmod 4$. In this case, we may factor $p=(a+ib)(a-ib)$ in $\mathbb{Z}[i]$ and so the fiber consists of the two points with residue fields $\mathbb{F}_p$,

$$f^{-1}((p)) = V((p)\mathbb{Z}[i]) = \{(a+ib), (a-ib)\}.$$

(iii) $p \equiv 3 \pmod 4$. Then the ideal $(p)$ stays prime in $\mathbb{Z}[i]$ and the fiber $f^{-1}((p))$ consists of the single point $x$ corresponding to $(p)\mathbb{Z}[i]$. The residue field $\kappa(x)$ is in fact isomorphic to $\mathbb{F}_p[x]/(x^2+1) \simeq \mathbb{F}_{p^2}$, the field with $p^2$ elements.

Note that the complex conjugation map $\sigma\colon \mathbb{Z}[i] \to \mathbb{Z}[i]$ acts on $\operatorname{Spec}\mathbb{Z}[i]$ and the fibers of the map $f$. For instance, the primes sitting over $(5)$ are $(2+i)$ and $(2-i)$, and complex conjugation sends one to the other. We therefore picture $\operatorname{Spec}\mathbb{Z}[i]$ as a curve lying above $\operatorname{Spec}\mathbb{Z}$, with $\sigma$ permuting the points in each fiber

△

**Example 2.42** (The affine line $\mathbb{A}^1_{\mathbb{Z}}$)**.** Consider the affine line $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[t]$ and the morphism $f\colon \operatorname{Spec}\mathbb{Z}[t] \to \operatorname{Spec}\mathbb{Z}$ induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}[t]$.

There are two cases for a fiber $f^{-1}(y)$ of a point $y \in \operatorname{Spec}\mathbb{Z}$. If $y$ corresponds to the closed

[1] Here is a quick proof for $p \neq 2$: The elements $\{1, \dots, p-1\}$ modulo $p$ can be partitioned into subsets of the form $C(x) = \{x, -x, x^{-1}, -x^{-1}\}$. If $|C(x)| \neq 4$, then either $x = x^{-1}$ (meaning $C(x) = \{1, -1\}$); or $x = -x^{-1}$ (meaning $C(x) = \{x, -x\}$ consists of the two solutions to $x^2+1 = 0 \mod p$). This implies that $p-1 = \sum |C(x)|$ is conguent to 2 mod 4 if $x^2+1=0$ has no solutions, and to 0 mod 4 otherwise.

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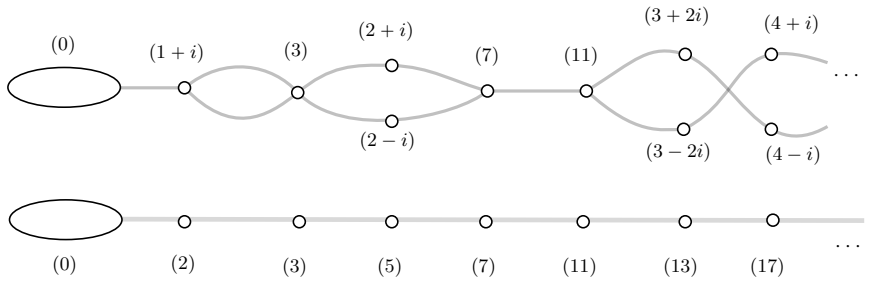


**Figure 2.3** The spectrum $\operatorname{Spec}\mathbb{Z}[i]$

point $(p) \in \operatorname{Spec}\mathbb{Z}$, the fiber $f^{-1}(y)$ consists of all primes $\mathfrak{p} \subset \mathbb{Z}[t]$ such that $\mathfrak{p} \cap \mathbb{Z} = (p)$. According to Proposition 2.36, this is given by

$$V((p)\mathbb{Z}[t]) = \operatorname{Spec}(\mathbb{Z}[t]/(p)\mathbb{Z}[t]) = \mathbb{A}^1_{\mathbb{F}_p}.$$

Likewise, if $y = \eta$ is the generic point of $\operatorname{Spec}\mathbb{Z}$, Proposition 2.36 tells us that the fiber $f^{-1}(\eta)$ is the spectrum of the localization $S^{-1}\mathbb{Z}[t] = \mathbb{Q}[t]$, where $S = \mathbb{Z} - (0)$. In other words,

$$f^{-1}(\eta) = \operatorname{Spec}\mathbb{Q}[t] = \mathbb{A}^1_{\mathbb{Q}}.$$

The affine line $\mathbb{A}^1_{\mathbb{Z}}$ is shown in Figure **??**. In the figure, we have depicted the two closed sets $V(6x+1)$ and $V(x^2+3)$. Note that $V(6x+1)$ is disjoint from the fibers above the primes 2 and 3 (why?). The closed subset $V(x^2+3)$ should be compared to Example 2.41). △

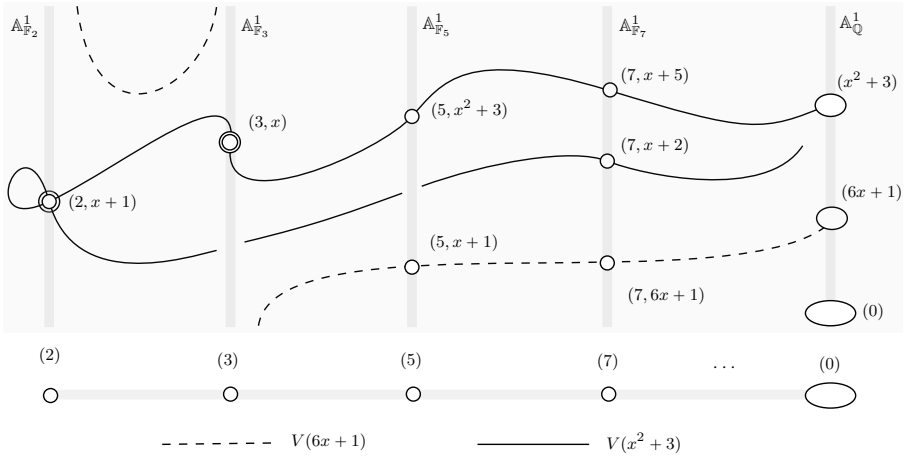


**Figure 2.4** The affine line $\operatorname{Spec}\mathbb{Z}[x]$

## 2.7 Exercises

**Exercise 2.7.1.** Show that $\operatorname{Spec} A$ is the empty set if and only if $A$ is the zero ring.

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**Exercise 2.7.2.** Describe $\operatorname{Spec} \mathbb{Z}[\frac{1}{255}]$.

**Exercise 2.7.3.** Consider the discrete valuation ring $R = \mathbb{C}[u]_{(u)}$. Describe $\operatorname{Spec} A[t]$ and the map $\operatorname{Spec} A[t] \to \operatorname{Spec} A$ induced by the inclusion $A \subset A[t]$. Which fibers are closed?

**Exercise 2.7.4.** Let $\mathfrak{p}$ be a prime ideal in a ring $A$. Show that there is a canonical inclusion $A/\mathfrak{p} \hookrightarrow A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and that this yields an identification of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ with the fraction field of $A/\mathfrak{p}$.

**Exercise 2.7.5** (Disconnected spectra)**.**

a) Show that a topological space $X$ is disconnected if and only if it has two *orthogonal idempotents*, i.e., continuous maps $e_1, e_2 \colon X \to \{0,1\}$ such that $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 e_2 = 0$ and $e_1 + e_2 = 1$. HINT: Try the characteristic functions of two open sets.

b) Let $A$ be a reduced ring. Show that $\operatorname{Spec} A$ is disconnected if and only if $A = A_1 \times A_2$ for two non-trivial rings $A_1$ and $A_2$. HINT: Find two ideals $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{a} + \mathfrak{b} = A$ and $\mathfrak{a} \cap \mathfrak{b} = (0)$. Use this to find two non-trivial idempotents.

**Exercise 2.7.6.** Let $A$ be an Artinian ring. Show that $X = \operatorname{Spec} A$ is a finite set, and the topology is the discrete topology. HINT: Write $A$ as a product of its localizations.

**Exercise 2.7.7.** With the notation of Proposition 2.32, show that the set $D$ can be written

$$D = \{\, \mathfrak{p} \mid s(\mathfrak{p}) \neq 0 \text{ in } \kappa(\mathfrak{p}) \text{ for all } s \in S \,\}$$

Find an example of an $S$ so that $D$ is not an open set.

**Exercise 2.7.8.** Show that $D(f) = \varnothing$ if and only if $f$ is nilpotent. HINT: Use the identity $\sqrt{(0)} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}$.

**Exercise 2.7.9.** Show that $D(f) = D(g)$ if and only if there are integers $m, n$ such that $g^m = u \cdot f^n$ for some unit $u \in A$.

**Exercise 2.7.10.** Show that the ideal $\mathfrak{m} = (x, y-1)$ in $A = \mathbb{R}[x,y]/(x^2+y^2-1)$ is not principal.

**Exercise 2.7.11.** Check that for a nested inclusion $D(h) \subset D(g) \subset D(f)$, we have $\rho_{fh} = \rho_{gh} \circ \rho_{fg}$.

**Exercise 2.7.12.** Let $A$ be a ring, let $\mathfrak{a}$ be an ideal in $A$ and let $\{f_i\}_{i \in I}$ be elements from $\mathfrak{a}$. Show that the open distinguished sets $D(f_i)$ cover $\operatorname{Spec} A - V(\mathfrak{a})$ if and only if some power of each element $f \in \mathfrak{a}$ lies in the ideal generated by the $f_i$'s.

**Exercise 2.7.13.** Let $k$ be a field and let $A = k[t_0, t_1, \dots]$ be a polynomial ring in countably many variables. Let $\mathfrak{m}$ be the maximal ideal $\mathfrak{m} = (t_0, t_1, \dots)$. Show that $U = \operatorname{Spec} A - \mathfrak{m}$ is not quasi-compact. Conclude that $U$ is not the spectrum of a ring. HINT: Consider the open cover $\{D(t_i)\}_{i \geqslant 0}$.

**Exercise 2.7.14.** Find an open set of $\mathbb{A}^1_{\mathbb{Z}}$ which is not a distinguished open set.

**Exercise 2.7.15.** Let $\phi \colon A \to B$ and $\psi \colon B \to C$ be two ring maps. Show that, with the notation of (2.12), we have $\operatorname{Spec}(\psi \circ \phi) = \operatorname{Spec}(\phi) \circ \operatorname{Spec}(\psi)$.

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**Exercise 2.7.16** (The Gaussian integers)**.** For a complex number $z = a + ib$, we let $N(z) = a^2 + b^2$ denote the *norm* of $z$.

a) Show that for any two $\alpha, \beta \in \mathbb{Z}[i]$, there exists $\mu$ and $\rho$ in $\mathbb{Z}[i]$ such that $\alpha = \mu\beta + \rho$ where $N(\beta) > N(\rho)$. HINT: Consider the lattice generated by $\beta, i\beta$ and consider the square nearest $\alpha$.
b) Show that $\mathbb{Z}[i]$ is a PID. HINT: If $\mathfrak{a} \subset \mathbb{Z}[i]$ is a non-zero ideal, show that $\mathfrak{a}$ is generated by a non-zero element in $\mathfrak{a}$ of minimal norm.
c) Conclude that $\mathbb{Z}[i]$ is a UFD.

**Exercise 2.7.17.** Show that the Zariski topology on $\operatorname{Spec} A$ is Hausdorff if and only if every prime ideal $\mathfrak{p}$ is maximal.

**Exercise 2.7.18.** Show that the closed points in $\operatorname{Spec} A$ form a dense set if and only if $\sqrt{0}$ equals the intersection $\bigcap_{\mathfrak{m} \subset A} \mathfrak{m}$ of all maximal ideals in $A$. HINT: Corollary 2.5 on page 27.

**Exercise 2.7.19.** Let $A$ be an integral domain and $U \subset \operatorname{Spec} A$ an open non-empty subset. Show that there is no closed point in $U$ if and only if there is an $f \in A$ such that $A_f$ is a field. HINT: Consider distinguished open subsets $D(f) \subset U$.

**Exercise 2.7.20.** Let $\{A_i\}_{i \in I}$ be an infinite sequence of non-trivial rings, and let $X$ be the disjoint union of the spectra $\operatorname{Spec} A_i$. Show that $X$ is not homeomorphic to a spectrum of a ring.

**Exercise 2.7.21.** Recall that a ring is *local* if there is only one maximal ideal.

a) Show that $A$ is local if and only if $\operatorname{Spec} A$ has a unique closed point.
b) Give examples of local rings $A$ so that $\operatorname{Spec} A$ consists of (i) one point; (ii) two points; (iii) infinitely many points.
c) A map of rings $\phi\colon A \to B$ is said to be local if $\phi(\mathfrak{m}_A) \subset \mathfrak{m}_B$. Show that $\phi$ is local if and only if the induced map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ maps the unique closed point of $\operatorname{Spec} B$ to that of $\operatorname{Spec} A$.
d) Give an example of a map of rings $f\colon A \to B$ which is not local. Describe your example in terms of the corresponding map on spectra.

**Exercise 2.7.22.** Show that $\operatorname{Spec} A$ has just one element if and only if $A$ is a local ring all whose non-units are nilpotent, i.e. the radical $\sqrt{(0)}$ of the ring is a maximal ideal. For Noetherian rings this is equivalent to the ring being an Artinian local ring.

**Exercise 2.7.23.** Perform the analysis of the fibers of the map in Example **??** on page ?? when the field $k$ has characteristic 2.

**Exercise 2.7.24.** With reference to Example 2.41 on page 40:

a) Show that the fiber of $\phi$ over a prime ideal $(p)$ is homeomorphic to

$$\operatorname{Spec} \mathbb{F}_p[x]/(x^2+1)$$

and that $\dim_{\mathbb{F}_p} \mathbb{F}_p[x]/(x^2+1)$ =2. HINT: Use that $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2+1)$.
b) Show that $\mathbb{F}_p[x]/(x^2+1)$ is a field if and only if $x^2+1$ does not have a root in $\mathbb{F}_p$.
c) Show that $\mathbb{F}_p[x]/(x^2+1)$ is a field if and only if $(p)\mathbb{Z}[i]$ is a prime ideal.

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**Exercise 2.7.25.** Consider the ring map

$$\phi\colon \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y,z]/(xz-y)$$

which induces $f\colon \operatorname{Spec}\mathbb{C}[x,y,z]/(xz-y) \to \mathbb{A}^2_{\mathbb{C}}$. Show that the map $f$ on the level of closed points sends $(a,ab,b)$ to $(a,ab)$, and the generic point to the generic point. Show that in this example, the image is neither open nor closed: it equals $D(x) \cup V(x,y)$.

**Exercise 2.7.26.** Let $p$ and $q$ be two different prime numbers and consider the morphism $\phi\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ induced from the map $k[x,y] \to k[t]$ which is defined by the assignments $x \mapsto t^p$ and $y \mapsto t^q$. Determine all fibers of $\phi$.

**Exercise 2.7.27.** Let $A = \mathbb{C}[x,y,z]/(xy,xz,yz)$ and $X = \operatorname{Spec} A$. Consider the map $f\colon X \to \mathbb{A}^1$ corresponding to the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[t] \to A$ that sends $t$ to $x+y+z$. Determine all fibers of $f$. HINT: The set of closed points of $X$ is the union of the three coordinate axes in $\mathbb{C}^3$, and the map sends a point to the sum of the coordinates.

**Exercise 2.7.28.** Describe all fibers of the following morphisms.

a) $f\colon \operatorname{Spec}\mathbb{C}[x,y]/(xy-1) \to \operatorname{Spec}\mathbb{C}[x]$
b) $f\colon \operatorname{Spec}\mathbb{C}[x,y]/(x^2-y^2) \to \operatorname{Spec}\mathbb{C}[x]$
c) $f\colon \operatorname{Spec}\mathbb{Z}[x,y]/(xy-n) \to \operatorname{Spec}\mathbb{Z}$, where $n$ is a non-zero integer.
d) $\operatorname{Spec}\mathbb{C}[x,y]/(x^2+xy+1) \to \operatorname{Spec}\mathbb{C}[y]$.

**Exercise 2.7.29.** Describe the fibers of the map

$$\mathbb{A}^1_k \to \operatorname{Spec} k[x,y]/(y^2-x^3-x^2)$$

induced by $x \mapsto t^2-1$, $y \mapsto t^3-t$.

**Exercise 2.7.30.** Determine all the fibers of the morphism

$$\operatorname{Spec}\mathbb{Z}[(1+\sqrt{5})/2] \longrightarrow \operatorname{Spec}\mathbb{Z}[\sqrt{5}]$$

induced by the natural inclusion $\mathbb{Z}[\sqrt{5}] \subset \mathbb{Z}[(1+\sqrt{5})/2]$.

**Exercise 2.7.31.** Let $A = \mathbb{Z}[x,y]/(x^2-y^2-5)$ and consider the morphism $f\colon \operatorname{Spec} A \to \operatorname{Spec}\mathbb{Z}$. Compute the fibers over $(0)$, $(2)$, $(3)$ and $(5)$. What happens if you replace $A$ with the ring $\mathbb{Z}[x,y]/(3x^2-3y^2-15)$?

**Exercise 2.7.32.** Describe the following prime spectra

a) $\operatorname{Spec}\mathbb{C}[x]/(x^3+x^2)$
b) $\operatorname{Spec}\mathbb{R}[x]/(x^3+x^2)$.

**Exercise 2.7.33.** Let $A$ and $B$ be finitely generated $k$-algebras, and let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be the map induced by a ring map $A \to B$. Show that $f$ maps closed points to closed points.

**Exercise 2.7.34.** Study the fibers of the morphisms

a) $\operatorname{Spec}\mathbb{Z}[t]/(t^2+t+1) \to \operatorname{Spec}\mathbb{Z}$;
b) $\operatorname{Spec}\mathbb{Q}(t)[x]/(x^3+3x+1) \to \operatorname{Spec}\mathbb{Q}(t)$.

**Exercise 2.7.35.** Let $A = \mathbb{R}[x,y]/(x^2+y^2+1)$.

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a) Show that for each $a, b, c \in \mathbb{R}$ with $(a, b) \neq (0, 0)$, the ideal $\mathfrak{m} = (ax+by+c)$ is a maximal ideal of $A$.
b) Show that every non-zero prime ideal is of this form. Use this to describe $\operatorname{Spec} A$. HINT: $A/\mathfrak{m}$ is a finite extension of $\mathbb{R}$. Show that $1, x, x^2$ are linearly dependent, and likewise $1, y, y^2$.

**Exercise 2.7.36.** Show that $\operatorname{Spec} \overline{\mathbb{Q}}[x]$ is homeomorphic to $\operatorname{Spec} \mathbb{Z}$.

**Exercise 2.7.37.** Let $X = \operatorname{Spec} A$, and let $U = \operatorname{Spec}(A_f) \subset X$ be a distinguished open set. If $V$ is a distinguished open set in $U$, show that $V$ is a distinguished open set in $X$ as well.

**Exercise 2.7.38.** Let $X = \operatorname{Spec} A$ show that the intersection $\bigcap_{U \subset X, U \neq \varnothing} U$ is equal to the set of generic points of $X$.

**Exercise 2.7.39.** Let $A = \mathbb{C}[a, b, c]$ and $R = A[x]/(ax^2 + bx + c)$. Study the fibers of the morphism $\operatorname{Spec} R \to \operatorname{Spec} A$.

**Exercise 2.7.40.** Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a map of spectra induced by a ring map $\phi\colon A \to B$.
a) Show that if $\phi$ is injective, then the image of $f$ is dense in $\operatorname{Spec} A$.
b) More generally, show that $f(\operatorname{Spec} B)$ is dense if and only if $\operatorname{Ker} \phi \subset \sqrt{(0)}$.
c) Define $\phi : \mathbb{Q}[x] \to \mathbb{C}$ by sending $x$ to $\pi$. Show that $\phi$ is injective. Describe the morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{A}^1_{\mathbb{Q}}$.

**Exercise 2.7.41.** Describe the spectrum of the ring

$$A = \{\, (a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a \equiv b \pmod 2 \,\}.$$

**Exercise 2.7.42.** Let $X$ be a topological space with an open cover $\{U_i\}_{i \in I}$, and let $\{U_j\}_{j \in J}$ be a finite subcover where $J \subset I$. Show that if the sheaf sequence for $\mathcal{F}$ with respect to the cover $\{U_i\}_{j \in J}$ is exact, then it is also exact for the full cover $\{U_i\}_{i \in I}$.

**Exercise 2.7.43.** Describe the following prime spectra:
a) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 + 2)$
b) $\operatorname{Spec} \mathbb{Z}[x]/(x^2 - 2)$
c) $\operatorname{Spec} \mathbb{Z}[x]/(x^3 + 2)$
d) $\operatorname{Spec} \mathbb{Z}[x, y]/(x^2 + y, y^2 + 1)$

**Exercise 2.7.44.** Let $X$ be the spectrum of a DVR, as described in Example 2.14. Show that any presheaf on $X$ is automatically a sheaf.

**Exercise 2.7.45.** Let $n$ be a positive integer. Study the fibers of the morphism $\operatorname{Spec} k[x] \to \operatorname{Spec} k[y]$ induced by the ring map $y \mapsto x^n$.

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# 3

# The Structure Sheaf

In this chapter, we define the *structure sheaf* on the spectrum of a ring $X = \operatorname{Spec} A$, which generalizes the notion of regular functions on an affine variety. As in Chapter 1, there is, for for every open set $U \subset X$, a ring $\mathcal{O}_X(U)$ whose elements may be thought of as functions on $U$. However, as we'll see, this interpretation needs to be carefully handled when working with general rings, where elements may not have a natural way to be evaluated at points.

More broadly, a *sheaf* is a mathematical structure which systematically assigns algebraic data to open sets of a space. Initially, it is helpful to think of sheaves as defining 'distinguished functions' on a space (regular, continuous, holomorphic, ..), but they can also represent more abstract mathematical objects, such as modules or sets of morphisms. The essential requirement is that this assignment satisfies the two sheaf axioms: locality and gluing.

The theory of sheaves is truly central to algebraic geometry. Every scheme carries with it a whole ecosystem of sheaves, and understanding a scheme often means understanding the sheaves it carries. In this chapter, we will lay the groundwork by introducing sheaves generally, leading up to the definition of the structure sheaf in Section ??. We will return to sheaf theory in greater detail in Chapter 11, where we will develop more advanced techniques and apply them to study sheaves on schemes.

### *Presheaves and sheaves*

The concept of a sheaf may be defined for any topological space, and the theory is best studied at this level of generality. We begin with the definition of a *presheaf*.

**Definition 3.1** (Presheaf)**.** Let $X$ be a topological space. A *presheaf of abelian groups* $\mathcal{F}$ on $X$ consists of the following two sets of data:

(i) For each open $U \subset X$, an abelian group $\mathcal{F}(U)$.

(ii) For each pair of nested opens $V \subset U$, a map of groups

$$\rho_{UV} \colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V).$$

called the *restriction map*. These data are required to satisfy the following two conditions:

(a) For any open $U \subset X$, we have $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$.

(b) For any three open subsets $W \subset V \subset U$, one has $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

We will usually write $s|_V$ for $\rho_{UV}(s)$ when $s \in \mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are usually called *sections* (or *sections* of $\mathcal{F}$ over $U$). It is also common to write $\Gamma(U, \mathcal{F})$ for the group $\mathcal{F}(U)$; $\Gamma$ is the 'global sections'-functor (it is functorial in both $U$ and $\mathcal{F}$).



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The notion of a presheaf is not confined to presheaves of abelian groups. One can also consider presheaves of sets, rings, vector spaces etc. In fact, for any category $\mathsf{C}$ one may define presheaves with values in $\mathsf{C}$. The definition is essentially the same as for presheaves of abelian groups, the only difference being that one requires that the $\mathcal{F}(U)$ are objects from $\mathsf{C}$, and of course, that restriction maps are all morphisms in $\mathsf{C}$. We are certainly going to meet sheaves with more structure beyond that of abelian groups, e.g. sheaves of rings, but they will usually have an underlying structure of abelian group, so we start with these. We will also encounter sheaves of sets. Most of the results we establish for sheaves of abelian groups can be proved for sheaves of sets as well, as long as they can be formulated in terms of sets, and the proofs are essentially the same.

**Definition 3.2** (Sheaf)**.** A presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the following two conditions for every open set $U \subset X$ and every open cover $\{U_i\}_{i\in I}$ of $U$:

(i) (Locality axiom) If $s, t \in \mathcal{F}(U)$ are sections such that
$$s|_{U_i} = t|_{U_i}$$
for all $i$, then $s = t$.

(ii) (Gluing axiom) Given a collection of sections $s_i \in \mathcal{F}(U_i)$ such that
$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \qquad \text{for all } i, j \in I,$$
then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i$.

These two axioms mirror the properties of continuous functions mentioned in the introduction. The Locality axiom says that sections are uniquely determined from their restrictions to smaller open sets. The Gluing axiom says that you are allowed to patch together local sections to a global one, provided they agree on overlaps.

There is a convenient way of formulating the two sheaf axioms simultaneously. For each open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of an open set $U \subset X$, there is a *sheaf exact sequence*

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathcal{F}(U_i) \xrightarrow{\beta} \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j), \tag{3.1}$$

where the maps $\alpha$ and $\beta$ are defined by $\alpha(s) = (s|_{U_i})_i$, and $\beta(s_i) = (s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j})_{i,j}$. Then $\mathcal{F}$ is a sheaf if and only if these sequences are exact. Indeed, exactness at $\mathcal{F}(U)$ means that $\alpha$ is injective, i.e. that $s|_{U_i} = 0$ for all $i$ implies that $s = 0$ (this is equivalent to the Locality axiom). Exactness in the middle means that $\operatorname{Ker}\beta = \operatorname{Im}\alpha$; that is, elements $s_i$ satisfying $s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j} = 0$ come from an element $s \in \mathcal{F}(U)$ (the Gluing axiom).

When $\mathcal{F}$ is a sheaf, since $\mathcal{F}(U) = \operatorname{Ker}\beta$, we can often use the sequence to compute $\mathcal{F}(U)$ if the $\mathcal{F}(U_i)$'s and the $\mathcal{F}(U_i \cap U_j)$'s are known.

*Remark* 3.3 (The empty set). For a sheaf $\mathcal{F}$ of abelian groups, it is common to include the condition $\mathcal{F}(\varnothing) = 0$. This is essentially forced upon us by the sheaf axioms: the empty set is covered by the empty covering, and the sheaf axioms requires $\mathcal{F}(\varnothing)$ to be the empty product $\prod_{\varnothing}$ in the category of abelian groups, which is the trivial group (the terminal object).

A *morphism* (or simply a *map*) *of (pre)sheaves* $\phi\colon \mathcal{F} \to \mathcal{G}$ is a collection of maps of abelian groups $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$, one for each open set in $X$, which are required to be compatible with the restriction maps, in the sense that the following diagram commutes for

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each inclusion $V \subset U$ of open sets:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow[\phi_U]{} & \mathcal{G}(U) \\ \big\downarrow{\scriptstyle \rho_{UV}} & & \big\downarrow{\scriptstyle \rho_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V). \end{array} \tag{3.2}$$

In this way, presheaves of abelian groups on $X$, together with their morphisms form a category, which is denoted by $\mathsf{PAb}(X)$. Likewise, the we have the category of sheaves of abelian groups, $\mathsf{Ab}(X)$, which forms a full subcategory.

As usual, a map $\phi$ between two (pre)sheaves $\mathcal{F}$ and $\mathcal{G}$ is an *isomorphism* if it has a two-sided inverse, i.e. a map $\psi\colon \mathcal{G} \to \mathcal{F}$ such that $\phi \circ \psi = \mathrm{id}_{\mathcal{G}}$ and $\psi \circ \phi = \mathrm{id}_{\mathcal{F}}$. This the same as saying that $\phi_U$ is an isomorphism for every open set $U$.

A presheaf $\mathcal{G}$ is a *subpresheaf* of a presheaf $\mathcal{F}$ if $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open $U \subset X$, and such that the restriction maps of $\mathcal{G}$ are the restrictons of those of $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, $\mathcal{G}$ is naturally called a *subsheaf*.

### *Examples*

**Example 3.4** (Continuous functions)**.** Take $X = \mathbb{R}^n$ and let $C(U, \mathbb{R})$ denote the presheaf whose sections over an open set $U$ is the ring of continuous real valued functions on $U$, and whose restriction maps $\rho_{UV}$ are the usual restrictions of functions. Then $C(-, \mathbb{R})$ is a sheaf of rings (functions can be added and multiplied), and both sheaf axioms are satisfied. Indeed, any function $f\colon X \to \mathbb{R}$ which restricts to zero on an open covering of $X$ takes the value zero at every point, so it is the zero function. Also, given continuous functions $f_i\colon U_i \to \mathbb{R}$ that agree on the overlaps $U_i \cap U_j$, we can form the continuous function $f\colon U \to \mathbb{R}$ by setting $f(x) = f_i(x)$ for any $i$ such that $x \in U_i$. △

**Example 3.5** (Holomorphic functions)**.** For a second familiar example, let $X = \mathbb{C}$ and consider the presheaf $\mathscr{A}_X$ of holomorphic functions. That is, for any open $U \subset X$, the sections $\mathscr{A}_X(U)$ is the ring of complex differentiable functions on $U$. Just like in the example above, one checks that $\mathscr{A}_X$ forms a sheaf. In fact, $\mathscr{A}_X$ is a *subsheaf* of the sheaf of continuous functions $U \to \mathbb{C}$. △

**Example 3.6.** More generally, for any two topological spaces $X$ and $Y$, the presheaf $C(-, Y)$ of continuous maps $f\colon U \to Y$ forms a sheaf (they are sheaves of sets, because we cannot in general add or multiply maps).

Furthermore, any presheaf of the form

$$\mathcal{F}(U) = \{\, f\colon U \to Y \mid f \text{ satisfies } (*) \,\}$$

is a subsheaf of $C(-, Y)$, provided $(*)$ is a condition that can be checked locally. For instance, we can take $(*)$ to mean 'differentiable', 'smooth', 'analytic', and so on. Locality holds because it holds in $C(-, Y)$. Given local sections $f_i \in \mathcal{F}(U_i)$, agreeing on the overlaps, we can glue them to an element $f \in C(U, Y)$, and the result lies in $\mathcal{F}(U)$, because the restriction of $f$ to $U_i$ is equal to $f_i$, which satisfies $(*)$. See Example 3.8 for an example where $(*)$ is not a local condition. △

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One of the main examples in this book will be the following:

**Example 3.7** (Affine varieties)**.** Let $k$ be an algebraically closed field, and let $X$ be an affine variety in $\mathbb{A}^n(k)$. As in Section 1.5, we define, for an open set $U \subset X$, the ring of all rational functions which are regular in $U$:

$$\mathcal{O}_X(U) = \{\, f \in K(X) \mid f \text{ is regular in } U \,\}.$$

Note that if $V \subset U$, then $\mathcal{O}_X(U) \subset \mathcal{O}_X(V)$, so this defines a presheaf by letting the restriction maps be the inclusions.

The sheaf axioms are also satisfied: The Locality axiom holds, because if $f \in K(X)$ restricts to zero in some $\mathcal{O}_X(U)$, then this simply means that $f = 0$ in $K(X)$. The Gluing axiom holds, because if $f_i \in \mathcal{O}_X(U_i)$ is a collection of rational functions that agree on the overlaps $U_i \cap U_j$ of an open covering, then they are all equal to the same element $f$ in $K(X)$. This rational function $f$ must in turn be regular in all of $U$, because if $p \in U$ is a point, then $p$ lies in some $U_j$, and hence $f = f_j$ can be written as $a/b$ with $b(p) \neq 0$. It follows that $\mathcal{O}_X$ defines a sheaf of rings, which is called the *sheaf of regular functions* on $X$.

We obtain subsheaves sheaves of $\mathcal{O}_X$ by imposing vanishing conditions on the regular functions. For example, if $p \in X$ is a point, one has the sheaf denoted $\mathfrak{m}_p$ of holomorphic functions vanishing at $p$. This is an example of an *ideal sheaf*: for each open $U \subset X$, $\mathfrak{m}_p(U)$ is an ideal of the ring $\mathcal{O}_X(U)$. △

**Example 3.8** (A presheaf which is not a sheaf)**.** Let us continue the set-up in Example 3.5 to construct an example of a presheaf which is not a sheaf. Let $X = \mathbb{C} - \{0\}$, and let $\mathscr{A}_X$ denote the sheaf of holomorphic functions. Inside $\mathscr{A}_X$ we find a subpresheaf given by

$$\mathcal{F}(U) = \{\, f \in \mathscr{A}_X(U) \mid f = g^2 \text{ for some } g \in \mathscr{A}_X(U) \,\}.$$

While $\mathcal{F}$ satisfies the Locality axiom (being a subpresheaf of a sheaf), the Gluing axiom fails. Consider for instance, the function $f(z) = z$ in $\mathcal{F}(X)$. For each point $a \in \mathbb{C} - \{0\}$, we can find an open neighbourhood $U_a$ and a local square root $g_a$ such that $g_a^2 = z$. The $g_a$ can moreover be chosen to agree on the overlaps $U_a \cap U_{a'}$ by consistently choosing signs. However, there is no continuous function $g(z)$ on $\mathbb{C} - \{0\}$ such that $g(z)^2 = z$, as any such function would change sign when $z$ transveses a loop around the origin. △

**Example 3.9** (Constant presheaves)**.** For any space $X$ and any abelian group $A$, one has the *constant presheaf* defined by $A(U) = A$ for any nonempty open set $U$ (and $A(\varnothing) = 0$).

This is not a sheaf in general. For instance, if $X = U_1 \cup U_2$ is a disjoint union, and $A = \mathbb{Z}$, then any choice of integers $a_1, a_2 \in \mathbb{Z}$ will give sections of $A(U_1)$ and $A(U_2)$, and they automatically agree over the intersection, which is empty. But if $a_1 \neq a_2$, they cannot be glued to an element in $A(X) = \mathbb{Z}$. In fact, the constant presheaf is a sheaf if and only if any two non-empty open subsets of $X$ have non-empty intersection. Algebraic varieties with the Zariski topology are examples of such spaces.

There is a quick fix for this. We can define the following sheaf $A_X$ by letting

$$A_X(U) = \{\, f \colon U \to A \mid f \text{ is locally constant} \,\}$$

As before, we also must put $A_X(\varnothing) = 0$. For a connected open set $U$, we then have

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$A_X(U) = A$. More generally, since $f$ must be constant on each connected component of $U$, we have $A_X(U) \simeq \prod_{\pi_0(U)} A$, where $\pi_0(U)$ denotes the set of connected components of $U$.

The new presheaf $A_X$ is called the *constant sheaf* on $X$ with value $A$. It is a sheaf (e.g. by Example 3.6). That being said, the sheaf $A_X$ is not quite worthy of its name, as it is not quite constant. △

## 3.1 Stalks

Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space $X$. For any point $x \in X$, we can associate an abelian group $\mathcal{F}_x$ called the *stalk* of $\mathcal{F}$ at $x$. The stalk captures the local behavior of sections of $\mathcal{F}$ near $x$ (ignoring differences that occur away from $x$). The elements of $\mathcal{F}_x$ are called *germs of sections*, or simply *germs*, near $x$; they represent the sections of $\mathcal{F}$ defined in some sufficiently small neighbourhood of $x$.

Formally, the group $\mathcal{F}_x$ is defined as the *direct limit* of the groups $\mathcal{F}(U)$ as $U$ runs through the directed set of open neighbourhoods $U$ of $x$ (ordered by inclusion)[1]:

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U).$$

Concretely, $\mathcal{F}_x$ can be constructed as follows. Consider the disjoint union $\coprod_{U \ni x} \mathcal{F}(U)$, where the elements are pairs $(s, U)$ with $U$ an open neighbourhood of $x$ and $s$ is a section in $\mathcal{F}(U)$. We identify sections that coincide near $x$ by declaring $(s, U)$ and $(s', U')$ to be equivalent, written $(s, U) \sim (s', U')$, if there is an open $V \subset U \cap U'$ with $x \in V$ such that

$$s|_V = s'|_V.$$

The relation $\sim$ is reflexive and symmetric by definition. For transitivity, suppose $(s, U) \sim (s', U')$ and $(s', U') \sim (s'', U'')$. Then there exist open neighbourhoods $V \subset U \cap U'$ and $V' \subset U' \cap U''$ of $x$ such that $s|_V = s'|_V$ and $s'|_{V'} = s''|_{V'}$. The restrictions of $s$ and $s''$ then coincide over the open set $V' \cap V$, and so $(s, U) \sim (s'', U'')$. Therefore, $\sim$ is an equivalence relation.

**Definition 3.10.** The *stalk* $\mathcal{F}_x$ at $x \in X$ is defined as the set of equivalence classes

$$\mathcal{F}_x = \coprod_{U \ni x} \mathcal{F}(U) / \sim .$$

When $\mathcal{F}$ is a presheaf of abelian groups, the stalks $\mathcal{F}_x$ are all abelian groups. This is not completely obvious, because sections over different open sets can not be added directly. However, if $(s, U)$ and $(s', U')$ are given, the restrictions $s|_V$ and $s'|_V$ to any open $V \subset U \cap U'$ can be added within $\mathcal{F}(V)$, and this suffices to define an abelian group structure on $\mathcal{F}_x$.

For any neighbourhood $U$ of $x \in X$, there is a natural map $\mathcal{F}(U) \to \mathcal{F}_x$ sending a section $s \in \mathcal{F}(U)$ to its equivalence class in $\mathcal{F}_x$. This class is called the *germ* of $s$ at $x$, and it is denoted $s_x$.

[1] For background on direct limits, see Appendix ??

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Taking the germ is compatible with restrictions: if $V \subset U$ is another open set containing $x$, we have $s_x = (s|_V)_x$. This is illustrated by the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}_x \\ {\scriptstyle \rho_{UV}}\downarrow & \nearrow & \\ \mathcal{F}(V). & & \end{array} \tag{3.3}$$

When working with sheaves and stalks, it is useful to keep in mind the following properties:

- The germ $s_x$ of a section $s$ vanishes if and only if $s$ vanishes on some neighbourhood of $x$.
- All elements of the stalk $\mathcal{F}_x$ are germs. That is, every element of $\mathcal{F}_x$ is of the form $s_x$ for some section $s$ over some open neighbourhood of $x$.
- A sheaf $\mathcal{F}$ is the zero sheaf if and only if all stalks are zero: $\mathcal{F}_x = 0$ for all $x \in X$.

A map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ induces a map between the stalks

$$\phi_x\colon \mathcal{F}_x \longrightarrow \mathcal{G}_x.$$

This map sends the equivalence class of $(s, U)$ to the equivalence class of $(\phi_U(s), U)$. As $\phi$ commutes with restrictions, this assignment is well-defined: if $(s, U) \sim (s', U')$ and $s$ and $s'$ coincide on an open set $V \subset U \cap U'$, then the diagram (3.2) gives

$$\phi_U(s)|_V = \phi_V(s|_V) = \phi_V(s'|_V) = \phi_{U'}(s')|_V.$$

One also verifies that $(\phi \circ \psi)_x = \phi_x \circ \psi_x$ and $(\mathrm{id}_{\mathcal{F}})_x = \mathrm{id}_{\mathcal{F}_x}$, so the assignments $\mathcal{F} \mapsto \mathcal{F}_x$ and $\phi \mapsto \phi_x$ define a functor from the category of sheaves to the category of abelian groups.

**Example 3.11.** Let $X = \mathbb{C}$, and let $\mathscr{A}_X$ be the sheaf of holomorphic functions in $X$. The stalk $\mathscr{A}_{X,p}$ at a point $p \in X$ is determined as follows: Two holomorphic functions $f$ and $g$ have the same germ at $p$ if they agree on some neighbourhood of $X$. This in particular implies that $f$ and $g$ must have the same Taylor series at $p$. The stalk $\mathscr{A}_{X,p}$ is therefore identified with the ring of power series that converge in a neighbourhood of $p$. △

## 3.2 The pushforward of a sheaf

Given a continuous map $f\colon X \to Y$ and a sheaf $\mathcal{F}$ on $X$, we define the *pushforward* $f_*\mathcal{F}$ to be the sheaf on $U$ defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U),$$

and the restriction maps $\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}V)$ to be those coming from $\mathcal{F}$.

**Definition 3.12.** The sheaf $f_*\mathcal{F}$ is called the *pushforward* or *the direct image* of $\mathcal{F}$.

It is straightforward to verify that $f_*\mathcal{F}$ is a sheaf and not merely a presheaf. Indeed, if $\{U_i\}$ is an open covering of $U$, then $\{f^{-1}U_i\}$ is an open covering of $f^{-1}U$. A set of gluing data for $f_*\mathcal{F}$ and the given covering consists of sections $s_i \in \Gamma(U_i, f_*\mathcal{F}) = \Gamma(f^{-1}U_i, \mathcal{F})$ that agree on the intersections. This means that they coincide in $\Gamma(U_i \cap U_j, f_*\mathcal{F})$, which equals $\Gamma(f^{-1}U_i \cap f^{-1}U_j, \mathcal{F})$, and they may therefore be glued together to a section in

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$\Gamma(f^{-1}U, \mathcal{F}) = \Gamma(U, f_*\mathcal{F})$, as $\mathcal{F}$ is a sheaf. The Locality axiom follows for $f_*\mathcal{F}$, because it holds for $\mathcal{F}$.

The pushfoward is also functorial in $f$: if $g\colon X \to Y$ and $f\colon Y \to Z$ are continuous maps between topological spaces, and $\mathcal{F}$ is a sheaf on $X$, one has an equality of sheaves on $Z$

$$(f \circ g)_*\mathcal{F} = f_*(g_*\mathcal{F}). \tag{3.4}$$

**Example 3.13** (Skyscraper sheaves)**.** Let $\iota\colon \{x\} \to X$ be the inclusion of a closed point in $X$ and let $\mathcal{A}$ be the constant sheaf of an abelian group $A$ on $\{x\}$. The *skyscraper sheaf* of $A$ at $x$, denoted $A(x)$, is the pushforward sheaf $\iota_*\mathcal{A}$. Explicitly, the sections are given by

$$A(x)(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

△

### 3.3 Sheaves defined on a basis

Recall that a *basis* for the topology on a space $X$ is a collection of open subsets $\mathscr{B}$ such that every open set of $X$ can be written as a union of members of $\mathscr{B}$. In many situations, it turns out to be convenient to define a sheaf by saying what it should be over the open sets in a specific basis for the topology on $X$. This leads to the concept of a $\mathscr{B}$-sheaf.

The definition is similar to that of a sheaf: a *$\mathscr{B}$-presheaf* $F$ consists of (i) for each $U \in \mathscr{B}$, an abelian group $F(U)$, and (ii) for each pair $V \subset U$ in $\mathscr{B}$, a restriction map $\rho_{UV}\colon F(U) \longrightarrow F(V)$ satisfying $\rho_{UU} = \mathrm{id}_{F(U)}$ and $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ for each triple $W \subset V \subset U$ of open sets in $\mathscr{B}$.

A $\mathscr{B}$-presheaf is a *$\mathscr{B}$-sheaf* if, for every open set $U \in \mathscr{B}$ and any open cover $\{U_i\}_{i\in I}$ with $U_i \in \mathscr{B}$, the following sequence is exact:

$$0 \longrightarrow F(U) \longrightarrow \prod_{i\in I} F(U_i) \longrightarrow \prod_{V\subset U_i\cap U_j,\ V\in\mathscr{B}} F(V) \tag{3.5}$$

where the maps are defined by restrictions and differences of restrictions, as before.

The whole point of introducing $\mathscr{B}$-sheaves is expressed in the following proposition.

**Proposition 3.14.** Let $X$ be a topological space and let $\mathscr{B}$ be a basis for the topology. Then:

(i) Every $\mathscr{B}$-sheaf $F$ extends to a sheaf $\mathcal{F}$ on $X$, which is unique up to isomorphism.

(ii) Every morphism of $\mathscr{B}$-sheaves $\phi_0\colon F \to G$ extends uniquely to a morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$.

The basic idea is to define $\mathcal{F}(U)$ as the set of all possible gluings of sections of $F$ over open sets in $\mathscr{B}$ that cover $U$. That is, a section $s \in \mathcal{F}(U)$ is given by a set of sections $s_i \in F(V_i)$ for some open cover $\{V_i\}_{i\in I}$ of $U$ with $V_i \in \mathscr{B}$, such that

$$s_i|_W = s_j|_W$$

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for any $W \in \mathcal{B}$ with $W \subset V_i \cap V_j$. If we define $\mathcal{F}(U)$ and $\mathcal{G}(U)$ as the sets of compatible sections of $F$ and $G$ in this way, we can define the extended sheaf map $\phi_U \colon \mathcal{F}(U) \to \mathcal{G}(U)$ by sending $(s_i)$ to $(\phi_0)_{V_i}(s_i)$.

This is indeed the best way to think of an element of $\mathcal{F}(U)$. The only drawback is that each section depends on the choice of a covering $V_i$. To define the group $\mathcal{F}(U)$ in a way that is independent of any particular cover, we consider the 'biggest cover' of $U$, where we include *every* $V \in B$ contained in $U$.

*Proof* Let $U \subset X$ be any open subset. Let $\mathscr{B}_U \subset \mathscr{B}$ denote the open sets in $\mathscr{B}$ which are contained in $U$ and define

$$\mathcal{F}(U) = \left\{ (s_V) \in \prod_{V \in \mathscr{B}_U} F(V) \;\middle|\; s_V|_W = s_W, \, \forall \, W \subset V \text{ in } \mathscr{B} \right\} \tag{3.6}$$

An element of $\mathcal{F}(U)$ is therefore given by a set of sections $s_V \in F(V)$, such that whenever $W \subset V$ in $\mathscr{B}_U$, we have the compatibility condition $s_V|_W = s_W$.

Note that if $U' \subset U$, then $\mathscr{B}_{U'} \subset \mathscr{B}_U$, and so the projection maps give us restriction maps $\mathcal{F}(U) \to \mathcal{F}(U')$. This makes $\mathcal{F}$ into a presheaf. We leave it as an exercise for the reader to check the sheaf axioms for $\mathcal{F}$.

Note that if $U$ is an open set in $\mathscr{B}$, there is an obvious map

$$F(U) \longrightarrow \mathcal{F}(U).$$

sending $t \in F(U)$ to the collection $s_V = t|_V$. This is in fact an isomorphism, with inverse defined by the projection $\mathcal{F}(U) \to F(U)$ to the '$U$-th component'.

(ii): Saying that $\phi_0 \colon F \to G$ is a map of $\mathscr{B}$-sheaves amounts to saying that the following diagram commutes for each pair $V' \subset V$ of opens in $\mathscr{B}$:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{(\phi_0)_V} & G(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(V') & \xrightarrow[(\phi_0)_{V'}]{} & G(V'). \end{array}$$

Taking the products over all open subsets $V \in \mathscr{B}_U$, we obtain a natural map $\mathcal{F}(U) \to \mathcal{G}(U)$ which extends $\phi_0$. These maps are moreover compatible with the restriction maps, so we get a map of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$.

The extension $\mathcal{F}$ is unique: If $\mathcal{G}$ is another extension, satisfying $\mathcal{F}(U) = \mathcal{G}(U)$ for every $U \in \mathscr{B}$, then since $\mathcal{F}$ and $\mathcal{G}$ are sheaves, we may use the expression (3.6) to compute $\mathcal{F}(U)$ and $\mathcal{G}(U)$ for any open set. Using this, we obtain a canonical isomorphism of sheaves $\mathcal{F} \simeq \mathcal{G}$. A similar argument shows that the extended morphism $\phi \colon \mathcal{F} \to \mathcal{G}$ in (ii) is unique. □

## 3.4 The structure sheaf on the spectrum of a ring

We now define the structure sheaf $\mathcal{O}_X$ on $X = \operatorname{Spec} A$. To motivate the definition, consider an affine variety $X \subset \mathbb{A}^n(k)$ be an affine variety with coordinate ring $A = A(X)$. For each open set $U \subset X$, the ring $\mathcal{O}_X(U)$ consists of the rational functions that are regular at every point of $U$. The elements of $\mathcal{O}_X(U)$ can be viewed either as elements of the function field

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$K(X)$, or as functions $f\colon U \to k$. Moreover, each restriction map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ can be seen as an inclusion of subrings of $K(X)$, or as an actual restriction map on functions.

For a distinguished open set $D(f)$, the ring of regular functions $\mathcal{O}_X(D(f))$ is simply the localization $A_f$ (Proposition 1.25). This is intuitive, because the elements of the form $a/f^n$ define regular functions on $D(f)$. Moreover, if $D(g) \subset D(f)$, the restriction map corresponds to the canonical localization map $A_f \to A_g$.

In fact, the sheaf $\mathcal{O}_X$ is completely determined by knowing $\mathcal{O}_X(D(f)) = A_f$ for each $f \in A$. Indeed, inside $K(X)$, we have

$$\mathcal{O}_X(U) = \bigcap_{D(f)\subset U} \mathcal{O}_X(D(f)). \tag{3.7}$$

If one tries to carry out the above construction for a general ring $A$, there are a few issues that arise. First, there may not exist a natural field $k$ for which we can view elements of $\mathcal{O}_X(U)$ as functions $U \to k$. More critically, the localization maps $A_f \to A_g$ may fail to be injective (so an intersection like (3.7) does not make sense). This happens already in the simple example when $A = k[x,y]/(xy)$, which corresponds to the union of $x$-axis and the $y$-axis in the affine plane. Here the nonzero element $x \in A$ maps to 0 via the localization map $A \to A_y$. So this behaviour is not a big surprise; it naturally appears once we allow reducible spaces into our discussion.

Motivated by this discussion, we consider the basis of distinguished open sets $D(f)$. We would like $\mathcal{O}_X$ to satisfy $\mathcal{O}_X(D(f)) \simeq A_f$ for every $f \in A$, with restriction maps $\mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(g))$ corresponding to the localization maps $A_f \to A_g$ whenever $D(f) \supset D(g)$. There is a small subtlety here, namely that two different $f$'s may define the same distinguished open set $U$ (for instance, $D(f) = D(f^2)$). A nice way to resolve this is to work with a more canonical localization:

**Definition 3.15.** For a distinguished open set $U$, we define

$$\mathcal{O}(U) = S(U)^{-1}A. \tag{3.8}$$

where $S(U) = \{\, s \in A \mid D(s) \supset U \,\}$ is the multiplicative subset of elements $s \in A$ that vanish nowhere on $U$.

The elements of $\mathcal{O}(U)$ are fractions $a/s$ so that $s(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in U$. If $U \supset V$ are two distinguished open sets, then $S(U) \subset S(V)$, so by the universal property of localization, there is a canonical map

$$S(U)^{-1}A \longrightarrow S(V)^{-1}A,$$

which we use to define the restriction maps $\mathcal{O}(U) \to \mathcal{O}(V)$.

Note that for $U = D(f)$, we have $f \in S(U)$, and so there is a canonical localization map $A_f \to S(U)^{-1}A$. We claim that this is an isomorphism:

**Lemma 3.16.** When $U = D(f)$, there is a canonical isomorphism $S(U)^{-1}A \simeq A_f$.

*Proof* This map is injective: if $a/f^n \in A_f$ maps to 0 in $S(U)^{-1}A$, there is an $s \in S(U)$ such that $sa = 0$. By definition, we have $D(f) \subset D(s)$, which is equivalent to being able

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to write $f^N = cs$ for some $c \in A$ and $N \in \mathbb{N}$. Multiplying $sa = 0$ by $c$ now shows that $f^N a = 0$ in $A$, and hence $a/f^n = 0$ in $A_f$. The map is also surjective: if $a/s \in S(U)^{-1}A$, then writing $f^N = cs$ as before, we have $a/s = (ca)/f^N$ in $S(U)^{-1}A$, which is in the image of $A_f \to S(U)^{-1}A$. □

In light of this, we will from now on always identify $\mathcal{O}(D(f)) = A_f$, and $\mathcal{O}(D(f)) \to \mathcal{O}(D(g))$ with $A_f \to A_g$ whenever $D(g) \subset D(f)$.

We now prove that $\mathcal{O}$ is a $\mathscr{B}$-sheaf. This will follow from the following algebraic lemma:

**Lemma 3.17.** Let $R$ be a ring and let $f_1, \ldots, f_r \in R$ be elements generating the unit ideal. For any $R$-module $M$, the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{\alpha} \prod_{i=1}^r M_{f_i} \xrightarrow{\beta} \prod_{i,j=1}^r M_{f_i f_j} \tag{3.9}$$

where $\alpha(s) = (s/1, \ldots, s/1)$ and $\beta(s_1, \ldots, s_r)_{i,j} = s_i/1 - s_j/1$.

*Proof* As (3.12) is a sequence of $R$-modules, it is exact if and only if it is exact after being localized at every prime ideal $\mathfrak{p}$. After reordering the indices, we may assume that $\mathfrak{p} \in D(f_1)$. As localization commutes with finite products, the localization of (3.10) at $\mathfrak{p}$ can be obtained via localizing the following sequence

$$0 \longrightarrow M_{f_1} \xrightarrow{a} \prod_{i=1}^r M_{f_1 f_i} \xrightarrow{b} \prod_{i,j=1}^r M_{f_1 f_i f_j} \tag{3.10}$$

It it is therefore enough to show that (3.10) is exact. We write this as

$$0 \longrightarrow M_{f_1} \xrightarrow{a} M_{f_1} \times \prod_{i=2}^r M_{f_1 f_i} \xrightarrow{b} \prod_{i=1}^r M_{f_1 f_i} \times \prod_{i,j=2}^r M_{f_1 f_i f_j} \tag{3.11}$$

Now it is clear that $a$ is injective, by projecting onto the first factor. Next, if $(s_1, \ldots, s_r) \in M_{f_1} \times \prod_{i=2}^r M_{f_1 f_i}$ lies in $\operatorname{Ker} b$, then by projecting onto the factor $\prod_{i=1}^r M_{f_1 f_i}$, we see that $s_i/1 = s_1/1$ in $M_{f_1 f_i}$ for all $i = 1, \ldots, r$. This means that $(s_1, \ldots, s_r) = (s_1/1, \ldots, s_1/1)$ is in the image of $a$. □

**Proposition 3.18.** $\mathcal{O}$ is a $\mathscr{B}$-sheaf of rings.

*Proof* *Step 1. Finite covers*. Suppose that $D(f)$ is a distinguished open set and $\{D(f_i)\}_{i=1}^r$ is a finite cover of $D(f)$. Then applying Lemma 3.17 to $R = A_f$ and $M = A_f$ gives an exact sequence

$$0 \longrightarrow A_f \xrightarrow{\alpha} \prod_{i \in J}^r A_{f_i} \xrightarrow{\beta} \prod_{i,j \in J}^r A_{f_i f_j} \tag{3.12}$$

The Locality and Gluing axioms follow immediately from this.

*Step 2. Arbitrary covers*. Suppose now that $\{D(f_i)\}_{i \in I}$ is a general cover. Since $D(f)$ is quasi-compact, we may pick a finite subset $J \subset I$, so that $\{D(f_j)\}_{j \in J}$ forms a subcover.

Locality axiom: If $s \in A_f$ maps to zero in $A_{f_i}$ for every $i \in I$, then in particular, it maps to zero in each $A_{f_i}$ for $i \in J$, so by Step 1, we have $s = 0$ in $A_f$.

Gluing axiom: Given compatible elements $s_i \in A_{f_i}$ for $i \in I$ (meaning $s_i/1 = s_j/1$ in $A_{f_i f_j}$ for all $i, j \in I$), then Step 1 provides a unique $s \in A_f$ such that $s_i = s/1 \in A_{f_i}$ for all

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$i \in J$. We claim that $s$ in fact induces $s_i$ for all $i \in I$. For this, fix an index $\alpha \in I$. Applying Step 1 to the finite covering consisting of $D(f_i)$, $i \in J \cup \{\alpha\}$, we get an element $s' \in A_f$ such that $s'/1 = s_i$ in $A_{f_i}$ for all $i \in J$ and also $s/1 = s_\alpha$. As $s$ and $s'$ coincide in each $A_{f_i}$ for $i \in J$, we must also have $s = s'$ in $A_f$ by uniqueness in Step 1, and hence $s/1 = s_\alpha$ holds in $A_{f_\alpha}$ as well. □

Using Proposition 3.14 on page 52, we may now make the following definition:

**Definition 3.19.** The *structure sheaf* $\mathcal{O}_{\operatorname{Spec} A}$ on $\operatorname{Spec} A$ is the unique sheaf extending the $\mathscr{B}$-sheaf $\mathcal{O}$.

The formula (3.6) tells us how to compute sections for a general open set $U \subset X$: cover $U$ by distinguished open sets $D(f_i)$, $i \in I$; then $\mathcal{O}_{\operatorname{Spec} A}(U)$ is identified with the group

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \left\{ (s_i) \in \prod_{i \in I} A_{f_i} \;\middle|\; s_i = s_j \text{ in } A_{f_i f_j} \text{ for all } i, j \in I \right\}. \tag{3.13}$$

That being said, we will basically never need to know the ring $\mathcal{O}_{\operatorname{Spec} A}(U)$ for other open sets than distinguished open sets $U = D(f)$.

We will an alternative construction of $\mathcal{O}_X$ in Example 11.18.

**Proposition 3.20.**
(i) $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$.
(ii) If $P \in \operatorname{Spec} A$ is a point corresponding to a prime ideal $\mathfrak{p}$, then

$$\mathcal{O}_{\operatorname{Spec} A, P} = A_{\mathfrak{p}}.$$

*Proof* We defined $\mathcal{O}_{\operatorname{Spec} A}$ so that $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f$ for every every $f \in A$. Taking $f = 1$, we see that (i) holds. For (ii), since the distinguished open sets form a basis for the topology, we may compute the stalk as:

$$\mathcal{O}_{\operatorname{Spec} A, P} = \varinjlim_{P \in D(f)} \mathcal{O}_{\operatorname{Spec} A}(D(f)) \simeq \varinjlim_{f \notin \mathfrak{p}} A_f.$$

We claim that the natural map $\varinjlim_{f \notin \mathfrak{p}} A_f \to A_{\mathfrak{p}}$ is an isomorphism. Surjectivity holds because any element $a/s \in A_{\mathfrak{p}}$, where $s \notin \mathfrak{p}$ is in the image of $a/s \in A_s$ under the canonical map $A_s \to A_{\mathfrak{p}}$. For injectivity: an element $a/f^n$ maps to 0 in $A_{\mathfrak{p}}$ precisely when there is an element $s \in A - \mathfrak{p}$ such that $as = 0$, but then $a/f^n = 0$ in $A_g$, where $g = sf$. Hence $a/f^n$ vanishes in the direct limit, and so we get (ii). □

*Remark* 3.21. The structure sheaf behaves particularly well when the base ring $A$ is an integral domain. In this case, all the localizations $A_f$ are subrings of the field of fractions $K$ of $A$, and for $D(g) \subset D(f)$, the localization maps $A_f \to A_g$ are simply inclusions of subrings. This implies that $\mathcal{O}_X(U)$ is a subring of $K$ for any open set $U$. Indeed, covering $U$ by distinguished open sets $\{D(f_i)\}_{i \in I}$, the formula (3.13) shows that $\mathcal{O}_X(U)$ is identified with the intersection $\bigcap_{i \in I} A_{f_i}$ inside $K$. Moreover, by Proposition A.5, we have

$$\mathcal{O}_X(U) = \bigcap_{i \in I} A_{f_i} = \bigcap_{i \in I} \bigcap_{\mathfrak{p} \in D(f_i)} A_{\mathfrak{p}} = \bigcap_{\mathfrak{p} \in U} A_{\mathfrak{p}}.$$

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Consequently, when $A$ is an integral domain, the intuition from affine varieties is valid: the rings $\mathcal{O}_X(U)$ are subrings of the fraction field of $A$, and the elements $f \in \mathcal{O}_X(U)$ are precisely those for which, for each point $P \in U$, we may write $f = a/b$ and $b(P) \neq 0$.

### *Examples*

**Example 3.22.** For a field $K$, the structure sheaf $\mathcal{O}_{\operatorname{Spec} K}$ is a constant sheaf with the value $K$ at the single point of $\operatorname{Spec} K$. △

**Example 3.23.** The structure sheaf of $\operatorname{Spec} \mathbb{Z}$ satisfies $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(n)) = \mathbb{Z}[\frac{1}{n}]$ for each natural number $n$. The stalks of $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}$ at the closed point $(p)$ is equal to $\mathcal{O}_{\operatorname{Spec} \mathbb{Z},(p)} = \mathbb{Z}_{(p)}$ and at the generic point the stalk is equal to $\mathcal{O}_{\operatorname{Spec} \mathbb{Z},(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$. △

**Example 3.24.** Let $X = \operatorname{Spec} \mathbb{C}[t]$. Then the stalk of $\mathcal{O}_X$ at the generic point $\xi = (0)$ is equal to $\mathcal{O}_{X,\xi} = \mathbb{C}(t)$. Each closed point $P \in X$ corresponds to a maximal ideal $(t - a)$, and the stalk of $\mathcal{O}_X$ at $P$ is equal to $\mathcal{O}_{X,P} = \mathbb{C}[t]_{(t-a)}$. △

**Example 3.25.** We continue Example 2.14 about the spectrum of a discrete valuation ring $A$. The space $X = \operatorname{Spec} A$ has two points, $s$ and $\xi$, and there are three open sets: $\varnothing$, $\{\xi\}$, and $X$. The maximal ideal of $A$ is principal, so we may write $\mathfrak{m} = (t)$ for $t \in A$. Then $\{\xi\} = D(t)$ is a distinguished open in $X$. The structure sheaf takes the following values at the open sets:

$$\mathcal{O}_X(\varnothing) = 0, \qquad \mathcal{O}_X(X) = A, \qquad \mathcal{O}_X(\{\xi\}) = A_t = K,$$

where $K$ denotes the fraction field of $A$. The stalks are given by $\mathcal{O}_{X,s} = A_{(t)} = A$ and $\mathcal{O}_{X,\xi} = A_{(0)} = K$. △

**Example 3.26** (Disconnected spectra)**.** Using the structure sheaf, we may now prove that if $A$ is a ring such that $X = \operatorname{Spec} A$ is not connected, then $A$ is a nontrivial product of two rings. Suppose $X = U_1 \cup U_2$, where $U_1$ and $U_2$ are open subsets with $U_1 \cap U_2 = \varnothing$. The sheaf exact sequence takes the form

$$0 \longrightarrow \mathcal{O}_X(X) = A \longrightarrow \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_1 \cap U_2) = 0,$$

and we deduce that there is an isomorphim of rings $A \simeq \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$. △

**Example 3.27.** Consider the open subset $U = \mathbb{A}^2_k - V(x, y)$ of $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$. We can use the sheaf sequence to compute $\mathcal{O}_{\mathbb{A}^2_k}(U)$. In analogy with Example 1.26, we will show that the restriction map $\mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k) \to \mathcal{O}_{\mathbb{A}^2_k}(U)$ is an isomorphism; that is, $\mathcal{O}_{\mathbb{A}^2_k}(U) = k[x, y]$.

To do this, we use the sheaf sequence (3.1) for the covering $\{D(x), D(y)\}$ of $U$. In view of the equalities $\mathcal{O}_{\mathbb{A}^2_k}(D(x)) = k[x, y]_x$ and $\mathcal{O}_{\mathbb{A}^2_k}(D(y)) = k[x, y]_y$, the sheaf sequence takes the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}(U) \xrightarrow{\alpha} k[x, y]_x \oplus k[x, y]_y, \xrightarrow{\beta} k[x, y]_{xy}$$

Here $\alpha(s) = (s|_{D(x)}, s|_{D(y)})$ and $\beta$ sends a pair $(a/x^n, b/y^m)$ to $a/x^n - b/y^m$. The group $\mathcal{O}_{\mathbb{A}^2_k}(U)$ is identified with the kernel of $\beta$, which consists of pairs $(a/x^n, b/y^m)$ such that $a/x^n = b/y^m$ in $k[x, y]_{xy}$. This equality implies that $ay^m = bx^n$ in $k[x, y]$. As the

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polynomial ring $k[x,y]$ is a UFD, we conclude that $a = cx^n$ and $b = cy^m$ for some $c \in k[x,y]$. But this means that any element in the kernel is of the form $(c,c)$ with $c \in k[x,y]$, and so $\mathcal{O}_{\mathbb{A}^2_k}(U) = k[x,y]$. △

### 3.5 The sheaf associated to an $A$-module

The contruction of the structure sheaf can be extended for arbitrary $A$-modules. For each $A$-module $M$, we define a presheaf $\widetilde{M}$ on $\operatorname{Spec} A$, whose sections over distinguished open sets $D(f)$ are given by

$$\widetilde{M}(D(f)) = M_f.$$

If $D(g) \subset D(f)$, the restriction map is given by the localization map $M_f \to M_g$. The same proof as for the structure sheaf (Proposition 3.18), shows that this is actually a $\mathscr{B}$-sheaf, which extends uniquely to a sheaf on $\operatorname{Spec} A$, which we will continue to denote by $\widetilde{M}$.

**Proposition 3.28.** (i) The global sections of $\widetilde{M}$ recovers the module:

$$\Gamma(\operatorname{Spec} A, \widetilde{M}) = M.$$

(ii) For a point $x \in \operatorname{Spec} A$ corresponding to a prime ideal $\mathfrak{p}$, the stalk is given by

$$\widetilde{M}_x = M_{\mathfrak{p}}.$$

This tilde-construction is functorial in $M$. For any $A$-linear map $\phi\colon M \to N$, there is an induced map of sheaves $\widetilde{\phi}\colon \widetilde{M} \to \widetilde{N}$. To define $\widetilde{\phi}$, it suffices, by Proposition 3.14, to specify it over the distinguished open sets $D(f)$. Here, we define $\tilde{\phi}_{D(f)}\colon \widetilde{M}(D(f)) \to \widetilde{N}(D(f))$ to be the localization of $\phi$, that is, the map

$$\phi_f\colon M_f \to N_f; \qquad m/f^n \mapsto \phi(m)/f^n$$

This defines a map of $\mathscr{B}$-sheaves because the following diagram commutes for each $f$ and $g$ with $D(g) \subset D(f)$:

$$\begin{array}{ccc} M_f & \xrightarrow{\phi_f} & N_f \\ \downarrow & & \downarrow \\ M_g & \xrightarrow{\phi_g} & N_g \end{array}$$

Indeed, writing $g^r = af$, the composition $M_f \to M_g \to N_g$ sends $m/f^n$ to $\phi(ma^n)/g^{rn} = a^n\phi(m)/g^{rn}$, which is is the same as the image via $M_f \to N_f \to N_g$.

We also have $\widetilde{\phi \circ \psi} = \widetilde{\phi} \circ \widetilde{\psi}$, whenever $\phi$ and $\psi$ are composable $A$-linear maps. Consequently the tilde-operation defines a functor from the category $\mathsf{Mod}_A$ of $A$-modules to the category of sheaves on $\operatorname{Spec} A$.

**Example 3.29.** Let $A$ be a ring and let $I \subset A$ be an ideal. Then $\widetilde{I}$ is an *ideal sheaf* in $\mathcal{O}_{\operatorname{Spec} A}$. This means that for each $U \subset \operatorname{Spec} A$, the space of sections $\widetilde{I}(U)$ is an ideal of $\mathcal{O}_X(U)$. For $U = D(f)$, we have $\widetilde{I}(D(f)) = IA_f$, the ideal generated by $I$ in $A_f$. △

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**Example 3.30.** Let $A = k[x,y]/(x^2+y^2-1)$ and $X = \operatorname{Spec} A$. Consider the $A$-module

$$M = (Ae_1 \oplus Ae_2)/(xe_1 + ye_2).$$

Let us determine the stalk of $\widetilde{M}$ at the point $P = (0,1) \in X$ corresponding to the prime ideal $\mathfrak{p} = (x, y-1)$. Since $y \notin \mathfrak{p}$, $y$ is invertible in $A_\mathfrak{p}$, and we can replace the relation $xe_1 + ye_2$ by $xy^{-1}e_1 + e_2$. This allows us to eliminate the component $A_\mathfrak{p}e_2$, and we find:

$$(\widetilde{M})_P = M_\mathfrak{p} = (A_\mathfrak{p}e_1 \oplus A_\mathfrak{p}e_2)/(xy^{-1}e_1 + e_2) \simeq A_\mathfrak{p} = \mathcal{O}_{X,P}.$$

As $X$ is covered by the two distinguished open sets $D(x)$ and $D(y)$, the same type of argument shows that $(\widetilde{M})_\mathfrak{p} \simeq A_\mathfrak{p}$ for every $\mathfrak{p}$. Therefore $\widetilde{M}$ and $\mathcal{O}_X$ have isomorphic stalks at every point. △

## 3.6 Exercises

**Exercise 3.6.1** (Differential operators)**.** Let $X = \mathbb{R}$ and let $C^r(X,\mathbb{R})$ be the sheaf of functions $f\colon U \to \mathbb{R}$ which are $r$ times continuously differentiable.

a) Show that $C^r(X,\mathbb{R})$ is a subsheaf of $C(X,\mathbb{R})$.
b) Show that the differential operator $D = d/dx$ defines a morphism of sheaves $D\colon C^r(X,\mathbb{R}) \to C^{r-1}(X,\mathbb{R})$.

**Exercise 3.6.2.** Let $X$ be the set with two elements with the discrete topology. Find a presheaf on $X$ for which the Locality axiom fails. HINT: Define $\mathcal{F}(X)$ to be much bigger than $\mathcal{F}(\{p\})$ and $\mathcal{F}(\{q\})$.

**Exercise 3.6.3.** In the notation of Example 3.5, the differential operator gives a map of sheaves $D\colon \mathscr{A}_X \to \mathscr{A}_X$, where as previously $X \subset \mathbb{C}$ is an open set.

a) Show that the assignment

$$\mathcal{F}(U) = \{\, f \in \mathscr{A}_X(U) \mid Df = 0 \,\}$$

defines a subsheaf $\mathcal{F}$ of $\mathscr{A}_X$.
b) Show that if $U$ is a connected open subset of $X$, one has $\mathcal{F}(U) = \mathbb{C}$.
c) For a not necessarily connected set $U$, show that $\mathcal{F}(U) = \prod_{\pi_0(U)} \mathbb{C}$ where the product is taken over the set $\pi_0(U)$ of connected components of $U$. Deduce that $\mathcal{F}$ is the constant sheaf with value $\mathbb{C}$.

**Exercise 3.6.4.** Let $X \subset \mathbb{C}$ be an open set.

a) For $U \subset X$ open, define $\mathcal{K}(U)$ to be the group of meromorphic functions on $U$ (these are functions holomorphic on all of $U$ except for a set of isolated points, where they have poles). Show that $\mathcal{K}(U)$ is a sheaf of groups which contains the sheaf of holomorphic functions $\mathcal{A}_X$ as a subsheaf.
b) For points $a_1, \dots, a_r \in X$ and $n_1, \dots, n_r$ natural numbers, define $\mathcal{F}(U)$ to be the set of meromorphic functions in $U$ which are holomorphic away from the $a_i$'s and having a pole order bounded by $n_i$ at $a_i$. Show that $\mathcal{F}$ is a sheaf of abelian groups.
c) Are $\mathcal{K}$ and $\mathcal{F}$ sheaves of rings?

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**Exercise 3.6.5.** Let $X$ be a topological space and let $Z \subset X$ be a closed subset with inclusion $\iota\colon Z \to X$. For a sheaf $\mathcal{F}$ on $Z$, describe the stalks of $\iota_*\mathcal{F}$.

**Exercise 3.6.6.** Let $X$ be a topological space and define $\mathcal{F}$ by setting $\mathcal{F}(X) = 0, \mathcal{F}(\varnothing) = 0$ and $\mathcal{F}(U) = \mathbb{Z}$ for every other open set, and with restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ equal to the identity if $U$ and $V$ are different from $\varnothing$ and $X$, and the zero map otherwise. Show that $\mathcal{F}$ is a presheaf. Is $\mathcal{F}$ a sheaf?

**Exercise 3.6.7** (The sheaf of homomorphisms)**.** Given two presheaves $\mathcal{F}$ and $\mathcal{G}$, we may form a presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ by letting the sections over an open $U$ be given by

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathsf{Ab}(X)}(\mathcal{F}|_U, \mathcal{G}|_U). \tag{3.14}$$

If $V \subset U$, the restriction map sends $\phi\colon \mathcal{F}|_U \to \mathcal{G}|_U$ to the restriction $\phi|_V\colon \mathcal{F}|_V \to \mathcal{G}|_V$. Show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf whenever $\mathcal{G}$ is a sheaf.

**Exercise 3.6.8.** Let $\mathcal{F}$ be a sheaf and let $s, t \in \mathcal{F}(U)$ be two sections. Show that $s = t$ if and only if $s_x = t_x$ for every $x \in U$.

**Exercise 3.6.9.** Let $\phi, \psi\colon \mathcal{F} \to \mathcal{G}$ be maps of presheaves and assume that $\mathcal{G}$ is a sheaf. Prove that $\phi = \psi$ if and only if $\phi$ and $\psi$ induce the same maps on all stalks, i.e. $\phi_x = \psi_x$ for every $x \in X$. HINT: Use Exercise 3.6.8.

**Exercise 3.6.10.** Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves. We say that $\phi$ is *injective* (resp. surjective) if $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective (resp. surjective) for every point $x \in X$.

a) Show that $\phi$ is injective if and only if $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open set $U$.
b) Give an example of a map of sheaves $\phi$ so that $\phi_x$ is surjective for all $x \in X$, but $\phi_X : \mathcal{F}(X) \to \mathcal{G}(X)$ is not surjective. HINT: Example 3.8.

**Exercise 3.6.11.** Verify the formula (3.4).

**Exercise 3.6.12.** Denote by $\{*\}$ a one point set. Let $X$ be a topological space and $f\colon X \to \{*\}$ be the one and only map. Show that $f_*\mathcal{F} = \Gamma(X, \mathcal{F})$ (where strictly speaking $\Gamma(X, \mathcal{F})$ stands for the constant sheaf on $\{*\}$ with value $\Gamma(X, \mathcal{F})$).

**Exercise 3.6.13.** Let $X$ be a topological space and $x \in X$ a point that is not necessarily closed. Let $\iota : \{x\} \to X$ be the inclusion. Let $\mathcal{A}$ be the constant sheaf on $\{x\}$ with value the group $A$. Show that the stalks of $\iota_*\mathcal{A}$ are

$$(\iota_*\mathcal{A})_y = \begin{cases} A & \text{if } y \in \overline{\{x\}} \\ 0 & \text{otherwise.} \end{cases}$$

**Exercise 3.6.14.** Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on a space $X$ and assume there is an open covering $\mathcal{U}$ of $X$ and isomorphisms $\theta_U\colon \mathcal{F}|_U \simeq \mathcal{G}|_U$ that match on intersections; that is, $\theta_U|_{U\cap U'} = \theta_{U'}|_{U\cap U'}$. Show that there is an isomorphism $\theta\colon \mathcal{F} \simeq \mathcal{G}$ extending the $\theta_U$'s. HINT: It is possible to do this via $\mathscr{B}$-sheaves.

**Exercise 3.6.15.** Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of a space $X$ and let $\mathcal{U}' = \{U_i\}_{i\in J}$ be a subcover ($J \subset I$). Show that the sequence (3.1) is exact for the covering $\mathcal{U}$ if and only if it is exact for $\mathcal{U}'$.

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**Exercise 3.6.16** (The restriction of a sheaf)**.** Let $U \subset X$ be an open subset of a topological space $X$ and let $\mathcal{F}$ be a sheaf on $X$. Define $\mathcal{F}|_U$ by the assignment $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for $V \subset U$.

a) Show that $\mathcal{F}|_U$ defines a sheaf on $U$, and that the stalks at points in $U$ are equal to the corresponding stalks of $\mathcal{F}$.
b) Let $\mathcal{G}$ be a sheaf on $U$. Let $\iota\colon U \to X$ be the inclusion map. Show that $\iota_*(\mathcal{G})|_U \simeq \mathcal{G}$.

**Exercise 3.6.17.** Let $X = \mathbb{R}$ with the standard topology.

a) Let $\mathcal{F}$ be the subpresheaf of $C(-, \mathbb{R})$ defined by

$$\mathcal{F}(U) = \{\, f\colon U \to \mathbb{R} \mid f \text{ is bounded} \,\}$$

Show that $\mathcal{F}$ is not a sheaf.
b) Let $\mathcal{G}$ be the subpresheaf of $C(-, \mathbb{R})$ defined by

$$\mathcal{G}(U) = \{\, f\colon U \to \mathbb{R} \mid f(0) = f(1) \,\}$$

Show that $\mathcal{G}$ is not a sheaf.

**Exercise 3.6.18.** Let $X$ be a topological space where every open cover of any open set has a finite subcover (i.e., if $U = \bigcup_{i\in I} U_i$, then there exists a finite $J \subset I$ such that $U = \bigcup_{i\in J} U_i$). Show that a presheaf $\mathcal{F}$ is a sheaf if and only if it satisfies the Locality and Gluing axioms for all finite covers.

**Exercise 3.6.19.** Let $\mathscr{B}$ be a basis and let $F$ be a $\mathscr{B}$ sheaf. Show that the stalk of the extended sheaf $\mathcal{F}$ at a point $x \in X$ is given by

$$\mathcal{F}_x = \varinjlim_{U\in\mathscr{B},\ x\in U} F(U). \tag{3.15}$$

**Exercise 3.6.20.** Let $X = \operatorname{Spec} A$ where $A = \mathbb{Q}[x, y, z]/(x^2, xy, xz)$. Show that if $U = X - \{(0, 0, 0)\}$, then

$$\mathcal{O}_X(U) \neq S(U)^{-1}A.$$

HINT: Write $U = D(y) \cup D(z)$ and imitate Example 3.27.

This example shows that the formula $\mathcal{O}_X(U) = S(U)^{-1}A$ does not hold for *all* open sets $U$.

**Exercise 3.6.21** (Alternative description of $\mathcal{O}_X$)**.** Let $X = \operatorname{Spec} A$. Consider the partial order on the elements of $A$ defined by $g \leqslant f$ if $D(g) \subset D(f)$.

a) Show that the localizations $A_f$ form a directed system of $A$-modules with the localization maps $A_f \to A_g$ whenever $g \leqslant f$.
b) For an open set $U \subset X$, define

$$\mathcal{O}_X(U) = \varprojlim_{f\in A,\ D(f)\subset U} A_f.$$

That is, a section of $\mathcal{O}_X(U)$ is a collection of elements $(s_f)_f$ where each $s_f \in A_f$ such that if $D(g) \subset D(f)$, then $s_f$ maps to $s_g$ under $A_f \to A_g$. If

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$V \subset U$ are two opens, then any $D(f)$ which is contained in $V$ is also contained in $U$, so there are canonical restriction maps $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.
Compare with the formula (3.7).

c) Show that $\mathcal{O}_X$ is a sheaf.
d) Show that $\mathcal{O}_X$ is isomorphic to the structure sheaf on $\operatorname{Spec} A$ as defined in the text.

**Exercise 3.6.22.** Describe the points of the prime spectrum $X = \operatorname{Spec} \mathbb{C}[x, y]_{(x,y)}$. Compute $\mathcal{O}_X(U)$ where $U = X - \{(x, y)\}$.

**Exercise 3.6.23.** Let $X \subset \mathbb{C}$ be an open set and let $\mathcal{A}$ be the sheaf of holomorphic functions on $X$. Show that for a point $P \in X$, the stalk $\mathcal{A}_P$ can be identified with the ring of power series that converge in a neighbourhood of $P$.

**Exercise 3.6.24.** Let $X$ be a topological space and let $\mathcal{F}$ be a sheaf of rings on $X$. Show that if $s \in \mathcal{F}(X)$, the map $\mathcal{F}(U) \to \mathcal{F}(U)$ given by multiplication by $s$ is a morphism of sheaves.

**Exercise 3.6.25.** Let $A$ be an Artinian ring and let $X = \operatorname{Spec} A$. Using the structure sheaf and the fact that $X$ is discrete, show that

$$A \simeq \prod_{\mathfrak{m}} A_{\mathfrak{m}}.$$

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# 4

# Schemes

## 4.1 Locally ringed spaces

We would like to define a scheme to be a space which is 'locally affine', that is, one where every point has a neighbourhood which is isomorphic to the spectrum of a ring. To be able to make such a definition precise, we need a suitable category of spaces to work with. To this end, we use the two pieces of data we have: $\operatorname{Spec} A$ and its sheaf of rings $\mathcal{O}_{\operatorname{Spec} A}$.

**Definition 4.1** (Locally ringed spaces)**.** A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of rings on $X$ such that all the stalks $\mathcal{O}_{X,x}$ are local rings.

To make this into a category, we need to specify what a morphism between two locally ringed spaces is. Reflecting the above definition, a morphism betwen $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ should have two components, one map between the underlying topological spaces $X$ and $Y$ and one on the level of sheaves. Note that it does not make sense to talk about morphisms $\mathcal{O}_Y \to \mathcal{O}_X$, as these sheaves live on different spaces. Instead, once a continuous map $f\colon X \to Y$ is specified, the sheaf map should be a map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on $Y$. This means that for all open subsets $U \subset Y$, there are ring maps

$$f_U^\sharp\colon \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(f^{-1}U),$$

compatible with the restriction maps; that is, such that the following diagrams commute:

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f_U^\sharp} & \mathcal{O}_X(f^{-1}U) \\ {\scriptstyle \rho_{UV}}\downarrow & & \downarrow{\scriptstyle \rho_{f^{-1}U\, f^{-1}V}} \\ \mathcal{O}_Y(V) & \xrightarrow[f_V^\sharp]{} & \mathcal{O}_X(f^{-1}V). \end{array} \tag{4.1}$$

The intuition again comes from the setting of affine varieties. As we saw on page **??**, a morphism of affine varieties $f\colon X \to Y$ is precisely a continuous map such that the pullback $g \circ f$ is a regular function on $f^{-1}(U)$ whenever $g$ is regular on $U$. In other words, the pullback is a ring map $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$, and the collection of all of these defines a map of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Note that if $g$ is a regular function which vanishes at a point $y \in Y$ then its pullback $f^\sharp(g) = g \circ f$ vanishes at every point of $x \in f^{-1}(y)$ in $U$. In other words, if $g \in \mathfrak{m}_y$, then $f^\sharp(g) \in \mathfrak{m}_x$ for each such $x$ and $y$.

For a general locally ringed space, we do not have the luxury of speaking about functions



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into some fixed field $k$, so the sheaf map $f^\sharp$ has to be specified as part of the definition of a morphism. Moreover, the condition of 'pulling back vanishing functions' translates into the following requirement. For a point $x \in X$ and $y = f(x)$, the morphism $f^\sharp$ induces a map of local rings:

$$f_x^\sharp \colon \mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}, \tag{4.2}$$

This map is defined as follows: pick an element $s_y \in \mathcal{O}_{Y,y}$ and represent it as the germ of a section $s \in \mathcal{O}_Y(W)$ over some open set $W \subset Y$. Then the section $t = f_W^\sharp(s)$ is a section of $\mathcal{O}_X(f^{-1}W)$. We define $f_x^\sharp(s_y)$ to be the germ of this section at $x$, i.e. $f_x^\sharp(s_y) = t_x \in \mathcal{O}_{X,x}$. This makes sense because $f^{-1}W$ contains $x$. By the definition of the stalk, this does not depend on the choice of open set $W$ containing $y$.

In the case of affine varieties, the map (4.2) sends a rational function $g \in \mathcal{O}_{Y,y}$ which is regular at $y = f(x)$ to $g \circ f$ which is regular at $x$. By what we said above, $f_x^\sharp$ sends the maximal $\mathfrak{m}_y$ into $\mathfrak{m}_x$ for every $x \in X$. In other words, $f_x^\sharp$ is a *map of local rings*.

**Definition 4.2** (Morphisms of locally ringed spaces)**.** A *morphism*, or simply *map*, of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$, where

(i) $f \colon X \to Y$ is a continuous map,

(ii) $f^\sharp \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves of rings on $Y$,

such that for every $x \in X$, the induced map on stalks

$$f_x^\sharp \colon \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$$

is a map of local rings; that is, $f_x^\sharp(\mathfrak{m}_{f(x)}) \subset \mathfrak{m}_x$.

For simplicity, we usually denote the morphism by $f \colon X \to Y$, omitting $f^\sharp$.

Morphisms of locally ringed spaces can be composed. Given $(f, f^\sharp) \colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $(g, g^\sharp) \colon (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$, we define the morphism $X \to Z$ by the composition $g \circ f$ on the level of topological spaces. The sheaf map $(g \circ f)^\sharp$ is defined over an open set $U \subset Z$ by the composition

$$\mathcal{O}_Z(U) \xrightarrow{g^\sharp} \mathcal{O}_Y(g^{-1}U) \xrightarrow{f^\sharp} \mathcal{O}_X((g \circ f)^{-1}U) = \Gamma(U, (g \circ f)_*\mathcal{O}_X).$$

An *isomorphism of locally ringed spaces* is a morphism $f \colon X \to Y$ which admits an inverse morphism. In other words, there is a morphism $g \colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In concrete terms, this boils down to: (i) $f$ being a homeomorphism, and (ii) $f_U^\sharp \colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$ being an isomorphism of rings for every open $U \subset Y$.

### $\operatorname{Spec} A$ *as a locally ringed space*

For a ring $A$, the pair $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is by design a locally ringed space. The set $\operatorname{Spec} A$ is endowed with the Zariski topology and the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is a sheaf of rings with stalks which are local rings of the form $A_\mathfrak{p}$. In this section, we show that ring maps induce maps of locally ringed spaces.

Let $\phi \colon A \to B$ be a map of rings and write $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. We define a

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morphism

$$(f, f^\sharp)\colon (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y). \tag{4.3}$$

as follows. On the level of topological spaces, we define $f\colon X \to Y$ by

$$f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}). \tag{4.4}$$

The sheaf map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is defined on the basis of distinguished open sets. Consider a distinguished open set $D(g) \subset Y = \operatorname{Spec} A$. Then,

$$f^{-1}(D(g)) = D(\phi(g)). \tag{4.5}$$

The sections involved are

$$\begin{aligned} \mathcal{O}_Y(D(g)) &= A_g, \\ f_*\mathcal{O}_X(D(g)) &= \mathcal{O}_X(D(\phi(g))) = B_{\phi(g)}. \end{aligned}$$

We define $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ over $D(g)$ by the canonical localization map

$$\begin{aligned} A_g &\longrightarrow B_{\phi(g)} \\ \frac{a}{g^n} &\longmapsto \frac{\phi(a)}{\phi(g)^n}. \end{aligned} \tag{4.6}$$

Note that if $\phi(g) = 0$, then $B_{\phi(g)} = 0$, and this is the zero map.

These maps define a morphism of $\mathscr{B}$-sheaves: if $D(h) \subset D(g)$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_Y(D(g)) & \longrightarrow & \mathcal{O}_X(D(\phi(g))) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(D(h)) & \longrightarrow & \mathcal{O}_Y(D(\phi(h))) \end{array} \quad = \quad \begin{array}{ccc} A_g & \longrightarrow & B_{\phi(g)} \\ \downarrow & & \downarrow \\ A_h & \longrightarrow & B_{\phi(h)} \end{array}$$

In the diagram on the left, the vertical maps on the left are restriction maps of sheaves, while on the right, the vertical maps are localization maps of rings.

Finally, we check that for each point $x \in X$, the induced map $f_x^\sharp\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a map of local rings. Let $\mathfrak{p}$ be the prime ideal corresponding to $x$, so that $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$ is the prime ideal corresponding to $f(x) \in \operatorname{Spec} A$. From the paragraph following (4.2), we can understand the map $f_x^\sharp$ as follows: For an element $t \in \mathcal{O}_{Y,f(x)} = A_\mathfrak{q}$, writing it as $t = a/g^n$ allows us to view it as the germ of a section $a/g^n \in \mathcal{O}_Y(D(g))$ for some open set $D(g)$ containing $\mathfrak{q}$. Note that $\mathfrak{q} \in D(g)$ means that $g \notin \mathfrak{q} = \phi^{-1}(\mathfrak{p})$. Then $f_x^\sharp(t)$ is the image of $t$ via the composition

$$A_g \longrightarrow B_{\phi(g)} \longrightarrow B_\mathfrak{p} = \mathcal{O}_{X,x}. \tag{4.7}$$

Now, if $t \in \mathfrak{q}A_\mathfrak{q}$ lies in the maximal ideal, then we can write $t = a/g^n$ with $a \in \mathfrak{q} = \phi^{-1}(\mathfrak{p})$. Then $\phi(a)/\phi(g)^n \in \mathfrak{p}B_{\phi(g)}$, and hence $f_x^\sharp(t) \in \mathfrak{p}B_\mathfrak{p}$ lies in the maximal ideal of $\mathcal{O}_{X,x}$. This shows that $(f, f^\sharp)$ is a morphism of locally ringed spaces.

The second part of the next example gives a pair $(f, f^\sharp)$ which is not a morphism of locally ringed spaces.

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**Example 4.3.** Let $X = \operatorname{Spec} \mathbb{C}(t)$ and $Y = \mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t]$. There is a natural map $f\colon X \to Y$ induced by the inclusion $\mathbb{C}[t] \subset \mathbb{C}(t)$. On the level of topological spaces, $X$ consists of a single point $\xi$, and $f$ maps $\xi$ to the generic point $\gamma$ of $Y$. The corresponding stalk map $f^\sharp_\xi\colon \mathcal{O}_{Y,\gamma} \to \mathcal{O}_{X,\xi}$ is the identity map

$$\mathcal{O}_{Y,\gamma} = \mathbb{C}[t]_{(0)} = \mathbb{C}(t) \longrightarrow \mathbb{C}(t) = \mathcal{O}_{X,\xi},$$

which is certainly a map of local rings.

On the other hand, we could try to define a strange map $g\colon X \to Y$ by sending $\xi$ to some other point $y \in \mathbb{A}^1_{\mathbb{C}}$ corresponding to a maximal ideal $(t-a) \subset \mathbb{C}[t]$. This map is certainly continuous. For the sheaf map, note that $(g_*\mathcal{O}_X)(U) = \mathbb{C}(t)$ if $U$ contains $y$ and 0 otherwise. We can therefore define $g^\sharp\colon \mathcal{O}_Y \to g_*\mathcal{O}_X$ as follows: for an open set $U \subset Y$, $g^\sharp_U$ is the inclusion map $\mathcal{O}_Y(U) \subset \mathbb{C}(t)$ if $y \in U$, and the zero map otherwise. This is clearly a map of sheaves of rings. However, the map on stalks is given by

$$g^\sharp_\xi\colon \mathcal{O}_{Y,y} = \mathbb{C}[t]_{(t-a)} \longrightarrow \mathbb{C}(t) = \mathcal{O}_{X,\xi}$$

and this sends the maximal ideal $(t-a)$ to the unit ideal in $\mathbb{C}(t)$. In other words, the element $t-a$ vanishes at $y \in Y$, but its pullback, the image of $t-a$ in $\mathbb{C}(t)$ does not vanish at $\xi$. Therefore, $g$ is not a morphism of locally ringed spaces. △

## 4.2 Schemes

We have now come to the definition of a scheme.

**Definition 4.4.** An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring $A$.

If $X$ is a locally ringed space and $U \subset X$ is an open subset, the restriction $\mathcal{O}_X|_U$ is a sheaf on $U$ making $(U, \mathcal{O}_X|_U)$ into a locally ringed space. Indeed, the stalks of $\mathcal{O}_X|_U$ are the same as those of $\mathcal{O}_X$ at every point of $U$. In particular, it makes sense to ask when a locally ringed space is locally isomorphic to an affine scheme:

**Definition 4.5** (Schemes)**.** A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme in the following sense: there is an open cover $\{U_i\}_{i\in I}$ of open subsets of $X$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to some affine scheme $(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$.

To summarize, a scheme consists of a topological space $X$, which has a covering consisting of open sets of the form $\operatorname{Spec} A_i$, and sheaf of rings $\mathcal{O}_X$ which restricts to the structure sheaves $\mathcal{O}_{\operatorname{Spec} A_i}$. For a point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring with maximal ideal $\mathfrak{m}_P$. The quotient field $\kappa(P) = \mathcal{O}_{X,P}/\mathfrak{m}_P$ is called the *residue field*. If $P$ is contained in an affine open set $U = \operatorname{Spec} A$ and corresponds to a prime ideal $\mathfrak{p}$, then we have

$$\mathcal{O}_{X,P} = A_{\mathfrak{p}}, \quad \mathfrak{m}_P = \mathfrak{p}A_{\mathfrak{p}}, \quad \kappa(P) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}.$$

A *morphism*, or *map* for short, between two schemes $X$ and $Y$ is simply a map $f$ between $X$

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and $Y$ regarded as locally ringed spaces. In this way, the schemes form a category, which we shall denote by $\mathsf{Sch}$. It contains the category of affine schemes $\mathsf{AffSch}$ as a full subcategory.

## 4.3 Morphisms into an affine scheme

The following is a fundamental result in the theory of schemes.

**Theorem 4.6.** Let $X$ be a scheme and let $A$ be a ring. Then there is a canonical bijection

$$\mathrm{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) = \mathrm{Hom}_{\mathsf{Rings}}(A, \mathcal{O}_X(X)). \tag{4.8}$$

defined by sending $(f, f^\sharp)\colon X \to \operatorname{Spec} A$ to the map of rings $f^\sharp_{\operatorname{Spec} A}\colon A \to \mathcal{O}_X(X)$.

*Proof* *Injectivity of* (4.8): Let $(f, f^\sharp)$ and $(g, g^\sharp)$ be two morphisms of schemes $X \to \operatorname{Spec} A$ and assume that $f^\sharp$ and $g^\sharp$ induce the same ring map $\phi\colon A \to \mathcal{O}_X(X)$.

Let $x \in X$ and write $\mathfrak{p} \subset A$ for the prime ideal corresponding to $y = f(x)$. If $\ell\colon A \to A_\mathfrak{p}$ denotes the localization map and $\rho_x\colon \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ denotes the germ map, then there is a commutative diagram

$$\begin{array}{ccccc} A & = & \mathcal{O}_Y(Y) & \xrightarrow{\phi} & \mathcal{O}_X(X) \\ \downarrow{\scriptstyle \ell} & & \downarrow & & \downarrow{\scriptstyle \rho_x} \\ A_\mathfrak{p} & = & \mathcal{O}_{Y,f(x)} & \xrightarrow{f^\sharp_x} & \mathcal{O}_{X,x} \end{array} \tag{4.9}$$

From the diagram, we see that

$$\mathfrak{p} = \ell^{-1}(\mathfrak{p}A_\mathfrak{p}) \overset{(*)}{=} \ell^{-1}((f^\sharp_x)^{-1}(\mathfrak{m}_x)) = \phi^{-1}(\rho_x^{-1}(\mathfrak{m}_x)). \tag{4.10}$$

where in $(*)$, we have $\mathfrak{p}A_\mathfrak{p} = (f^\sharp_x)^{-1}(\mathfrak{m}_x)$, because $f^\sharp_x$ is a map of local rings. As the right-hand side of (4.10) only depends on $\phi$ and $x$, we see that $\mathfrak{p}$ is determined by $\phi$ and hence $f(x)$ and $g(x)$ must be equal, as they correspond to the same prime ideal in $A$. Therefore, $f$ and $g$ are equal on the level of topological spaces. We will from now on write $f$ for this map.

Next we show that $f^\sharp = g^\sharp$ as maps of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$. As the sheaves involved are on $Y = \operatorname{Spec} A$, it suffices to show that they agree over the distinguished open sets. For $h \in A$, consider the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(Y) & \xrightarrow{\phi} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(D(h)) & \xrightarrow{f^\sharp_{D(h)}} & \mathcal{O}_X(f^{-1}D(h)) \end{array} \quad = \quad \begin{array}{ccc} A & \xrightarrow{\phi} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ A_h & \xrightarrow{f^\sharp_{D(h)}} & \mathcal{O}_X(f^{-1}D(h)) \end{array}$$

By the universal property of localization, a morphism $A_h \to \mathcal{O}_X(f^{-1}D(h))$ exists if and only if $h$ becomes invertible $\mathcal{O}_X(f^{-1}D(h))$, in which case it is unique. This means that $f^\sharp_{D(h)} = g^\sharp_{D(h)}$ as ring maps. As this happens for every $h \in A$, we find that $f^\sharp = g^\sharp$.

*Surjectivity of* (4.8): Let $\phi\colon A \to \mathcal{O}_X(X)$ be a ring map. We will define a morphism of schemes $f\colon X \to Y$ such that $f^\sharp_Y = \phi$.

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In light of (4.4), there is a natural choice for the map on topological spaces. For $x \in X$, we set $f(x)$ to be the prime ideal in $A$ that is the preimage of $\mathfrak{m}_x$ under the composition

$$A \xrightarrow{\phi} \mathcal{O}_X(X) \xrightarrow{\rho_x} \mathcal{O}_{X,x}.$$

First we check that this assignment is continuous. For a section $s \in \mathcal{O}_X(X)$, define the set

$$D(s) = \{\, x \in X \mid s_x \notin \mathfrak{m}_x \,\}.$$

This set is open in $X$, because if $s_x \notin \mathfrak{m}_x$ for some point $x \in X$, then $s_x$ is invertible in $\mathcal{O}_{X,x}$, but then $s|_V$ is also invertible in some neighbourhood $V$ of $x$, and $V \subset D(s)$ (see Lemma 4.7).

Consider a distinguished open set $D(g) \subset \operatorname{Spec} A$. Then

$$\begin{aligned} f^{-1}(D(g)) &= \{\, x \in X \mid f(x) \in D(g) \,\} \\ &= \{\, x \in X \mid g \notin (\rho_x \circ \phi)^{-1}(\mathfrak{m}_x) \,\} \\ &= \{\, x \in X \mid \rho_x(\phi(g)) \notin \mathfrak{m}_x \,\} \\ &= \{\, x \in X \mid (\phi(g))_x \notin \mathfrak{m}_x \,\} \\ &= D(\phi(g)). \end{aligned}$$

Since $D(\phi(g))$ is open in $X$, and the sets $D(g)$ form a basis for the topology on $\operatorname{Spec} A$, we conclude that $f$ is continuous.

By Lemma 4.7, the restriction $\phi(g)|_{D(\phi(g))}$ is invertible in $\mathcal{O}_X(D(\phi(g)))$. Therefore, by the universal property of localization, the ring map $\phi$ induces a ring map

$$A_g \longrightarrow \mathcal{O}_X(D(\phi(g))). \tag{4.11}$$

Explicitly, (4.11) sends $ag^{-n}$ to $\phi(a)|_{D(\phi(g))} \cdot (\phi(g)|_{D(\phi(g))})^{-n}$. Moreover, if $D(h) \subset D(g)$, then we have the following diagram

$$\begin{array}{ccc} A_g & \longrightarrow & \mathcal{O}_X(D(\phi(g))) \\ \downarrow & & \downarrow \\ A_h & \longrightarrow & \mathcal{O}_X(D(\phi(h))) \end{array} \tag{4.12}$$

which commutes by the uniqueness part in the universal property of localization.

We define $f^\sharp_{D(g)}$ to be the ring map (4.11). By (4.12), these maps are compatible with the restriction maps, and so we get a map of sheaves $f^\sharp \colon \mathcal{O}_Y \to f_*\mathcal{O}_X$.

To see that $(f, f^\sharp)$ is a morphism of locally ringed spaces, we need to check that for each $x \in X$, the induced map $f^\sharp_x \colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a map of local rings. That is, we need to check that $f^\sharp(\mathfrak{p}A_\mathfrak{p}) \subset \mathfrak{m}_x$. But this follows by looking at the commutative diagram (4.9), because $\mathfrak{p}$ is exactly the inverse image of $\mathfrak{m}_x$ under the composition $\rho_x \circ \phi$.

Finally, if we set $g = 1$ in the definition of $f^\sharp$, then $\phi(g) = 1$ and (4.11) is simply given by the original ring map $\phi \colon A \to \mathcal{O}_X(X)$. This means that $\phi$ is induced by $(f, f^\sharp)$. □

For the surjectivity part of the above proof, we used the following little lemma:

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**Lemma 4.7.** Let $X$ be a scheme and let $s \in \mathcal{O}_X(U)$ be a section.
(i) If $s_x$ is invertible in $\mathcal{O}_{X,x}$ for some point $x \in U$, then there is an open set $V$ containing $x$ such that $s|_V$ is invertible in $\mathcal{O}_X(V)$.
(ii) If $s_x$ is invertible in $\mathcal{O}_{X,x}$ for every $x \in U$, then $s$ is invertible in $\mathcal{O}_X(U)$.

*Proof* (i): Let $t_x \in \mathcal{O}_{X,x}$ be the inverse of $s_x$, so that $s_x \cdot t_x = 1$ in $\mathcal{O}_{X,x}$. As all elements of $\mathcal{O}_{X,x}$ are germs, we may find an open set $V$ containing $x$ and a section $t_V \in \mathcal{O}_X(V)$ such that $(t_V)_x = t_x$. Then $(t_V)_x \cdot s_x = 1$ in $\mathcal{O}_{X,x}$, so shrinking $V$ if neccessary, we may assume that $s_V \cdot t|_V = 1$ in $\mathcal{O}_X(V)$.

(ii): By the first part, we may find an open covering $\{V_i\}_{i\in I}$ of $U$ and sections $t_i \in \mathcal{O}_X(V_i)$ such that $t_i \cdot s|_{V_i} = 1$ in $\mathcal{O}_X(V_i)$. As inverses are unique, we must have $t_i|_{V_i\cap V_j} = t_j|_{V_i\cap V_j}$ for every $i$ and $j$, and so the $t_i$ glue to a section $t \in \mathcal{O}_X(U)$. By the Locality axiom, the section $t$ has the property that $st = 1$ in $\mathcal{O}_X(U)$, because $st$ and '1' both restrict to the same section over each $V_i$. Hence $s$ is invertible in $\mathcal{O}_X(U)$. □

In the above proof, we never used the fact that $X$ had an affine covering consisting of affine schemes. This means that the theorem and the proof is valid even in the case where $X$ is a general locally ringed space.

**Corollary 4.8.** For any scheme, there is a canonical morphism $\theta\colon X \to \operatorname{Spec}\mathcal{O}_X(X)$. If $f\colon X \to \operatorname{Spec} A$ is any morphism to an affine scheme, then there is a unique morphism $h\colon \operatorname{Spec}\mathcal{O}_X(X) \to \operatorname{Spec} A$ such that $f = h \circ \theta$.

*Proof* The morphism $\theta$ is obtained by taking the identity map in the right-hand side of (4.8). Morphisms $f\colon X \to \operatorname{Spec} A$ are in one-to-one correspondence with ring maps $A \to \mathcal{O}_X(X)$; the factorization $f = h \circ \theta$ comes from the composition $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(X)$. □

**Example 4.9** (Morphisms to $\mathbb{A}^1$ and $\mathcal{O}_X(X)$)**.** For a ring $A$, there is a bijection between ring maps $\phi\colon \mathbb{Z}[t] \to A$ and elements of $A$ ($\phi$ is determined uniquely by the image of $t$). Therefore, by Theorem 4.10, there is a bijection

$$\operatorname{Hom}_{\mathsf{Sch}}(X, \mathbb{A}^1) = \mathcal{O}_X(X).$$

That is, an element of $\mathcal{O}_X(X)$ is the same thing as a morphism

$$f\colon X \longrightarrow \mathbb{A}^1.$$

Thus the global sections of the structure sheaf $\mathcal{O}_X$ do indeed correspond to some sort of 'regular functions' on $X$, not into a field $k$, but into the affine line over $\mathbb{Z}$. △

### *The main theorem of affine schemes*

The main consequence of Theorem 4.6 is that the category of affine schemes, AffSch, is anti-equivalent to the category of rings, Rings. To formulate this precisely, we note that the assignment $A \mapsto \operatorname{Spec} A$ defines a functor

$$\operatorname{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \longrightarrow \mathsf{AffSch}. \tag{4.13}$$

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For each ring map $\phi\colon A \to B$, this sends $\phi$ to the morphism $(f, f^\sharp)\colon \operatorname{Spec} B \to \operatorname{Spec} A$. It follows by the definitions that $\operatorname{Spec}\phi \circ \operatorname{Spec}\psi = \operatorname{Spec}(\psi \circ \phi)$, whenever $\phi$ and $\psi$ are composable ring maps (Exericse 4.11.60).

In the opposite direction, there is the global sections functor $\Gamma$ which sends each affine scheme $X$ to the ring $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$. For a morphism of affine schemes $f\colon X \to Y$ the associated sheaf map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a ring map on global sections:

$$f^\sharp_Y\colon \mathcal{O}_Y(Y) \longrightarrow \Gamma(Y, f_*\mathcal{O}_X) = \mathcal{O}_X(X).$$

By Theorem 4.6, these two functors give an equivalence of categories, analogous of Theorem 1.29 for affine varieties.

**Theorem 4.10 (Main Theorem of Affine Schemes).** The two functors

$$\operatorname{Spec}\colon \mathsf{Rings}^{\mathrm{op}} \to \mathsf{AffSch} \quad \text{and} \quad \Gamma\colon \mathsf{AffSch} \to \mathsf{Rings}^{\mathrm{op}}$$

define an equivalence of categories between $\mathsf{Rings}^{\mathrm{op}}$ and $\mathsf{AffSch}$.

*Proof* As $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) = A$ for any ring $A$, we have an equality of functors $\Gamma \circ \operatorname{Spec} = \mathrm{id}_{\mathsf{Rings}^{\mathrm{op}}}$. Conversely, starting with an affine scheme $X$, Theorem 4.6 shows that $X$ is canonically isomorphic to $\operatorname{Spec}(\mathcal{O}_X(X))$. Therefore $\operatorname{Spec} \circ \Gamma$ is equivalent to $\mathrm{id}_{\mathsf{AffSch}}$. □

To summarize, affine schemes $X$ are completely characterized by their rings of global sections $\mathcal{O}_X(X)$, both on the level of topological spaces and the structure sheaf. Morphisms between affine schemes $X \to Y$ are in bijective correspondence with ring homomorphisms $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. In particular, a morphism between two affine schemes is an isomorphism if and only if the corresponding ring map is an isomorphism.

**Example 4.11.** There is one and only one morphism of schemes $\operatorname{Spec}\mathbb{Z} \to \operatorname{Spec}\mathbb{Z}$. Indeed, ring maps are required to send 1 to 1, so there is only one ring map $\mathbb{Z} \to \mathbb{Z}$.

More generally, for any scheme $X$, there is a unique map of rings $\mathbb{Z} \to \mathcal{O}_X(X)$, so by the theorem, any scheme admits a unique morphism $X \to \operatorname{Spec}\mathbb{Z}$. In categorical terms, this means that $\operatorname{Spec}\mathbb{Z}$ is the *terminal object* in the category of schemes. △

**Example 4.12.** In order to check that a morphism of schemes $f\colon X \to Y$ is an isomorphism, it is not enough to check that $f$ is injective and surjective – also the structure sheaves have to be isomorphic. For instance, $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ is a homeomorphism, but the two affine schemes are not isomorphic. For another example, see Example 1.32. △

If $k$ is a field, then any $n$-tuple of polynomials $f_1, \dots, f_n$ in $k[x_1, \dots, x_m]$ determines a morphism $\mathbb{A}^m_k \to \mathbb{A}^n_k$ via the ring map $\phi\colon k[y_1, \dots, y_n] \to k[x_1, \dots, x_m]$ which sends $y_1$ to $f_1$, $y_2$ to $f_2$ etc. Conversely, any morphism $\mathbb{A}^m_k \to \mathbb{A}^n_k$ is of this form; the polynomials $f_1, \dots, f_n$ can be recovered as the images $\phi(y_1), \dots, \phi(y_n)$.

For brevity, we will continue to denote such morphisms by

$$\begin{aligned} f\colon \mathbb{A}^m &\longrightarrow \mathbb{A}^n \\ (x_1, \dots, x_n) &\longmapsto (f_1(x), \dots, f_n(x)) \end{aligned}$$

This is a useful notation, but one should keep in mind that the indicated mapping on sets is only valid for the 'traditional points' of the form $(a_1, \dots, a_n) \in k^n$.

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## 4.4 Relative schemes

For a ring $A$, a *scheme over* $A$ or simply $A$*-scheme*, is a scheme $X$ together with a morphism $X \to \operatorname{Spec} A$. A *morphism of* $A$*-schemes* is a morphism $f\colon X \to Y$ which is compatible with the two morphisms to $\operatorname{Spec} A$, that is, so that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} A & \end{array} \tag{4.14}$$

The schemes over $A$ form a category $\mathsf{Sch}/\mathrm{A}$, with the category of affine schemes over $A$, $\mathsf{AffSch}/A$, as a full subcategory.

The Main Theorem of Affine Schemes (Theorem 4.10) has the following analogue:

**Theorem 4.13.** Let $A$ be a ring. Then the category $\mathsf{AffSch}/A$ of affine schemes over $A$ is equivalent to the opposite category of the category of $A$-algebras $\mathsf{Alg}/A$.

One motivation for discussing relative schemes is that it allows us to precisely say what it means for a scheme or a morphism of schemes to be 'defined' over a field or a ring. If $X$ is a scheme over a field $k$, then all of the rings $\mathcal{O}_X(U)$ and the local rings $\mathcal{O}_{X,x}$ are $k$-algebras. Moreover, for a morphism $f\colon X \to Y$ of $k$-schemes, the ring maps $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$ are maps of $k$-algebras. So this is a scheme-analogue of being an 'affine variety over $k$'.

**Example 4.14.** An affine scheme $\operatorname{Spec} B$ is an $A$-scheme precisely when $B$ is an $A$-algebra. A morphism $\operatorname{Spec} B' \to \operatorname{Spec} B$ is a morphism over $A$ precisely when then corresponding ring map $B \to B'$ is a map of $A$-algebras. △

**Example 4.15.** $\mathbb{A}^1_{\mathbb{Q}}$ is a scheme over $\mathbb{Q}$. It is however not a scheme over $\mathbb{R}$. △

**Example 4.16.** By Example 4.11, every scheme has a unique structure as a $\mathbb{Z}$-scheme. Hence there is an equivalence of categories $\mathsf{Sch}/\mathbb{Z} = \mathsf{Sch}$. △

By the requirement (4.14), the set of morphisms between two fixed schemes $X$ and $Y$ depends on the category of schemes in which we consider them. The next example illustrates this.

**Example 4.17.** The scheme $X = \operatorname{Spec} \mathbb{C}$ can be regarded as a scheme over $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$. The set of self-morphisms $X \to X$ to depends on the context as follows:

(i) As a $\mathbb{C}$-scheme, the only morphism $X \to X$ is the identity, so $\operatorname{Hom}_{\mathsf{Sch}/\mathbb{C}}(X, X)$ has a single element.

(ii) As an $\mathbb{R}$-scheme, there are two morphisms: the identity and the one induced by complex conjugation $\mathbb{C} \to \mathbb{C}$. Hence $\operatorname{Hom}_{\mathsf{Sch}/\mathbb{R}}(X, X)$ has two elements.

(iii) As a scheme over $\mathbb{Z}$, the set of self-morphisms is considerably larger, as it can be identified with the set of field automorphisms of $\mathbb{C}$. In particular,

$$\operatorname{Hom}_{\mathsf{Sch}/\mathbb{Z}}(\operatorname{Spec} \mathbb{C}, \operatorname{Spec} \mathbb{C})$$

is an uncountable group. △

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There is an important generalization of relative schemes where we replace the base scheme $\operatorname{Spec} A$ by an arbitrary scheme. Given a scheme $S$, a *scheme over* $S$, or simply $S$*-scheme*, is a scheme $X$ together with a morphism $X \to S$. As above, the schemes over $S$ form a category $\mathsf{Sch}/\mathrm{S}$, where the morphisms between two $S$-schemes $X \to S$ and $Y \to S$ are the morphisms $X \to Y$ that are compatible with the two morphisms to $S$.

The power of this definition comes from thinking of $X$ as a family of schemes parametrized by the points of $S$. This perspective turns out to be conceptually very useful e.g. when we discuss fiber products and base change in Chapter 7.

**Example 4.18.** The Möbius strip scheme

$$X = \operatorname{Spec} \mathbb{R}[x, y, u, v]/(vx - uy, x^2 + y^2 - 1)$$

from Example 2.39 can be viewed as a scheme over $\mathbb{R}$, but one can also view it as a scheme over $S = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 - 1)$. The latter perspective offers extra geometric insight, as all the fibers of $X \to S$ are affine lines. △

## 4.5 Open embeddings and open subschemes

If $X$ is a scheme and $U \subset X$ is an open subset, the restriction $\mathcal{O}_X|_U$ is a sheaf on $U$ making $(U, \mathcal{O}_X|_U)$ into a locally ringed space. This is moreover a scheme, because if $X$ is covered by affines $V_i = \operatorname{Spec} A_i$, then each $U \cap V_i$ is open in $V_i$, hence can be covered by distinguished open subsets, which are all affine schemes. We call $(U, \mathcal{O}_X|_U)$ an *open subscheme* of $X$ and refer to the scheme structure on $U$ has the *induced scheme structure*.

A morphism of schemes $\iota\colon V \to X$ is called an *open embedding* if it is an isomorphism onto an open subscheme of $X$.

When referring to an 'open affine subset' in $X$ or an 'open affine covering' of $X$, we will always implicitly assume that the open sets are endowed with their canonical scheme structures, so that they are open *subschemes*.

**Example 4.19.** The open set $U = \mathbb{A}^1_k - V(x)$ is an open subscheme of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$. Note the isomorphism $U \simeq \operatorname{Spec} k[x, x^{-1}] = \operatorname{Spec} k[x, y]/(xy - 1)$ of schemes. △

**Example 4.20** (Distinguished open subsets)**.** More generally, each distinguished open set $D(f)$ in an affine scheme $\operatorname{Spec} A$ is an open subscheme. It is affine, isomorphic to $\operatorname{Spec} A_f$. Indeed, by Proposition 2.32 the map $\iota\colon \operatorname{Spec} A_f \to \operatorname{Spec} A$ corresponding to the localization map $A \to A_f$ is a homeomorphism onto $D(f)$, and it follows readily from the definition of the sheaf $\mathcal{O}_X$ that the restriction $\mathcal{O}_X|_{D(f)}$ coincides with the structure sheaf on $\operatorname{Spec} A_f$. △

**Example 4.21.** $\operatorname{Spec} k \to \operatorname{Spec} k[x]/(x^2)$ is not an open affine subscheme of $\operatorname{Spec} k[x]/(x^2)$, even though the subset is open and the scheme is affine. △

A word of warning: an open subscheme of an affine scheme might not itself be affine, as the next example shows.

**Example 4.22** (A non-affine scheme)**.** The open subset $U = \mathbb{A}^2_k - V(x, y)$ of $\mathbb{A}^2_k =$

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$\operatorname{Spec} k[x,y]$ is not an affine scheme. This is a consequence of Example 3.27 which showed that the restriction map

$$\iota^\sharp\colon \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k) \longrightarrow \mathcal{O}_{\mathbb{A}^2_k}(U) \tag{4.15}$$

is an isomorphism. Indeed, if $U$ were affine, the inclusion $\iota\colon U \to \mathbb{A}^2_k$, being induced by the ring map $\iota^\sharp$, would be an isomorphism by the Main Theorem of Affine Schemes (Theorem 4.10), but obviously this is not the case. △

## 4.6 Closed embeddings and closed subschemes

In this section, we define *closed subschemes* of a scheme. Intuitively, a closed subscheme of a scheme $X$ is a scheme $Z$ which is embedded as a closed subset $Z \subset X$. Here there are many possibilities for choosing the scheme structure on the same underlying closed set, and this makes the definition slightly more subtle than the one for open subscheme.

The main example to have in mind is $\operatorname{Spec}(A/\mathfrak{a})$, which naturally embeds as the closed subset $V(\mathfrak{a})$ of $\operatorname{Spec} A$ (Proposition 2.29 on page 34). In general, a closed subscheme is a scheme $(Z, \mathcal{O}_Z)$ with a morphism $\iota\colon Z \to X$, which locally looks like the map $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$. We formalize this in the following definition.

**Definition 4.23** (Closed embeddings and closed subschemes)**.** A morphism $\iota\colon Z \to X$ is called a *closed embedding* if there is an affine open cover $\{U_i\}_{i\in I}$ of $X$ such that for each $i \in I$:

(i) $\iota^{-1}(U_i)$ is affine, and

(ii) the ring map

$$\iota^\sharp\colon \mathcal{O}_X(U_i) \longrightarrow \mathcal{O}_Z(\iota^{-1}(U_i))$$

is surjective.

In this case, we say that $Z$ is a *closed subscheme* of $X$. Two closed subschemes $Z, Z'$ are considered *equal* if there is an isomorphism $\phi : Z \to Z'$ such that $\iota = \iota' \circ \phi$.

In concrete terms, $X$ and $Z$ can be covered by affine schemes $U_i = \operatorname{Spec}(A_i)$, and $\iota^{-1}(U_i) = \operatorname{Spec} B_i$, so that for each $i$, the induced ring map $A_i \to B_i$ is surjective, which means that $B_i \simeq A_i/\mathfrak{a}_i$ for some ideal $\mathfrak{a}_i$. Moreover, the morphism $\iota^{-1}U_i \to U_i$ is identified with the morphism $\operatorname{Spec}(A_i/\mathfrak{a}_i) \to \operatorname{Spec}(A_i)$.

Even if a closed subscheme $Z$ is defined as an abstract scheme which maps into $X$, we usually view it as a closed subset of $X$. This is justified because the image $V = \iota(Z)$ is a closed subset (the $U_i$'s form an open cover of $X$, and each subset $\iota(Z) \cap U_i$ is closed being equal to $V(\mathfrak{a}_i)$). Moreover, we can equip $V$ with a structure sheaf by defining $\mathcal{O}_V$ to be $\iota_*\mathcal{O}_Z$.

For notational convenience, we will henceforth use $V(\mathfrak{a})$ to refer to both the closed subscheme associated with $\mathfrak{a}$ and the underlying closed subset. The intended meaning will usually be clear from the context.

**Example 4.24.** The schemes $\operatorname{Spec} k[x]/(x^n)$ with $n \in \mathbb{N}$ and $k$ a field, give different subschemes of $\mathbb{A}^1_k$. Still, the underlying topological spaces are identical (a single point).

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However, they are not isomorphic as schemes, as the underlying structure sheaves are not isomorphic. △

**Example 4.25.** Consider the affine 4-space $\mathbb{A}^4_k = \operatorname{Spec} k[x,y,z,w]$ over a field $k$. The three ideals

$$I_1 = (x,y), \quad I_2 = (x^2,y) \text{ and } I_3 = (x^2,xy,y^2,xw-yz),$$

have the same radical $(x,y)$, and hence give rise to the same closed subset $V(x,y) \subset \mathbb{A}^4_k$, but they define different closed subschemes of $\mathbb{A}^4_k$. △

While the above definition allows us to specify and study a closed subscheme, it is not immediately clear how to classify all possible closed subschemes of a given scheme – even in the case of affine schemes. Although every ideal $\mathfrak{a} \subset A$ gives rise to a closed subcheme $\operatorname{Spec}(A/\mathfrak{a}) \to \operatorname{Spec} A$, the definition a priori also allows for closed subschemes defined in terms of other affine coverings. Nevertheless, it is a theorem that all closed subchemes of $\operatorname{Spec} A$ arise from ideals. We will defer the proof of this fact to Section **??**, as it is more naturally proved using sheaves.

**Proposition 4.26.** Let $X = \operatorname{Spec} A$ be an affine scheme. The map $\mathfrak{a} \mapsto \operatorname{Spec}(A/\mathfrak{a})$ defines a one-to-one correspondence between the set of ideals of $A$ and the set of closed subschemes of $X$. In particular, every closed subscheme of an affine scheme is also affine.

For later use, we include the following definition, which combines the two types of embeddings we have seen:

**Definition 4.27** (Locally closed embeddings)**.** A morphism $f\colon Z \to X$ is said to be a *locally closed embedding* if it factors as

$$Z \xrightarrow{\iota} U \xrightarrow{j} X$$

where $\iota\colon Z \to U$ is a closed embedding and $j\colon U \to X$ is an open embedding.

## 4.7 $R$-valued points

Until now, we have considered morphisms from a scheme *to* an affine scheme. If we instead study the set of morphisms *from* an affine scheme we arrive at the concept of 'ring-valued points'.

A basic motivation for studying $R$-valued points comes from the viewpoint that spaces in algebraic geometry are defined in terms of solutions of polynomial equations. Concretely, given a set of polynomials $f_1,\dots,f_r \in \mathbb{Z}[x_1,\dots,x_n]$, we would like to study the set of common solutions

$$f_1(x_1,\dots,x_n) = \cdots = f_r(x_1,\dots,x_n) = 0 \tag{4.16}$$

Of course, one has to specify the ring in which the variables $x_1,\dots,x_n$ take their values. In fact, since the polynomials have integer coefficients, the equations are meaningful over

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any ring $R$. The scheme $X = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_r)$ encodes the all possible solution sets in a single geometric object. More precisely, the solutions of (4.16) over $R$ will be the set of *$R$-valued points* of $X$.

**Definition 4.28.** Let $X$ be a scheme and let $R$ be a ring.
An *$R$-valued point* is a morphism of schemes

$$\operatorname{Spec} R \longrightarrow X.$$

The set of $R$-valued points is denoted $X(R)$, that is,

$$X(R) = \operatorname{Hom}_{\mathsf{Sch}}(\operatorname{Spec} R, X). \tag{4.17}$$

Note that if $f\colon X \to Y$ is morphism, then composition induces a map of sets $X(R) \to Y(R)$. The sets $X(R)$ also depend functorially on $R$: to each ring map $R \to S$ there is a corresponding morphism $\operatorname{Spec} S \to \operatorname{Spec} R$, which induces a map of sets $X(R) \to X(S)$ via composition. Therefore the scheme $X$ determines a *functor* from the category of rings to Sets. We will explore the link between a scheme and its associated functor in Section 8.1.

**Example 4.29.** For the affine $n$-space $\mathbb{A}^n = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$, we have

$$\mathbb{A}^n(R) = R^n.$$

Indeed, by Theorem 4.10 on page 70, the $R$-valued points $f\colon \operatorname{Spec} R \to \mathbb{A}^n$ correspond bijectively to ring maps

$$\phi\colon \mathbb{Z}[x_1, \dots, x_n] \longrightarrow R, \tag{4.18}$$

and these in turn, are in bijection with the set of $n$-tuples $(\phi(x_1), \dots, \phi(x_n)) \in R^n$; there is one for every element of $R^n$.

In particular, for fields $k$ it holds that $\mathbb{A}^n(k) = k^n$, which explains the notation $\mathbb{A}^n(k)$ used in Chapter 1.

Likewise, if $X \subset \mathbb{A}^n$ is the closed subscheme defined by an ideal $\mathfrak{a} = (g_1, \dots, g_r) \subset \mathbb{Z}[x_1, \dots, x_n]$, the set of $R$-valued points $X(R)$ are in a one-to-one correspondence with ring maps $\phi\colon \mathbb{Z}[x_1, \dots, x_n]/\mathfrak{a} \to R$. These are in turn in bijection with ring maps $\phi$ as in (4.18) that vanish on the ideal $\mathfrak{a}$. That is, they are in bijection with $n$-tuples $a = (a_1, \dots, a_n) \in R^n = \mathbb{A}^n(R)$ such that $g_i(a) = 0$ for all $i$, or in other words, the solutions of the system $g_1 = \cdots = g_r = 0$ in $R$. △

**Example 4.30.** For a concrete example, consider the scheme

$$X = \operatorname{Spec} \mathbb{Z}[u, v]/(u^2 + v^2 - 1).$$

Then the set $X(\mathbb{R})$ of $\mathbb{R}$-points consists of the points of the unit circle in $\mathbb{R}^2$, and the $\mathbb{Z}$-points $X(\mathbb{Z})$ consists of the four points $(\pm 1, 0)$ and $(0, \pm 1)$. (See Exercise 4.11.28 for a description of $X(\mathbb{Q})$.) △

Looking at maps from spectra of fields helps us understand the points of a scheme. Every point of $X$ is a $K$-point for some field $K$. Namely, if $x \in X$ is a point, there is a canonical morphism

$$\iota_x\colon \operatorname{Spec} \kappa(x) \longrightarrow X$$



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which maps the unique point of $\operatorname{Spec}\kappa(x)$ to $x$. To see this, suppose $x$ is contained in an open affine subset $U = \operatorname{Spec} A$ and corresponds to a prime ideal $\mathfrak{p} \subset A$. Then the residue field is given by $\kappa(x) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$, and the morphism $\iota_x : \operatorname{Spec}\kappa(x) \to X$ is defined by the composition

$$\operatorname{Spec}\kappa(x) \to \operatorname{Spec}(A_{\mathfrak{p}}) \to U \to X.$$

The residue fields $\kappa(x)$ allow us to understand $K$-valued points of $X$ in general:

**Proposition 4.31.** Let $X$ be a scheme and let $K$ be a field. Then there is a bijection

$$X(K) = \left\{ (x, \alpha) \;\middle|\; \begin{array}{l} x \in X \text{ is a point, and} \\ \alpha\colon \kappa(x) \to K \text{ is a map of fields} \end{array} \right\}.$$

*Proof* Fix a point $x \in X$, and let $U = \operatorname{Spec} A$ be an open affine subscheme containing $x$, so that $x$ corresponds to a prime ideal $\mathfrak{p}$ in $A$.

Write $o \in \operatorname{Spec} K$ for the unique point in $\operatorname{Spec} K$. Since $\operatorname{Spec} K$ has a single point, any $K$-valued point $f\colon \operatorname{Spec} K \to X$ with image $x$, factors uniquely via $U \to X$. We therefore reduce to considering the $K$-valued points of $U$. By Theorem 4.6, these are in bijection with ring maps $\phi\colon A \to K$. Under this correspondence, we have $f(o) = x$ precisely when $\phi^{-1}(0) = \mathfrak{p}$, which means that $\phi$ factors uniquely as $A \to A/\mathfrak{p} \to K$. Since $K$ is a field, any ring map $A/\mathfrak{p} \to K$ in turn factors uniquely via the residue field $A/\mathfrak{p} \to \kappa(\mathfrak{p}) \to K$. Hence the $K$-valued points $f$ are are in bijection with the maps of fields $\kappa(x) \to K$. □

There is also the relative version of the notion of $R$-valued points. Let $X$ be a scheme over a ring $A$ and let $R$ be an $A$-algebra. Then an $R$-valued point of $X$ is a morphism of $A$-schemes $\operatorname{Spec} R \to X$. The set of $R$-valued points is denoted by

$$X(R) = \operatorname{Hom}_{\mathsf{Sch}/A}(\operatorname{Spec} R, X).$$

Working over a base ring $A$ is in fact very important here. The reason is that considering *all* morphisms $\operatorname{Spec} R \to X$ typically leads to a set that is too large and does not capture the geometric picture we want. For instance, for $X = \operatorname{Spec}\mathbb{C}$, the set $X(\mathbb{C})$ has 1, 2 or an infinite number of elements depending on whether we view it as a scheme over $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Z}$ respectively (see Example 4.17).

Even though the base ring $A$ is not present in the notation $X(R)$, it will usually be clear from the context which category we are working in.

## 4.8 Gluing two schemes together

By definition, a scheme is a space which is locally built from affine open subschemes. Over the next few sections, we will explain how to conversely construct schemes by gluing together schemes along open subschemes. These gluing techniques are fundamental in scheme theory. As we will see in Chapter **??**, this gives a vast collection of new examples of schemes. The gluing of schemes is also an important part in many general existence proofs, such as the construction of the fiber product.

We begin by examining the most basic case, where we glue together *two* schemes $X_1$ and $X_2$ along a common open subscheme. Precisely, this means that we are given open

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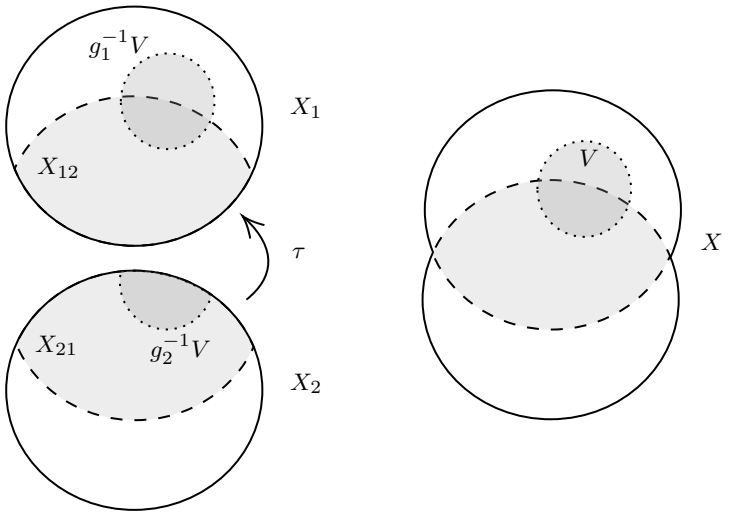


**Figure 4.1** Gluing two schemes together

subschemes $X_{12} \subset X_1$ and $X_{21} \subset X_2$ respectively, and we use an isomorphism $\tau\colon X_{21} \to X_{12}$ to identify the points of $X_{12}$ and $X_{21}$.

The new scheme $X$ has an underlying topological space given by the quotient space

$$X = (X_1 \sqcup X_2) / \sim$$

where $\sim$ is the equivalence relation defined by $x \sim \tau(x)$ for all $x \in X_{21} \subset X_2$.

$q\colon X_1 \sqcup X_2 \to X$ denote the quotient map, and define $g_i = q|_{X_i}\colon X_i \to X$ for $i = 1, 2$. Then the maps $g_i$ are open, and their images $U_i = g_i(X_i)$ form an open cover of $X$. By the definition of the quotient topology, a set $V \subset X$ is open if and only if $g_i^{-1}V$ is open in $X_i$ for $i = 1, 2$.

We define the sections of $\mathcal{O}_X$ over an open set $V \subset X$ as the set of compatible pairs of sections from $X_1$ and $X_2$:

$$\mathcal{O}_X(V) = \left\{ (s_1, s_2) \in \mathcal{O}_{X_1}(g_1^{-1}V) \times \mathcal{O}_{X_2}(g_2^{-1}V) \;\middle|\; s_1\big|_{g_1^{-1}V \cap X_{12}} = \tau^\sharp(s_2\big|_{g_2^{-1}V \cap X_{21}}) \right\}$$

For $V' \subset V$, the restriction $\mathcal{O}_X(V) \to \mathcal{O}_X(V')$ sends $(s_1, s_2)$ to $(s_1\big|_{g_1^{-1}V'}, s_2\big|_{g_2^{-1}V'})$. It is not hard to check that $\mathcal{O}_X$ is a sheaf and that $(X, \mathcal{O}_X)$ is a locally ringed space. Moreover, the maps $g_i\colon X_i \to X$ are open embeddings onto $U_i \subset X$ for $i = 1, 2$, so every point of $X$ has an open neighbourhood which is an affine scheme. (We will prove a more general statement below).

A basic example to keep in mind is when $X_1$ and $X_2$ are both affine, say $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$, glued together along distinguished open subsets $D(a)$ and $D(b)$ for some $a \in A$ and $b \in B$. The gluing map $\tau$ is specified by a ring isomorphism between the localizations

$$\phi : A_a \longrightarrow B_b.$$

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We picture this by the following diagram

$$\operatorname{Spec} A \supset \operatorname{Spec} A_a = D(a) \xleftarrow[\simeq]{\tau} D(b) = \operatorname{Spec} B_b \subset \operatorname{Spec} B$$

The scheme $X$ will not be affine in general. Still, it can be concretely studied in terms of $A$, $B$ and the ring map $\phi$. For instance, the global sections have the following description:

$$\mathcal{O}_X(X) = \{\, (f,g) \in A \times B \mid g/1 = \phi(f/1) \text{ in } B_b \,\}. \tag{4.19}$$

This basic construction produces a surprisingly rich set of examples (see the examples at the end of this chapter.)

## 4.9 Gluing sheaves

We will now study the more general situation where we glue together an arbitrary collection of schemes. For this, we also need general results on gluing sheaves and morphisms of sheaves.

Let $X$ be a topological space $X$ and let $\{U_i\}_{i\in I}$ be an open cover of $X$. For convenience, we will denote the pairwise and triple intersections by

$$U_{ij} = U_i \cap U_j \quad \text{and} \quad U_{ijk} = U_i \cap U_j \cap U_k, \quad i,j,k \in I.$$

The simplest case of gluing involves morphisms of sheaves. The following proposition gives the precise conditions under which this can be done:

**Proposition 4.32 (Gluing conditions for maps for sheaves).** Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $X$. Suppose we are given, for each $i \in I$, a morphism of sheaves

$$\phi_i \colon \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i},$$

such that for all $i, j \in I$, the restrictions agree on overlaps:

$$\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}. \tag{4.20}$$

Then there exists a unique morphism of sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ satisfying $\phi|_{U_i} = \phi_i$ for all $i \in I$.

*Proof* Let $V \subset X$ be an open set and let $s \in \mathcal{F}(V)$ be a section. We want to define $\phi(s)$. The open sets $V_i = U_i \cap V$ form a cover of $V$. Consider the sections $\phi_i(s|_{V_i}) \in \mathcal{G}(V_i)$. By (4.20), we have

$$\phi_i(s)|_{V_{ij}} = \phi_i(s|_{V_{ij}}) = \phi_j(s|_{V_{ij}}) = \phi_i(s)|_{V_{ij}} \quad \text{for all } i,j \in I.$$

Since $\mathcal{G}$ is a sheaf, the sections $\phi_i(s|_{V_i})$ glue to a unique section in $\mathcal{G}(V)$, and we define $\phi(s)$ to be this section. This assignment is additive and compatible with restrictions, so $\phi$ defines a map of sheaves. Moreover, by construction, $\phi|_{U_i} = \phi_i$ for all $i$.

For uniqueness, suppose $\phi$ and $\psi$ are two morphisms of sheaves so that $\phi|_{U_i} = \psi|_{U_i}$ for all $i \in I$. Then $\phi_V(s) = \psi_V(s)$ for all $s \in \mathcal{F}(V)$, by the Locality axiom for $\mathcal{G}$, and consequently $\phi = \psi$. □

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For gluing sheaves, the set-up is as follows: given a sheaf $\mathcal{F}_i$ on each $U_i$, the goal is to construct a global sheaf $\mathcal{F}$ on $X$ that restricts to $\mathcal{F}_i$ for every $U_i$. A necessary condition for such an $\mathcal{F}$ to exist is that the $\mathcal{F}_i$'s should be isomorphic over the intersections $U_{ij}$. In fact, by specifying the precise conditions that these isomorphisms must satisfy (the *gluing data*), we get not just a necessary but also a sufficient condition.

**Proposition 4.33 (Gluing conditions for sheaves).** Let $\{U_i\}_{i\in I}$ be a covering of $X$. Suppose we are given, for each $i \in I$, a sheaf $\mathcal{F}_i$ on $U_i$, and for each $i, j \in I$ an isomorphism

$$\tau_{ji}\colon \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}},$$

satisfying the three conditions

(i) $\tau_{ii} = \mathrm{id}_{\mathcal{F}_i}$
(ii) $\tau_{ji} = \tau_{ij}^{-1}$
(iii) $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$ over the triple intersection $U_{ijk}$.

Then there exists a sheaf $\mathcal{F}$ on $X$ and isomorphisms

$$\nu_i\colon \mathcal{F}|_{U_i} \longrightarrow \mathcal{F}_i$$

such that $\nu_j = \tau_{ji} \circ \nu_i$ over each $U_{ij}$. The sheaf $\mathcal{F}$, equipped with the isomorphisms $\nu_i$, is unique up to isomorphism.

*Proof* If $W \subset X$ is an open set, we will write $W_i = U_i \cap W$ and $W_{ij} = U_{ij} \cap W$.

We first define $\mathcal{F}$ as a presheaf. A section of $\mathcal{F}$ over an open set $V$ is given by a collection of compatible sections $s_i \in \mathcal{F}_i(V_i)$. Precisely, for each $i, j \in I$, we want to identify $s_j$ and $\tau_{ji}(s_j)$ over $V_{ij}$, and define

$$\mathcal{F}(V) = \big\{(s_i)_{i\in I} \;\big|\; \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}}\big\} \subset \prod_{i\in I} \mathcal{F}_i(V_i). \tag{4.21}$$

The restriction maps are induced componentwise: if $W \subset V$, the map $\mathcal{F}(V) \to \mathcal{F}(W)$ sends $(s_i)_{i\in I}$ to $(s_i|_{W_i})_{i\in I}$. This is well-defined, because the $\tau_{ji}$ are compatible with restrictions, which ensures that $\tau_{ji}(s_i|_{W_{ij}}) = s_j|_{W_{ij}}$ if $\tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}}$.

We next check the two sheaf axioms.

Locality: Let $s = (s_i) \in \mathcal{F}(V)$ be a section, and suppose that $s|_{V_\alpha} = 0$ over every open set in a cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of $V$. Then also $s_i|_{U_i\cap V_\alpha} = 0$ in $\mathcal{F}_i(V_\alpha \cap U_i)$ for all $\alpha$ and $i$. As $V_\alpha \cap U_i$ forms an open cover of $V \cap U_i$, and $\mathcal{F}_i$ is a sheaf on $U_i$, this means that $s_i = 0$ in $\mathcal{F}(V \cap U_i)$. Since this holds for every $i$, we see that $s = 0$ in $\mathcal{F}(V)$.

Gluing: Let $s^\alpha \in \mathcal{F}(V_\alpha)$ be a set of compatible sections over the opens of a covering $\{V_\alpha\}_{\alpha\in\Lambda}$ of $V$. This means that $s^\alpha$ and $s^\beta$ are equal when restricted to $V_{\alpha\beta} = V_\alpha \cap V_\beta$. For a fixed $i \in I$, we then have a compatible collection of sections $s_i^\alpha \in \mathcal{F}(U_i \cap V_\alpha)$, which, since $\mathcal{F}_i$ is a sheaf, glue to a unique element $s_i \in \mathcal{F}(U_i)$. We have $\tau_{ij}(s_j) = s_i$ in $\mathcal{F}(V_{ij})$ because this holds when restricted to each $V_\alpha \cap U_{ij}$, and since $s^\alpha \in \mathcal{F}(V_\alpha)$. The tuple $s = (s_i)$ therefore defines an element of $\mathcal{F}(V)$, which by construction restricts to $s^\alpha$ on each $W_\alpha$.

Note that we have not used the condition (iii) yet. It will be needed in order to construct the isomorphisms $\nu_i\colon \mathcal{F}|_{U_i} \to \mathcal{F}_i$. To avoid getting confused by the indices, we work with a

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fixed index $0 \in I$. Suppose $V \subset U_0$ is an open set. The projection map

$$\nu_0 \colon \mathcal{F}|_{U_0}(V) \longrightarrow \mathcal{F}_0(V), \qquad (s_i)_{i\in I} \mapsto s_0$$

induces a well-defined sheaf map $\nu_0 \colon \mathcal{F}|_{U_0} \longrightarrow \mathcal{F}_0$. We claim that $\nu_0$ is an isomorphism.

To begin with, we note requirement $\nu_\beta = \tau_{\beta 0} \circ \nu_0$ is fulfilled. Indeed, by the compatibility condition in (4.21), we have for $V \subset U_{0\beta}$,

$$\tau_{\beta 0}(\nu_0(s)) = \tau_{\beta 0}(s_0) = s_\beta = \nu_\beta.$$

$\nu_0$ is injective: if $s = (s_i) \in \mathcal{F}(V)$ is a section such that $s_0 = 0 \in \mathcal{F}_0(V)$, then we also have

$$s_i = s_i|_{V_{i0}} = \tau_{i0}(s_0) = 0$$

for all $i \in I$, and hence $s = 0$.

$\nu_0$ is surjective: take any section $\sigma \in \mathcal{F}_0(V)$ over some $V \subset U_0$ and define $s = (\tau_{i0}(\sigma|_{V_{i0}}))_{i\in I}$. Note that for every $i, j \in I$, we have

$$\tau_{ji}(\tau_{i0}(\sigma|_{V_{ji0}})) = \tau_{j0}(\sigma|_{V_{ji0}})$$

Therefore, $s$ satisfies the condition in (4.21), and defines an element of $\mathcal{F}(V)$. As $\tau_{00}(\sigma|_{V_{00}}) = \sigma$ by the first gluing condition, we also have $\nu_0(s) = \sigma$. □

**Example 4.34.** If $X$ is the scheme obtained by gluing together $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$, Proposition 4.33 tells us that giving a sheaf $\mathcal{F}$ on $X$ is equivalent to specifying (i) a sheaf $\mathcal{F}_1$ on $X_1$; (ii) a sheaf $\mathcal{F}_2$ on $X_2$; (iii) a sheaf isomorphism

$$\nu_{12} \colon \mathcal{F}_2|_{D(b)} \longrightarrow \mathcal{F}_1|_{D(a)},$$

where we use the isomorphism $\tau$ to identify $D(b)$ and $D(a)$. In the special case that $\mathcal{F}_1 = \widetilde{M}$ and $\mathcal{F}_2 = \widetilde{N}$ for modules $M$ and $N$ over $A$ and $B$ respectively, it is equivalent to specify an isomorphism of $A_a$-modules

$$v_{12} \colon A_a \longrightarrow B_b. \qquad \triangle$$

## 4.10 Gluing schemes

When we talk about gluing schemes, the set-up is as follows. We are given a collection of schemes $\{X_i\}_{i\in I}$, and in each of the schemes $X_i$ we are given a collection of open subschemes $X_{ij}$, one for each $j \in I$. The goal is to produce a new scheme $X$ by gluing together all the $X_i$'s along these open subschemes. This is done by identifying the open sets $X_{ij} \subset X_i$ and $X_{ji} \subset X_j$ using scheme isomorphisms $\tau_{ji} \colon X_{ij} \to X_{ji}$. If we let $X_{ijk} = X_{ik} \cap X_{ij}$ (these correspond to the triple intersections before the gluing has been done), we require that $\tau_{ji}(X_{ijk}) = X_{jik}$. Notice that $X_{ijk}$ is an open subscheme of $X_i$.

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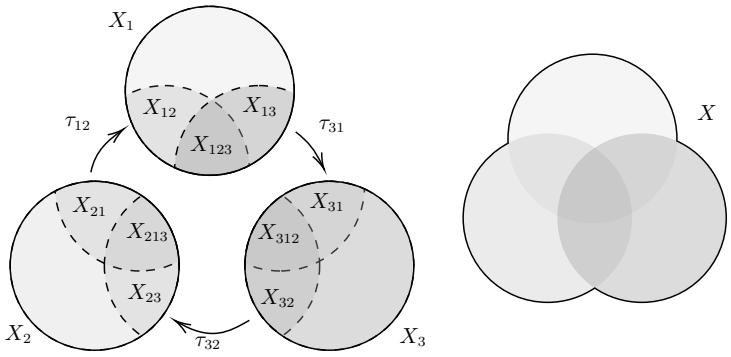


There are three gluing conditions, similar to the ones we saw for sheaves, which must be satisfied for the gluing to be possible.

**Proposition 4.35 (Gluing conditions for schemes).** Suppose that we are given a collection of schemes $\{X_i\}_{i\in I}$; for each $i, j$ an open subschemes $X_{ij} \subset X_i$ and scheme isomorphisms $\tau_{ji}\colon X_{ij} \to X_{ji}$ satisfying

(i) $\tau_{ii} = \mathrm{id}_{X_i}$
(ii) $\tau_{ij} = \tau_{ji}^{-1}$
(iii) $\tau_{ij}$ takes $X_{ijk}$ into $X_{jik}$ and $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$ over $X_{ijk}$.

Then there exists a scheme $X$ with open embeddings $g_i\colon X_i \to X$ onto an open subscheme $U_i = g_i(X_i) \subset X$ such that

- $\{U_i\}_{i\in I}$ is an open cover of $X$.
- For each $i, j \in I$, $g_i(X_{ij}) = U_i \cap U_j$ and the following diagram commutes:

$$\begin{array}{ccc} X_{ij} & \xrightarrow{\quad\tau_{ij}\quad} & X_{ji} \\ & {\scriptstyle g_i} \searrow \quad \swarrow {\scriptstyle g_j} & \\ & U_i \cap U_j & \end{array}$$

The scheme $X$ with the $g_i$ is uniquely characterized by these properties up to unique isomorphism.

For the latter, we rely on the gluing technique for sheaves explained in Proposition 4.33. The fact that $X$ is locally affine will follow immediately once the embeddings $g_i$ are in place, because the $X_i$'s are schemes and therefore locally affine.

*Proof* We construct the scheme $X$ in two steps: first the underlying topological space $X$ and then the structure sheaf.

Define the topological space $X$ as the quotient of the disjoint union $\coprod_i X_i$ by the equivalence relation generated by $x \sim \tau_{ji}(x)$ for $x \in X_{ij}$. The three gluing conditions enusure that this is an equivalence relation. The first requirement means that the relation is reflexive, the second that it is symmetric, and the third ensures it is transitive. Here $X$ has the quotient topology: if $\pi\colon \coprod_i X_i \to X$ denotes the quotient map, a subset $U$ of $X$ is open if and only if $\pi^{-1}(U)$ is open.

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Topologically, the maps $g_i\colon X_i \to X$ are simply the maps induced by the open inclusions $X_i \to \coprod_i X_i$. They are clearly injective, because a point $x \in X_i$ is never equivalent to another point in $X_i$. Now, by definition of the quotient topology, a subset $U$ of $X$ is open if and only if $g_i^{-1}(U) = X_i \cap \pi^{-1}(U)$ is open for all $i$. In view of the formula

$$\pi^{-1}(g_i(U)) = \bigcup_j \tau_{ji}(U \cap X_{ij}),$$

we conclude that each $g_i$ is an open map, hence a homeomorphism onto its image.

Write $U_i = g_i(X_i)$, $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. On $X_{ij}$, we have the isomorphisms $\tau_{ji}^\sharp\colon \mathcal{O}_{X_j}|_{X_{ij}} \to \mathcal{O}_{X_i}|_{X_{ij}}$, the sheaf maps of the scheme isomorphisms $\tau_{ji}\colon X_{ij} \to X_{ji}$. These satisfy $\tau_{ii}^\sharp = \mathrm{id}$, $\tau_{ji}^\sharp = (\tau_{ij}^\sharp)^{-1}$ and $\tau_{ki}^\sharp = \tau_{ji}^\sharp \circ \tau_{kj}^\sharp$ (on $X_{ijk}$) by the gluing conditions for the morphisms $\tau_{ji}$. Applying Proposition 4.33 the structure sheaves $\mathcal{O}_{U_i}$ glue to a sheaf of rings $\mathcal{O}_X$. By construction, this sheaf of rings restricts to $\mathcal{O}_{U_i}$ on each of the open subsets $U_i$, and therefore the stalks are local rings. So $(X, \mathcal{O}_X)$ is a locally ringed space which is locally affine, hence a scheme.

We leave it to the reader to check the uniqueness statement in the proposition. □

Finally, we consider conditions under which we can glue morphisms of schemes.

**Proposition 4.36 (Gluing morphisms of schemes).** Let $X$ and $Y$ be schemes and let $\{U_i\}_{i\in I}$ be an open cover of $X$. Given morphisms $f_i\colon U_i \longrightarrow Y$ such that

$$f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$$

for each $i$ and $j$, there exists a unique morphism

$$f\colon X \longrightarrow Y$$

such that $f|_{U_i} = f_i$ for every $i \in I$.

*Proof* On the level of topological spaces, we set $f(x) = f_i(x)$ if $x \in U_i$. This is well-defined because $f_i(x) = f_j(x)$ for $x \in U_i \cap U_j$, and continuity follows from the continuity of each $f_i$.

Next we define the sheaf map $f^\sharp$. If $V \subset Y$ is an open set, we construct the ring map $f^\sharp\colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}V)$ as follows. Take any section $s \in \mathcal{O}_Y(V)$. Using the sheaf maps $f_i^\sharp$ over $U_i$, we get sections $t_i = f_i^\sharp(s)$ in $\mathcal{O}_X(f^{-1}V \cap U_i)$. As $f_i^\sharp$ and $f_j^\sharp$ restrict to the same map on $U_{ij}$, we have

$$t_i|_{f^{-1}V\cap U_{ij}} = t_j|_{f^{-1}V\cap U_{ij}}$$

in $\mathcal{O}_X(f^{-1}V \cap U_{ij})$. The $t_i$ therefore glue together to a section $t \in \mathcal{O}_X(f^{-1}V)$, and we can define $f^\sharp(s)$ to be $t$. This assignment is compatible with restriction maps, and define a sheaf of rings $f^\sharp$ and that $f_V^\sharp = (f_i)_V^\sharp(s)$ when $V \subset U_i$. The pair $(f, f^\sharp)$ is therefore a map of locally ringed spaces because it is locally given by the $f_i$.

The uniqueness of $f$ follows from the fact that morphisms of schemes are determined by their restrictions to any open cover, and the restrictions $f|_{U_i}$ are equal to $f_i$ by design. □

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### *The affine line with two origins*

Consider the scheme $X$ obtained by gluing together two copies $X_1$ and $X_2$ of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[u]$ over a field $k$ along their common open subset $X_{12} = \operatorname{Spec} k[u, u^{-1}]$ with the identity morphism $\phi : k[u, u^{-1}] \to k[u, u^{-1}]$ on the open set. The resulting scheme $X$ is covered by two $\mathbb{A}^1_k$'s which overlap outside the origin. However, as the gluing process does nothing over the origins of each $\mathbb{A}^1_k$, there are now *two* points in $X$ that replace the origin. $X$ is called the *affine line with two origins*.

This scheme is not affine. To see this, we use the sheaf sequence, which takes the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X, \mathcal{O}_X) & \longrightarrow & \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) \oplus \Gamma(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) & \longrightarrow & \Gamma(X_{12}, \mathcal{O}_{X_{12}}) \\ & & & & \downarrow \| & & \downarrow \| \\ & & & & k[u] \oplus k[u] & \xrightarrow{\rho} & k[u, u^{-1}] \end{array}$$

where now $\rho(p(u), q(u)) = p(u) - q(u)$, and it follows that $\Gamma(X, \mathcal{O}_X) \simeq k[u]$. Moreover, both inclusions $\iota \colon \mathbb{A}^1_k \to X$ induce the identity map $\iota^\sharp \colon k[u] \to k[u]$ on global sections. If $X$ were affine, this would imply that $\iota \colon \mathbb{A}^1_k = \operatorname{Spec} k[u] \to X$ is an isomorphism, which is clearly not the case (it is not surjective, since the image misses one of the two origins).

### *Semi-local rings*

Semi-local rings are rings with finitely many maximal ideals. In the next two examples we give a few examples of such rings and how they can be described as local rings glued together.

**Example 4.37** (Semi-local rings)**.** Consider the two rings $\mathbb{Z}_{(2)}$ and $\mathbb{Z}_{(3)}$. These are both discrete valuation rings with with a common fraction field $\mathbb{Q}$ and maximal ideals $(2)$ and $(3)$ respectively. As described in Example 2.14 on page 29, $X_1 = \operatorname{Spec} \mathbb{Z}_{(2)}$ consists of two points: the closed point $x_1$ corresponding to $(2)$, and the generic point $\eta_1$ corresponding to $(0)$. Likewise, $\operatorname{Spec} \mathbb{Z}_{(3)}$ consists of two points $x_2$ and $\eta_2$. The generic points given the open embeddings $\operatorname{Spec} \mathbb{Q} \to X_i$ for $i = 1, 2$. Hence we can glue together $X_1$ and $X_2$ along the two generic points and we obtain a scheme $X$ with one open point $\eta$ and two closed points $x_1$ and $x_2$.

By (4.19), we can compute $\mathcal{O}_X(X)$ as follows:

$$\mathcal{O}_X(X) = \{\, (a, b) \in \mathbb{Z}_{(2)} \times \mathbb{Z}_{(3)} \mid a = b \text{ in } \mathbb{Q} \,\} \simeq \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}.$$

The ring $\mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$ is a semi-local ring with the two maximal ideals $(2)$ and $(3)$. By the Main Theorem of Affine Schemes (Theorem 4.10) there is a map $X \to \operatorname{Spec} \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$, and it is left as an exercise to show that this is an isomorphism. △

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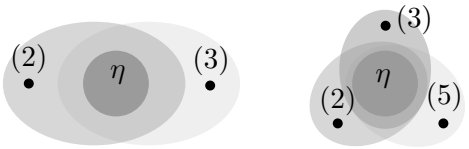


**Example 4.38** (More semi-local rings)**.** More generally, if $\mathcal{P} = \{p_1, \ldots, p_r\}$ is a finite set of distinct prime numbers, we can consider the schemes $X_p = \operatorname{Spec} \mathbb{Z}_{(p)}$ for $p \in \mathcal{P}$ and glue them together to form a new scheme $X$. Indeed, each $X_p$ has an open point $\eta_p$, which is the image of the canonical open embedding $\operatorname{Spec} \mathbb{Q} \to X_p$, and we glue $X_p$ and $X_q$ by identifying $\eta_p$ and $\eta_q$ using the identity map on $\operatorname{Spec} \mathbb{Q}$.

We may compute the global sections of the structure sheaf of the new scheme using the sheaf sequence

$$\begin{array}{ccccc} 0 \to \Gamma(X, \mathcal{O}_X) & \longrightarrow & \prod_{p\in\mathcal{P}} \Gamma(X_p, \mathcal{O}_X) & \longrightarrow & \prod_{p,q\in\mathcal{P}} \Gamma(X_p \cap X_q, \mathcal{O}_X) \\ & & \| \downarrow & & \downarrow \| \\ & & \prod_{p\in\mathcal{P}} \mathbb{Z}_{(p)} & \xrightarrow[\rho]{} & \prod_{p,q\in\mathcal{P}} \mathbb{Q}. \end{array}$$

Here, the map $\rho$ sends an $r$-tuple $(a_p)_{p\in\mathcal{P}}$ to the sequence $(a_p - a_q)_{p,q\in\mathcal{P}}$, and it follows that the kernel of $\rho$ is equal to the intersection $A_{\mathcal{P}} = \bigcap_{p\in\mathcal{P}} \mathbb{Z}_{(p)}$. This is a semi-local ring whose maximal ideals are the $(p)A_{\mathcal{P}}$'s for $p \in \mathcal{P}$. There is a canonical morphism $X \to \operatorname{Spec} A_{\mathcal{P}}$, and again we leave it to the reader to verify that this is an isomorphism. △

## 4.11 Exercises

**Exercise 4.11.1.** Prove the uniqueness part in Proposition 4.35.

**Exercise 4.11.2.** Let $X$ and $Y$ be schemes and let $\mathscr{B}$ be a basis for the topology on $X$. Suppose that there is a collection of morphisms $f_U \colon U \to Y$, one for each $U \in \mathscr{B}$, such that if $V \in \mathscr{B}$ satisfies $V \subset U$, we have

$$f_U|_V = f_V.$$

Show that there exists a unique morphism of schemes $f \colon X \to Y$ such that $f|_U = f_U$.

**Exercise 4.11.3** (A finite non-affine scheme)**.** Consider a set $X$ consisting of three elements, $x, y, z$. Define a topology on $X$ where the open sets are defined as $\varnothing, \{x\}, \{x, y\}, \{x, z\}$, and $X$ itself. On $X$, define a sheaf of rings $\mathcal{O}_X$ where $\mathcal{O}_X(\{x\})$ corresponds to the field $\mathbb{Q}(t)$, while for the open sets $\mathcal{O}_X(\{x, y\}), \mathcal{O}_X(\{x, z\})$, and $\mathcal{O}_X(X)$ we assign the localized ring $\mathbb{Q}[t]_{(t)}$. Show that $(X, \mathcal{O}_X)$ is a scheme which is not isomorphic to an affine scheme. HINT: An affine scheme is isomorphic to $\operatorname{Spec} \mathcal{O}_X(X)$.

**Exercise 4.11.4** (The cuspidal cubic)**.** Let $k$ be an algebraically closed field, and consider the morphism

$$\begin{array}{rcl} f \colon \mathbb{A}^1_k & \longrightarrow & \operatorname{Spec} k[x, y]/(x^3 - y^2) \\ t & \mapsto & (t^2, t^3). \end{array}$$

induced by the ring map $\phi \colon k[x, y] \mapsto k[t]$ sending $x$ to $t^2$ and $y$ to $t^3$.

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a) Show that $f$ is a homeomorphism.
b) Let $p \in \mathbb{A}^1_k$ be the origin. Compute the map $f_p^\sharp \colon \mathcal{O}_{X,(0,0)} \to \mathcal{O}_{\mathbb{A}^1_k,p}$ and show that it is not an isomorphism. Conclude that the cuspidal cubic is not even 'locally isomorphic' to $\mathbb{A}^1_k$ near the origin.

**Exercise 4.11.5.** Consider the ring $A = \mathbb{Z}[\sqrt{-5}]$ and the ideal $\mathfrak{a} = (2, 1 + \sqrt{-5})$. Show that $\mathfrak{a}$ is not principal as follows. For an element $\alpha = a + b\sqrt{-5}$, define the *norm* of $\alpha$ as $N(\alpha) = a^2 + 5b^2$.

a) Show that $A/\mathfrak{a} \simeq \mathbb{Z}/2$.
b) Show that $N(xy) = N(x)N(y)$ for all $x, y \in A$.
c) Show that $N(x) = 1$ if and only if $x$ is a unit.
d) Show that there is no element $x$ with $N(x) = 2$.
e) Show that $\mathfrak{a}$ is not principal. HINT: If $\mathfrak{a} = (x)$, show that $x$ must be a unit.

**Exercise 4.11.6.** Let $X$ be a scheme whose underlying topological space is a finite and discrete set. Show that $X$ is affine.

**Exercise 4.11.7.** Let $f \colon X \to \operatorname{Spec} A$ be a morphism of schemes. Show that the closure of the image $f(X)$ is given by $V(\operatorname{Ker} \phi)$ where $\phi \colon A \to \mathcal{O}_X(X)$ is the ring map inducing $f$.

**Exercise 4.11.8.** Let $k$ be a field and let $X = \operatorname{Spec} A$, where $A = k[x, y, z, w]/(xz, xw, yz, yw)$.

a) Show that $X$ is the union of two affine planes intersecting at the origin.
b) Let $U \subset X$ be the open subset obtained by removing the origin. Define the subset $S \subseteq A$ by

$$S := \{f \in A \mid f(p) \neq 0 \text{ for all } p \in U\}$$

where $\mathfrak{m}_p$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$ at the point $p \in U$.

Prove that the natural homomorphism $S^{-1}A \to \mathcal{O}_X(U)$ is *not* an isomorphism.
c) Explain why the analogous statement in part does hold for the union of two affine lines intersecting at a single point.

**Exercise 4.11.9** (The reduction of a scheme)**.** Let $X$ be a scheme and let $\mathscr{B}$ be the basis of affine open subschemes of $X$. For a ring $R$, we let $R_{\mathrm{red}} = R/\sqrt{0}$, where $\sqrt{0}$ is the ideal of nilpotent elements in $R$.

a) Show that $V \mapsto \mathcal{O}_X(V)_{\mathrm{red}}$ defines a $\mathscr{B}$-sheaf of rings on $X$. If $X = \operatorname{Spec} A$, show that this sheaf is isomorphic to the sheaf associated to the $A$-module $A_{\mathrm{red}} = A/\sqrt{0}$.
b) Let $\mathcal{O}_{X_{\mathrm{red}}}$ denote the sheaf associated to the $\mathscr{B}$-sheaf from *a*). Show that $(X, \mathcal{O}_{X_{\mathrm{red}}})$ is a scheme, which is isomorphic to $\operatorname{Spec}(A_{\mathrm{red}})$ if $X = \operatorname{Spec} A$ is affine.
c) Show that there is a closed embedding $\iota \colon X_{\mathrm{red}} \to X$.
d) Show that $X_{\mathrm{red}}$ and $\iota$ satisfies the following universal property: for any reduced scheme $Y$ and a morphism $f : Y \to X$, there is a unique morphism $g \colon Y \to X_{\mathrm{red}}$ so that $f = \iota \circ g$.
e) If $f \colon X \to Y$ is a morphism, show that there is a unique morphism $f_{\mathrm{red}} \colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$ so that $f_{\mathrm{red}} \circ r_X = r_Y \circ f_{\mathrm{red}}$. Show that assignments $X \mapsto X_{\mathrm{red}}$ and

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$f \mapsto f_{\mathrm{red}}$ defines a functor Sch to RedSch which is adjoint to the inclusion functor RedSch $\to$ Sch, where RedSch is the full subcategory of Sch whose objects are the reduced schemes.

**Exercise 4.11.10.** Let $f\colon X \to Y$ be a morphism of schemes. Show that the following are equivalent.

(i) $f$ is dominant.
(ii) $f(D)$ is dense for one dense subset $D \subset X$.
(iii) $f(D)$ is dense for all dense subsets $D \subset X$.

If $X$ is irreducible, they are also equivalent to

$Y$ is irreducible and the generic point of $X$ maps to the generic point of $Y$.

**Exercise 4.11.11.** Let $X$ be an integral scheme and let $f \in K(X)$. Show that the set of points $p \in X$ such that $f \in \mathcal{O}_{X,p}$ is open.

**Exercise 4.11.12.** Fill in the details for the proof of Proposition **??**.

**Exercise 4.11.13.** Let $X$ be an integral scheme and let $f\colon X \dashrightarrow Y$ be a rational map. Show that there is a unique maximal open set $U$ for which the $f$ is defined. This is another way of resolving the ambiguity in the definition of a rational map. HINT: Proposition 4.36.

**Exercise 4.11.14.** Let $X \subset \mathbb{P}^2_k$ denote the *Klein quartic curve* defined by the equation $x^3y + y^3z + z^3x = 0$. Compute the number of $k$-points on $X$ for $k = \mathbb{F}_2, \mathbb{F}_4$ and $\mathbb{F}_8$.

**Exercise 4.11.15.** Let $X$ be a scheme. Show that the map $p \mapsto \overline{\{p\}}$ defines a bijection

$$X \longrightarrow \{\text{irreducible closed subsets of } X\}$$

**Exercise 4.11.16.** Let $X = \operatorname{Spec} A$ and let $M$ be an $A$-module. Show that $\widetilde{M}$ is an $\mathcal{O}_X$-*module*, that is, $\Gamma(U, \widetilde{M})$ is an $\mathcal{O}_X(U)$-module, and the module structure is compatible with restriction maps. HINT: Show this first for $U = D(f)$, then cover a general open by destinguished open sets.

**Exercise 4.11.17.** Let $f\colon X \to Y$ be a morphism. Show that if $g \in \mathcal{O}_Y(Y)$ corresponds to the morphism $g\colon Y \to \mathbb{A}^1$, then $f^\sharp(g) \in \mathcal{O}_X(X)$ corresponds to the composition $X \to Y \to \mathbb{A}^1$.

**Exercise 4.11.18.** Show that $\mathbb{A}^1_{\mathbb{C}}$ embeds as a closed subscheme of $\mathbb{A}^2_{\mathbb{R}}$. Can $\mathbb{A}^1_{\mathbb{R}}$ be a closed subscheme of some $\mathbb{A}^n_{\mathbb{C}}$?

**Exercise 4.11.19.** Let $f\colon Z \to X$ be a morphism and let $U \subset X$ be an open subscheme such that $f(Z) \subset U$ (as sets). Show that there exists a unique morphism $g\colon Z \to U$ such that $f = \iota \circ g$ where $\iota\colon U \to X$ is the inclusion map.

**Exercise 4.11.20.** Consider $X = \operatorname{Spec} A$, where $A = \mathbb{R}[u, v]/(u^2 + v^2 + 1)$. Show that $X(\mathbb{R}) = \varnothing$, but $X(\mathbb{C})$ is infinite. Show that $A$ contains infinitely many maximal ideals, so the underlying topological space of $X$ is infinite.

**Exercise 4.11.21.** Let $X$, $\mathcal{F}_i$, $\tau_{ij}$ be as in Proposition 4.33.

a) Let $\mathcal{B}$ be the collection of all open sets $U \subset X$ such that $U \subset U_i$ for at least one index $i \in I$. Show that $\mathscr{B}$ is a basis for the topology on $X$.

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b) For each $U \in \mathcal{B}$, choose an index $i = i(U) \in I$ such that $U \subset U_i$, and set $\mathcal{F}(U) = \mathcal{F}_i(U)$. For an inclusion $V \subset U$ in $\mathcal{B}$, let $i = i(U)$ and $j = i(V)$, so that also $V \subset U_i \cap U_j = U_{ij}$. Define the restriction maps by

$$\rho_{U,V} \colon \mathcal{F}(U) = \mathcal{F}_i(U) \longrightarrow \mathcal{F}_i(V) \xrightarrow{\tau_{ji}|_V} \mathcal{F}_j(V) = \mathcal{F}(V) \tag{4.22}$$

Show that this makes $\mathcal{F}$ into a $\mathscr{B}$-presheaf.

c) Show that $\tau_{ij}$ defines an isomorphism $\nu_{i,V} \colon \mathcal{F}(V) \to \mathcal{F}_i(V)$. Use this to define an isomorphism of presheaves $\mathcal{F}|_{U_i} \to \mathcal{F}_i$. Deduce that $\mathcal{F}_i$ is a $\mathscr{B}$-*sheaf*.

d) Show that the extended sheaf $\mathcal{F}$ and $\nu_i$ satisfy the conclusion in Proposition 4.33.

**Exercise 4.11.22.** Show that the sections of $\mathcal{O}_{\operatorname{Spec} A}$ over an open set $U \subset X = \operatorname{Spec} A$, are given by the inverse limit of the localizations

$$\mathcal{O}_X(U) = \varprojlim_{D(f)\subset U} \mathcal{O}(D(f)) = \varprojlim_{D(f)\subset U} A_f. \tag{4.23}$$

**Exercise 4.11.23.** Let $A = k[x,y,z]/(xyz)$ and $X = \operatorname{Spec} A$. Compute $\mathcal{O}_{X,p}$ where $x$ corresponds to the prime ideal $\mathfrak{p} = (x-1, y, z)$. Show that $yz \neq 0$ in $\mathcal{O}_{X,p}$, but takes the value 0 for all points in a neighbourhood of $p$.

**Exercise 4.11.24.** Show that if $f \colon X \to Y$ is a morphism of locally ringed spaces, the stalk maps $f_x^\sharp \colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induce maps between the residue fields $\kappa(f(x))$ and $\kappa(x)$. What happens when $X$ and $Y$ are affine varieties?

**Exercise 4.11.25.** Let $X = \operatorname{Spec} \mathbb{Z}$. Compute $X(\mathbb{F}_p)$, $X(\mathbb{Q})$ and $X(\mathbb{C})$.

**Exercise 4.11.26.** Show that $\operatorname{Spec} \overline{\mathbb{Q}}[x]$ and $\operatorname{Spec} \mathbb{Z}$ are homeomorphic, but not isomorphic as schemes.

**Exercise 4.11.27.** Is $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ a closed embedding?

**Exercise 4.11.28.** Show that the $\mathbb{Q}$-points of the scheme $\operatorname{Spec} \mathbb{Z}[u,v]/(u^2+v^2-1)$ are given by

$$X(\mathbb{Q}) = \left\{ \left( \frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right) \,\middle|\, t \in \mathbb{Q} \right\} \cup \{(0,-1)\}.$$

HINT: Consider lines trough $(0,1)$ and the 'second intersection point' with the circle.

**Exercise 4.11.29.** With reference to Example 4.30, show that one may interpret $X(\mathbb{Q})$ as the set of Pythagorean triples:

$$X(\mathbb{Q}) = \{\, (a,b,c) \in \mathbb{Z}^3 \mid a^2 + b^2 = c^2 \text{ and } a, b, c \text{ relatively prime} \,\}.$$

**Exercise 4.11.30.** With reference to Example 4.30, let $p$ be a prime such that $p \not\equiv 1 \mod 4$. Show that the description in Example 4.30 also is valid for $X(\mathbb{F}_p)$.

**Exercise 4.11.31.** With reference to Example **??**, consider the natural inclusion

$$A = \mathbb{R}[u,v]/(u^2+v^2+1) \subset \mathbb{C}[u,v]/(u^2+v^2+1) = A_{\mathbb{C}}.$$

For each point $z = (a,b) \in X(\mathbb{C})$ consider the ideal $\mathfrak{n}_z = \mathfrak{m}_z \cap A$. Show that $\mathfrak{n}_x$ is maximal

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and that $\mathfrak{n}_z = \mathfrak{n}_w$ if and only if $w = (\bar{a}, \bar{b})$ with $z = (a, b)$. Conclude that $A$ has infinitely many maximal ideals.

**Exercise 4.11.32.** Let $X$ be a scheme and let $x \in X$ be a point.

a) Show that there is a canonical morphism
$$f\colon \operatorname{Spec} \mathcal{O}_{X,x} \longrightarrow X$$
b) Show that $f$ induces a homeomorphism between $\operatorname{Spec}(\mathcal{O}_{X,x})$ and the subset $W \subset X$ of points $w \in X$ such that $w \in \overline{x}$.
c) Show that the map $\iota_x\colon \operatorname{Spec} \kappa(x) \to X$ defined in the text factors via $f$.
d) Show that on the level of topological spaces, the image of $f$ is the intersection of all open neighbourhoods containing $x$.
e) Compute the image of $f$ when:
   (i) $x$ is the generic point of an irreducible scheme.
   (ii) $x$ is a closed point of $\mathbb{A}^2_{\mathbb{C}}$.

**Exercise 4.11.33.** Deduce Theorem 4.13 from Theorem 4.10.

**Exercise 4.11.34.** An inclusion $\mathbb{Q} \subset \overline{\mathbb{Q}}$ induces a morphism $\mathbb{A}^n_{\overline{\mathbb{Q}}} \to \mathbb{A}^n_{\mathbb{Q}}$. Compute the images of the following points under the morphism $\mathbb{A}^2_{\overline{\mathbb{Q}}} \to \mathbb{A}^2_{\mathbb{Q}}$:

a) $(x - \sqrt{2}, y - \sqrt{2})$
b) $(x - \omega, y - \omega^2)$ where $\omega$ is a cube root of unity.
c) $(\sqrt{2}x + \sqrt{3}y)$

**Exercise 4.11.35.** Let $(f, f^\sharp)\colon X \to Y$ be a morphism of locally ringed spaces. Show that $(f, f^\sharp)$ is an isomorphism if and only if $f$ is a homeomorphism and the sheaf map $f^\sharp$ is an isomorphism (that is, $f^\sharp_U$ is an isomorphism for every open set $U \subset Y$).

**Exercise 4.11.36.** Show that being a closed embedding is a property which is 'local on the target'. In clear text: given a morphism $f\colon Z \to X$ and an open cover $\{U_i\}$ of $X$. Let $V_i = f^{-1}U_i$ and assume that each restriction $f|_{V_i}\colon V_i \to U_i$ is a closed embedding. Prove that then also $f$ is a closed embedding.

**Exercise 4.11.37.** Show that being a locally closed embedding is 'local on the image'. Assume that $f\colon Z \to X$ is a morphism and that $\{U_i\}$ is a collection of open subsets of $X$ covering the image $f(Z)$. Assume further that each restriction $f|_{f^{-1}U_i}\colon f^{-1}U_i \to U_i$ is a closed embedding, then $f$ is a locally closed embedding.

**Exercise 4.11.38.** Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of schemes. Prove that if both $f$ and $g$ are closed embeddings, then $g \circ f$ is one as well.

**Exercise 4.11.39.** Let $f\colon X \to Y$ be a morphism which is both an open embedding and a closed embedding. Show that $f$ is an isomorphism.

**Exercise 4.11.40.** Consider the ring $R = \mathbb{Z}[t]$ and let $X = \operatorname{Spec} R$.

a) For a prime number $p$, show that $\mathfrak{m} = (t, p)$ is a maximal ideal of $R$.
b) Let $U = X - \{\mathfrak{m}\}$. Show that $U = D(p) \cup D(t)$ and that
$$\mathcal{O}_X(U) = \mathbb{Z}[t]$$

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c) Deduce that $U$ is not affine.

**Exercise 4.11.41.** Let $X = \{a, b, c\}$ be a set with three elements. Let $X$ have the coarsest topology so that the two subsets $U = \{a, b\}$, and $V = \{a, c\}$ are open. Define a presheaf $\mathcal{O}_X$ by $\mathcal{O}_X(U) = \mathcal{O}_X(V) = \mathbb{C}[t]_{(t)}$ and $\mathcal{O}_X(U \cap V) = \mathbb{C}(t)$ with the restriction map given by the inclusion $\mathbb{C}[t]_{(t)} \subset \mathbb{C}(t)$.

a) Show that $\mathcal{O}_X$ is a sheaf.
b) Show that $(X, \mathcal{O}_X)$ is a scheme.
c) Show that $(X, \mathcal{O}_X)$ is not affine.

**Exercise 4.11.42.** Let $X$ be a scheme.

a) Show that any irreducible and closed subset $Z \subset X$ has a unique generic point. HINT: Reduce to the affine case.
b) Show that in general schemes are not Hausdorff. What are the possible underlying topologies of affine schemes that are Hausdorff?
c) Show that $X$ satisfies the zeroth separation axiom (they are $T_0$); that is, given two points $x$ and $y$ in $X$, there is an open subset of $X$ containing one of them but not the other.

**Exercise 4.11.43.** a) Show that $\mathrm{Hom}_{\mathsf{Rings}}(\mathbb{Z}, \mathbb{Q})$ has only one element.
b) Define continuous maps $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$, sending $(0)$ to $(p)$. Is it possible to make these into morphisms of schemes?

**Exercise 4.11.44** (The sheaf of units)**.** Let $X$ be a scheme with structure sheaf $\mathcal{O}_X$. We say that $s \in \mathcal{O}_X(U)$ is a *unit* if there exists a multiplicative inverse $s^{-1} \in \mathcal{O}_X(U)$.

a) Show that $s \in \mathcal{O}_X(U)$ is a unit if and only if for all $x \in U$, the germ $s_x$ is a unit in the ring $\mathcal{O}_{X,x}$; that is, if and only if $s_x$ does not lie in the maximal ideal of $\mathcal{O}_{X,x}$.
b) We let $\mathcal{O}_X^\times(U)$ denote the subgroup of units in $\mathcal{O}_X(U)$. Show that $\mathcal{O}_X^\times(U)$ is a sub*sheaf* of $\mathcal{O}_X$.

**Exercise 4.11.45.** In the same vein as Example 2.33, show that a ring $A$ is a $\mathbb{Q}$-algebra (that is, it contains a copy of $\mathbb{Q}$) if and only if the canonical map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ factors through the generic point $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$.

**Exercise 4.11.46.** For every ring $A$, there is a canonical map $\mathbb{Z} \to A$ which sends 1 to 1. Hence there is a canonical map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$. Show that map factors through the canonical map $\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$ if and only if $A$ is of characteristic $p$.

**Exercise 4.11.47** (The Frobenius morphism)**.** Let $p$ be a prime number and let $A$ be a ring of characteristic $p$. The ring map $F_A : A \to A$ given by $a \mapsto a^p$ is called the *Frobenius map* on $A$.

a) Show that $F_A$ induces the identity map on $\operatorname{Spec} A$.
b) Show that if $A$ is local, then $F_A$ is a map of local rings.
c) For a scheme $X$ over $\mathbb{F}_p$, define the *Frobenius morphism* $F_X \colon X \to X$ by the identity on the underlying topological space and with $F_X^\sharp : \mathcal{O}_X \to \mathcal{O}_X$ given by $g \mapsto g^p$. Show that $F_X$ is a morphism of schemes.

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d) Show that $F_X$ is natural in the sense that if $f : X \to Y$ is a morphism of schemes over $\mathbb{F}_p$, we have $f \circ F_X = F_Y \circ f$.

In particular, this exercise shows that for a morphism of schemes $f\colon X \to Y$, in order to check that $f$ is an isomorphism, is not enough to check that $f$ is a homeomorphism; also the map $f^\sharp$ must be an isomorphism.

**Exercise 4.11.48.** Let $X$ be an integral scheme over a ring $A$. Let $f \in K(X)$ and let $U_f \subset X$ be the open set of points $x \in X$ such that $f \in \mathcal{O}_{X,x}$. Show that there is a morphism $\phi\colon U_f \to \mathbb{A}^1_A$ such that $\phi^\sharp\colon A[t] \to \Gamma(U_f, \mathcal{O}_X)$ is given by $t \mapsto f$.

**Exercise 4.11.49.** Prove Proposition 6.5. That is, prove that a scheme $X$ is integral if and only if $\mathcal{O}_X(U)$ is an integral domain for each open $U \subset X$.

**Exercise 4.11.50.** Let $X = \operatorname{Spec} k[x,y,z,w]/(xw - yz)$ and consider the open set $U = X - V(x,y)$. Use the above strategy as in Example 4.22 to compute $\mathcal{O}_X(U)$. Conclude that $U$ is not affine.

**Exercise 4.11.51.** Prove that a composition of two closed embeddings is a closed embedding.

**Exercise 4.11.52.** Let $X$ be a scheme so that the underlying topological space is finite and discrete. Show that $X$ is an affine scheme. HINT: Consider the case where $X$ has one point first.

**Exercise 4.11.53.** Prove that a morphism $f\colon X \to Y$ is a closed embedding if and only if $f$ induces a homeomorphism from $X$ onto a closed subset of $Y$, and for each $x \in X$, the map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective. HINT: Consider the stalks of $f_*\mathcal{O}_X$. You need to use the fact that $f$ is a homeomorphism onto $f(X)$.

**Exercise 4.11.54.** Describe the following schemes and the structure sheaf on them.

a) $\operatorname{Spec} \mathbb{C}[t]/(t^2+1)$
b) $\operatorname{Spec} \mathbb{R}[t]/(t^3 - t^2)$
c) $\operatorname{Spec} \mathbb{F}_3[t]/(t^3 - 1)$.

**Exercise 4.11.55.** Let $K$ be a finite field extension of $\mathbb{Q}$ and let $X = \operatorname{Spec} K$. Show that $\operatorname{Hom}_{\mathsf{Sch}}(X,X)$ can be identified with the Galois group $\operatorname{Gal}(K/\mathbb{Q})$.

**Exercise 4.11.56.** Let $A$ be a ring and consider a morphism $g\colon \mathbb{A}^m_A \to \mathbb{A}^n_A$ given by polynomials $g_1, \dots, g_n \in A[x_1, \dots, x_m]$. Consider the morphism $f\colon \mathbb{A}^m_A \to \mathbb{A}^{m+n}_A$ given by $(x_1, \dots, x_n, g_1, \dots, g_n)$. Show that $f$ is a closed embedding.

**Exercise 4.11.57.** Show that the $\operatorname{Spec}$-functor preserves inverse limits of rings. That is, if $\{R_i\}_{i\in I}$ is a directed system of rings, then

$$\operatorname{Spec}(\varinjlim R_i) = \varprojlim \operatorname{Spec} R_i.$$

HINT: The $\operatorname{Spec}$-functor is a right adjoint to the global sections functor.

**Exercise 4.11.58.** Let $A$ be a ring and let $M$ be an $A$-module. Show that for each $f \in A$, there is a natural isomorphism of sheaves on $D(f) = \operatorname{Spec}(A_f)$

$$\widetilde{M}_f = \widetilde{M}|_{D(f)}.$$

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**Exercise 4.11.59.** Describe the schemes $\operatorname{Spec} A$ for

a) $A = \mathbb{Z}[x]/(2x-1)$
b) $A = \mathbb{Z}[x]/(3x-1)$
c) $A = \mathbb{Z}[x]/(9x-3)$
d) $A = \mathbb{Z}[x]/(2x^2-1)$

Which of these are isomorphic/homeomorphic?

**Exercise 4.11.60.** Check that $\operatorname{Spec}\phi \circ \operatorname{Spec}\psi = \operatorname{Spec}(\psi \circ \phi)$, whenever $\phi$ and $\psi$ are composable ring maps.

**Exercise 4.11.61.** Prove that $\mathbb{A}^n_k - \{(0,\dots,0)\}$ is not affine for any $n \geqslant 2$.

**Exercise 4.11.62.** Let $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ be the morphism associated to a ring map $\phi\colon A \to B$. Show that the sheaf map $f^\sharp\colon \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_{\operatorname{Spec} B}$ is given by $\widetilde{\phi} : \widetilde{A} \to \widetilde{B}$, where we regard $B$ as an $A$-module via $\phi$.

**Exercise 4.11.63.** Let $X$ be a scheme and let $X_{\mathrm{red}}$ be its reduction. Show that if $k$ is a field, then there is a natural bijection $X(k) = X_{\mathrm{red}}(k)$. Generalize the statement to reduced rings.

**Exercise 4.11.64.** Show that the morphism

$$\operatorname{Spec}\mathbb{C}[x,b,c]/(x^2+bx+c) \longrightarrow \operatorname{Spec}\mathbb{C}[b,c]$$

is finite and describe its fibers.

What about the morphism $\operatorname{Spec}\mathbb{C}[x,a,b,c]/(ax^2+bx+c) \to \operatorname{Spec}\mathbb{C}[a,b,c]$?

**Exercise 4.11.65.** Let $\mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$. Compute the scheme-theoretic intersection of the two subschemes $V(z-xy^2)$ and $V(z)$. Is it irreducible? Reduced?

**Exercise 4.11.66.** Show that any closed embedding $\iota\colon X \to Y$ is of finite type.

**Exercise 4.11.67.** Show that open subschemes $U \subset X$ satisfy the following 'universal property': for any scheme $Z$, to give a morphism $Z \to U$ is equivalent to giving a morphism $f\colon Z \to X$ such that $f(Z) \subset X$ (as sets).

**Exercise 4.11.68** ($\mathbb{A}^1$ minus a point)**.** Consider the affine line $\mathbb{A}^1 = \operatorname{Spec}\mathbb{Z}[t]$.

a) For a prime number $p$, show that $\mathfrak{m} = (t,p)$ defines a closed point of $\mathbb{A}^1$.
b) Let $X = \mathbb{A}^1 - \{\mathfrak{m}\}$. Show that $U = D(p) \cup D(t)$ and that

$$\mathcal{O}_X(U) = \mathbb{Z}[t]$$

c) Is $U$ an affine scheme?
d) What happens if we replace $\mathbb{Z}$ by a field $k$ in the above?

**Exercise 4.11.69.** Let $X$ and $Y$ be schemes and define for an open set $U \subset X$,

$$\mathcal{F}(U) = \operatorname{Hom}_{\mathsf{Sch}}(U, Y)$$

Show that $\mathcal{F}$ is a sheaf of sets on $X$.

**Exercise 4.11.70.** Consider the weighted projective space $\mathbb{P}(p,q) = \operatorname{Proj} R$, where $R = k[x_0,x_1]$ with $\deg(x_0) = p$ and $\deg(x_1) = q$. Show that

$$\mathbb{P}(p,q) \simeq \mathbb{P}^1_k.$$

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HINT: Consider the Veronese subring $R^{(d)}$ where $d = pq$, and define a ring map $\phi\colon\ k[u, v] \to R^{(d)}$ by $u \mapsto x_0^q$, $v \mapsto x_1^p$.

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# 5

# Projective space

## 5.1 Projective space

Traditionally, the projective $n$-space over a field $k$ is defined as the quotient space

$$\mathbb{P}^n(k) = \big(k^{n+1} - \{0\}\big) \,/\, \sim \tag{5.1}$$

where we say that two $(n+1)$-tuples $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ are equivalent if there is some $\lambda \in k^\times$ so that $b_i = \lambda a_i$ for all $i$. Geometrically, this means that $\mathbb{P}^n(k)$ parameterizes the 1-dimensional subspaces in $k^{n+1}$, or in other words, the lines through the origin.

The equivalence class of an $(n+1)$-tuple $(x_0, \ldots, x_n)$ is usually written $(x_0 : \cdots : x_n)$. So for instance, $(1:3) = (-1:-3)$ and $(1:2:3) = (2:4:6)$. The $x_0, \ldots, x_n$ are called *homogeneous coordinates*. It is important to note that $x_0, \ldots, x_n$ are not coordinates in the usual sense, as they are not even functions. However, the ratios $x_j/x_i$ define functions on the subset

$$U_i = \{\, (x_0 : \cdots : x_n) \in \mathbb{P}^n(k) \mid x_i \neq 0 \,\}.$$

Note that $\mathbb{P}^n(k)$ is covered by these sets:

$$\mathbb{P}^n(k) = \bigcup_{i=0}^{n} U_i$$

For each $i = 0, \ldots, n$, there is a bijection

$$\begin{aligned} \phi_i \colon \mathbb{A}^n(k) &\longrightarrow U_i \\ (y_1, \ldots, y_n) &\longmapsto (y_1 : \cdots : y_{i-1} : 1 : y_i : \cdots : y_n). \end{aligned} \tag{5.2}$$

The inverse is defined by sending a point $(x_0 : \cdots : x_n) \in U_i$ with $x_i \neq 0$ to the point

$$\left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_n}{x_i}\right) \in \mathbb{A}^n(k). \tag{5.3}$$

Thus $\mathbb{P}^n(k)$ is a union of $n+1$ affine spaces $\mathbb{A}^n(k)$ with affine coordinates given by the ratios in (5.3). We can define a topology on $\mathbb{P}^n(k)$ by defining a set $W \subset \mathbb{P}^n(k)$ to be open if and only if $W \cap U_i$ is open in $U_i$ for all $i = 0, \ldots, n$.

The ratios $x_0/x_i, \ldots, x_n/x_i$ also serve as natural candidates for defining regular functions on $\mathbb{P}^n(k)$. A regular function $f \colon U_i \to k$, however it is defined, should pull back to a regular function $D(x_i) \subset \mathbb{A}^{n+1}(k)$. This pullback corresponds to an element $g \in k[x_0, \ldots, x_n, x_i^{-1}]$ which is invariant under scaling $x_i \mapsto t x_i$ for $t \in k^\times$. When $k$ is infinite,



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such polynomials must be homogeneous of degree 0, or equivalently, they are polynomials in the ratios $x_0/x_i, \ldots, x_i/x_i$.

Note that the set $\mathbb{P}^n_k - U_0$ consists of all the points $(x_0 : \cdots : x_n)$ where $x_0 = 0$. This can be naturally be identified with $\mathbb{P}^{n-1}(k)$ via the map $(y_0 : \cdots : y_{n-1}) \mapsto (0 : y_1 : \cdots : y_n)$. We may write

$$\mathbb{P}^n(k) = U_0 \cup \mathbb{P}^{n-1}(k).$$

This way of thinking of $\mathbb{P}^n(k)$, as adding to $\mathbb{A}^n(k)$ a $\mathbb{P}^{n-1}(k)$ 'at infinity' is often a more useful way of thinking about the points of $\mathbb{P}^n(k)$.

**Example 5.1.** The projective line $\mathbb{P}^1(k)$ is a union

$$\mathbb{P}^1(k) = U_0 \cup U_1$$

We identify $U_0 = \{\, (x_0 : x_1) \mid x_0 \neq 0 \,\}$ with $\mathbb{A}^1(k)$ using the map $z \mapsto (1 : z)$, so informally $z = x_1/x_0$ defines an affine coordinate over $U_0$. Likewise, $U_1$ is identified with $\mathbb{A}^1(k)$ using the map $w \mapsto (w : 1)$ and $w = x_0/x_1$ is the affine coordinate over $U_1$. The intersection $U_0 \cap U_1$ consists of the points $(x_0 : x_1)$ where both $x_0$ and $x_1$ are non-zero. Over this open set, the coordinates $z$ and $w$ are related by $w = z^{-1}$. One can then view $\mathbb{P}^1(k)$ as a space obtained by gluing together two copies of $\mathbb{A}^1(k)$, using the homeomorphism $z \mapsto z^{-1}$ over the subset $\mathbb{A}^1(k) - \{0\}$.

Note that $\mathbb{P}^1(k) = U_0 \cup \{(0 : 1)\}$, so we can think of $\mathbb{P}^1(k)$ as adding to $\mathbb{A}^1(k)$ a 'point at infinity'. △

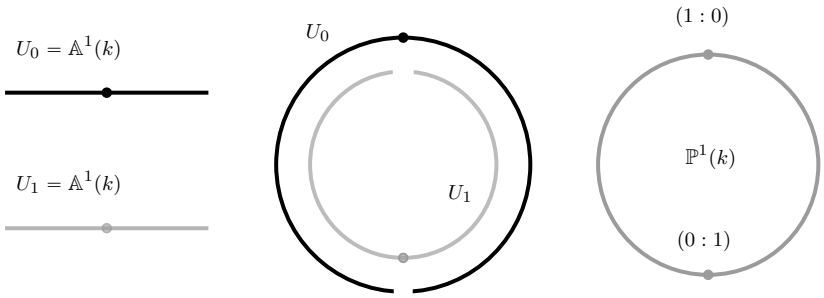


**Figure 5.1** $\mathbb{P}^1(k)$ obtained by gluing two affine lines.

As $\mathbb{P}^n(k)$ is defined from $\mathbb{A}^{n+1}(k) - \{(0, \ldots, 0)\}$ the closed sets of $\mathbb{P}^n(k)$ are closely related to closed sets in $\mathbb{A}^{n+1}(k)$, which are in turn defined by ideals in $k[x_0, \ldots, x_n]$. However, because of the equivalence relation, one cannot directly speak of the zero set of any polynomial $F(x_0, \ldots, x_n)$ in $\mathbb{P}^n(k)$. For instance, on $\mathbb{P}^1(k)$ the polynomial $x_0^2 - x_1$ takes the value 0 for $(x_0, x_1) = (1, 1)$, but not at $(x_0, x_1) = (2, 2)$. However, if $F$ is homogeneous, that is, all the terms have the same degree, then the zero set of $F$ is well-defined, as $F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n)$ where $d$ is the degree of $F$. Then it makes sense to define the zero set

$$Z_+(F) = \{\, P \in \mathbb{P}^n(k) \mid F(P) = 0 \,\}.$$

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More generally, if $\mathfrak{a} \subset k[x_0, \ldots, x_n]$ is a *homogeneous ideal*, e.g., generated by homogeneous polynomials, we can define

$$Z_+(\mathfrak{a}) = \{\, P \in \mathbb{P}^n(k) \mid F(P) = 0 \text{ for all } F \in \mathfrak{a} \text{ homogeneous} \,\}.$$

Every closed set of $\mathbb{P}^n(k)$ is of this form (see Exercise **??**).

**Example 5.2.** The projective plane $\mathbb{P}^2(k)$ is covered by 3 open sets

$$\mathbb{P}^2(k) = U_0 \cup U_1 \cup U_2$$

and each $U_i$ can be identified with an affine plane $\mathbb{A}^2(k)$. For instance, $U_0$ is identified with $\mathbb{A}^2(k)$ with affine coordinates $x = x_1/x_0$ and $y = x_2/x_0$ and $U_1$ is $\mathbb{A}^2(k)$ with affine coordinates $u = x_0/x_1$ and $v = x_2/x_1$. Over the intersection $U_0 \cap U_1$ these are related by $u = x^{-1}$ and $v = x^{-1}y$.

We have $\mathbb{P}^2(k) = U_0 \cup \mathbb{P}^1(k)$, where $\mathbb{P}^1(k)$ sits inside $\mathbb{P}^2(k)$ as the closed set $\{x_0 = 0\}$. Thinking about $\mathbb{P}^2_k$ as $\mathbb{A}^2_k$ with the 'line at infinity added' leads to a space where the theory of intersections is much nicer than in affine space. For instance, consider two paralell lines $L_1, L_2$ in $\mathbb{A}^2_k$ defined by the linear equations

$$L_1 \colon ay_1 + by_2 + c = 0$$
$$L_2 \colon ay_1 + by_2 + d = 0$$

where $c \neq d$. If we identify $\mathbb{A}^2_k$ with the open set $U_0 \subset \mathbb{P}^2_k$, the closures $\overline{L}_1$ and $\overline{L}_2$ are given by the equations

$$\overline{L}_1 \colon ax_1 + bx_2 + cx_0 = 0$$
$$\overline{L}_2 \colon ax_1 + bx_2 + dx_0 = 0$$

These lines *do* intersect, namely in the point $(0 : -b : a)$, which lies in $\mathbb{P}^2 - U_0$. In fact, any two lines in $\mathbb{P}^2(k)$, that is, closed subsets defined by a linear equation, will always intersect in $\mathbb{P}^2_k$. This is the most classical example illustrating that projective varieties have better geometric properties than affine varieties. △

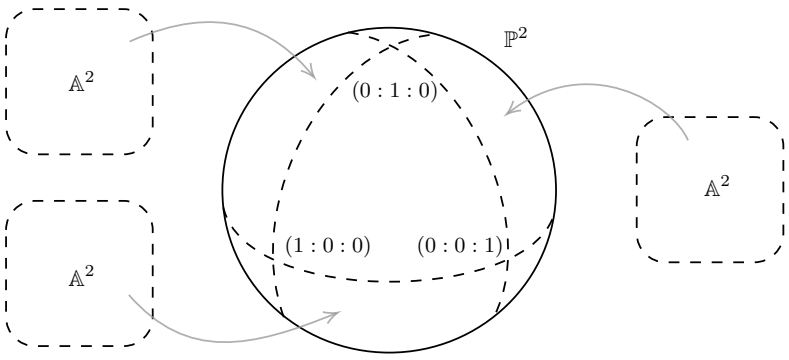


**Figure 5.2** The projective plane

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### *Projective space as a scheme*

We would now like to construct a scheme $\mathbb{P}^n$, so that for every field $k$, the $k$-points of $\mathbb{P}^n$ is given by the set (5.1), that is, the 1-dimensional subspaces in $k^{k+1}$. In fact, the construction works over any base ring $A$. The basic idea is to construct $\mathbb{P}^n_A$ from $n+1$ copies of affine $n$-space $\mathbb{A}^n_A$ using similar gluing formulas in the previous section.

Fix a ring $A$ and variables $x_0, \ldots, x_n$. For each $i = 0, \ldots, n$, define the ring

$$R_i = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]. \tag{5.4}$$

Since $R_i$ is a polynomial ring in $n$ variables, each $U_i = \operatorname{Spec} R_i$ is isomorphic to an affine space $\mathbb{A}^n_A$. We will also, for $0 \leqslant i, j, k \leqslant n$ consider the rings

$$R_{ij} = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right] \quad \text{and} \quad R_{ijk} = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{x_i}{x_j}, \frac{x_i}{x_k}\right].$$

If we consider these rings as subrings of the ring $A[x_0, x_0^{-1}, \ldots, x_n, x_n^{-1}]$, we have for each pair $i, j$ an equality of subrings

$$R_{ij} = R_{ji}.$$

This follows from the identities $x_l/x_i = x_l/x_j \cdot x_j/x_i$, for all $i$, $j$ and $l$. Similarly,

$$R_{ijk} = R_{jik} = R_{jki}.$$

Note that we may view $R_{ij}$ as the localization of $R_i$ in the element $x_j/x_i$. Hence $U_{ij} = \operatorname{Spec} R_{ij}$ can be identified with the distinguished open subscheme $D(x_j/x_i) \subset U_i$. Since $R_{ij} = R_{ji}$, using the identity maps $\tau_{ij} \colon U_{ij} \to U_{ji}$ as gluing maps, we see that the gluing conditions are satisfied, and the $U_i$ glue together to a scheme, which we denote by $\mathbb{P}^n_A$.

**Definition 5.3.** The scheme $\mathbb{P}^n_A$ is called the *projective $n$-space over $A$*.
The projective space over $\mathbb{Z}$ is called the *projective $n$-space*, and it is denoted by $\mathbb{P}^n$.

Note that all rings $R_i$ are $A$-algebras, so each $U_i$ is a scheme over $A$ and comes with a structure map $U_i \to \operatorname{Spec} A$. The same is true for the localizations $R_{ij}$, and we see that the structure maps glue together to a morphism $\pi \colon \mathbb{P}^n_A \to \operatorname{Spec} A$, making $\mathbb{P}^n_A$ an $A$-scheme.

If $\operatorname{Spec} A$ is irreducible, then so is $\mathbb{P}^n_A$. This is because $\mathbb{P}^n_A$ contains $U_0 \simeq \mathbb{A}^n_A$ as a dense open subset and $\mathbb{A}^n_A$ is irreducible if $A$ is. Likewise, $\mathbb{P}^n_A$ is reduced if $A$ is, because it has the same local rings as $\mathbb{A}^n_R$, and $\mathbb{A}^n_A$ is reduced if $A$ is. This means that $\mathbb{P}^n_A$ is integral if $A$ is an integral domain.

**Example 5.4.** The projective line $\mathbb{P}^1_A$ over $A$ is constructed by gluing two copies of the affine line $\mathbb{A}^1_A$, $U_0 = \operatorname{Spec} A[u]$ and $U_1 = \operatorname{Spec} A[u^{-1}]$, along $U_{01} = \operatorname{Spec} A[u, u^{-1}]$ which is a distinguished open set in both $U_0$ and $U_1$. △

**Example 5.5.** The *projective plane* $\mathbb{P}^2_A$ is constructed by gluing together the three affine planes $U_i = D_+(x_i) = \operatorname{Spec} R_i$ for $i = 0, 1, 2$. It is sometimes convenient to rewrite these charts using the '$U_0$-coordinates', i.e., writing $x = x_1/x_0$ and $y = x_2/x_0$. We can then

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express the other ratios in terms of $x$ and $y$. For instance $x_2/x_1 = x^{-1}$. With this convention, the three affine opens become

$$U_0 = \operatorname{Spec} A[x, y], \quad U_1 = \operatorname{Spec} A[x^{-1}, yx^{-1}], \quad U_2 = \operatorname{Spec} A[y^{-1}, xy^{-1}].$$

△

There is a morphism of schemes

$$\pi\colon \mathbb{A}^{n+1}_A - V(x_0, \dots, x_n) \longrightarrow \mathbb{P}^n_A. \tag{5.5}$$

which can be seen as a scheme-analogue of the quotient map $k^{n+1} - \{0\} \to \mathbb{P}^n(k)$.

To define it, we work locally over $U_i$. The inclusions $R_i \subset A[x_0, \dots, x_n, x_i^{-1}]$ determine morphisms of affine schemes

$$\pi_i\colon \operatorname{Spec} A[x_0, \dots, x_n, x_i^{-1}] \longrightarrow U_i.$$

Note that the scheme of the left can be identified with the distinguished open set $D(x_i) \subset \mathbb{A}^{n+1}_A$, and the collection of these form an affine cover of $D(x_0) \cup \cdots \cup D(x_n) = \mathbb{A}^{n+1}_A - V(x_0, \dots, x_n)$. To check that the morphisms $\pi_i$ glue, we need only check that they agree over the intersections. This follows by applying $\operatorname{Spec}$ to the following diagram:

$$\begin{array}{ccccc} A[x_0, \dots, x_n, x_i^{-1}] & \longrightarrow & A[x_0, \dots, x_n, x_i^{-1}, x_j^{-1}] & \longleftarrow & A[x_0, \dots, x_n, x_j^{-1}] \\ \uparrow & & \uparrow & & \uparrow \\ R_i & \longrightarrow & R_i\left[\tfrac{x_i}{x_j}\right] = R_j\left[\tfrac{x_j}{x_i}\right] & \longleftarrow & R_j \end{array}$$

The projective space over $\mathbb{Z}$ is especially important, because for every field $k$, the $k$-valued points of $\mathbb{P}^n_{\mathbb{Z}}$ coincides with the projective space $\mathbb{P}^n(k)$ as defined in the introduction.

**Proposition 5.6.** For any field $k$, we the $k$-valued points of $\mathbb{P}^n$ is given by (5.1), i.e.,

$$\mathbb{P}^n(k) = \left(k^{n+1} - \{0\}\right) / \sim . \tag{5.6}$$

*Proof* The image of a $k$-point $\iota\colon \operatorname{Spec} k \to \mathbb{P}^n$ is a single point in $\mathbb{P}^n$, so it must be contained in one of the $U_i$. Therefore, the $k$-points of $\mathbb{P}^n(k)$ is the union of the $k$-points of the $U_i$. Identifying $U_i(k) = \mathbb{A}^n(k) = k^n$, we define $\tau\colon k^{n+1} - \{0\} \to \mathbb{P}^n(k)$ by the map

$$\tau(a_0, \dots, a_n) = \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i}\right) \in U_i(k) \quad \text{if } a_i \neq 0$$

Then $\tau$ is well-defined, surjective, and $\tau(a_0, \dots, a_n) = \tau(b_0, \dots, b_n)$ precisely when $(b_0, \dots, b_n) = \lambda \cdot (a_0, \dots, a_n)$ for some $\lambda \in k^\times$. □

More generally, if $R$ is any ring, and $s = (s_0, \dots, s_n)$ is an $(n+1)$-tuple of elements which generate the unit ideal in $R$, there is a corresponding $R$-point

$$\iota_a : \operatorname{Spec} R \longrightarrow \mathbb{P}^n. \tag{5.7}$$

We define this morphism using the gluing theorem for morphisms. By our assumption, the distinguished open sets $D(s_0), \dots, D(s_n)$ cover $\operatorname{Spec} R$. Over the distinguished open

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sets $D(s_i) = \operatorname{Spec} R_{s_i}$, we have ring maps $\phi_i : \mathbb{Z}[x_0/x_i, \dots, x_n/x_i] \to R_{s_i}$ defined by $x_k/x_i \mapsto s_k/s_i$ which make the following diagrams commute:

$$\begin{array}{ccccc} \mathbb{Z}\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] & \hookrightarrow & \mathbb{Z}\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right] & \hookleftarrow & \mathbb{Z}\left[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}\right] \\ \downarrow & & \downarrow & & \downarrow \\ R_{s_i} & \longleftarrow & R_{s_i s_j} & \longrightarrow & R_{s_j} \end{array}$$

It follows that the scheme maps $\operatorname{Spec} R_{s_i} \to U_i$ glue to the desired morphism (5.7).

Observe that if $\lambda \in R$ is a unit, then $(\lambda s_0, \dots, \lambda s_n)$ determines the same $R$-point as $(s_0, \dots, s_n)$. This is because multiplying each $s_i$ by a unit do not change the expressions $s_k/s_i$ used in the construction of $\phi_i$.

It is natural to ask whether the formula $\mathbb{P}^n(R) = (R^n - \{0\}) / \sim$ holds for all rings $R$, that is, whether every $R$-valued point is of 'homogeneous form'. While this is true if $R$ is a local ring (Exercise B.5.27) or $R = \mathbb{Z}$ (Exercise B.5.28), it does not hold in general. In fact, it fails for an interesting reason, as the following example shows:

**Example 5.7.** Let us consider the $R$-valued points of $\mathbb{P}^1$ for $R = \mathbb{Z}[\sqrt{-5}]$. The ring $\mathbb{Z}[\sqrt{-5}]$ is famously not a UFD, as

$$2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}). \tag{5.8}$$

We will exploit this observation to define a non-trivial $R$-point of $\mathbb{P}^1_R$.

Note that 2 and 3 generate the unit ideal of $R$, so $\operatorname{Spec} R$ is covered by the two distinguished open sets $D(2) = \operatorname{Spec} R[\frac{1}{2}]$ and $D(3) = \operatorname{Spec} R[\frac{1}{3}]$, and these two intersect along the distinguished open set $D(6) = \operatorname{Spec} R[\frac{1}{6}]$. Let $\alpha = \frac{1}{2}(1 + \sqrt{5})$ and $\beta = \frac{1}{3}(1 - \sqrt{5})$ and consider the ring maps $\phi_0 : \mathbb{Z}[x] \to R[\frac{1}{2}]$ and $\phi_1 : \mathbb{Z}[x^{-1}] \to R[\frac{1}{3}]$ sending $x \mapsto \alpha$ and $x^{-1} \mapsto \beta$. By the factorization (5.8), we see that $\alpha\beta = 1$, so the following diagram commutes:

$$\begin{array}{ccccc} & & \mathbb{Z}[x] & \xrightarrow{\phi_0} & R[\frac{1}{2}] \\ & \swarrow & & & \searrow \\ \mathbb{Z}[x, x^{-1}] & & \longrightarrow & & R[\frac{1}{6}] \\ & \nwarrow & & & \nearrow \\ & & \mathbb{Z}[x^{-1}] & \xrightarrow{\phi_1} & R[\frac{1}{3}] \end{array}$$

Applying $\operatorname{Spec}$, we see that the morphisms $\operatorname{Spec}(\phi_0)$ and $\operatorname{Spec}(\phi_1)$ glue to a morphism $\operatorname{Spec} R \to \mathbb{P}^1$. The corresponding $R$-point in $\mathbb{P}^1(R)$ is *not* of the form $(a : b)$ with $a, b \in R$ generating the unit ideal. If it were, it would correspond to $(1 + \sqrt{-5} : 2)$ (this is what it is over $D(2)$). However, the ideal $(1 + \sqrt{-5}, 2)$ is not the unit ideal in $R$, in fact it is not even a principal ideal in $R$. (For a proof of this, see Exercise 4.11.5.)

What makes this example work is the fact that $R$ is not a UFD and has many non-principal ideals. We will consider this example later in Chapter 16 when we discuss invertible sheaves and maps to projective space. △

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### *The structure sheaf of* $\mathbb{P}^n_A$

The following computation is very important.

**Proposition 5.8.** For each ring $A$, we have $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$.

*Proof* By the sheaf sequence associated to the covering $U_0, \ldots, U_n$, a global section of $\mathcal{O}_{\mathbb{P}^n_A}$ consists of elements $s_i \in \Gamma(U_i, \mathcal{O}_{\mathbb{P}^n_A})$ that agree on the overlaps $U_i \cap U_j$. Over $U_i$ we have $\Gamma(U_i, \mathcal{O}_{\mathbb{P}^n_A}) = R_i$ and a section $s_i$ is a polynomial of the form

$$g_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) \tag{5.9}$$

The compatability condition on the overlaps means that

$$g_j\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) = g_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right). \tag{5.10}$$

The only way (5.10) can hold is that $g_i$ and $g_j$ are of degree 0 as polynomials in the ratios $x_k/x_i$ and $x_k/x_j$. That is, $g_i$ and $g_j$ are equal to the same element in $A$. Conversely, if we choose any element $a \in A$ and set $g_i = a$ for every $i = 0, \ldots, n$, then the sections clearly glue to a global section of $\mathcal{O}_{\mathbb{P}^n_A}$. □

In particular, for a field $k$, the space of global sections of $\mathcal{O}_{\mathbb{P}^n_k}$ is just the 'constants', $k$. When $k = \mathbb{C}$, this can be seen as an analogue of Liouville's theorem, that the only global analytic functions on $\mathbb{CP}^1$ are the constants. This example also shows that the group $\mathcal{O}_X(X)$ may not give much information about $X$ for general schemes.

We note that we also have got yet another example of a scheme which is not affine: if $\mathbb{P}^1_{\mathbb{C}}$ were affine, it would have to be isomorphic to $\operatorname{Spec}\mathbb{C}$ according to Theorem 4.10 on page 70. But this is clearly not the case, as $\mathbb{P}^1_{\mathbb{C}}$ contains infinitely many closed points. $\mathbb{P}^1_{\mathbb{C}}$ is in fact related to the first example of a non-affine scheme of Example 4.22 on page 72, namely the affine plane $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[u, v]$ with the 'origin' $V(u, v)$ removed, see (5.5).

### *Closed subschemes of* $\mathbb{P}^n_A$

In this section, we will consider closed subchemes of the projective space $\mathbb{P}^n_A$. We say that a polynomial $F \in A[x_0, \ldots, x_n]$ is homogeneous of degree $d$ if the monomials in the $x_i$ appearing in $F$ all have degree $d$. For such elements, we can consider their 'dehomogenization' with respect to the variable $x_i$, defined by

$$F_{(i)} = x_i^{-d} F(x_0, \ldots, x_n) = F\left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}\right) \in R_i = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right].$$

Conversely, if $f \in R_i$ is a polynomial of degree $d$ in the $x_0/x_i, \ldots, x_n/x_i$, then the polynomial

$$F(x_0, \ldots, x_n) = x_i^d f\left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_n}{x_i}\right)$$

will be homogeneous of degree $d$.

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The homogenization dehomogenization operations will in most cases be inverses. There are exceptions however: any power $x_0^d$ will dehomogenize to 1, and there is no way of recovering $x_0^d$ without knowing $d$. The best we can say is that there is a bijection between polynomials in $A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right]$ of degree $d$ and homogeneous polynomials in $A[x_0, \ldots, x_n]$ of degree $d$ not divisible by $x_i$.

More generally, if $\mathfrak{a} \subset R$ is a homogeneous ideal, we define the *dehomogenization of* $\mathfrak{a}$ *with respect to* $x_i$ by

$$\mathfrak{a}_{(i)} = \{\, F_{(i)} \in A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \mid F \in \mathfrak{a} \,\}.$$

It is not difficult to check that if $\mathfrak{a}$ is a prime ideal, then so is $\mathfrak{a}_{(i)}$ for each $i = 0, \ldots, n$.

Consider the subschemes defined by the $\mathfrak{a}_{(i)}$:

$$X_i = \operatorname{Spec}(R_i/\mathfrak{a}_{(i)}) \longrightarrow \operatorname{Spec}(R_i).$$

If $F_{(i)}$ and $F_{(j)}$ denote the dehomogenizations with respect to $x_i$ and $x_j$, we have

$$F_{(i)} = \left(\frac{x_j}{x_i}\right)^d F_{(j)} \quad \text{in } R_i\left[\frac{x_i}{x_j}\right] = R_j\left[\frac{x_j}{x_i}\right]$$

This implies that the two ideals $\mathfrak{a}_{(i)}$ and $\mathfrak{a}_{(j)}$ become equal in the localization $A_i[x_i/x_j] = A_j[x_j/x_i]$. Consequently, the subschemes $X_i$ coincide over the intersections $U_{ij}$, and consequently they glue together to a closed subscheme $X = V_+(\mathfrak{a}) \subset \mathbb{P}^n_A$.

**Definition 5.9.** We say that an $A$-scheme $X$ is *projective over* $A$ if it is isomorphic to one of the form $V_+(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a} \subset A[x_0, \ldots, x_n]$.

**Example 5.10.** For $d \in \mathbb{N}$, the homogeneous polynomial $F = x_0^d + x_1^d + x_2^d$ defines a closed subscheme $X = V_+(F) \subset \mathbb{P}^2_A$. When $A = k$ is a field, $X$ is covered by three affine charts, each isomorphic to the plane curve $V(x^d + y^d + 1) \subset \mathbb{A}^2_k$, so we may picture as a plane curve of degree $d$. The geometry of this scheme is however more intricate for other rings, such as $A = \mathbb{Z}$ or $A = \mathbb{C}[t]$. △

**Example 5.11.** If $\mathfrak{a} = (x_0, \ldots, x_n)$, then the corresponding closed subscheme is empty. Indeed, over $U_i$, the dehomogenized ideal contains $x_i/x_i = 1$, so $\mathfrak{a}$ defines the empty scheme in each $U_i$. The same argument applies to any ideal $\mathfrak{a}$ such that $\sqrt{\mathfrak{a}} = (x_0, \ldots, x_n)$. For this reason, $(x_0, \ldots, x_n)$ is called the *irrelevant ideal*.

This also implies that two ideals can correspond to the same closed subscheme. For instance, in $\mathbb{P}^1_k$ the two ideals $(x_0)$ and $(x_0^2, x_0x_1) = (x_0) \cap (x_0^2, x_1)$ both define the point $(0 : 1)$. △

**Example 5.12** (Homogeneous coordinates)**.** For an $(n+1)$-tuple $(a_0, \ldots, a_n)$ of elements in $A$ which generate the unit ideal, we define a closed subscheme defined by the ideal $\mathfrak{a}$ generated by the $2 \times 2$-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_n \\ a_0 & a_1 & \ldots & a_n \end{pmatrix}. \tag{5.11}$$

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That is, the homogeneous ideal $\mathfrak{a} = (a_i x_j - a_j x_i \,|\, 0 \leqslant i, j \leqslant n)$. One can check that the corresponding closed embedding is exactly the morphism $\iota_a \colon \operatorname{Spec} A \to \mathbb{P}^n_A$ defined in (5.7) (see Exercise B.5.8). △

**Proposition 5.13.** Every closed subset $Z \subset \mathbb{P}^n_A$ is of the form $V_+(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a} \subset A[x_0, \dots, x_n]$.

*Proof* By construction, a subset $Z$ is closed if and only if $Z \cap U_i$ for each $i = 0, \dots, n$. As each $U_i = \operatorname{Spec} R_i$ is affine, we have $Z \cap U_i = V(\mathfrak{a}_i)$ for some radical ideal $\mathfrak{a}_i \subset R_i$. Let us define

$$\mathfrak{a} = \{\, F \in A[x_0, \dots, x_n] \mid F \text{ homogeneous and } x_i^{-\deg F} F \in \mathfrak{a}_i \text{ for all } i = 0, \dots, n \,\}.$$

Then $\mathfrak{a}$ is homogeneous. Let us check that $Z = V_+(\mathfrak{a})$.

Let $p \in Z$ be a point. We may assume that $p \in Z \cap U_0 = V(\mathfrak{a}_0)$. Therefore $f(p) = 0$ for every $f \in \mathfrak{a}_0$. Consider $F \in \mathfrak{a}$ of degree $d$. Then $x_0^{-d} F \in \mathfrak{a}_0$, so $(x_0^{-d} F)(p) = 0$. As $p \in U_0$, we must have $F(p) = 0$. Hence $p \in V_+(\mathfrak{a})$.

Conversely, let $p \in V_+(\mathfrak{a})$. As before, we may assume that $p \in U_0$. By assumption, for every $F \in \mathfrak{a}$, we have $F(p) = 0$ and hence $x_0^{-\deg F} F \in \mathfrak{a}_0$. Therefore, $p \in V(\mathfrak{a}_0) = Z \cap U_0$. This shows that $p \in Z$, so $V_+(\mathfrak{a}) = Z$. □

If $I \subset A[x_1/x_0, \dots, x_n/x_0]$ is an ideal, we define the homogenization of $I$ as the ideal $\mathfrak{a} \subset A[x_0, \dots, x_n]$ obtained by homogenizing all the elements of $I$. Geometrically, the subscheme $V_+(\mathfrak{a}) \subset \mathbb{P}^n_A$ represents the closure of $V(I) \subset U_i$ in $\mathbb{P}^n_A$.

### *A family of sheaves on* $\mathbb{P}^n_A$

The projective space $\mathbb{P}^n_A$ carries a special family of sheaves, known as the *twisting sheaves* $\mathcal{O}_{\mathbb{P}^n_A}(m)$, where $m$ is an integer. These sheaves will play an important role in the study of projective schemes.

These sheaves are closely related to homogeneous polynomials. While homogeneous polynomials of degree $m \geqslant 1$ in $k[x_0, \dots, x_n]$ do not define regular functions on $\mathbb{P}^n_k$, they do define global sections of the sheaf $\mathcal{O}_{\mathbb{P}^n_k}(m)$. Intuitively, over an open set $U \subset \mathbb{P}^n_k$, the sections of $\mathcal{O}_{\mathbb{P}^n_k}(m)$ can be identified with the set of quotients

$$\frac{f(x_0, \dots, x_n)}{g(x_0, \dots, x_n)} \quad \text{where } \deg f - \deg g = m$$

where $f, g$ are homogeneous and $g(p) \neq 0$ for all $p \in U$.

To define these sheaves by gluing, we use the covering of $\mathbb{P}^n_A$ by $U_i = \operatorname{Spec} R_i$ where $R_i = A[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$, as before. Over $U_i$, we define the $R_i$-module

$$M_i = A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] x_i^m$$

Note that since $x_i^m = (x_i/x_j)^m x_j^m$, there is an equality of $A$-modules

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right] x_i^m = A\left[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}, \frac{x_j}{x_i}\right] x_j^m \tag{5.12}$$

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In other words, we may identify the localizations

$$(M_i)_{x_j/x_i} = (M_j)_{x_i/x_j} \tag{5.13}$$

as modules over the ring $R_i[\frac{x_i}{x_j}] = R_j[\frac{x_j}{x_i}]$.

Note that $M_i$ is a free $R_i$-module of rank 1, with $x_i^m$ serving as a generator. For each $i$ and $j$, one can pass from $M_i$ to $M_j$ by multiplying by $(x_j/x_i)^m$:

$$g\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) x_i^m = \left(\frac{x_i}{x_j}\right)^m g\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) x_j^m. \tag{5.14}$$

By the equality (5.13), the two sheaves $\widetilde{M_i}$ and $\widetilde{M_j}$ on $U_i$ restrict to the same sheaf on $U_{ij} = \operatorname{Spec}(A_i[x_i/x_j])$, namely the tilde of the module (5.12) (see Exercise 4.11.58). As the isomorphisms are identity maps, the gluing conditions are satisfied, and the sheaves $\widetilde{M_i}$ over $U_i$ glue to a sheaf on $\mathbb{P}^n_A$. We denote this sheaf by $\mathcal{O}_{\mathbb{P}^n_A}(m)$.

In the special case $m = 0$, we have $M_i = A_i$ and the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(0) = \mathcal{O}_{\mathbb{P}^n_A}$ is simply the structure sheaf of $\mathbb{P}^n_A$. The following computation generalizes Proposition 5.8:

**Proposition 5.14.** $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ can be identified with the free $R$-module of homogeneous polynomials of degree $m$ in the $x_0, \ldots, x_n$.

*Proof* By the sheaf sequence associated to the covering $U_0, \ldots, U_n$, a global section of $\mathcal{O}_{\mathbb{P}^n_A}(m)$ consists of sections $s_i \in \Gamma(U_i, \mathcal{O}_{\mathbb{P}^n_A}(m))$ that agree on the overlaps $U_i \cap U_j$. Over $U_i$ we have $\Gamma(U_i, \mathcal{O}_{\mathbb{P}^n_A}(m)) = M_i$ and a section $s_i$ is an element of the form

$$g_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) x_i^m \in M_i. \tag{5.15}$$

The compatability condition on the overlaps implies that

$$g_j\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right) x_j^m = g_i\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) x_i^m. \tag{5.16}$$

If $m < 0$, then clearly there is only one tuple satisfying (5.16), namely where $g_0 = \cdots = g_n = 0$.

If $m \geqslant 0$, we see that the $g_j$ are determined by $g_0$. Comparing degrees in (5.16), we see that $g_0$ must have degree $\leqslant m$ as a polynomial in the $x_1/x_0, \ldots, x_n/x_0$. This means that $F = g_0 x_0^m$ is a homogeneous polynomial of degree $m$ in $x_0, \ldots, x_n$ with coefficients in $A$, and $s_i = g_i x_i^m = F$ for all $i = 0, \ldots, n$. Conversely, any such $F$ determines a valid $(n+1)$-tuple $(s_0, \ldots, s_n)$. □

## 5.2 The Proj construction

In this section, we discuss a generalization of the construction of $\mathbb{P}^n_A$, known as the *Proj-construction*. Starting with any graded ring

$$R = \bigoplus_{d \geqslant 0} R_d \tag{5.17}$$

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we will construct a scheme $\operatorname{Proj} R$, called the *projective spectrum*. See Appendix ??? for background and notation on graded rings and modules.

The construction is somewhat parallel to that in the spectrum of a ring in Chapter 2. In both constructions, the topological space is defined in term of prime ideals in the ring and the structure sheaf is defined in term of localizations of the ring. There are however several notable differences between the two constructions. For instance, $\operatorname{Proj} R$ does not depend functorially on the ring $R$, in the sense that graded ring maps do not always induce maps between the projective spectra. Also, different graded rings $R$ may yield isomorphic projective spectra.

**Definition 5.15.** Let $R$ be a graded ring. The *projective spectrum*, $\operatorname{Proj} R$ is defined as the set of homogeneous prime ideals of $R$ that do not contain the irrelevant ideal $R_+$, that is,

$$\operatorname{Proj}(R) = \big\{\mathfrak{p} \in \operatorname{Spec} R - V(R_+) \mid \mathfrak{p} \text{ homogeneous}\big\}.$$

The reason for excluding primes containing the irrelevant ideal $R_+$ comes from Example 5.11. In that example, when $R = k[x_0, \ldots, x_n]$ with the standard grading, the irrelevant ideal $(x_0, \ldots, x_n)$ corresponds to the origin $(0, \ldots, 0)$ in $\mathbb{A}^{n+1}(k)$, which does not give a well-defined point of $\mathbb{P}^n$.

The set $\operatorname{Proj} R$ is endowed with the *Zariski topology* where the closed sets are of the form

$$V_+(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Proj} R \mid \mathfrak{a} \subset \mathfrak{p} \,\}$$

with $\mathfrak{a}$ a homogeneous ideal. The three topology axioms follow from the identities in the following lemma.

**Lemma 5.16.** Let $\mathfrak{a}$, $\mathfrak{b}$ and $\{\mathfrak{a}_i\}_{i\in I}$ be homogeneous ideals. Then:

(i) If $\mathfrak{a} \subset \mathfrak{b}$, then $V_+(\mathfrak{b}) \subset V_+(\mathfrak{a})$.
(ii) $V_+(\sum \mathfrak{a}_i) = \bigcap V_+(\mathfrak{a}_i)$.
(iii) $V_+(\mathfrak{a}\mathfrak{b}) = V_+(\mathfrak{a} \cap \mathfrak{b}) = V_+(\mathfrak{a}) \cup V_+(\mathfrak{b})$.
(iv) $V_+(\sqrt{\mathfrak{a}}) = V_+(\mathfrak{a})$.

*Proof* Note that sums, products and radicals remain homogeneous when the involved ideals are homogeneous. The proofs of the four statements are exactly the same as in Lemma 2.3 for $\operatorname{Spec} R$ (the arguments there remain valid under the additional constrains that the prime ideals are homogeneous and do not contain the irrelevant ideal). □

Just as for the spectrum of a ring, $\operatorname{Proj} R$ has a collection of distinguished open sets, which form a basis for the topology.

For each $f \in R$ which is homogeneous of positive degree, we define the *distinguished open set* $D_+(f)$ as

$$D_+(f) = \{\mathfrak{p} \in \operatorname{Proj} R \,|\, \mathfrak{p} \not\ni f\}.$$

In other words, $D_+(f)$ is the set of homogeneous prime ideals in $R$ that do not contain the irrelevant ideal $R_+$, and do not contain $f$. It is clear that $D_+(f)$ is an open set, being the complement of the closed set $V_+(f)$.

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**Lemma 5.17.** Let $R$ be a graded ring.
(i) For any two homogeneous elements $f$ and $g$ of positive degree, we have $D_+(f) \cap D_+(g) = D_+(fg)$.
(ii) The sets $D_+(f)$, where $f$ ranges over homogeneous elements of $R$ of positive degree, form a basis for the topology on $\operatorname{Proj} R$.

*Proof* These statements follow as in the affine case. For (i), for a homogeneous prime ideal $\mathfrak{p}$, the conditions $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$ are equivalent to $fg \notin \mathfrak{p}$.

For (ii), note that a closed set $V_+(\mathfrak{a})$ is equal to the intersection of the $V_+(f)$'s for the homogeneous $f \in \mathfrak{a} \cap R_+$. Taking complements we see that any open set $\operatorname{Proj} R - V_+(\mathfrak{a})$ is the union of the corresponding $D_+(f)$'s. □

In analogy with the formulas for defining $\mathbb{P}^n_A$, we will use the degree 0 localizations of $R$ to define the structure sheaf on $\operatorname{Proj} R$. More precisely, for a nonzero homogeneous element $f \in R$, the ring $R_f$ is naturally a $\mathbb{Z}$-graded ring, by defining

$$\deg(a/f^n) = \deg a - n \deg f.$$

Thus it makes sense to consider the subring of degree 0 elements $(R_f)_0$.

Recall that for a distinguished open set $D(f)$ of an affine scheme $\operatorname{Spec} A$, there is a canonical homeomorphism between $D(f)$ and $\operatorname{Spec} A_f$ which sends a prime $\mathfrak{p} \in D(f)$ to the prime ideal $\mathfrak{p}A_f$. In analogy with this, we will show below that the map $\mathfrak{p} \mapsto (\mathfrak{p}R_f)_0$ defines a homeomorphism between $D_+(f)$ and $\operatorname{Spec}(R_f)_0$, where $(R_f)_0$ denotes the degree 0 part of the localization $R_f$.

If $f, g \in R$ are homogeneous elements of positive degree, the canonical localization map $R_f \to R_{fg}$ induces a map of rings

$$(R_f)_0 \longrightarrow (R_{fg})_0 \tag{5.18}$$

The ring on the right can be viewed as a localization of the ring on the left in the degree 0 element $g^{\deg f} f^{-\deg f}$. Indeed, in $(R_f)_u$, both $f$ and $g$ are invertible, so we may identify $(R_f)_u = R_{fg}$. As the element $g^{\deg f} f^{-\deg f}$ has degree 0, this identification preserves the $\mathbb{Z}$-grading. Looking in degree 0 part, we find

$$((R_f)_0)_{g^{\deg f} f^{-\deg g}} = (R_{fg})_0. \tag{5.19}$$

**Proposition 5.18.** For each nonzero homogeneous $f \in R_+$, the map

$$\phi(\mathfrak{p}) = (\mathfrak{p}R_f)_0$$

defines a homeomorphism

$$\phi\colon D_+(f) \longrightarrow \operatorname{Spec}(R_f)_0. \tag{5.20}$$

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The homeomorphisms (5.20) are compatible in the sense that the following diagram

$$\begin{array}{ccc} D_+(fg) & \hookrightarrow & D_+(f) \\ {\scriptstyle \phi_{fg}}\downarrow & & {\scriptstyle \phi_f}\downarrow \\ \operatorname{Spec}(R_{fg})_0 & \hookrightarrow & \operatorname{Spec}(R_f)_0 \end{array} \quad . \tag{5.21}$$

commutes for each $f, g \in R_+$ homogeneous.

*Proof* Note that $\phi$ is defined as the composition

$$D_+(f) \to D(f) \simeq \operatorname{Spec}(R_f) \to \operatorname{Spec}(R_f)_0. \tag{5.22}$$

Therefore, $\phi$ is continuous. We next show that it is bijective.

The following observation will be useful. If $x \in R_{nd}$, and $\mathfrak{p}$ is a homogeneous prime ideal such that $\mathfrak{p} \not\ni f$, then

$$a \cdot f^{-n} \in \phi(\mathfrak{p}) \iff a \in \mathfrak{p}. \tag{5.23}$$

This follows because $af^{-n} \in \phi(\mathfrak{p})$ is equivalent to $a/1 \in \mathfrak{p}R_f$, and this is equivalent to $a \in \mathfrak{p}$, because $f \notin \mathfrak{p}$.

Injectivity of $\phi$: If $\phi(\mathfrak{p}) = \phi(\mathfrak{p}')$, and $a \in R_n$, then (5.23) shows that $a^d \in R_{nd}$ is contained in $\mathfrak{p}$ if and only it is contained in $\mathfrak{p}'$. But as $\mathfrak{p}$ is prime, it contains $a^d$ if and only if it contains $a$, and the same applies to $\mathfrak{p}'$. Hence $\mathfrak{p} = \mathfrak{p}'$,

Surjectivity of $\phi$: let $\mathfrak{q} \subset (R_f)_0$ be a prime ideal. We consider the homogeneous prime ideal $\mathfrak{p} = \bigoplus_{n \geqslant 0} \mathfrak{p}_n$ where

$$\mathfrak{p}_n = \left\{ a \in R_n \;\middle|\; a^d \cdot f^{-n} \in \mathfrak{q} \right\}.$$

In other words, $\mathfrak{p}_n$ consists of the homogenous elements of degree $n$ that end up in $\mathfrak{q}$ when 'dehomogenized'.

The first thing to check is that $\mathfrak{p}$ is a prime ideal. Once we know it is an ideal, it will be homogeneous by definition. It is clear that it is closed under multiplication by elements of $R$: if $t \in R_m$, and $a^d f^{-n} \in \mathfrak{q}$, then $ta^d f^{-m-n} \in \mathfrak{q}$ in $(R_f)_0$, and hence $ta \in \mathfrak{p}_{n+m}$. It is also an additive subgroup: assume we are given two elements $a, b \in \mathfrak{p}_n$. By definition, we have $a^d \cdot f^{-n} \in \mathfrak{q}$ and $b^d \cdot f^{-n} \in \mathfrak{q}$. Expanding $(a+b)^{2d} \cdot f^{-2n}$, we obtain a sum where every term is a multiple of either $a^d \cdot f^{-n}$ or $b^d \cdot f^{-n}$. Therefore $(a+b)^{2d} \cdot f^{-2n} \in \mathfrak{q}$, and hence $(a+b)^d \cdot f^{-n} \in \mathfrak{q}$, because $\mathfrak{q}$ is prime. Consequently $\mathfrak{p}$ is closed under addition.

To show that $\mathfrak{p}$ is prime, it suffices to check that $ab \in \mathfrak{p}$ implies either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ for $a$ and $b$ homogeneous. That $a \in R_m$ and $b \in R_n$ satisfy $ab \in \mathfrak{p}$ means that $(ab)^d f^{-(m+n)} \in \mathfrak{q}$. Since $f$ is a unit in $R_f$, it holds that $(ab)^d \in \mathfrak{q}$, and hence either $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. Therefore we have either $a^d f^{-m} \in \mathfrak{p}_m$ or $b^d f^{-n} \in \mathfrak{p}_n$, and so $\mathfrak{p}$ is prime.

Next we verify that $\phi(\mathfrak{p}) = \mathfrak{q}$. This follows from the implications

$$af^{-n} \in \mathfrak{q} \iff a^d f^{-nd} \in \mathfrak{q} \iff a \in \mathfrak{p}_{nd} \iff af^{-n} \in \phi(\mathfrak{p}) \tag{5.24}$$

which follow from (5.23) and the primeness of $\mathfrak{p}$ and $\mathfrak{q}$.

Hence $\phi$ is bijective. To prove that $\phi$ is a homeomorphism, it suffices to show that it is

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open. Since the open sets of the form $D_+(fg)$ form a basis for the open sets contained in $D_+(f)$, it suffices to show that for any homogeneous element $g \in R$, we have

$$\phi(D_+(fg)) = D(g^d f^{-\deg f}).$$

For $\mathfrak{p} \in D_+(f)$, the following series of equivalences hold because $\mathfrak{p}R_f$ is a prime ideal:

$$\begin{aligned} \mathfrak{p} \in D_+(fg) &\iff g \notin \mathfrak{p} \\ &\iff g^d f^{-\deg g} \notin (\mathfrak{p}R_f)_0 = \phi(\mathfrak{p}). \\ &\iff \phi(\mathfrak{p}) \in D(g^d f^{-\deg g}) \end{aligned}$$

Hence $\phi(D_+(fg)) = D(g^d f^{-\deg g})$.

Finally, we show that the diagram (5.21) commutes. The lower horizontal map is induced by the localization map $\ell\colon R_f \to R_{fg}$. If $\mathfrak{p} \in D_+(fg)$ is a homogeneous prime ideal such that $\mathfrak{p} \not\ni f$ and $\mathfrak{p} \not\ni g$, then we have $\ell^{-1}(\mathfrak{p}R_{fg}) = \mathfrak{p}R_f$. Since $\ell$ preserves the gradings, we have

$$\ell^{-1}(\mathfrak{p}R_{fg} \cap (R_{fg})_0) = \mathfrak{p}R_f \cap (R_f)_0.$$

This shows that $\mathfrak{p}$ maps to $\mathfrak{p}R_f \cap (R_{fg})_0$ via both compositions in (5.21). □

In light of (5.19), we may identify $\operatorname{Spec}(R_{fg})_0$ with a distinguished open subset of both $\operatorname{Spec}(R_f)_0$ and $\operatorname{Spec}(R_g)_0$. We now glue $\operatorname{Spec}(R_f)_0$ to $\operatorname{Spec}(R_g)_0$ along $\operatorname{Spec}(R_{fg})_0$ using the ring maps

$$(R_f)_0 \longrightarrow (R_{fg})_0 \longleftarrow (R_g)_0.$$

Here the 'triple intersections' can be identified with the affine scheme $\operatorname{Spec}(R_{fgh})_0$. The gluing conditions for schemes are satisfied (taking the gluing isomorphisms to be the identity maps), and hence we obtain a scheme, which we denote by $\operatorname{Proj} R$. The resulting structure sheaf has the property that for each $f \in R$ homogeneous of positive degree,

$$\mathcal{O}_{\operatorname{Proj} R}(D_+(f)) = (R_f)_0. \tag{5.25}$$

The scheme $\operatorname{Proj} R$ is in a natural way a scheme over $\operatorname{Spec} R_0$. Indeed, for each open set $U$, the group $\mathcal{O}_{\operatorname{Proj}}(U)$ is an $R_0$-algebra (this is clear if $U = D_+(f)$ since $\mathcal{O}(D_+(f)) = (R_f)_0$ is an $R_0$-algebra, and the general case follows by covering $U$ by $D_+(f)$'s). In particular, $\mathcal{O}_{\operatorname{Proj} R}(\operatorname{Proj} R)$ is an $R_0$-algebra, so applying Theorem 4.6 to $R_0 \to \mathcal{O}_{\operatorname{Proj} R}(\operatorname{Proj} R)$ we get a canonical morphism $\pi\colon \operatorname{Proj} R \to \operatorname{Spec} R_0$. When restricted to a distinguished open $D_+(f)$, this morphism can be identified with the canonical map $\operatorname{Spec}((R_f)_0) \to \operatorname{Spec}(R_0)$.

**Example 5.19.** For a ring $A$ and each non-negative integer $n$, we have

$$\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \dots, x_n].$$

To see this, note that for $R = A[x_0, \dots, x_n]$, then $D_+(x_i) = (R_{x_i})_0 = \operatorname{Spec} R_i$ with $R_i = A[x_0/x_i, \dots, x_n/x_i]$ and the intersections $D_+(x_i) \cap D_+(x_j)$ are equal to $D_+(x_i x_j) = \operatorname{Spec}(R_{x_i x_j})_0$. Thus $\operatorname{Proj} R$ is obtained as exactly the same gluing construction as $\mathbb{P}^n_A$. △

**Example 5.20.** The projective 0-space $\mathbb{P}^0_A = \operatorname{Proj} A[t]$ is particularly simple: since the irrelevant ideal is generated by $t$, we have

$$\operatorname{Proj} A[t] = D_+(t) = \operatorname{Spec}(A[t, t^{-1}])_0 = \operatorname{Spec} A.$$

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Over a field, we recover the expected geometric picture where $\mathbb{P}^0_k = \operatorname{Spec} k$ is a point. $\triangle$

**Example 5.21.** Consider $R = k[x_0, x_1]/(x_0x_1)$ with the natural grading. Geometrically, $\operatorname{Spec} R - V(x_0, x_1)$ represents the union of the $x_0$- and $x_1$-axes, excluding the origin. Therefore, we expect $\operatorname{Proj} R$ to consist of only two points. Besides the irrelevant ideal $R_+ = (x_0, x_1)$, there are only two homogeneous prime ideals, $(x_0)$ and $(x_1)$. Hence, $\operatorname{Proj} R$ indeed consists of two points. $\triangle$

## 5.3 Functoriality

In contrast to the Spec-construction, the Proj-construction is not entirely functorial. While a map of graded rings $\phi\colon R \to S$ induces a map on homogeneous prime ideals via $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$, this does not always yield a morphism. The issue is that the preimage $\phi^{-1}(\mathfrak{p})$ of a homogeneous prime ideal $\mathfrak{p} \subset S$ may contain the irrelevant ideal $R_+$.

To remedy this, we define the *base locus* of $\phi$ as the closed set

$$V(\phi(R_+)) \subset \operatorname{Proj} S.$$

This consists of precisely the prime ideals $\mathfrak{p}$ in $S$ for which $\phi^{-1}(\mathfrak{p}) \supset R_+$. This means that for prime ideals $\mathfrak{p} \notin V(\phi(R_+))$, the preimage $\phi^{-1}(\mathfrak{p})$ defines a point of $\operatorname{Proj} R$.

**Proposition 5.22.** Let $\phi\colon R \to S$ be a map of graded rings. Then there is a morphism of schemes

$$F\colon \operatorname{Proj} S - V(\phi(R_+)) \longrightarrow \operatorname{Proj} R.$$

which on the level of topological spaces is given by $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$.

*Proof* Note that the complement $U = \operatorname{Proj} S - V(\phi(R_+))$ is open in $\operatorname{Proj} S$ and has a canonical scheme structure. By the above remarks, the mapping $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$ which defines $F$ is well-defined. It is also continuous, because it is the restriction of the continuous map $\operatorname{Spec} S \to \operatorname{Spec} R$.

Next, we define the map of sheaves of rings $F^\sharp\colon \mathcal{O}_{\operatorname{Proj} R} \to F_*\mathcal{O}_U$. We define this map using the $\mathscr{B}$-sheaf construction on the open sets $D_+(f)$. Note that as $f$ runs through the homogeneous elements of $R_+$, the $D_+(\phi(f))$ cover $U$. Moreover,

$$F^{-1}(D_+(f)) = D_+(\phi(f)) \tag{5.26}$$

Indeed, for $\mathfrak{p} \in U$:

$$F(\mathfrak{p}) \in D_+(f) \iff f \notin \phi^{-1}(\mathfrak{p}) \iff \phi(f) \notin \mathfrak{p} \iff \mathfrak{p} \in D_+(\phi(f))$$

We have $\mathcal{O}_{\operatorname{Proj} R}(D_+(f)) = (R_f)_0$ and $(F_*\mathcal{O}_U)(D_+(f)) = (S_{\phi(f)})_0$. We define the map $F^\sharp$ over $D_+(f)$ to be the degree part 0 part of the localization

$$(R_f)_0 \longrightarrow (S_{\phi(f)})_0. \tag{5.27}$$

By the diagram below, these ring maps are compatible with the restriction maps, and define a

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map of sheaves $\mathcal{O}_{\operatorname{Proj} R} \to F_*\mathcal{O}_U$.

$$\begin{array}{ccc} (R_f)_0 & \longrightarrow & (S_{\phi(f)})_0 \\ \downarrow & & \downarrow \\ (R_{fg})_0 & \longrightarrow & (S_{\phi(fg)})_0 \end{array}$$

If we restrict $(F, F^\sharp)$ to a distinguished open set $D_+(f) \subset \operatorname{Proj} S$, we obtain the morphism of affine schemes

$$\operatorname{Spec}(S_{\phi(f)})_0 \longrightarrow \operatorname{Spec}(R_f)_0.$$

In particular, this implies the map induced by $F^\sharp$ on local rings is a map of local rings. (In fact, the induced map is given by $(R_{\phi^{-1}(\mathfrak{p})})_0 \longrightarrow (S_{\mathfrak{p}})_0$.) Hence $(F, F^\sharp)$ is a morphism of schemes. □

**Example 5.23** (Projection from a linear subspace)**.** If we consider the graded ring map given by the inclusion $\mathbb{Z}[x_0, \ldots, x_r] \hookrightarrow \mathbb{Z}[x_0, \ldots, x_n]$, the base locus equals the subscheme $V_+(x_0, \ldots, x_r)$. We get a corresponding morphism

$$\mathbb{P}^n - V_+(x_0, \ldots, x_r) \longrightarrow \mathbb{P}^r.$$

On the level of $k$-points, the induced map $\mathbb{P}^n(k) - V_+(x_0, \ldots, x_r) \to \mathbb{P}^1(k)$ is just the projection $(a_0 : \cdots : a_n) \mapsto (a_0 : \ldots, a_r)$. △

### *Closed embeddings*

Let $\mathfrak{a}$ be a homogeneous ideal in the graded ring $R$. The quotient map $\phi\colon R \to R/\mathfrak{a}$ is a map of graded rings, and it satisfies $\phi(R_+) = (R/\mathfrak{a})_+$. To see this, note that a prime $\mathfrak{q} \in \operatorname{Proj}(R/\mathfrak{a})$ contains $(R/\mathfrak{a})_+$ if and only if $\phi^{-1}(\mathfrak{q})$ contains $\phi^{-1}((R/\mathfrak{a})_+) = R_+$. The base locus is therefore empty, and we obtain a morphism

$$\iota\colon \operatorname{Proj} R/\mathfrak{a} \longrightarrow \operatorname{Proj} R$$

whose image is $V_+(\mathfrak{a})$.

We claim that $\iota$ is a closed embedding. It will suffice to check this over each $D_+(f)$ for $f \in R$ homogeneous. Over this open set, we have $\iota^{-1}D_+(f) = D_+(\phi(f))$, and the restriction of $\iota$ to $\iota^{-1}D_+(f)$ is identified with the morphism

$$\operatorname{Spec}\left((R/\mathfrak{a})_f\right)_0 \longrightarrow \operatorname{Spec}\left(R_f\right)_0$$

induced by the degree zero part of the localization $R_f \to (R/\mathfrak{a})_f$. Since this map is surjective, $\iota|_{\iota^{-1}D_+(f)}$ is a closed embedding.

In fact, under some mild assumptions on the graded ring $R$, every closed embedding into $\operatorname{Proj} R$ arises in this way. We will prove this in Chapter **??**.

**Example 5.24.** Let $R = k[x_0, \ldots, x_n]$ with the standard grading, and let $\mathfrak{a} \subset R$ be a homogeneous ideal. Consider the closed subscheme $Y = V_+(\mathfrak{a}) \subset \mathbb{P}^n_k = \operatorname{Proj}(R)$. For

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each $i = 0, \ldots, n$, the distinguished open subset $D_+(x_i) \subset \mathbb{P}^n_k$ is isomorphic to $\mathbb{A}^n_k$ via the map

$$(R_{x_i})_0 \simeq k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right].$$

The ideal $(\mathfrak{a}R_{x_i})_0$ in $(R_{x_i})_0$ is generated by the dehomogenizations of elements of $\mathfrak{a}$ with respect to $x_i$, i.e.,

$$(\mathfrak{a}R_{x_i})_0 = \left\{ f_{(i)} \mid x_i^d f \in \mathfrak{a} \text{ for some } d \in \mathbb{N} \right\},$$

The intersection $Y \cap D_+(x_i)$ is then defined by

$$Y \cap D_+(x_i) = V((\mathfrak{a}R_{x_i})_0) \subset D_+(x_i) \simeq \mathbb{A}^n_k.$$

Thus $Y$ coincides with the closed subscheme $V_+(\mathfrak{a})$ defined earler. △

### *Veronese subrings*

Let $R$ be a graded ring and let $n$ be a positive integer. The *Veronese subring*

$$R^{(n)} = \bigoplus_{d \geqslant 0} R_{nd}$$

is a graded ring. The inclusion $R^{(n)} \subset R$ induces a morphism

$$v_n \colon \operatorname{Proj}(R) \longrightarrow \operatorname{Proj}(R^{(n)}).$$

To see that the base locus is empty, note that the irrelevant ideal of $R^{(n)}$ is generated by all elements in $R$ whose degree is positive and divisible by $n$. Then $\phi(R^{(n)}_+)$ defines the empty set, since any prime $\mathfrak{p} \subset R$ such that $R_+ \cap R^{(n)} \subset \mathfrak{p}$ must contain all of $R_+$: for any $a \in R_+$, we have $a^n \in R_+ \cap R^{(n)}$, which forces $a \in \mathfrak{p}$ as well.

**Proposition 5.25.** The morphism $v_n$ is an isomorphism.

*Proof* Both $\operatorname{Proj}(R)$ and $\operatorname{Proj}(R^{(n)})$ are covered by distinguished open sets of the form $D_+(f^n)$ where $f \in R_+$ is homogeneous. Over such an open set, the morphism $v_n$ is induced by the inclusion

$$(R^{(n)}_{f^n})_0 \longrightarrow (R_{f^n})_0$$

This inlcusion is actually an equality: if $g/f^{ns} \in (R_{f^n})_0$, then $\deg g = ns \cdot \deg f$, so $g \in R^{(n)}$ and $g/f^{ns}$ belongs to the left-hand side.

This means that $v_d$ restricts to an isomorphism of schemes over an open covering of $\operatorname{Proj}(R^{(d)})$, and hence it is an isomorphism. □

### *More examples*

An important special case is when $R = A[x_0, \ldots, x_n]$ and $S = A[y_0, \ldots, y_m]$ with $\phi \colon R \to S$ is defined by

$$\phi \colon A[y_0, \ldots, y_m] \longrightarrow A[x_0, \ldots, x_n]; \quad y_i \mapsto f_i$$

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where $f_0, \ldots, f_m$ are homogeneous polynomials of the same degree $d$. In that case, the base locus of $\phi$ is the closed subscheme $V_+(f_0, \ldots, f_m)$. Strictly speaking, to make $\phi$ into a map of graded rings, we need to adjust the grading by defining $\deg(y_i) = d$ for each $i$. However, doing this does not change the Proj by Proposition 5.25.

We will for brevity write

$$\begin{aligned} \mathbb{P}^m_A - V_+(f_0, \ldots, f_n) &\longrightarrow \mathbb{P}^n_A \\ (x_0 : \cdots : x_m) &\longmapsto (f_0(x) : \cdots : f_n(x)) \end{aligned}$$

for the morphism defined by $f_0, \ldots, f_n$, even if the indicated map is only valid on the level of $k$-points, when $A = \mathbb{Z}$ or $A = k$.

In good cases, the subscheme $V_+(f_0, \ldots, f_n)$ is empty, and we obtain an actual morphism $\mathbb{P}^m_A \to \mathbb{P}^n_A$. This happens if and only if the radical of $(f_0, \ldots, f_n)$ contains irrelevant ideal $(x_0, \ldots, x_n)$.

**Example 5.26.** Let $n \in \mathbb{N}$ and consider the morphism

$$\begin{aligned} f \colon \mathbb{P}^1_A &\longrightarrow \mathbb{P}^1_A \\ (x_0 : x_1) &\longmapsto (x_0^n : x_1^n) \end{aligned}$$

Here the base locus is indeed empty, as the radical of $(x_0^n, x_1^n)$ is equal to the irrelevant ideal $(x_0, x_1)$. The morphism $f$ is called the *$n$-th power map*. If $k$ is a field, the map on $k$-points is given by $(a : b) \mapsto (a^n : b^n)$. In general, $f$ is a finite morphism, as over the two standard affines it coincides with the morphism $\mathbb{A}^1_A \to \mathbb{A}^1_A$ induced by $u \mapsto u^n$. △

**Example 5.27.** Consider the morphism

$$\begin{aligned} \iota \colon \mathbb{P}^1_A &\longrightarrow \mathbb{P}^2_A \\ (x_0 : x_1) &\longmapsto (x_0^2 : x_0x_1 : x_1^2) \end{aligned}$$

This is actually a closed embedding. Over the open set $D_+(y_0)$, we have $\iota^{-1}(D_+(y_0)) = D_+(x_0)$ and the morphism $\iota$ is given by

$$\operatorname{Spec} A\left[\frac{x_1}{x_0}\right] \longrightarrow \operatorname{Spec} A\left[\frac{y_1}{y_0}, \frac{y_2}{y_0}\right] \tag{5.28}$$

induced by the ring map $y_1/y_0 \mapsto (x_0x_1)/x_0^2 = x_1/x_0$ and $y_2/y_0 \mapsto x_1^2/x_0^2$. As this ring map is surjective, we obtain a closed embedding. Similarly, $\iota$ is a closed embedding over $D_+(y_1)$.

Note the relation $(x_0x_1)^2 = (x_0^2)(x_1^2)$ between the three monimials. This translates into the equation $y_1^2 - y_0y_2 = 0$ for the image of $\iota$. In fact, $\iota$ is exactly the closed subscheme associated with the homogeneous ideal $\mathfrak{a} = (y_1^2 - y_0y_2)$. △

**Example 5.28** (Rational normal curves)**.** More generally, the $d + 1$ monomials of degree $d$, $x_0^d, x_0^{d-1}x_1, \ldots, x_1^d$ define a closed embedding

$$\nu \colon \mathbb{P}^1_A \longrightarrow \mathbb{P}^d_A.$$

onto the closed subscheme $V_+(\mathfrak{a}) \subset \mathbb{P}^n_A$ defined by the ideal $\mathfrak{a}$ generated by the $2 \times 2$-minors

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of the matrix

$$\begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}. \tag{5.29}$$

When $A = k$ is a field, this closed subscheme is called the *rational normal curve of degree $d$*.

The case $d = 3$ is perhaps especially interesting: it is the *twisted cubic curve*, which is the projective version of the curve appearing in Example 1.33. It is the curve in $\mathbb{P}^3_k$ defined by the $2 \times 2$ minors of the matrix

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix}.$$

△

**Example 5.29** (The rational quartic curve)**.** Consider the morphism $\iota\colon \mathbb{P}^1_k \to \mathbb{P}^3_k$ defined by the four monomials

$$x_0^4, x_0^3x_1, x_0x_1^3, x_1^4$$

Note that the monomial $x_0^2x_1^2$ is missing. By looking in the four affine charts of $\mathbb{P}^3_k$ one can check that $\iota$ is a closed embedding. Writing $y_0, y_1, y_3, y_4$ for the coordinates on $\mathbb{P}^3_k$, the image $X = \iota(\mathbb{P}^1_k)$ is defined by ideal

$$I = \big(y_1y_3 - y_0y_4, y_3^3 - y_1y_4^2, y_0y_3^2 - y_1^2y_4, y_1^3 - y_0^2y_3\big)$$

Geometrically, $X$ arises as the projection of the rational normal curve in $\mathbb{P}^4_k$ from the closed point $(0 : 0 : 1 : 0 : 0)$. △

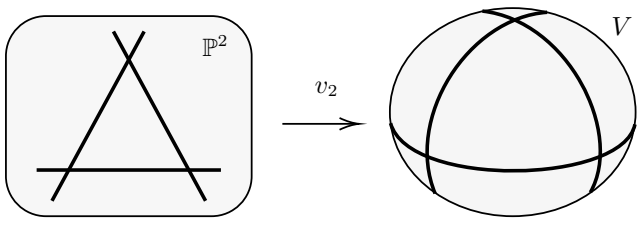
**Example 5.30** (Veronese varieties)**.** The morphism $v_2\colon \mathbb{P}^2_k \to \mathbb{P}^5_k$ defined by

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2),$$

defines an isomorphism between $\mathbb{P}^2_k$ and the *Veronese surface* $V_+(\mathfrak{a}) \subset \mathbb{P}^5_k$. Here $\mathfrak{a}$ is the ideal generated by the $2 \times 2$-minors of the matrix

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_3 & y_4 \\ y_2 & y_4 & y_5 \end{pmatrix}.$$

Under this embedding, lines in $\mathbb{P}^2_k$ are mapped into conics on $V$.




△

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### *Weighted projective spaces*

A rich source of examples is given by the *weighted projective spaces*, which are defined in terms of polynomial rings with non-standard gradings:

**Definition 5.31.** For a field $k$ and integers $d_0, \dots, d_n \in \mathbb{N}$, we define the *weighted projective space* over $k$ as

$$\mathbb{P}(d_0, \dots, d_n) = \operatorname{Proj} k[x_0, \dots, x_n]$$

where $\deg(x_i) = d_i$ for each $i$.

While this definition resembles that of a traditional projective space (where all $d_i$ are equal to 1), the weighted projective spaces give a surpringly diverse and rich class of examples.

**Example 5.32** (The weighted projective space $\mathbb{P}(1,1,d)$)**.** Consider the graded ring $R = k[x,y,z]$ where $\deg(x) = \deg(y) = 1$ and $\deg(z) = d$, and let $X = \mathbb{P}(1,1,d) = \operatorname{Proj} R$. The scheme $X$ is covered by the affine open subsets $D_+(x)$, $D_+(y)$, and $D_+(z)$.

The first two open subsets are easily understood: we have $(R_x)_0 = k[y/x, z/x^d]$ and $(R_y)_0 = k[x/y, z/y^d]$, both isomorphic to the polynomial ring in two variables, so that $D_+(x)$ and $D_+(y)$ are each isomorphic to $\mathbb{A}^2_k$. The third open subset is more subtle: the degree 0 part of $R_z$ is the $k$-subalgebra generated by the monomials $x^{d-i}y^i z^{-1}$ for $0 \leqslant i \leqslant d$, so that

$$(R_z)_0 = k[x^{d-i}y^i z^{-1} \mid 0 \leqslant i \leqslant d].$$

This ring is not a polynomial ring when $d > 1$, and in fact $D_+(z)$ is singular if $d \geqslant 2$.

The inclusion $S = k[x,y] \hookrightarrow R$ defines a rational map $f\colon X \dashrightarrow \mathbb{P}^1_k$ whose base locus is the closed point $p = V_+(x,y)$. On the open subset $D_+(x)$, the map $f$ is given by projection onto the first factor, i.e., by the map $(y/x, z/x^d) \mapsto y/x$, and similarly on $D_+(y)$ it is given by $(x/y, z/y^d) \mapsto x/y$. The gluing of $D_+(x)$ and $D_+(y)$ over $D_+(xy)$ is determined by the transition function $z/x^d = (x/y)^d z/y^d$, which shows that $X - \{p\}$ is isomorphic to the line bundle $L_d$ from Section B.3. Thus, $X$ is obtained from this line bundle by contracting the section to the point $p$. △

**Example 5.33.** The weighted projective plane $\mathbb{P}(1,2,3)$ is the Proj of the ring $R = k[x_0, x_1, x_2]$ where the variables $x_0, x_1, x_2$ have degrees $1, 2, 3$ respectively. The affine opens are

$$(R_{x_0})_0 = k\left[\frac{x_1}{x_0^2}, \frac{x_2}{x_0^3}\right]$$
$$(R_{x_1})_0 = k\left[\frac{x_0^2}{x_1}, \frac{x_0x_2}{x_1^2}, \frac{x_1^2}{x_1^3}\right]$$
$$(R_{x_2})_0 = k\left[\frac{x_0^3}{x_2}, \frac{x_0x_1}{x_2}, \frac{x_1^3}{x_2^2}\right].$$

This means that $D_+(x_0) = \operatorname{Spec}(R_{x_0})_0$ is isomorphic to an affine plane $\mathbb{A}^2_k$. The open sets $D_+(x_1)$ and $D_+(x_2)$ are not. For instance,

$$(R_{x_2})_0 \simeq k[U,V,W]/(UW - V^3).$$

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We can embed $\mathbb{P}(1,2,3)$ into a standard projective space by looking at the Veronese subring generated by degree 6 elements

$$R^{(6)} = k[x_0^6, x_1^3, x_2^2, x_0^4x_1, x_0^3x_2, x_0^2x_1^2, x_0x_1x_2].$$

The graded surjection $k[y_0, \ldots, y_6] \to R^{(6)}$ defined by $y_0 \mapsto x_0^6, y_1 \mapsto x_1^3, \ldots, y_6 \mapsto x_0x_1x_2$ induces a closed embedding

$$\iota\colon \mathbb{P}(1,2,3) \longrightarrow \mathbb{P}^6_k.$$

△

One of the advantages of weighted projective spaces is that they provide natural ambient spaces for projective schemes. Given a graded ring $R$ generated by homogeneous elements $x_0, \ldots, x_n$ of degrees $d_0, \ldots, d_n$ respectively, there is a graded surjection

$$R_0[x_0, \ldots, x_n] \longrightarrow R$$

which induces a closed embedding $\operatorname{Proj} R \to \mathbb{P}(d_0, \ldots, d_n)$ over $R_0$. Here is a concrete example:

**Example 5.34.** Recall the hyperelliptic curves from Section B.4. While the equation

$$y^2 = f(x_0, x_1)$$

where $f$ is homogeneous of degree $2d$, does not define a closed subscheme of $\mathbb{P}^2_k$, it does define a closed subscheme in the weighted projective space $\mathbb{P}(1,1,d) = \operatorname{Proj} k[x_0, x_1, y]$ with $\deg(y) = d$. In fact, every hyperelliptic curve constructed earlier embed as closed subschemes of $\mathbb{P}(1,1,d)$ defined by this equation.

The affine covering of $\mathbb{P}(1,1,d)$ consists of three open sets $D_+(x_0)$, $D_+(x_1)$, and $D_+(y)$. However, for the hyperelliptic curve $X$ defined by $y^2 = f(x_0, x_1)$, only the first two charts are needed since $X$ is entirely contained in $D_+(x_0) \cup D_+(x_1) = \mathbb{P}(1,1,d) - V_+(x_0, x_1)$. △

**Exercise 5.3.1.** Let $\mathfrak{a}$ be a homogeneous ideal. Then $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supset \mathfrak{a}} \mathfrak{p}$ where the intersection is taken over all the *homogeneous* ideals that contain $\mathfrak{a}$. HINT: If $x \notin \sqrt{\mathfrak{a}}$, then pick a homogeneous prime ideal $\mathfrak{p}$ so that $x^n \notin \mathfrak{p}$ for all $n$, and choose $\mathfrak{p}$ maximal with respect to this property. Show that $\mathfrak{p}$ is a prime ideal.

**Exercise 5.3.2.** Let $\mathfrak{a}$ and $I$ be homogeneous ideals in the graded ring $R$. Show the following:

a) If $\sqrt{I} = R_+$, then $V_+(\mathfrak{a}) = V_+(\mathfrak{a} \cap I)$. Hence, when constructing the closed sets $V_+(\mathfrak{a})$, it suffices to work with ideals contained in the irrelevant ideal.
b) $V_+(\mathfrak{a}) = \varnothing$ if and only if $R_+ \subset \sqrt{\mathfrak{a}}$. HINT: Use Exercise 5.4.11.
c) Show that the Zariski topology on $\operatorname{Proj}(R) \subset \operatorname{Spec}(R)$ is the induced topology from $\operatorname{Spec}(R)$. HINT: Any ideal $\mathfrak{a}$ has a corresponding 'homogenization', the ideal generated by all homogeneous components of the elements in $\mathfrak{a}$.

**Exercise 5.3.3.** Let $R$ be a graded ring and let $S \subset R$ be a subring such that for some $N \in \mathbb{N}$, $S_d = R_d$ for all $d \geqslant N$. Show that $\operatorname{Proj}(S) \simeq \operatorname{Proj}(R)$.

**Exercise 5.3.4.** Let $\phi\colon R \to S$ be a map of graded rings.

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a) Assume that $R_d \to S_d$ is surjective for all sufficiently large $d$. Show that $V(\phi(R_+)) = \varnothing$.
b) If $R_d \to S_d$ is an isomorphism for all sufficiently large $d$, show that $\operatorname{Proj} S \to \operatorname{Proj} S$ is an isomorphism.

**Exercise 5.3.5.** Let $k$ be an infinite field. Show that a Laurent polynomial $f \in k[x_0, \dots, x_n, x_i^{-1}]$ is invariant under the scaling operation $x_i \mapsto tx_i$ if and only if it belongs to the subring $k[x_0/x_i, \dots, x_n/x_i]$.

**Exercise 5.3.6.** Consider the projective twisted cubic curve $Y \subset \mathbb{P}^3_k$ as described in Example 5.28. Show that $Y \cap D_+(y_0)$ can be identified with the affine twisted cubic, as defined in Chapter 1. Describe the 'points at infinity' $V(y_0) \cap Y$.

## 5.4 Exercises

**Exercise 5.4.1.** Let $\mathfrak{a} \subset k[x, y, z]$ be the ideal $(xy, xz, yz)$. Show that $A = R/\mathfrak{a}$ is graded ring and describe each homogeneous component $A_n$.

**Exercise 5.4.2.** Consider the weighted projective space $\mathbb{P}(p, q) = \operatorname{Proj} R$, where $R = k[x_0, x_1]$ with $\deg(x_0) = p$ and $\deg(x_1) = q$. Show that

$$\mathbb{P}(p, q) \simeq \mathbb{P}^1_k.$$

HINT: Consider the Veronese subring $R^{(d)}$ where $d = pq$, and define a ring map $\phi\colon k[u, v] \to R^{(d)}$ by $u \mapsto x_0^q$, $v \mapsto x_1^p$.

**Exercise 5.4.3.** A polynomial ring $k[t_0, \dots, t_n]$ can be given a non-standard grading by declaring the degree of each $t_i$ to be any given natural number $d_i$. For instance, give $R = k[t_0, t_1]$ a grading by letting $\deg t_0 = 2$ and $\deg t_1 = 3$.
a) Describe the homogeneous pieces $R_n$ of degree $n$.
b) Let $k[u]$ have standard grading and define a map $\phi\colon R \to k[u]$ by the assignments $t_0 \mapsto u^3$ and $t_1 \mapsto u^2$. Show that $\phi$ is a map of graded rings.
c) Describe the kernel and the cokernel of $\phi$ as graded modules.

**Exercise 5.4.4.** Show that an ideal $\mathfrak{a}$ in a graded ring $R$ is homogeneous if and only if it is generated by homogeneous elements.

**Exercise 5.4.5.** Let $R$ be a graded ring which is not necessarily positively graded. Assume that a homogeneous element $f$ of $R$ is expressed as a combination $f = \sum a_i g_i$ where the $g_i$'s are homogeneous. Show that $f$ may be expressed as $f = \sum_i b_i g_i$, where each $b_i$ is homogeneous of degree $\deg f - \deg g_i$. HINT: Homogeneous components are unique.

**Exercise 5.4.6.** Let $R$ be a graded ring and $\mathfrak{a}$, $\mathfrak{b}$ and $\{\mathfrak{a}_i\}_{i \in I}$ be homogeneous ideals. Show that the ideals $\sqrt{\mathfrak{a}}$, $\mathfrak{ab}$, $(\mathfrak{a} : \mathfrak{b})$, $\sum \mathfrak{a}_i$, $\bigcap_i \mathfrak{a}_i$ are homogeneous.

**Exercise 5.4.7.** Let $R$ be a graded ring and let $f \in R_d$ be homogeneous of degree $d$. Show that there is an isomorphism of rings

$$(R_f)_0 = R^{(d)}/(f - 1). \tag{5.30}$$

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**Exercise 5.4.8.** Let $R$ be a graded ring and $\mathfrak{p}$ a homogeneous prime ideal. Show that $(R_{\mathfrak{p}})_0$ is a local ring with maximal ideal equal to

$$\mathfrak{m} = \{\, fg^{-1} \mid f \in \mathfrak{p}, g \in S(\mathfrak{p}) \text{ and } \deg f = \deg g \,\}.$$

where $S(\mathfrak{p})$ is the multiplicative subset of homogeneous elements in $R - \mathfrak{p}$.

**Exercise 5.4.9.** Let $R$ be a graded ring and $\mathfrak{p}$ a homogeneous ideal in $R$. Show that $\mathfrak{p}$ is prime if and only if $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ for all homogeneous elements $x$ and $y$.

**Exercise 5.4.10.** Let $R$ and $S$ be graded rings and $\phi\colon R \to S$ a map of graded rings. Show that the inverse image $\phi^{-1}(\mathfrak{p})$ of an ideal $\mathfrak{p} \subset S$ is homogeneous whenever $\mathfrak{p}$ is.

**Exercise 5.4.11.** Let $\mathfrak{a}$ be a homogeneous ideal. Then $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p}\supset\mathfrak{a}} \mathfrak{p}$ where the intersection is taken over all the *homogeneous* ideals that contain $\mathfrak{a}$. HINT: If $x \notin \sqrt{\mathfrak{a}}$, then pick a homogeneous prime ideal $\mathfrak{p}$ so that $x^n \notin \mathfrak{p}$ for all $n$, and choose $\mathfrak{p}$ maximal with respect to this property. Show that $\mathfrak{p}$ is a prime ideal.

**Exercise 5.4.12.** Let $R$ be a graded ring and let $f$ and $\{f_i\}_{i\in I}$ be homogenous elements from $R$ all of positive degree. Show that the distinguished open sets $D_+(f_i)$ cover $D_+(f)$ if and only if a power of $f$ lies in the ideal generated by the $f_i$'s.

**Exercise 5.4.13.** Let $R$ be a graded ring and let he $\pi\colon \operatorname{Proj} R \to \operatorname{Spec} R_0$ be the structure map. Show that for each $f \in R_0$, the inverse image $\pi^{-1}D(f)$ is isomorphic to $\operatorname{Proj} R_f$.

**Exercise 5.4.14.** Let $R$ be a 1-dimensional graded ring, with $R_0 = k$ a field, and assume that $R$ is finitely generated as a $k$-algebra. Show that $\operatorname{Proj} R$ is a finite set. HINT: the maximal ideal $R_+$ contains all homogeneous prime ideals.

**Exercise 5.4.15.** If $R$ is a graded integral domain, show that the function field of $X = \operatorname{Proj} R$ is given by

$$K(X) = \left\{ \frac{g}{h} \mid g \in R, h \in R, \deg g = \deg h \right\} \subset k(R) \tag{5.31}$$

**Exercise 5.4.16.** Show that $\operatorname{Proj} R$ is empty if and only if every element in $R_+$ is nilpotent.

**Exercise 5.4.17.** Let $R$ a graded ring. Show $R$ is Noetherian if and only if $R_0$ is Noetherian and $R_+$ is finitely generated.

**Exercise 5.4.18.** Find a non-Noetherian graded ring $R$ such that

a) $\operatorname{Proj} R$ is Noetherian.
b) $R$ is not of finite type over a field $k$, but $\operatorname{Proj} R$ is.
c) $R$ is not an integral domain, but $\operatorname{Proj} R$ is an integral scheme.

**Exercise 5.4.19.** Consider the $\mathbb{Q}$-scheme $X = \operatorname{Proj}(\mathbb{Q}[x,y,z]/(2x^2 + y^2 - 5z^2))$. Show that $X(\mathbb{Q}) = \varnothing$.

**Exercise 5.4.20.** Consider the projective scheme

$$X = \operatorname{Proj} \mathbb{Z}[x,y,z]/(2x^2 + 2y^2 - 3z^2).$$

Describe the scheme-theoretic fibers of the canonical map $\pi\colon X \to \operatorname{Spec} \mathbb{Z}$. Which ones are irreducible, reduced, integral? What are their dimensions?

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**Exercise 5.4.21** (Cremona transformation)**.** Let $A$ be a ring and consider the map of graded rings $\phi\colon \mathbb{Z}[u_0,u_1,u_2] \to \mathbb{Z}[x_0,x_1,x_2]$ defined by the three assignments $u_i \to x_j x_k$ where the indices satisfy $\{i,j,k\} = \{1,2,3\}$.

Determine the base locus $\mathrm{Bs}(\phi)$ and describe the $k$-points of $V(\mathrm{Bs}(\phi))$ when $k$ is a field.

**Exercise 5.4.22.** Show that the inverse of the Veronese map $\nu_d$ is not induced by a map of graded rings $R \to R^{(d)}$.

**Exercise 5.4.23.** Show that $\operatorname{Proj} k[x_0,x_1]/(x_0^2,x_1^2) = \varnothing$.

**Exercise 5.4.24.** Show that $\mathbb{P}(1,\dots,1,d)$ is isomorphic to the cone over the Veronese variety $V_{n,d}$.

**Exercise 5.4.25.** Let $t_1,\dots,t_r$ be a finite set of generators for the graded ring $R$ and let $d_i = \deg t_i$.

a) Let $D$ be the least common multiple of the $d_i$ and set $D_i = D/d_i$. Show that the Veronese ring $R^{(D)}$ is generated by elements of degree $D$.
b) Show that $\operatorname{Proj} R$ embeds as a closed subscheme of the weighted projective space $\mathbb{P}(d_0,\dots,d_n)$ over $R_0$.

**Exercise 5.4.26.** Let $x$ and $y$ be two points in $\mathbb{P}^n_k$. Prove there is an open affine $U \subset \mathbb{P}^n_k$ containing both $x$ and $y$.

**Exercise 5.4.27.** Show that equation (B.39) holds.

**Exercise 5.4.28** (The weighted projective space $\mathbb{P}(1,1,p)$)**.** Let $R$ be as in the Example 5.32 above, and let $A = k[x,y,w]$ with the usual grading. Furthermore, let $\alpha\colon R \to A$ be the map of graded rings that sends $z$ to $w^p$, while leaving $x$ and $y$ unchanged.

a) Show that $\alpha$ is a map of graded rings and induces a morphism $\pi\colon \mathbb{P}^2_k \to \operatorname{Proj} R$.
b) Describe the fibers of $\pi$ over closed points in case $k$ is algebraically closed.

**Exercise 5.4.29.** Let $R = k[x,y,z]$ be the polynomial ring with grading given by $\deg x = 1$, $\deg y = 2$ and $\deg z = 3$, and consider $\operatorname{Proj} R$ (which also is denoted $\mathbb{P}(1,2,3)$). The aim of the exercise is to describe the three covering distinguished subschemes $D_+(x)$, $D_+(y)$ and $D_+(z)$.

a) Show that $(R_x)_0 = k[yx^{-2}, zx^{-3}]$ and that $D_+(x) \simeq \mathbb{A}^2_k$.
b) Show that $(R_y)_0 \simeq k[x^2y^{-1}, z^2y^{-6}, xzy^{-2}]$. Show that the map of graded rings $k[u,v,w] \to (R_y)_0$ given by the assignments $x \mapsto yx^{-2}$, $v \mapsto z^2y^{-6}$ and $w \mapsto xzy^{-2}$ induces an isomorphism $k[u,v,w]/(w^2-uv) \simeq (R_y)_0$. Hence $D_+(y)$ is a hypersurface in $\mathbb{A}^3_k$. Show it is not isomorphic to $\mathbb{A}^2_k$. HINT: Check the local ring at the origin.
c) Show that $R_z = k[x^3z^{-1}, y^3z^{-2}, xyz^{-1}]$ and that the map $k[u,v,w] \to (R_z)_0$ defined by the assignments $x \mapsto x^3z^{-1}$, $v \mapsto y^3z^{-2}$ and $w \mapsto xyz^{-1}$ induces an isomorphism $k[u,v,w]/(w^3-uv) \simeq (R_z)_0$. Show that it is not isomorphic to $\mathbb{A}^2_k$.
d) Show that the map $R \to k[U,V,W]$ sending $x \mapsto U$, $y \to V^2$ and $z \mapsto W^3$ induces a map $\mathbb{P}^2_k \to \operatorname{Proj} R$, and describe the fibers over closed points.

**Exercise 5.4.30.** Let $k$ be a field. Show that:

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a) Any closed subscheme of $\mathbb{A}^n_k$ of dimension $n-1$ is a hypersurface.
b) Any closed subscheme of $\mathbb{P}^n_k$ of dimension $n-1$ is a hypersurface.

**Exercise 5.4.31.** Let $R = k[x_0, x_1]$ where $k$ is a field, and consider the morphism

$$\pi\colon \operatorname{Spec} R - V(x_0, x_1) \to \mathbb{P}^1_k = \operatorname{Proj} R$$

defined in (5.5).

a) Show that $\pi$ maps a $k$-point $(a, b)$ to $(a : b)$
b) Show that $\pi$ maps each height 1 prime $\mathfrak{p} = (f(x_0, x_1))$ to the generic point in $\mathbb{P}^1_k$.

**Exercise 5.4.32.** Let $R = k[x, y, z]/(xz, yz, z^2)$. Consider the map $\phi\colon R \to R$ given by

$$\phi(x) = x, \phi(y) = 0, \phi(z) = 0.$$

a) Show that $\phi$ induces a morphism $f : \operatorname{Proj} R \to \operatorname{Proj} R$.
b) Show that $f$ is the identity morphism.

In particular, different graded ring maps can induce the same map on Proj's.

**Exercise 5.4.33.** Let $R = \mathbb{C}[x_0, x_1, x_2]$ and let the symmetric group $S_3$ act on $R$ by permuting the variables.

a) Show that the invariant polynomials of $R$ are generated by $s_1 = x_0 + x_1 + x_2$, $s_2 = x_0x_1 + x_0x_2 + x_1x_2$ and $s_3 = x_0x_1x_2$.
b) Show that $s_1, s_2, s_3$ define a morphism $\pi\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}(1, 2, 3)$.
c) Describe the fibers of $\pi$.
d) Show that $\mathbb{P}(1, 2, 3)$ is homeomorphic to the quotient space $\mathbb{P}^2/S_3$.

**Exercise 5.4.34** (Nakayama's lemma for graded modules)**.** Let $R$ be a graded ring and let $M = \bigoplus_{d\in\mathbb{Z}} M_d$ be a finitely generated graded $R$-module. Let $x_1, \ldots, x_n$ be a set of homogeneous elements. Show that $x_1, \ldots, x_n$ generate $M$ if and only if their classes generate $M/R_+M$ as a $R_0$-module.

**Exercise 5.4.35.** Let $R$ be a graded ring and let $p_1, \ldots, p_r \in \operatorname{Proj} R$ be points. Show that there exists a homogeneous $f \in R_+$ such that $D_+(f)$ contains all the $p_i$. HINT: Prime avoidance.

**Exercise 5.4.36.** Let $U = \mathbb{P}^2_k - \{(0 : 0 : 1)\}$.

a) Show that $\mathcal{O}_{\mathbb{P}^2_k}(U) \simeq k$.
b) Is $U = \operatorname{Proj} R$ for a $k$-algebra $R$?

**Exercise 5.4.37.** Let $R$ be a Noetherian, normal graded integral domain.

a) Show that $\operatorname{Proj} R$ is a normal scheme. HINT: Show that each $(R_f)_0$ is normal.
b) Find a graded ring $R$ so that $\operatorname{Proj} R$ is normal, but $R$ is not.

**Exercise 5.4.38** (Quotients)**.** Let $G$ be a group acting by automorphisms on a locally ringed space $X$. Define $Y$ as the quotient space $X/G$ and let $\pi\colon X \to Y$ be the quotient map. Define a presheaf of rings on $Y$ by $\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}U)^G$ (the ring of invariants of $\mathcal{O}_X(U)$). Show that $\mathcal{O}_Y$ is a sheaf of rings and that $(Y, \mathcal{O}_Y)$ is a locally ringed space.

**Exercise 5.4.39.** Let $k$ be an infinite field.

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a) Consider the group $G = k^\times$ acting on $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ by scaling the variables: $t \cdot (x,y) = (tx,ty)$. Let $U = \mathbb{A}^2_k - \{(0,0)\}$. Show that $U/G$ is isomorphic, as a locally ringed space, to $\mathbb{P}^1_k$.
b) Consider now the action of $G = k^\times$ on $U$ given by $t \cdot (x,y) = (tx, t^{-1}y)$. Show that $Y = U/G$ is a scheme, isomorphic to the 'affine line with two origins'. Show that the corresponding quotient $\mathbb{A}^2_k/G$ is not a scheme.

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# 6

# Properties of schemes

## 6.1 Reduced schemes

A ring $A$ is said to be reduced if it has no nilpotent elements. In general, if $A$ is a ring, the *reduction of* $A$ is the ring $A_{\mathrm{red}} = A/\sqrt{0}$ where $\sqrt{0}$ is the ideal of nilpotent elements of $A$.

We define a scheme $X$ to be *reduced* if, for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ is reduced, that is, it has no nilpotent elements. While the condition is on stalks, in fact every $\mathcal{O}_X(U)$ is reduced:

**Lemma 6.1.** A scheme $X$ is reduced if and only if for every open $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nonzero nilpotent elements.

*Proof* If $X$ is reduced, and $s \in \mathcal{O}_X(U)$ is a nonzero element, then it as non-zero germ in at least one local ring $\mathcal{O}_{X,P}$. Therefore, if $s$ is nilpotent then so is its germ, and hence $O_{X,P}$ would then not be reduced. Conversely, let $P \in X$ be a point and let $s \in \mathcal{O}_{X,P}$ be any element. We may write $s$ as the germ of some section $t \in \mathcal{O}_X(U)$. By assumption $t$ is not nilpotent, and hence $s$ is not nilpotent either. □

**Example 6.2.** An affine scheme $X = \operatorname{Spec} A$ is reduced precisely when $A$ is a reduced ring. So for instance $\operatorname{Spec} \mathbb{Z}$ and $\mathbb{A}^n_k$ are reduced, but $\operatorname{Spec}(\mathbb{Z}/18)$ and $\operatorname{Spec} k[x]/(x^3)$ are not. △

For any scheme $X$, there is an associated reduced scheme $X_{\mathrm{red}}$ called the *reduction* of $X$. The two schemes $X$ and $X_{\mathrm{red}}$ have the same underlying topological space, but the structure sheaves are different; $\mathcal{O}_{X_{\mathrm{red}}}$ is obtained from $\mathcal{O}_X$ by locally modding out by nilpotent elements.

More precisely, consider the basis $\mathscr{B}$ of affine open sets $U \subset X$. We define a $\mathscr{B}$-presheaf by setting

$$\mathcal{O}_{X_{\mathrm{red}}}(U) = \mathcal{O}_X(U)_{\mathrm{red}}. \tag{6.1}$$

If $V \subset U$ are two open affines, and $s \in \mathcal{O}_X(U)$ is a nilpotent element, then so is the restriction $s|_V$. It follows that the restriction maps of $\mathcal{O}_X$ induce restriction maps $\mathcal{O}_{X_{\mathrm{red}}}(U) \to \mathcal{O}_{X_{\mathrm{red}}}(V)$. Moreover, by Exercise 4.11.9 the $\mathscr{B}$-sheaf axioms are satisfied, so this gives rise to a sheaf of rings on $X$ which we denote by $\mathcal{O}_{X_{\mathrm{red}}}$. We let $X_{\mathrm{red}}$ denote the locally ringed space $(X, \mathcal{O}_{X_{\mathrm{red}}})$.

If $X = \operatorname{Spec} A$, the reduction of $X$ is simply the scheme $\operatorname{Spec}(A_{\mathrm{red}})$. We saw a glimpse of this scheme in Corollary 2.31.

In general, if $X$ is a scheme, then also $X_{\mathrm{red}}$ is a scheme, because if $U_i = \operatorname{Spec} A_i$ is



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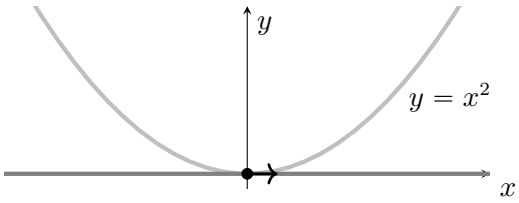
an affine cover of $X$, then $X_{\mathrm{red}}$ is covered by the affine schemes $(V_i)_{\mathrm{red}} = \operatorname{Spec}(A_i)_{\mathrm{red}}$. Moreover, $X_{\mathrm{red}}$ is naturally a closed subscheme of $X$, as for each $i$, the ring $(A_i)_{\mathrm{red}} = A_i/\sqrt{0}$ is a quotient of $A_i$.

Non-reduced schemes appear naturally when we consider intersections of subschemes. Suppose for instance we have an affine scheme $X = \operatorname{Spec} A$ and two closed subschemes $Y$ and $Z$, defined by ideals $\mathfrak{a}$ and $\mathfrak{b}$ in $A$, respectively. Then we define the *scheme-theoretic intersection* of $Y$ and $Z$ to be the closed subscheme of $\operatorname{Spec} A$ defined by the ideal $\mathfrak{a} + \mathfrak{b}$. Note that there is no reason to expect that the ideal $\mathfrak{a} + \mathfrak{b}$ is a prime ideal in $A$, even if $\mathfrak{a}$ and $\mathfrak{b}$ are. In fact, the scheme-theoretic intersection fail to be both reduced and irreducible even if both $Y$ and $Z$ are are. This is very natural and important: the scheme-theoretic intersection more accurately captures the multiplicities of an intersection, e.g. as in Bezout's theorem. This point is yet another reason for transitioning from varieties to schemes.

**Example 6.3.** Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, and consider the two subschemes $Y = V(y - x^2)$ and $Z = V(y)$. The scheme-theoretic intersection of these is given by the ideal $(y - x^2, y) = (x^2, y)$, which is not a radical ideal in $k[x, y]$. The nilpotent elements of $k[x, y]/(x^2, y) = k[x]/(x^2)$ in some sense account for the 'tangency' of the intersection $X \cap Y$. △




**Example 6.4.** If we consider the parabola $V(y - x^2)$ and the circle $V(x^2 + (y-1)^2 - 1)$ in $\mathbb{A}^2_k$, the calculation of Example 1.17 shows that the scheme-theoretic intersection is defined by the ideal

$$(y, x^2) \cap (y - 1, x - 1) \cap (y - 1, x + 1).$$

This scheme is also non-reduced, and the ideal $(y, x^2)$ reflects the fact that the two curves intersect at the origin $(0, 0)$ with 'multiplicity 2', unlike the two other intersection points, which have multiplicity 1. Geometrically, this corresponds to the fact that the parabola and the circle have a common tangent at $(0, 0)$. △

## 6.2 Integral schemes

A scheme $X$ is defined to be *integral* if $\mathcal{O}_X(U)$ is an integral domain for every affine open set $U \subset X$. As the name suggests, an affine scheme $X = \operatorname{Spec} A$ is integral if and only if $A$ is an integral domain. This follows from the following proposition:

**Proposition 6.5.** A scheme $X$ is integral if and only if it is irreducible and reduced.

*Proof* Assume that $X$ is integral. To show $X$ is reduced, let $P \in X$ be a point and choose an affine open $U = \operatorname{Spec} A$ containing it so that $P$ corresponds to a prime ideal $\mathfrak{p}$. Then by

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assumption, $A$ is an integral domain, so the local ring $\mathcal{O}_{X,P} = A_{\mathfrak{p}}$ has no zerodivisors. For irreducibility: if $X$ is not irreducible, then there exist two nonempty open set $U, V$ such that $U \cap V = \varnothing$. But then the sheaf sequence implies that $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$, which is not an integral domain.

Conversely, suppose that $X$ is irreducible and reduced. Let $U = \operatorname{Spec} A \subset X$ be an affine open set. Suppose that $f, g \in \mathcal{O}_X(U) = A$ are elements so that $fg = 0$; we have to show that $f = 0$ or $g = 0$. Note that $U = \operatorname{Spec} A = V(0) = V(fg) = V(f) \cup V(g)$. As $\overline{U} = X$ is irreducible, we must have, say $V(f) = U$. This means that $f$ is nilpotent in $A$. But as $X$ is reduced, there are no nilpotents of $A$, and so $f = 0$. □

**Example 6.6.** $\mathbb{A}^n_k$ and $\operatorname{Spec}\mathbb{Z}$ are integral schemes. The schemes $\operatorname{Spec}(\mathbb{C}[x]/x^2)$ and $\operatorname{Spec}(\mathbb{Z}/60)$ are not. △

Many of the examples in Chapter 1 correspond to non-integral schemes (e.g., Example 1.17 and Exercise 1.8.28).

## 6.3 Function fields

An important property of integral schemes $X$ is that they have a *function field*, $K(X)$, analogous to the field of rational functions on an affine variety (as discussed in Chapter 1).

To define $K(X)$, we note that any integral scheme $X$ has a unique generic point $\xi$. This point is the unique point which dense in $X$, that is, it belongs to every open non-empty subset of $X$ (see Exercise 4.11.42).

**Definition 6.7.** For an integral scheme $X$, we define the *function field*, or the *field of rational functions*, to be the local ring at the generic point $\xi \in X$:

$$K(X) = \mathcal{O}_{X,\xi},$$

Concretely, if $U = \operatorname{Spec} A$ is any affine open subset in $X$, then $A$ is an integral domain, and $\xi$ corresponds to the zero ideal $(0)$ of $A$. Then the local ring $\mathcal{O}_{X,\xi}$ is canonically identified with the field of fractions $K(A)$ of $A$.

**Example 6.8.** The function field of $\operatorname{Spec}\mathbb{Z}$ is equal to $\mathcal{O}_{\operatorname{Spec}\mathbb{Z},(0)} = \mathbb{Z}_{(0)} = \mathbb{Q}$. △

**Example 6.9.** The function field of $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ is equal to the field $k(x_1, \ldots, x_n)$ of rational functions in $x_1, \ldots, x_n$. △

**Example 6.10** (The quadratic cone)**.** The quadratic cone $X = \operatorname{Spec} k[x, y, z]/(y^2 - xz)$ is an integral scheme, as $y^2 - xz$ is irreducible. To compute the function field $K(X)$, we express $z$ as $z = x^{-1}y^2$, and we find

$$K(X) = k(x, y, z)/(y^2 - xz) \simeq k(x, y).$$ △

**Example 6.11.** The 'affine line with two origins' $X$ is both irreducible and reduced. The function field is isomorphic to $K = k(x)$. The two local rings $\mathcal{O}_{X,0_1}$ and $\mathcal{O}_{X,0_2}$ are equal as subrings of $K$; they are both equal to $k[x]_{(x)}$. This is somewhat unsettling: any rational function which is regular at $0_1$ is automatically regular at $0_2$ and it takes the same value there. This is related to the property of 'separatedness', which we will discuss in Chapter **??**. △

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We showed in that each $\mathcal{O}_X(U)$ is a subring of $K(X)$ when $X$ was an integral affine scheme. This extends to integral schemes in general.

**Proposition 6.12.** Let $X$ be an integral scheme.
(i) For an open set $U \subset X$, there is a natural inclusion $\mathcal{O}_X(U) \subset K(X)$.
(ii) For a point $P \in X$, there is a natural inclusion $\mathcal{O}_{X,P} \subset K(X)$.
With these identifications, we have

$$\mathcal{O}_X(U) = \bigcap_{P \in U} \mathcal{O}_{X,P} \subset K(X). \tag{6.2}$$

*Proof* (i): For $V = \operatorname{Spec} A$ affine, the germ map $\mathcal{O}_X(V) \to \mathcal{O}_{X,\xi}$ corresponds to the inclusion $A \subset K(A)$. This implies that for an arbitrary $U \subset X$, the germ map $\mathcal{O}_X(U) \to \mathcal{O}_{X,\xi}$ is injective: if $s \in \mathcal{O}_X(U)$ satisfies $s_\xi = 0$, then for any affine $V \subset U$, the factorization $\mathcal{O}_X(U) \to \mathcal{O}_X(V) \hookrightarrow \mathcal{O}_{X,\xi}$ forces $s|_V = 0$ and hence $s = 0$ by the Locality axiom. Hence each $\mathcal{O}_X(U)$ embeds as a subring of $K(X)$ via the germ map. The statement (ii) then follows by taking direct limits of the $\mathcal{O}_X(U) \subset K(X)$ for $U$ containing $P$.

We now prove the equality (6.2). The '$\subset$'-inclusion is clear. For the reverse inclusion, suppose $f \in K(X)$ lies in $\mathcal{O}_{X,P}$ for all $P \in U$. For each $P \in U$, we may pick an affine open neighbourhood $V_P$ of $P$ and sections $h_P \in \mathcal{O}_X(V_P)$ so that the germ of $h_P$ at $\xi$ is equal to $f$. By the injectivity of $\mathcal{O}_X(V_P \cap V_Q) \to \mathcal{O}_{X,\xi}$, the $h_P$ must agree on the overlaps $V_P \cap V_Q$, and hence they glue to a section of $\mathcal{O}_X(U)$ which maps to $f$. □

If $P \in X$ is a point and $f \in K(X)$ is a rational function, we say that $f$ is *defined* at $P$ if $f \in \mathcal{O}_{X,P}$. The subset of points $U \subset X$ where $f$ is defined is open (Exercise 4.11.11). The set $X - U$ is called the *indeterminacy locus* of $f$.

**Example 6.13.** If $A$ is an integral domain, then a rational function $a/b \in K(A)$ is defined on the open set $D(b) \subset \operatorname{Spec} A$. However, it may be defined on a larger open set (see Example 1.23). △

A *rational map* $f\colon X \dashrightarrow Y$ is a morphism defined on some open dense subset $U \subset X$. To avoid the ambiguity in the domain of definition, we say that two morphisms $f\colon U \to Y$ and $g\colon V \to Y$ define the same rational map if $f|_W = g|_W$ for some dense open subset $W \subset U \cap V$. This defines an equivalence relation on pairs $(U, f)$, and formally, a rational map is an equivalence class of such pairs.

**Proposition 6.14.** For an integral scheme $X$, there is a bijection

$$K(X) = \{\text{rational maps } f\colon X \dashrightarrow \mathbb{A}^1\}.$$

*Proof* This is essentially a consequence of Theorem 4.6, which implies that for an open set $U \subset X$ there is a bijection

$$\operatorname{Hom}_{\mathsf{Sch}}(U, \mathbb{A}^1) = \mathcal{O}_X(U).$$

Taking the direct limit over all open sets $U$ (which all contain the generic point $\xi$), the left side gives the set of rational maps to $\mathbb{A}^1$, whereas the right side gives $\mathcal{O}_{X,\xi} = K(X)$. □

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## 6.4 Dominant morphisms

We say that a morphism of schemes $f\colon X \to Y$ is *dominant* if the image of $f$ is a dense subset of $Y$. This means that any non-empty open set of $Y$ intersects the image $f(X) \subset Y$. Note that this is a purely topological condition.

**Example 6.15.** The morphism $\operatorname{Spec} k[x,y]/(xy-1) \to \operatorname{Spec} k[x]$ is dominant. More generally, any open embedding $U \to X$ of a dense open subset is dominant. △

**Example 6.16.** The map $\operatorname{Spec} k[x,y] \to \operatorname{Spec} k[u,v]$ given by $(u,v) \mapsto (x,xy)$ is dominant, as the image is $D(v) \cup \{V(u,v)\}$ (see Example 1.35) △

**Example 6.17** (Dominant morphisms into affine schemes)**.** A morphism of affine schemes $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$, induced by a ring map $\phi\colon A \to B$, is dominant if and only if every element of $\operatorname{Ker}\phi$ is nilpotent. To see this, note that by (iii) of Proposition 2.29, the closure of $f(\operatorname{Spec} B) = f(V(0))$ equals $V(\phi^{-1}(0)) = V(\operatorname{Ker}\phi)$. So $f(\operatorname{Spec} B)$ is dense if and only if $V(\operatorname{Ker}\phi) = \operatorname{Spec} A$. This condition holds precisely when $\operatorname{Ker}\phi \subset \mathfrak{p}$ for all $\mathfrak{p}$, or equivalently when $\operatorname{Ker}\phi \subset \sqrt{(0)}$.

The same argument applies more generally to morphisms into an affine scheme $f\colon X \to \operatorname{Spec} A$. By Theorem 4.6, $f$ is induced by a unique ring map $\phi\colon A \to \mathcal{O}_X(X)$ and Exercise 4.11.7 shows that the closure of the image of $f$ is given by $V(\operatorname{Ker}\phi)$. In particular, if $A$ is reduced, then $f\colon X \to \operatorname{Spec} A$ is dominant if and only if $\phi\colon A \to \mathcal{O}_X(X)$ is injective. △

The previous example generalizes as follows:

**Proposition 6.18.** Let $f\colon X \to Y$ be a morphism of integral schemes. Then the following are equivalent:

(i) $f$ is dominant
(ii) For all affine open subsets $U \subset X$ and $V \subset Y$ with $f(U) \subset V$, the induced ring map $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is injective.
(iii) $f$ maps the generic point of $X$ to the generic point of $Y$.

*Proof* (i) ⇒ (ii): $U$ and $V$ are dense, so also the restriction $f|_U\colon U \to V$ is dominant. Then the injectivity of $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ follows from Example 6.17.

(ii) ⇒ (iii): Write $\xi \in X$ and $\gamma \in Y$ for the two generic points. Note that for any subset $S \subset X$, we have $f(\overline{S}) \subset \overline{f(S)}$. Applying this to $S = \{\xi\}$, we see that $f(X) = f(\overline{\xi}) \subset \overline{f(\xi)}$. Therefore, if $f(X)$ is dense, then so is $f(\xi)$, and hence $f(\xi) = \gamma$ (there is only one point which is dense in $Y$).

(iii) ⇒ (i): If $f(\xi) = \gamma$, then $\overline{f(X)} \supset \overline{f(\xi)} = \overline{\gamma} = Y$, so $f$ is dominant. □

A key property of dominant morphisms is that they induce pullback maps between the corresponding function fields. More precisely, if $X$ and $Y$ are integral schemes with generic points $\xi$ and $\gamma$ respectively, and $f\colon X \to Y$ is dominant, then by Proposition 6.18, $f(\xi) = \gamma$ and so $f^\sharp$ induces a map between the local rings $\mathcal{O}_{Y,\gamma} \to \mathcal{O}_{X,\xi}$, hence a map

$$f^\sharp\colon K(Y) \longrightarrow K(X). \tag{6.3}$$

The same holds if $f\colon X \dashrightarrow Y$ is merely a dominant rational map.

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**Example 6.19.** If $k$ is a field, and $X \subset \mathbb{A}^n_k$ and $Y \subset \mathbb{A}^n_k$ are integral closed subschemes over $k$, then the pullback $f^\sharp(g)$ of a rational function $g \in K(Y)$ coincides with the pullback $f^\sharp(g) = g \circ f$, as defined in Section 1.6. △

If the image $f(X)$ is not dense in $Y$, there is no hope of defining a pullback map as in (6.3). For instance, if $f\colon \mathbb{A}^1_k \to \mathbb{A}^2_k$ is the closed embedding of the '$x$-axis', i.e., induced by the ring map $k[x,y] \to k[x,y]/y$, then there is no reasonable way to pull back the element $y^{-1} \in k(x,y)$ to $K(\mathbb{A}^1_k) = k(x)$. In fact there are no maps of fields $k(x,y) \to k(x)$ at all.

Typically, for a dominant morphism $f\colon X \to Y$, one expects $X$ to be 'as large as' or 'larger than' $Y$. For instance, in many good cases $X$ will have dimension at least the dimension of $Y$. On the other hand, there are also dominant morphisms with non-intuitive behaviour, as in the following examples:

**Example 6.20.** The morphism $\operatorname{Spec}\mathbb{Q} \to \operatorname{Spec}\mathbb{Z}$ is dominant. More generally, if $A$ is an integral domain with fraction field $K$, the canonical map $\operatorname{Spec} K \to \operatorname{Spec} A$ is dominant. △

**Example 6.21.** Let $X$ be the disjoint union of an infinite set of closed points of the affine line $\mathbb{A}^1_{\mathbb{C}}$. Then the embedding $X \to \mathbb{A}^1_{\mathbb{C}}$ is dominant. △

## 6.5 Covering intersections of affines by distinguished opens

A common situation in scheme theory is that if a property holds for the open sets of an affine cover of a scheme $X$, then the property holds for every affine open subset of $X$. For instance, we encountered such a situation in our discussion of closed subschemes in Section **??**. In this section, we explain the following basic, but extremely useful, lemma, which will allow us to systematically prove such statements.

**Lemma 6.22.** Let $X$ be a scheme and let $U = \operatorname{Spec} A$ be an affine open subscheme.

(i) If $V = \operatorname{Spec} B$ be is another open affine subscheme, then for any point $P \in U \cap V$, there exist an open set $W \subset U \cap V$ containing $P$, which is distinguished in both $U$ and $V$.

(ii) If $\{U_i\}_{i\in I}$ is an affine cover of $X$, then there exist elements $f_1, \dots, f_r \in A$ such that $D(f_1), \dots, D(f_r)$ cover $U$ and for each $j = 1, \dots, r$, $D(f_j)$ is a distinguished open set in both $U$ and some $U_{i(j)}$.

*Proof* (i): Let $P \in U \cap V$. Since the distinguished open sets form a basis for $V = \operatorname{Spec} B$, we may find a $g \in B$ such that $P \in D(g) \subset U \cap V$. As a distinguished open set in a distinguished open set is a distinguished open, we may replace $V$ by $D(g)$, and hence assume that $V \subset U$.

The inclusion map $\iota\colon V \to U$ is induced by a ring map $\phi\colon A \to B$. Let $a \in A$ be such that $D(a) \subset V$ and set $W = D(a) \subset U$. Then by Proposition 2.29, we have $W = \iota^{-1}(D(a)) = D(\phi(a))$ and so $W$ is distinguished in both $U$ and $V$.

(ii): Since $\{U_i\}_{i\in I}$ covers $X$, for each $P \in U$, there exists $U_i$ such that $P \in U_i$. By (i), there exists an open subset $W_P \subset U \cap U_i$ containing $P$ which is distinguished in both $U$ and $U_i$. Since $U$ is quasi-compact, we may reduce this to a finite subcover, and we obtain finitely

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many such open sets $D(f_1), \ldots, D(f_r)$ for $f_1, \ldots, f_r \in A$ which cover $U$. By construction each $D(f_j)$ is distinguished in some $U_{i(j)}$. □

## 6.6 Schemes of finite type

Building on the analogy with affine varieties, which arise from finitely generated $k$-algebras, we can extend the notion of 'finite generation' to schemes over a general ring $A$. Recall that an $A$-algebra $B$ is called *finitely generated*, or *of finite type* over $A$ if there exists a finite set of elements $x_1, \ldots, x_r \in B$ such that every element $f \in B$ can be expressed as a polynomial in the $x_i$'s with coefficients in $A$. Equivalently, $B$ is isomorphic to a quotient of a polynomial ring $A[x_1, \ldots, x_n]$.

**Definition 6.23.** An $A$-scheme $X$ is of *finite type over* $A$ if it admits a finite open cover $\{U_i\}_{i=1}^r$ where each $U_i = \operatorname{Spec} B_i$ and each $B_i$ is a finitely generated $A$-algebra.

The prototypical example of an $A$-scheme of finite type is $X = \operatorname{Spec} B$ where $B$ is a finitely generated $A$-algebra. This is not immediately clear from the definition, because $\operatorname{Spec} B$ could a priori be covered in some other way by affine open subschemes which are spectra of finitely generated $A$-algebras.

**Proposition 6.24.** Let $X$ be a scheme of finite type over $A$. Then for any affine open $U \subset X$, the ring $\mathcal{O}_X(U)$ is a finitely generated $A$-algebra.

*Proof* Write $U = \operatorname{Spec} B$, and let $U_i = \operatorname{Spec} B_i$ be the affine covering of $X$, where each $B_i$ are finitely generated $A$-algebras. By Lemma 6.22, we may refine the cover $\{U \cap U_i\}$ to a finite cover by distinguished open sets $D(f_1), \ldots, D(f_r)$ that cover $U$, such that each $D(f_j)$ is distinguished in some $U_{i(j)}$, say $D(f_j) = \operatorname{Spec}((B_{i(j)})_{g_j})$. As $B_{i(j)}$ is a finitely generated $A$-algebra, the localization $B_{f_j} = (B_{i(j)})_{g_j}$ is a finitely generated as well. We want to prove that $B$ itself is finitely generated.

Choose elements $a_{jk} \in B$ for $j = 1, \ldots, r$, $k = 1, \ldots, s$ such that for each $j$, $B_{f_j}$ is generated as an $A$-algebra by elements of the form $a_{jk}/f_j^{n_{jk}}$ for some $n_{jk} \in \mathbb{N}$.

We claim that $B$ is generated by $f_1, \ldots, f_r, a_{11}, \ldots, a_{rs}$. Consider the map of $A$-algebras

$$\phi\colon A[x_1, \ldots, x_r, y_{11}, \ldots, y_{rs}] \longrightarrow B$$

defined by $x_j \mapsto f_j$ and $y_{jk} \mapsto a_{jk}$ for all $j$ and $k$. To show that $\phi$ is surjective, it suffices to show that $\phi_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in \operatorname{Spec} B$. After reordering, we may assume that $\mathfrak{p} \in D(f_1)$. By our choice of generators the localization of $\phi$ at $x_i$ is surjective

$$A[x_1, \ldots, x_r, y_{11}, \ldots, y_{rs}]_{x_1} \longrightarrow B_{f_1} \tag{6.4}$$

As $f_1 \in A - \mathfrak{p}$, the ring map $\phi_{\mathfrak{p}}$ is a localization of the ring map (6.4). Since localization is exact, $\phi_{\mathfrak{p}}$ is surjective as well. Since this happens for all $\mathfrak{p}$, $B$ is finitely generated. □

**Example 6.25.** Any scheme of the form

$$X = \operatorname{Spec} A[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

is of finite type over $A$. △

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**Example 6.26.** The morphism $\coprod_{i=1}^{\infty} \mathbb{A}^1_k \to \mathbb{A}^1_k$ which is the identity on each component, is not of finite type, even though it is 'locally of finite type'. △

**Example 6.27.** An open embedding $\iota\colon U \to X$ need not be of finite type. For instance, the open embedding

$$\bigcup_{t\in\mathbb{N}} D(t_i) \longrightarrow \operatorname{Spec} k[t_1, t_2, \dots]$$

is not of finite type, because the scheme on the left is not quasi-compact (and so cannot be covered by finitely many affine subschemes).

However, if $U$ is quasi-compact, then an open embedding $\iota\colon U \to X$ is of finite type (see Exercise 6.11.33). △

A morphism of schemes $f\colon X \to S$ is of *finite type* if for every affine open set $V = \operatorname{Spec} A \subset S$, the inverse image $f^{-1}(V)$ is of finite type over $A$.

Although this definition requires the condition to hold for every affine open in $S$, the following proposition shows that it suffices to verify it for an affine cover of $S$.

**Proposition 6.28.** Let $f\colon X \to S$ be a morphism. If there exists an affine cover $S_i = \operatorname{Spec} A_i$ of $S$ such that for each $i$, the scheme $f^{-1}(S_i)$ is of finite type over $A_i$, then $f$ is of finite type.

*Proof* See Exercise 6.11.34. □

## 6.7 Finite morphisms

Recall that an $A$-module $B$ is said to be *finite over* $A$ if it is finitely generated as an $A$-module. In other words, there is a finite set of elements $b_1, \dots, b_r$ so that each $b$ is an $A$-linear combination $b = \sum a_i b_i$ with $a_i \in A$.

Even though the names 'finite type' and 'finite' are similar, the two notions are very different. To say that $B$ is of finite type is to say that $B$ is a ring quotient of a polynomial ring $A[x_1, \dots, x_r]$, where as $B$ being finite means that $B$ is a quotient module of a free module $A^r$ of finite rank. So for instance, $\mathbb{C}[t]$ is of finite type over $\mathbb{C}$, but it is not a finite $\mathbb{C}$-module.

**Definition 6.29.** A morphism $f\colon X \to Y$ is said to be *finite* if for every open affine $U = \operatorname{Spec} A \subset Y$, the preimage $f^{-1}(U)$ is affine, say, $f^{-1}(U) \simeq \operatorname{Spec} B$ and the induced ring map $A \to B$ makes $B$ a finite $A$-module.

If $Y = \operatorname{Spec} A$, we say that an $A$-scheme $X$ is *finite over* $A$ if the structure morphism $f\colon X \to \operatorname{Spec} A$ is finite.

**Proposition 6.30.** Let $f\colon X \to Y$ be a morphism and assume that there is an open affine covering $V_i = \operatorname{Spec} A_i$ of $Y$ such that $f^{-1}(V_i) \to V_i$ is finite for every $i$. Then $f$ is finite.

*Proof* Let $V = \operatorname{Spec} A$ be any affine. By Proposition **??**, $f$ is affine, so $f^{-1}V = \operatorname{Spec} R$ for some ring $R$, and it remains to prove that $R$ is a finite $A$-module.

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Using Lemma **??**, we find elements $g_1, \dots, g_r \in A$, so that $D(g_1), \dots, D(g_r)$ cover $V$ and each $D(g_i)$ is distinguished in some $V_{i(j)}$. By assumption, $R_{g_{i(j)}} = B_{i(j)}$ is a finite $A_{f_i}$-module for every $j = 1, \dots, r$.

Assume that finitely many $D(g_i)$'s cover $V$ and that $f^{-1}D(g_i) = \operatorname{Spec} B_{g_i}$ with each $B_i$ a finite modules over $A_{g_i}$. Let $x_{ij}$ be generators of $B_{g_i}$ over $A_{g_i}$, which we may choose to be images of elements $b_{ij}$ in $B$. We contend that the $b_{ij}$'s generate $B$ over $A$.

Given an element $b \in B$, it holds that $g_i^n b = \sum_j a_{ij} b_{ij}$ for some $n \in \mathbb{N}$ independent of $i$ and with $a_{ij} \in A$. Since the $D(g_i)$'s cover $V$, there is a relation of the form

$$1 = c_1 g_1^n + \cdots + c_r g_r^n,$$

which yields

$$b = \sum_j c_j g_j^n b = \sum c_j a_{ij} b_{ij}.$$

□

**Example 6.31.** Let $f\colon \operatorname{Spec} k[x] \to \operatorname{Spec} k[y]$ be the map induced by a ring map $\phi\colon k[y] \to k[x]$ sending $y \mapsto p(x)$, where $p$ is a nonconstant polynomial. Then $f$ is a finite morphism. Indeed, if $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then as a $k[x]$-module,

$$k[y] \simeq k[x][y]/(y - p(x)) \simeq k[x] \oplus k[x]x \oplus \cdots \oplus k[x]x^{n-1}.$$

△

**Example 6.32.** The morphism $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ is finite, while $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Q}$ is not. △

**Example 6.33.** Consider the 'blow-up' morphism

$$f\colon \operatorname{Spec} k[x, y, z]/(x - yz) \to \operatorname{Spec} k[x, y]$$

Then $f$ is not finite, as the ring on the left is not finitely generated as a $k[x, y]$-module. △

To underline the huge difference between the two finiteness conditions of this section, we observe the following: $X$ is of finite type over a field $k$ simply means it can be covered by open affine subschemes of the form $\operatorname{Spec} k[x_1, \dots, x_r]/\mathfrak{a}$. On the other hand, for $X$ is to be finite over a field $k$ means that $X = \operatorname{Spec} A$ is affine, and $A$ is a $k$-algebra of finite dimension over $k$. Such a ring $A$ is Artinian and has only finitely many prime ideals all being maximal. Hence the spectrum $\operatorname{Spec} A$ is a finite set, and the underlying topology is discrete.

**Example 6.34.** For $n \geqslant 1$, the structure morphisms $\mathbb{A}^n_k \to \operatorname{Spec} k$ and $\mathbb{P}^1_k \to \operatorname{Spec} k$ are of finite type, but not finite. △

**Example 6.35.** The embedding $\operatorname{Spec} A_g \hookrightarrow \operatorname{Spec} A$ of a distinguished open subscheme is of finite type, but typically not finite. For instance, the morphism $\operatorname{Spec} k[x, x^{-1}] \to \operatorname{Spec} k[x]$ is not finite, as $k[x, x^{-1}]$ is not a finite $k[x]$-module. △

**Example 6.36.** Consider 'the hyperbola' $X = V(xy - 1) \subset \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ and the projection $X \to \mathbb{A}^1_k = \operatorname{Spec} k[x]$ onto the $x$-axis, which is induced by the inclusion $k[x] \subset k[x, x^{-1}]$. The algebra $k[x, x^{-1}]$ is not finite over $k[x]$, as it requires all the

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negative powers $x^{-n}$ as generators. However, for any elements $a, b$ of $k$ with $ab \neq 0$, the ring $k[x, x^{-1}]$ is finite over $k[ax+bx^{-1}]$. Indeed, $k[x, x^{-1}]$ is generated by $x$ over $k[ax+bx^{-1}]$, and $x$ satisfies the monic equation

$$T^2 - a^{-1}(ax + bx^{-1})T + ba^{-1} = 0 \qquad \triangle.$$

**Example 6.37** (Noether's Normalization)**.** Let $X = \operatorname{Spec} A$ be an affine scheme of finite type over a field $k$. Then by Noether's Normalization Theorem (A.34), we may find algebraically independent elements $x_1, \ldots, x_n \in A$ such that $k[x_1, \ldots, x_n] \subset A$ is a finite extension. In particular, there is a finite dominant morphism

$$f\colon X \longrightarrow \mathbb{A}^n_k. \qquad \triangle$$

### *Properties of finite morphisms*

An important fact about finite morphisms is that they have finite fibers. Thus there is an elegant connection between the algebraic condition of 'being a finite module' and the geometric condition 'having finite fibers'.

**Proposition 6.38.** Let $f\colon X \to Y$ be a finite morphism. Then:
(i) $f$ is a closed map.
(ii) If $f$ is dominant, then it is surjective.
(iii) Each scheme-theoretic fiber $X_y$ has an underlying topological space which is finite and discrete.

*Proof* (i): As $f$ is affine, and 'closedness' of a subset can be checked over an open cover, we reduce to the case $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and $f$ is induced by a ring map $\phi\colon A \to B$.

By Proposition 2.29 on page 34, the closure $\overline{f(V(\mathfrak{b}))}$ of the image of a closed subset $V(\mathfrak{b}) \subset \operatorname{Spec} B$ equals $V(\phi^{-1}(\mathfrak{b}))$. On the other hand, note that the restriction of $f$ to $V(\mathfrak{b})$ can be identified with the map on spectra induced by $A/\phi^{-1}(\mathfrak{b}) \to B/\mathfrak{b}$. As this ring map is both injective and finite (since $\phi$ is finite), it is integral, and hence the Lying-Over Theorem (Theorem 6.40 (i) below) implies that $f$ is closed.

(ii): Again we may reduce to the affine case. Note that if $B$ is an $A$-algebra which is finite as an $A$-module, then $B_{\mathrm{red}}$ is an $A_{\mathrm{red}}$-algebra which is finite as a $A_{\mathrm{red}}$-module. Hence replacing $\phi_{\mathrm{red}}\colon A \to B$ with $\phi_{\mathrm{red}}\colon A_{\mathrm{red}} \to B_{\mathrm{red}}$, we do not change the induced map on spectra, and we may assume to the case where $A$ and $B$ are reduced. In that case, $f$ is dominant if and only if $\phi$ is injective. But an injective and finite ring map is integral, so we conclude by the Going-Up theorem (Theorem 6.40 (ii) below)

(iii): If $y \in Y$ is a point, choose an affine $U = \operatorname{Spec} A$ containing it. As $f$ is finite, $f^{-1}(U) = \operatorname{Spec} B$ is also affine, so we reduce to the case where $X$ and $Y$ are affine, and $f$ is induced by a ring map $A \to B$, making $B$ into a finite $A$-module.

In this situation, $y$ corresponds to a prime ideal $\mathfrak{p} \subset A$, and it follows that $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is a finite vector space over $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ (images of generators persist being generators). In other words, $B_{\mathfrak{b}}/\mathfrak{p}B_{\mathfrak{p}}$ is an Artinian ring, and hence its spectrum $X_y = \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ is finite and discrete. $\square$

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**Example 6.39.** The converse of Proposition 6.38 does not hold. The open embedding $\mathbb{A}^1_k - \{0\} \hookrightarrow \mathbb{A}^1_k$ has at most one point in each fiber, but it is not a finite morphism. △

In order to complete the proof of Proposition 6.38 (ii), we will prove a slightly more general result, for affine morphisms which are locally of the form $\operatorname{Spec} B \to \operatorname{Spec} A$ where $A \subset B$ is an integral ring extension. Recall that a ring extension $A \subset B$ is said to be *integral* if any element $y \in B$ satisfies some monic polynomial

$$y^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0 \tag{6.5}$$

with coefficients in $A$. (See Section A.10 for more background).

**Theorem 6.40.** Let $A \subset B$ be an integral ring extension and let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced map. Then:

(i) (Lying-Over) $f$ is surjective with discrete fibers.
(ii) (Going-Up) $f$ is closed

In concrete terms, $f$ being surjective means that for every prime ideal $\mathfrak{p} \subset A$ there is a prime ideal $\mathfrak{q} \subset B$ such that $\mathfrak{p} = \mathfrak{q} \cap A$.

*Proof* (i): Let $\mathfrak{p} \in \operatorname{Spec} A$ be a point. If $A \subset B$ is integral, then so is $A_\mathfrak{p} \subset B_\mathfrak{p}$. We want to show that the fiber $f^{-1}(\mathfrak{p}) = \operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$ is non-empty. In other words, we need to show that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is not the zero ring, or equivalently, that $\mathfrak{p}B_\mathfrak{p} \neq B_\mathfrak{p}$. Assuming the opposite, let $x \in \mathfrak{p}$ be an element so that $x/1$ is invertible in $B_\mathfrak{p}$. Then there is an element $y \in B_\mathfrak{p}$ such that $xy = 1$ in $B_\mathfrak{p}$. As $A_\mathfrak{p} \subset B_\mathfrak{p}$ is integral, $y$ satisfies an integral equation of the form (6.5) with coeffients in $A_\mathfrak{p}$. Multiplying this relation by $x^{n-1}$ shows that $y \in A_\mathfrak{p}$, contradicting the fact that $x$ is not invertible in $A_\mathfrak{p}$.

The fibers being discrete means that each point $\mathfrak{q} \in f^{-1}(\mathfrak{p})$ is closed in $\operatorname{Spec} B$. In other words, if $\mathfrak{q}, \mathfrak{q}' \in f^{-1}(\mathfrak{p})$ and $\mathfrak{q}' \in \overline{\{\mathfrak{q}\}} = V(\mathfrak{q})$, then $\mathfrak{q} = \mathfrak{q}'$. Replacing $A$ and $B$ by $A/(\mathfrak{q} \cap A) \subset B/\mathfrak{q}$, we reduce to the case where $A$ and $B$ are integral domains and $\mathfrak{p} = (0)$. If there is a nonzero prime ideal $\mathfrak{q}'$ such that $\mathfrak{q}' \cap A = (0)$, take any $y \in \mathfrak{q}'$. Then $y$ satisfies an integral equation of the form (6.5) and we may choose $n$ minimal. Note that the equation (6.5) implies that $a_0 \in \mathfrak{q}' \cap A = \mathfrak{p} = (0)$. Therefore, $y(y^{n-1} + \cdots + a_1) = 0 \in B$. As $n$ is chosen minimal, $y^{n-1} + \cdots + a_1 \neq 0$, contradicting the fact that $B$ is an integral domain.

(ii): Let $\mathfrak{b} \subset B$ be an ideal. We claim that $f(V(\mathfrak{b})) = V(\mathfrak{b} \cap A)$. The containment '$\subset$' is clear. Conversely, applying (i) to the extension $A/(\mathfrak{b} \cap A) \to B/\mathfrak{b}$ (which is again integral), we see that the restriction $V(\mathfrak{b}) \to V(\mathfrak{b} \cap A)$ is surjective, as we want. □

**Example 6.41.** Let $k$ be an algebraically closed field and consider the closed subscheme $X = V(y^2 + P(x))$ in $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, where $P(x)$ is a polynomial in $k[x]$. Let $\pi\colon \mathbb{A}^2_k \to \mathbb{A}^1_k$ denote the projection onto the $x$-axis (induced by the inclusion $k[x] \subset k[x, y]$). Then the restriction $\pi|_X$ will be finite. Indeed, its algebraic counterpart is the ring extension $k[x] \subset k[x, y]/(y^2 + P(x))$, and the latter ring has a basis as module over $k[x]$ consisting of 1 and $y$.

On the other hand, if $Y = V(xy^2 + P(x))$ and $P(0) \neq 0$, then $\pi|_Y$ is not a finite morphism. Indeed, the origin $0 \in \mathbb{A}^1(k)$ does not belong to its image, and so $\pi|_Y$ is not surjective. △

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**Proposition 6.42.** Let $f\colon X \to Y$ be a finite morphism. Then

$$\dim X \leqslant \dim Y$$

with equality if and only if $f$ is dominant.

*Proof* We first prove the statement when $X$ and $Y$ are affine, say $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $f$ is induced by a ring map $\phi\colon A \to B$.

We have $\dim \overline{f(X)} \subset \dim Y$ (Lemma 6.59). Therefore, we may as well replace $Y$ by $\overline{f(X)}$ and assume $f$ is dominant. On the level of rings, this corresponds to replacing $A$ by $A/\operatorname{Ker}\phi$, and $\phi\colon A \to B$ injective. Then $B$ is an integral extension of $A$, and $f$ is surjective by Theorem 6.40. Moreover, $\dim A = \dim B$ by Proposition A.19.

For the general case, cover $Y$ by affines $\{U_i\}_{i\in I}$. Then the collection $f^{-1}U_i$ form an affine cover of $X$. By the affine case and (iii) of Lemma 6.59 we find

$$\dim X = \sup_i \dim f^{-1}(U_i) \leqslant \sup_i U_i = \dim Y.$$

□

**Proposition 6.43.** Let $f\colon X \to Y$ be a finite surjective morphism of Noetherian integral schemes. Then:

(i) $K(X)$ is a finite field extension of $K(Y)$.
(ii) There is a nonempty open set $V \subset Y$ such that the fiber $f^{-1}(y)$ consists of exactly $d$ points, where $d = [K(X) : K(Y)]$.

*Proof* As $f$ is affine, we may reduce to the affine case where $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ induced by a ring map $\phi\colon A \to B$, making $B$ into a finite $A$-module. By the Generic Freeness Theorem (Theorem A.55) there is $g \in A$ such that $B_g$ is *free* as an $A_g$-module, i.e., $B_g \simeq A_g^d$ for some integer $d$. This implies that $K(B_g)$ is a finite extension of $K(A_g)$ of degree $d$. Over the open set $V = D(g)$, we have for each $\mathfrak{p} \in \operatorname{Spec} A$,

$$B \otimes_A \kappa(\mathfrak{p}) \simeq \prod_{i=1}^{d} \kappa(\mathfrak{p}).$$

This implies that $f^{-1}(\mathfrak{p}) = \operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ consists of exactly $d$ points, as required. □

**Definition 6.44** (The degree of a finite morphism)**.** Let $f\colon X \to Y$ be a finite morphism of integral schemes. The degree of the field exension $K(Y) \subset K(X)$ is called the *degree* of $f$ and is denoted $\deg f$.

Note that if $f\colon X \to Y$ and $g\colon Y \to Z$ are finite morphisms of integral schemes, the composition $g \circ f$ is finite and $\deg(g \circ f) = \deg f \cdot \deg g$. This follows because $[M : K] = [M : L][L : K]$ for a tower of field extensions $K \subset L \subset M$.

**Example 6.45.** The morphism $f\colon \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ is finite of degree 2, but of course there is only a single point in the preimage. △

**Example 6.46.** The '$n$-th power map' $f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ of Example 12.33 has degree $n$. Indeed,

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locally it is given by the ring map $k[u] \to k[x]$ sending $u$ to $x^n$, and $k(x) = k(u)[x]/(u - x^n)$ has dimension $n$ as a $k(u)$-vector space. △

**Example 6.47.** If $X$ is a hyperelliptic curve, as discussed in Section B.4, then the morphism $X \to \mathbb{P}^1_k$ has degree 2. In this case, $K(X)$ is obtained from $k(x)$ by adjoining a square root $y$ of a polynomial $f(x) \in k[x]$, so it is spanned by 1 and $y$ as a $k(x)$-vector space. △

## 6.8 Noetherian schemes

Recall that a ring $A$ is Noetherian if every ideal is finitely generated. This is a strong requirement, which has many important consequences for the ring.

**Definition 6.48.** A scheme $X$ is *Noetherian* if it has a finite open cover $\{U_i\}_{i=1}^r$ where $U_i = \operatorname{Spec} A_i$ where each $A_i$ is a Noetherian ring.

It is important to note that the cover is required to be finite. This in particular implies that a Noetherian scheme is quasi-compact, as it can be covered by finitely many affine schemes, each of which is quasi-compact.

**Proposition 6.49.** Let $X$ be a scheme. Then the following are equivalent:
(i) $X$ is Noetherian.
(ii) $X$ is quasi-compact, and for every open affine $U = \operatorname{Spec} A \subset X$, the ring $A$ is Noetherian.

*Proof* (ii)⇒(i) is immediate, so we focus on the implication (i)⇒(ii).

Let $\{U_i\}_{i=1}^r$ be an affine cover of $X$, where $U_i = \operatorname{Spec} A_i$ and each $A_i$ is a Noetherian ring. Since $X$ is covered by finitely many affine open sets, it is quasi-compact.

Now let $U = \operatorname{Spec} A$ be any affine open in $X$. We need to show that $A$ is a Noetherian ring. For each $i$, the intersection $U \cap U_i$ is an open set in $U_i = \operatorname{Spec} A_i$, and since $U_i$ is quasi-compact, it can be covered by distinguished open sets $D(f_{ij}) \subset U_i$, where $f_{ij} \in A_i$. The $D(f_{ij})$'s form a cover of $U$, so since $U$ is quasi-compact, we may reduce the cover to a finite one. Since $A_i$ is Noetherian, the localization $(A_i)_{f_{ij}}$ is also Noetherian. Therefore, $U$ can be covered by finitely many spectra of Noetherian rings, and hence it is a Noetherian scheme. Therefore, we reduce to considering $X = U = \operatorname{Spec} A$, and proceed to show that if $\operatorname{Spec} A$ is a Noetherian scheme then $A$ is Noetherian.

We may refine the covering $U_i$ to finitely many distinguished open sets, so that $\operatorname{Spec} A$ is covered by finitely many distinguished opens $D(f_1), \ldots, D(f_r)$, where $f_1, \ldots, f_r$ generate the unit ideal in $A$.

To conclude, we need to show that every ideal $\mathfrak{a}$ in $A$ is finitely generated. By assumption, the ideals $\mathfrak{a}A_{f_i}$ are all finitely generated, and since $f_i$ is invertible in $A_{f_i}$, we can find finitely many elements $a_{ij} \in A$ which map to generators of $\mathfrak{a}A_{f_i}$ in $A_{f_i}$. Consider the $A$-linear map

$$\phi\colon \bigoplus_{i,j} A \to \mathfrak{a}$$

which sends the standard basis vector $e_{ij}$ to $a_{ij}$. Since the $D(f_i)$'s cover $\operatorname{Spec} A$, the local-

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ization $\phi_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in \operatorname{Spec} A$, and from this it follows that $\phi$ is surjective as well. Consequently, $\mathfrak{a}$ is finitely generated and hence $A$ is Noetherian. □

**Corollary 6.50.** The scheme $\operatorname{Spec} A$ is Noetherian if and only if $A$ is a Noetherian ring.

The concept of a Noetherian scheme is closely related to being of finite type.

**Proposition 6.51.** Let $f\colon X \to Y$ be a morphism of finite type and suppose $Y$ is Noetherian. Then $X$ is also Noetherian.

*Proof* Let $V = \operatorname{Spec} A$ be any affine open set in $Y$. Since $Y$ is Noetherian, $A$ is a Noetherian ring. As $f$ is of finite type, $X$ can be covered by finitely many spectra $V = \operatorname{Spec} B$ where $B$ is a finitely generated $R$-algebra. As $A$ is Noetherian, Hilbert's basis theorem then implies that $B$ is Noetherian and hence $X$ is Noetherian as well. □

On the other hand, $\operatorname{Spec} \mathbb{Q}$ is not of finite type over $\mathbb{Z}$, but it is Noetherian.

**Proposition 6.52.** Let $X$ be a Noetherian scheme. Then any open or closed subscheme of $X$ is also Noetherian.

*Proof* Let $\{U_i\}_{i\in I}$ be a finite affine cover with $U_i = \operatorname{Spec} A_i$ for $A_i$ Noetherian. It suffices to prove that if $Y \subset X$ is a closed or open subscheme, then $Y \cap U_i$ is Noetherian. In particular, since $Y \cap U_i$ is closed or open subscheme of an affine scheme, it suffices to consider the case where $X = \operatorname{Spec} A$ and $A$ is Noetherian.

When $Y$ is an open subscheme, then there are elements $g_1, \ldots, g_n \in A$ such that $Y = \bigcup_{i=1}^n \operatorname{Spec} A_{g_i}$. If $A$ is Noetherian, then so is each of the localizations $A_{g_i}$, and consequently $Y$ is Noetherian.

When $Y$ is a closed subscheme, we may assume $Y = \operatorname{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$. If $A$ is Noetherian, then so is $A/\mathfrak{a}$, and hence $Y = \operatorname{Spec}(A/\mathfrak{a})$ is also Noetherian. □

Another nice property of Noetherian schemes is that the underlying topological space is Noetherian (that is, descending chains of closed subsets stabilize, see page 9).

**Proposition 6.53.** Let $X$ be a Noetherian scheme. Then

(i) The topological space of $X$ is Noetherian.

(ii) Any closed subset $Y \subset X$ admits a unique decomposition into irreducible components

$$Y = Y_1 \cup \cdots \cup Y_r.$$

*Proof* (i): By definition, $X$ may be covered by a finite number of open affine subsets. A descending chain stabilizes if and only if the intersection with each of those open sets stabilizes, so we reduce the proof to showing the proposition for $X = \operatorname{Spec} A$ where $A$ is a Noetherian ring, in which case the statement is clear.

(ii): See Proposition 1.15. □

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### *Examples*

**Example 6.54.** For a field $k$, the disjoint union $X = \coprod_{i=1}^{\infty} \operatorname{Spec} k$ is not Noetherian (it is not even quasi-compact). △

**Example 6.55.** The scheme $X = \operatorname{Spec}\left(\prod_{i=1}^{\infty} k\right)$ is affine, hence quasi-compact. However it is not Noetherian, because the ring is not Noetherian. In fact, the set of prime ideals in infinite products of fields is remarkably complicated: it is described by the set of so-called 'ultrafilters' on $\mathbb{N}$. (See also Exercise 6.11.10 for a related example.) △

The topological space $\operatorname{Spec} A$ can be Noetherian even without $A$ being Noetherian: the condition is equivalent to the weaker condition that ascending chains of *radical ideals* eventually stabilize, and there are many rings which satisfy this without being Noetherian. Here are two examples:

**Example 6.56.** Consider the polynomial ring $k[x_1, x_2, x_3, \dots]$ and the maximal ideal $\mathfrak{m} = (x_1, x_2, \dots)$. The ring

$$A = k[x_1, x_2, x_3, \dots]/\mathfrak{m}^2$$

has only one prime ideal, the maximal ideal $\mathfrak{m}$. Therefore, $\operatorname{Spec} A$ consists of a single point, and is therefore Noetherian as a topological space. The ring $A$ however is not Noetherian, as $\mathfrak{m}$ requires infinitely many generators, namely all the $x_i$'s. △

**Example 6.57.** Let $A = \mathbb{C}[[t]]$ be the ring of formal power series over $\mathbb{C}$, with fraction field $K = \mathbb{C}((t))$. Let $\overline{K}$ denote the algebraic closure of $K$, that is,

$$\overline{K} = \bigcup_{n \geqslant 1} \mathbb{C}((t))(t^{1/n}).$$

The integral closure of $A$ in $\overline{K}$ is given by

$$\overline{A} = \bigcup_{n \geqslant 1} \mathbb{C}[[t]](t^{1/n}).$$

Note that $\overline{A}$ is not Noetherian, as we have an infinite ascending chain of ideals:

$$(t) \subset (t^{1/2}) \subset (t^{1/4}) \subset \cdots .$$

However, the spectrum $\operatorname{Spec} \overline{A}$ is Noetherian as a topological space. In fact, it consists of only two points. To see this, observe that each $\mathbb{C}[[t^{1/n}]]$ is a discrete valuation ring, hence its spectrum consists of two points: the generic point and the closed point. Moreover, the inclusions $\mathbb{C}[[t^{1/2^n}]] \subset \mathbb{C}[[t^{1/2^{n+1}}]]$ induce a directed system

$$\cdots \to \operatorname{Spec} \mathbb{C}[[t^{1/4}]] \to \operatorname{Spec} \mathbb{C}[[t^{1/2}]] \to \operatorname{Spec} \mathbb{C}[[t]]$$

and $\operatorname{Spec} \overline{A}$ is the inverse limit of this system (see Exercise 4.11.57). As all the maps involved take the generic point to the generic point and the closed point to the closed point, they are are homeomorphisms, and hence $\operatorname{Spec} \overline{A}$ also consists of two points. △

While the first example may seem a bit artificial, the second one is one that could appear 'in nature'. That being said, most of classical algebraic geometry primarily deals with Noetherian rings. For instance, all of the examples from Chapter **??** are Noetherian, as they are all

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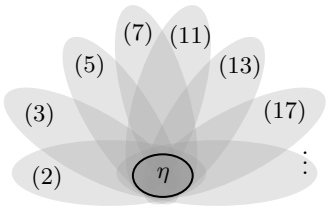


**Figure 6.1** Gluing together all the $\operatorname{Spec}\mathbb{Z}_{(p)}$'s

constructed by explicitly gluing together finitely many schemes of the form $\operatorname{Spec} A$ where $A$ is a Noetherian ring. Here is another example on the 'exotic side'.

**Example 6.58.** In Example 4.38 on page 84, we glued together the affine schemes $X_p = \operatorname{Spec}\mathbb{Z}_{(p)}$ with $p$ from a finite set $\mathcal{P}$ of prime numbers. However, in the gluing conditions for schemes, there are no restrictions on the number of schemes to be glued together, and we are free to glue together $\mathcal{P}$ infinite. For example, we can let $\mathcal{P}$ be the set of all primes.

The resulting scheme $X_{\mathcal{P}}$ is rather peculiar: it is neither affine nor Noetherian, but it is locally Noetherian. As a scheme over $\mathbb{Z}$, the canonical map $\pi\colon X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$ is bijective and continuous, but it is not a homeomorphism. Moreover, for all open subsets $U \subset \operatorname{Spec}\mathbb{Z}$ the map induced on sections $\pi^\sharp\colon \Gamma(U, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}) \to \Gamma(\pi^{-1}U, \mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism. In other words, $\pi^\sharp\colon \mathcal{O}_{\operatorname{Spec}\mathbb{Z}} \to \pi_*(\mathcal{O}_{X_{\mathcal{P}}})$ is an isomorphism of sheaves!

As in Example 4.38 the scheme $X_{\mathcal{P}}$ is constructed by gluing the different $\operatorname{Spec}\mathbb{Z}_{(p)}$'s together along the generic points. However, when computing the global sections, we see things changing. As in Example 4.38 the global sections are computed with the help of the sheaf sequence

$$\begin{array}{ccccc} 0 \longrightarrow \Gamma(X, \mathcal{O}_X) & \longrightarrow & \prod_{p\in\mathcal{P}} \Gamma(X_p, \mathcal{O}_X) & \longrightarrow & \prod_{p,q\in\mathcal{P}} \Gamma(X_p \cap X_q, \mathcal{O}_X) \\ & & \| \downarrow & & \downarrow \| \\ & & \prod_{p\in\mathcal{P}} \mathbb{Z}_{(p)} & \xrightarrow[\rho]{} & \prod_{p,q\in\mathcal{P}} \mathbb{Q}, \end{array}$$

and the kernel of $\rho$ is still $\bigcap_{p\in\mathcal{P}} \mathbb{Z}_{(p)}$, but now this intersection equals $\mathbb{Z}$. Indeed, a rational number $\alpha = a/b$ lies in $\mathbb{Z}_{(p)}$ precisely when the denominator $b$ does not have $p$ as factor, so lying in all $\mathbb{Z}_{(p)}$, means that $b$ has no non-trivial prime-factor. That is, $b = \pm 1$, and hence $\alpha \in \mathbb{Z}$.

One can understand the canonical map $\pi\colon X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$ as follows. Each of the schemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ maps in a natural way into $\operatorname{Spec}\mathbb{Z}$, by the map induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}_{(p)}$. Here the generic point of $\operatorname{Spec}\mathbb{Z}_p$ map to generic point of $\operatorname{Spec}\mathbb{Z}$, and the closed point maps to $(p) \in \operatorname{Spec}\mathbb{Z}$. As the maps agree on the generic points, they glue to the canonical map $\pi\colon X_{\mathcal{P}} \to \operatorname{Spec}\mathbb{Z}$. This is a continuous bijection by construction, but it is not a homeomorphism. Indeed, the subsets $\operatorname{Spec}\mathbb{Z}_{(p)}$ are open in $X_{\mathcal{P}}$ by the gluing construction, but they are not open in $\operatorname{Spec}\mathbb{Z}$, as their complements are infinite.

The underlying topological space of $X_{\mathcal{P}}$ is not Noetherian, as the subschemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ form an open cover that obviously cannot be reduced to a finite cover. However, it is locally Noetherian as the open subschemes $\operatorname{Spec}\mathbb{Z}_{(p)}$ are Noetherian. The sets $U_p = X_{\mathcal{P}} - \{(p)\}$

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map bijectively to $D(p) \subset \operatorname{Spec}\mathbb{Z}$ and $\Gamma(U_p, \mathcal{O}_{X_\mathcal{P}}) = \mathbb{Z}_p$, but $U_p$ and $D(p)$ are not isomorphic. △

## 6.9 The dimension of a scheme

Recall from Chapter 1 that the dimension of a topological space is the supremum of all integers $n$ such that there exists a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n \tag{6.6}$$

of distinct irreducible closed subsets of $X$. If no such finite upper bound exists, we say that $\dim X = \infty$.

We define the dimension of a scheme $X$, denoted $\dim X$, as the dimension of the underlying topological space. For a point $P \in X$, we will also consider the *dimension of $X$ at $P$*, denoted $\dim_P X$, which is defined as the suprenum of lengths of chains as in (6.6) with $P \in Z_0$.

**Lemma 6.59.** Let $X$ be a topological space.
(i) If $Y \subset X$ is any subset, then $\dim Y \leqslant \dim X$.
(ii) Assume that $\dim X < \infty$. If $Y \subset X$ is a closed and irreducible subset and $\dim Y = \dim X$, then $Y$ is an irreduible component of $X$.
(iii) If $\{U_i\}_{i\in I}$ is an open cover of $X$, then $\dim X = \sup_{i\in I} \dim U_i$.

*Proof* (i): As the closure of an irreducible subset is again irreducible, and since any closed $Z \subset Y$ satisfies $\overline{Z} \cap Y = Z$, a chain $\{Z_i\}$ of distinct irreducible closed subsets of $Y$ will yield a chain $\{\overline{Z_i}\}$ of distinct irreducible closed subsets of $X$.

(ii): Assume $\dim X < \infty$ and let $Y \subset X$ be a closed irreducible subset with $\dim Y = \dim X$. If $Y$ is not an irreducible component, there would exist a strictly larger irreducible closed subset $Z$ containing Y. But this would contradict the maximality of the dimension of $Y$.

(iii): First of all, $\dim X \geqslant \sup_{i\in I} \dim U_i$ by part (i). To prove the reverse inequality, we first remark that if $Z \subset X$ is closed and irreducible and $Z \cap U \neq \emptyset$, then $\overline{Z \cap U} = Z$. Indeed, $\overline{Z \cap U} \subset Z$ holds because $Z$ is closed, and if the inclusion is strict, $Z$ would be the union of two proper closed subsets, namely $\overline{Z \cap U}$ and $Z - U$.

Therefore, if $Z_0 \subset \ldots \subset Z_n$ is a chain in $X$ and $U$ an element of the open cover such that $U \cap Z_0 \neq \emptyset$, then $\{Z_i \cap U\}$ is a chain of distinct closed subsets in $U$, and consequently $n \leqslant \dim U$. This shows that when $\dim X = \infty$, the supremum $\sup_{i\in I} \dim U_i$ will be infinite as well. When $\dim X$ is finite, taking the chain to be maximal (that is, not extendable to a longer chain), we see that $\dim X = n \leqslant \dim U$, which proves the desired reverse inequality. □

If $X$ is a scheme, we define the dimension of $X$ as the dimension of the underlying topological space. In particular, $\dim X = \dim X_{\mathrm{red}}$ (see Exercise 4.11.9).

When $X = \operatorname{Spec} A$ is affine, the closed irreducible subsets are of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of $A$. Using this observation we find:

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**Proposition 6.60.** The dimension of $\operatorname{Spec} A$ equals the Krull dimension of $A$.

**Example 6.61.** The spectrum of the integers, $\operatorname{Spec} \mathbb{Z}$ has dimension 1. The maximal chains of prime ideals in $\mathbb{Z}$ have the form $(0) \subset (p)$ where $p$ is a prime number. △

**Example 6.62.** The affine line $\mathbb{A}^1_{\mathbb{Z}}$ has dimension 2. The maximal chains of prime ideals in $\mathbb{Z}[x]$ are of the form $(0) \subset (p) \subset (f(x), p)$, where $p$ is a prime number and $f(x)$ a polynomial which is irreducible mod $p$.

△

More generally, we have $\dim \mathbb{A}^n_{\mathbb{Z}} = n + 1$, by the following proposition:

**Proposition 6.63.** If $A$ is a Noetherian ring, then

$$\dim \operatorname{Spec} A[x_1, \ldots, x_n] = n + \dim A.$$

In particular, when $A = k$ is a field, $\mathbb{A}^n_k$ has dimension $n$. A maximal chain of irreducible closed subsets is given by

$$V(x_1, \ldots, x_n) \subset \ldots \subset V(x_1, x_2) \subset V(x_1) \subset \mathbb{A}^n_k.$$

If $A$ is not Noetherian, then the Krull dimension of $A[x_1, \ldots, x_n]$ is harder to control; it can be any integer between $\dim A + n$ and $\dim A + 2n$, inclusive.

**Example 6.64** (Zero-dimensional schemes)**.** The schemes

$$\operatorname{Spec} \mathbb{Z}/p\mathbb{Z}, \qquad \operatorname{Spec} \mathbb{C}[x]/(x^n), \qquad \operatorname{Spec} \mathbb{C}[x, y]/(x^2, xy, y^3),$$

have dimension zero.

More generally, the spectrum of an Artinian ring has dimension zero, and for Noetherian rings $A$, $\operatorname{Spec} A$ has dimension zero if and only if $A$ is Artinian. △

Having finite dimension does not guarantee that a scheme is Noetherian (see for instance Example 6.56). More surprisingly, there are also Noetherian rings whose Krull dimension is infinite. Even though each maximal chain of prime ideals in a Noetherian ring will be of finite length (prime ideals satisfy the descending chain condition) there can be arbitrary long ones. See Exercise 6.11.22 for an explicit example, due to Nagata.

### *Codimension*

For a closed subset $Y \subset X$ the dimensions $\dim Y$ and $\dim X$ are defined in terms of closed irreducible subsets contained in $Y$ and $X$ respectively. If we fix the subset $Y \subset X$, there is also a relative notion, the *codimension* of $Y$ in $X$, denoted by $\operatorname{codim}(Y, X)$, which is defined in terms of closed irreducible subsets of $X$ containing $Y$. These three numbers will in some important cases be related by the equality $\dim Y + \operatorname{codim}(Y, X) = \dim X$ (which justifies the name 'codimension'). However this formula does not hold in general (see Example 6.67 below).

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**Definition 6.65** (Codimension)**.** Let $Y \subset X$ be an irreducible closed subset of $X$. The *codimension* of $Y$ is the supremum of all integers $n$ such that there exists a chain

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

of distinct irreducible closed subsets of $X$.

When $X = \operatorname{Spec} A$, the bijective correspondence between irreducible closed subsets of $\operatorname{Spec} A$ and prime ideals in $A$, shows that the codimension of the subset $V(\mathfrak{p})$ will be equal to the height of the prime ideal $\mathfrak{p}$, that is, the maximal length of a chain of distinct prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r = \mathfrak{p}$. Equivalently,

$$\operatorname{codim} V(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}). \tag{6.7}$$

More generally, we have:

**Proposition 6.66.** Let $X$ be a scheme and $P \in X$ be a point. Set $Y = \overline{\{P\}}$. Then

$$\operatorname{codim}(Y, X) = \dim \mathcal{O}_{X,P}. \tag{6.8}$$

*Proof* Given a chain $Y \subset Y_1 \subset \cdots \subset Y_n$ of distinct irreducible closed subsets, the generic points $\xi_1, \ldots, \xi_n$ of the $Y_i$'s will be contained in any affine open neighbourhood of $P$. We may therefore assume that $X = \operatorname{Spec} A$ is affine, in which the claim follows by (6.7). □

A chain of distinct irreducible closed subsets of $Y$ may be extended to a chain in $X$ by appending a chain betwen $Y$ and $X$. Taking the suprenum over all such chains, we find the inequality

$$\dim Y + \operatorname{codim}(Y, X) \leqslant \dim X.$$

As mentioned above, equality does not hold in general. In fact, there are quite simple examples where the inequality is strict.

**Example 6.67.** Consider the discrete valuation ring $R = \mathbb{C}[x]_{(x)}$ with maximal ideal $\mathfrak{m} = (x)$. Consider the principal ideal $\mathfrak{n} = (tx - 1)$ in the polynomial ring $R[t]$. It is a maximal ideal, because $\mathbb{C}[x]_{(x)}/(tx - 1) \simeq \mathbb{C}(x)$, and one easily checks that it does not properly contain any non-zero prime ideal, so it is of height 1. Letting $Y = V(\mathfrak{n})$ and $X = \operatorname{Spec} R[t]$, we find $\dim Y = 0$ and $\operatorname{codim} Y = 1$, but it holds that $\dim X = 2$. △

### *Krull's Principal ideal theorem*

TODO.

## 6.10 Affine varieties as schemes

In Chapter 1, we defined an affine variety as an irreducible closed algebraic set $V \subset \mathbb{A}^n_k$ where $k$ is an algebraically closed field. The coordinate ring $A(V)$ of $V$ is a finitely generated $k$-algebra and an integral domain. We now present the more general scheme-theoretic version:

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**Definition 6.68.** An *affine variety over a field* $k$ is an affine scheme of the form $X = \operatorname{Spec} A$, where $A$ is a finitely generated $k$-algebra which is an integral domain. A *morphism of affine varieties over* $k$ is a morphism of the underlying $k$-schemes.

If we choose a presentation $A = k[x_1, \ldots, x_n]/\mathfrak{a}$, we see that any affine variety in this sense admits a closed embedding into an affine space $\mathbb{A}^n_k$ over $k$. Conversely, any closed subscheme $V(\mathfrak{a})$ of $\mathbb{A}^n_k$ is an affine variety, as long as the defining ideal $\mathfrak{a}$ is a prime ideal.

This definition naturally extends the classical notion when $k$ is algebraically closed. In this setting, then any affine variety $V \subset \mathbb{A}^n(k)$ as defined in 1 gives rise to an affine variety in the new sense by taking $X = \operatorname{Spec} A(V)$. The scheme $X$ is irreducible as $A(V)$ is an integral domain. Furthermore, the Nullstellensatz implies that the $k$-valued points of $X$ are in bijection with the points of $V$, allowing us to view $V$ as a subset of $X$. Under this inclusion, the Zariski topology on $V$ is the restriction of the Zariski topology on $X$.

The scheme $X$ however contains many additional points. These correspond to the non-maximal ideals of $A(V)$, or equivalently, the subvarieties of $V$ of dimension at least 1.

The assignment $V \to X$ extends to morphisms. Morphisms of 'classical' affine varieties $V \to W$ correspond to maps of $k$-algebras $A(W) \to A(V)$, and these in turn are in bijection with morphisms of $k$-schemes $X \to Y$. This means that we have defined a fully faithful functor from the category of affine varieties $\mathsf{AffVar}/k$ to the category of affine schemes over $k$, $\mathsf{AffSch}/k$.

By Example 3.21, the structure sheaf on $X$ coincides with the sheaf of regular functions (as in Chapter 1). Over an open set $U \subset X$, the ring $\mathcal{O}_X(U)$ consists of the rational functions $f \in K(X)$ which are regular at every point of $U$.

**Example 6.69.** The following schemes are affine varieties:

$$\mathbb{A}^n_{\mathbb{C}}, \quad \operatorname{Spec} \mathbb{R}[x,y]/(x^2+y^2), \quad \operatorname{Spec} \mathbb{F}_p[x,y,z]/(y^2 - xz).$$

The following schemes are not:

$$\operatorname{Spec} \mathbb{Z}, \quad \operatorname{Spec} \mathbb{C}[x,y]/(x^2+y^2), \quad \mathbb{F}_p[x,y]/(y^2).$$

△

The next proposition tells us that for a morphism of affine varieties, closed points map to closed points, as in the classical situation:

**Proposition 6.70.** Let $f\colon X \to Y$ be a morphism of affine varieties over a field $k$. If $P \in X$ is a closed point, then $f(P)$ is closed in $Y$.

*Proof* Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, where $A$ and $B$ are finitely generated $k$-algebras, and suppose $f$ is induced by a map of $k$-algebras $\phi\colon A \to B$. The point $P$ corresponds to a maximal ideal $\mathfrak{m}$ in $B$, and $\kappa(P) = B/\mathfrak{m}$ is a finite extension of $k$, by Theorem 1.7. Let $\mathfrak{p} = \phi^{-1}(\mathfrak{m}) \subset A$ be a prime ideal corresponding to $f(P)$. The map $\phi$ induces an injective map of $k$-algebras:

$$A/\mathfrak{p} \hookrightarrow B/\mathfrak{m} = \kappa(P).$$

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Since $\kappa(P)$ is a finite field extension of $k$, the domain $A/\mathfrak{p}$ also a finite-dimensional $k$-algebra. To show that $\mathfrak{p}$ is maximal, it suffices to show that $A/\mathfrak{p}$ is a field.

Take any nonzero $\overline{a} \in A/\mathfrak{p}$. Multiplication by $\overline{a}$ defines an injective $k$-linear map $A/\mathfrak{p} \to A/\mathfrak{p}$. Because $A/\mathfrak{p}$ is finite-dimensional, this map is also surjective. Hence, there exists $\overline{b} \in A/\mathfrak{p}$ such that $\overline{a}\overline{b} = 1$. Hence $A/\mathfrak{p}$ is a field and $\mathfrak{p}$ is a maximal ideal. □

## 6.11 Exercises

**Exercise 6.11.1.** Find an example of a connected scheme $X$ with a disconnected open subset $U \subset X$.

**Exercise 6.11.2.** Let $A$ be a Noetherian ring such that $\operatorname{Spec} A$ is of dimension 0. Show that $A$ is Artinian.

**Exercise 6.11.3.** Show that if $A$ is a Noetherian ring, then each local ring $A_{\mathfrak{p}}$ is also Noetherian.

**Exercise 6.11.4.** Describe $X = \operatorname{Spec} \mathbb{Z}[x]/(5x - 15)$. Is $X$ irreducible? Reduced? What are the fibers of the canonical map $X \to \operatorname{Spec} \mathbb{Z}$?

**Exercise 6.11.5.** Let $X$ be an integral scheme and $U \subset X$ an open subset. Show that $x \in U$ if and only if $\mathcal{O}_X(U) \subset \mathcal{O}_{X,x}$ inside $K(X)$.

**Exercise 6.11.6.** Let $X$ be a topological space. Show that the following two conditions are equivalent.

(i) $X$ is Noetherian.
(ii) Every open subset of $X$ is quasi-compact.

**Exercise 6.11.7.**
a) Find an open subset of an affine scheme which is not quasi-compact.
b) Let $U, V$ be quasi-compact open subsets of $\operatorname{Spec} A$ for some ring $A$. Show that $U \cap V$ is also quasi-compact.

**Exercise 6.11.8.** For a Noetherian scheme $X$, the ring $\mathcal{O}_X(X)$ may fail to be Noetherian in general. Consider $\mathbb{P}^3_k$ with homogeneous coordinates $x_0, x_1, x_2, x_3$. Let $H_0 = V_+(x_0)$ and $H_1 = V_+(x_1)$ and let $l = V_+(x_0, x_2) \subset D$ be a line on $H_0$ different from $H_0 \cap H_1$. Let $Y = H_0 \cup H_1$ and $X = Y - l$.

a) Show that $X$ is Noetherian.
b) Show that $U = H_0 - l$ is an open set of $X$ and that $\mathcal{O}_X(U) = k[x, y]$ where $x = \frac{x_1}{x_2}$ and $y = \frac{x_3}{x_2}$.
c) Show that $V = H_1 - H_1 \cap L$ is an open set of $X$ and that $\mathcal{O}_X(V) = k$
d) Show that $\mathcal{O}_X(X) = k + xk[x, y] \subset k[x, y]$.
e) Show that the ring in (*d*)) is not Noetherian. HINT: $(x) \subset (x, xy) \subset (x, xy, xy^2) \subset \ldots$.

**Exercise 6.11.9** (A 1-dimensional non-Noetherian domain)**.** Let $R$ be the subring of $\mathbb{C}(x, y)$ consisting of rational functions $f(x, y)$ that are defined and constant along the $y$-axis. The elements of $R$, when written in lowest terms, have a denominator not divisible by $x$, and $f(0, y)$, which is then meaningful, is constant.

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a) Show that the ideals $\mathfrak{a}_r = (x, xy^{-1}, \dots, xy^{-r})$ with $r \in \mathbb{N}$ form an ascending chain that does not stabilize. Conclude that $R$ is not Noetherian.
b) Show that $R$ is local with the set $\mathfrak{m}$ of elements $f \in R$ that vanish along the $y$-axis as the maximal ideal.
c) Prove that there are no other primes than $\mathfrak{m}$ and $(0)$ in $R$. HINT: Show first that any element in $R$ is of the form $x^i y^j \alpha$ where $i \geqslant 0$, $j \in \mathbb{Z}$ and $\alpha$ is a unit in $R$.

**Exercise 6.11.10** (The ring of eventually constant sequences)**.** Consider the subring $A$ of $\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$ consisting of sequences $(e_i)_{i \geqslant 1}$ which are eventually constant, that is, sequences with $e_i = e_{i+1}$ for $i \gg 0$.

a) Show that all elements of $A$ are idempotents and conclude that every prime ideal is maximal. HINT: the only idempotents in a domain are 0 and 1.
b) Let $\mathfrak{m}_n$ denote the ideal generated by $1 - a_n$ where $a_n = (0, \dots, 0, 1, 0, \dots)$ with a '1' in the $n$-th factor. Show that $\mathfrak{m}_n$ is a maximal ideal.
c) Show that $D(a_n) = \{m_n\}$ and conclude that the one-point set $\{\mathfrak{m}_n\}$ is both open and closed in $\operatorname{Spec} A$.
d) Let $\mathfrak{m}_\infty$ denote the ideal consisting of sequences which are eventually zero, i.e., $e_i = 0$ for all $i \gg 0$. Show that $\mathfrak{m}_\infty$ is a maximal ideal. HINT: Consider the 'limit map' $A \to \mathbb{Z}/2$.
e) Show that $A$ is not Noetherian. HINT: Show that $\mathfrak{m}_\infty$ is not finitely generated.
f) Show that these are all the prime ideals of $A$, i.e., that $\operatorname{Spec} A = \{\, \mathfrak{m}_i \mid i \in \mathbb{N} \,\} \cup \{\mathfrak{m}_\infty\}$. HINT: Consider the cases $a_i \notin \mathfrak{m}$ for some $i$ and $a_i \in \mathfrak{m}$ for all $i$ separately. Use the identity $a_i(1 - a_i) = 0$.
g) Show that $\operatorname{Spec} A$ is homeomorphic to the set $\{\, \frac{1}{n} \mid n \in \mathbb{N} \,\} \cup \{0\}$ (with the standard topology).

**Exercise 6.11.11.** For each of the following rings $A$, decide whether the corresponding morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is finite or finite type.

$$\mathbb{Z}[i],\ \mathbb{Z}[1/p],\ \mathbb{Z}_{(p)},\ \mathbb{Z} \times \mathbb{Z},\ \mathbb{Z}[x].$$

**Exercise 6.11.12.** Let $f\colon X \to Y$ be a morphism of finite type. Show that if $X$ is Noetherian, then so is $Y$.

**Exercise 6.11.13.** Show that the composition of two morphisms (locally) of finite type is (locally) of finite type. Show that if $S$ is quasi-compact and $f\colon X \to S$ is of finite type, then $X$ will be quasi-compact.

**Exercise 6.11.14.** Assume that $\iota\colon \operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$ is an open embedding. Show that $B$ is of finite type over $A$.

**Exercise 6.11.15.** Assume that $S$ is a Noetherian scheme and that $f\colon X \to S$ is of finite type. Prove that $X$ is Noetherian. HINT: Hilbert's Basis Theorem.

**Exercise 6.11.16** (The "Lying Over Theorem")**.** Let $A \subset B$ an integral extension of integral domains.

a) Show that $A$ is a field if and only if $B$ is a field.

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b) If $\mathfrak{p}$ is a prime in $A$, show that $\mathfrak{p}$ lies in the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ if and only if $\mathfrak{p}A_{\mathfrak{p}}$ lies in the image of $\operatorname{Spec} B_{\mathfrak{p}} \to \operatorname{Spec} A_{\mathfrak{p}}$.
c) Conclude that $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.

**Exercise 6.11.17.** a) Show that there exist surjective morphisms $X \to Y$ with $Y$ integral and such that all fibers $X_y$ are irreducible, without $X$ being irreducible.
b) Let $k$ be an algebraically closed field. Show that there exist morphisms $X \to \mathbb{A}^1_k$ with $X$ integral, the generic fiber $X_\eta$ non-empty and integral, but no closed fiber integral.
c) Show that there exists morphisms $X \to \mathbb{A}^1_{\mathbb{Q}}$ with $X$ integral and infinitely many irreducible and inifintiely many reducible closed fibers. What happens for the geometric closed fibers?
d) Show that there exists morphisms $f\colon X \to Y$ with $X$ and $Y$ integral whose geometric generic fiber is not reduced.

**Exercise 6.11.18.** Consider the ring $R = \mathbb{Q}[x, x^{-1}] \times \mathbb{Q}$. Show that the induced map $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Q}[x]$ is surjective, of finite type, and has finite fibers, but not finite.

**Exercise 6.11.19.** Consider the map

$$\begin{aligned} \phi : \mathbb{A}^1_k &\longrightarrow \mathbb{P}^1_k \\ u &\mapsto (u : 1 - u^2). \end{aligned}$$

Show that $\phi$ is finite and surjective.

**Exercise 6.11.20.** Let $f\colon X \to Y$ be an affine morphism and let $V \subset Y$ be an open set. Show that $f^{-1}(V) \to V$ is affine.

More generally, if $V \to Y$ is any morphism, show that the base change morphism $X \times_Y V \to V$ is affine. Thus affine morphisms are stable under base change.

**Exercise 6.11.21.** Show that the composition of two finite morphisms is finite.

**Exercise 6.11.22** (Nagata's example)**.** Let $R = k[x_1, x_2, \dots]$ be the polynomial ring in countably many variables, and decompose $\mathbb{N}$ as $\mathbb{N} = \bigcup_i J_i$ be a disjoint union of finite sets $J_i$ whose cardinality tends to infinity with $i$. For instance, we could let $J_i$ be the set of integers with $i$ digits in their binary expansion.

Let $\mathfrak{n}_i \subset R$ be the ideal generated by the $x_j$'s for which $j \in J_i$, and let $S \subset R$ be the multiplicatively closed subset $\bigcap_i (R - \mathfrak{n}_i)$, that is, the set of elements in $R$ not lying in any of the $\mathfrak{n}_i$'s. Nagata's example is the localized ring $A = S^{-1}R$. The aim of the exercise is to prove that $A$ is Noetherian, but of infinite Krull dimension. We let $\mathfrak{m}_i$ denote $\mathfrak{n}_i A$; the ideal in $A$ generated by the $x_j$'s with $j \in J_i$.

Consider the rational function field $K_i = k(x_j | j \notin J_i)$ in the variables $x_j$ whose index does not lie in $J_i$, and the polynomial ring $K_i[x_j | j \in J_i]$ over $K_i$ in the remaining variables. Also, define $\mathfrak{a}_i = (x_j | j \in J_i)$ in $K_i[x_j | j \in I_i]$.

a) Show that $B_{\mathfrak{n}_i} \simeq K_i[x_j | j \in I_i]_{\mathfrak{a}_i}$.
b) Show that $A_{\mathfrak{m}_i} = B_{\mathfrak{n}_i}$ and conclude that each local ring $A_{\mathfrak{m}_i}$ is Noetherian with $\dim A_{\mathfrak{m}_i} = |J_i|$ and hence that $\dim A = \infty$.
c) * Show that $A$ is Noetherian. HINT: Any ideal is contained in finitely many of the of the $\mathfrak{m}_i$'s, and is therefore finitely generated.

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**Exercise 6.11.23.** Let $X$ be a quasi-compact scheme. Show that for any $x \in X$, there exists a closed point in $\overline{\{x\}} \subset X$.

**Exercise 6.11.24.** Let $X$ be a quasi-compact scheme. Show that $X$ is reduced if and only if $\mathcal{O}_{X,p}$ is reduced for each closed point $p$. (See Exercise 6.11.23.)

**Exercise 6.11.25.** Let $X$ denote the disjoint union of the schemes $\coprod_{n\geqslant 1}(\operatorname{Spec} \mathbb{Q}[x]/x^n)$. Compute $X_{\mathrm{red}}$. Show that the formula (6.1) does not hold in general for non-affine open sets in $X$.

**Exercise 6.11.26** (Criterion for affineness)**.**
a) Let $f\colon X \to Y$ be a morphism and let $V_i$ be an open cover of $Y$ such that for each $i$, the restriction $f^{-1}(V_i) \to V_i$ is an isomorphism. Show that $f$ is an isomorphism.
b) Show that a scheme is affine if and only if there is a finite set of elements $f_1, \ldots, f_n \in \mathcal{O}_X(X)$ such that each of the open sets $X_{f_i} = \{x \in X | f_i(x) \neq 0\}$ are affine, and $f_1, \ldots, f_n$ generate the unit ideal. HINT: Show that $X = \bigcup X_{f_i}$. Glue the morphisms $X_{f_i} \to \operatorname{Spec} \mathcal{O}_X(X)$ to a morphism $X \to \operatorname{Spec} \mathcal{O}_X(X)$.

**Exercise 6.11.27.** Show that the Frobenius morphism is finite.

**Exercise 6.11.28.** Let $C \subset \mathbb{R}^2$ denote the cone generated by $(1,0)$ and $(1,\sqrt{2})$. Show that the toric algebra $k[C]$ is not a Noetherian ring.

**Exercise 6.11.29.** Let $X$ be a scheme and let $x \in X$ be a point. Show that $x$ is a closed point if and only if the corresponding morphism $\operatorname{Spec} \kappa(x) \to X$ is finite.

**Exercise 6.11.30.** Let $X$ be a Noetherian scheme. Show that $X_{\mathrm{red}}$ is affine if and only if $X$ is affine.

**Exercise 6.11.31.** Let $X$ be a quasi-compact scheme. Prove that $X$ has a closed point.

**Exercise 6.11.32** (Properties of Noetherian spaces)**.**
a) Show that any Noetherian topological space $T$ is quasi-compact. HINT: If $\{U_i\}_{i\in I}$ is a cover, start with any $U_{i_1}$, and pick $U_{i_2}$ so that $U_{i_1} \subsetneq U_{i_1} \cup U_{i_2}$, and so on.
b) Show that if $T$ is Noetherian, then every subset $S \subset T$ is also Noetherian. HINT: The closed subsets of $S$ are of the form $W \cap S$ where $W \subset T$ is closed.

**Exercise 6.11.33.** Let $X$ be a scheme
a) Let $\iota\colon U \to X$ be an open embedding. Show that if $U$ is quasi-compact, then $\iota$ is of finite type. HINT: For $\operatorname{Spec} A \subset X$ cover $U \cap \operatorname{Spec} A$ by distinguished open sets.
b) Suppose that $X$ is Noetherian. Show that any open embedding $\iota\colon U \to X$ is of finite type.

**Exercise 6.11.34.** Let $f\colon X \to S$ be a morphism of schemes. Show that $f$ is of finite type if there exists an affine open cover $\{S_i = \operatorname{Spec} A_i\}$ of $S$ such that each $f^{-1}(S_i)$ is of finite type over $A_i$. HINT: Cover an affine open set $V = \operatorname{Spec} A \subset S$ by finitely many $D(g_i)$'s such that each $D(g_i)$ is distinguished in both $V$ and some $S_j$. Apply Proposition 6.24 to each $f^{-1}(D(g_i)) \to D(g_i)$.

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**Exercise 6.11.35.** a) Show that the composition of two morphisms of finite type is of finite type.

b)

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# 7

# Fiber products

## 7.1 Introduction

Given two scheme morphisms $X \to S$ and $Y \to S$, the *fiber product* $X \times_S Y$ is a scheme equipped with projection morphisms $p_X \colon X \times_S Y \to X$ and $p_Y \colon X \times_S Y \to Y$. This construction generalizes the product of two affine varieties as discussed in Example 1.36. However, the fiber product is far more than a mere generalization; it is an indispensable tool in algebraic geometry and it takes on remarkably versatile roles.

The main result of the chapter is the construction of the fiber product $X \times_S Y$ as a scheme. We first carry out this construction when $X$, $Y$ and $S$ are affine schemes, and then extend it to general schemes using gluing techniques.

An important observation is that the fiber product of two varieties is not, in general, a variety, but rather a scheme. In fact, the fact that fiber products exist is one of the most important properties of the category of schemes, and one can argue that it is the definitive reason for transitioning from varieties to schemes.

Towards the end, we will treat the main applications and study a series of examples. We also explain some of the various contexts where fiber products appear, including *base change* and *scheme-theoretic fibers*.

### *Fiber products*

The notion of a fiber product is meaningful in any category $\mathsf{C}$. Although our main concern will be the category of schemes, we give the definition in a general setting.

A *fiber product* of two arrows $f_X \colon X \to S$ and $f_Y \colon Y \to S$ in a category $\mathsf{C}$, is an object $X \times_S Y$ together with two arrows $p_X \colon X \times_S Y \to X$ and $p_Y \colon X \times_S Y \to Y$ such that

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow[f_X]{} & S \end{array} \tag{7.1}$$

commutes, and satisfying the following universal property: for any two arrows $g_X \colon Z \to X$ and $g_Y \colon Z \to Y$ in $\mathsf{C}$ such that $f_X \circ g_X = f_Y \circ g_Y$, there is a *unique* arrow $g \colon Z \to X \times_S Y$ such that $p_X \circ g = g_X$ and $p_Y \circ g = g_Y$.

We call a commutative diagram (7.1) a *Cartesian diagram* or a *Cartesian square*. The universal property can be visualized in the following diagram:



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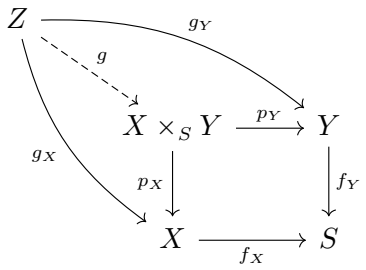


(7.2)

When the fiber product exists, it is unique up to a *unique isomorphism*. This follows directly from the universal property: given two candidates for the fiber product, $W$ and $W'$, with projections $p_X, p_Y$ and $p'_X, p'_Y$ respectively, then there exists a *unique* isomorphism $\theta\colon W \to W'$ so that $p_X = p'_X \circ \theta$ and $p_Y = p'_Y \circ \theta$. The uniqueness of $\theta$ follows because $W'$ is a fiber product: it is the *unique* morphism $W \to W'$ arising from the universal property of $W'$. For this reason, we allow ourselves to speak about *the* fiber product.

It is not so hard to come up with examples of categories where fiber products do *not* exist. For instance, the fiber product does not exist in the familiar category of differentiable manifolds, and also not in the category of affine varieties, as we will see later. This is yet another reason why we need to make the transition from varieties to schemes.

Given two morphisms over $S$, $\phi\colon Z \to X$ and $\psi\colon W \to Y$, we can also form their product

$$\phi \times \psi\colon Z \times_S W \longrightarrow X \times_S Y, \tag{7.3}$$

which is the unique morphism induced by the compositions $\phi \circ p_Z$ and $\psi \circ p_W$ where $p_Z, p_W$ are the projections of $Z \times_S W$. By construction, we have $p_X \circ (\phi \times_S \psi) = \phi \circ p_Z$ and $p_Y \circ (\phi \times_S \psi) = \psi \circ p_W$.

### *Fiber products of sets.*

In the category of sets, the fiber product has a concrete description. Given two maps of sets $f_X\colon X \to S$ and $f_Y\colon Y \to S$, their fiber product $X \times_S Y$ is the subset of $X \times Y$ consisting of pairs $(x, y)$ with equal images in $S$:

$$X \times_S Y = \{\, (x, y) \in X \times Y \mid f_X(x) = f_Y(y) \,\}.$$

The projections $p_X$ and $p_Y$ are the usual projection maps $p_X(x, y) = x$ and $p_Y(x, y) = y$.

We can check directly that $X \times_S Y$ satisfies the universal property: given any two maps $g_X\colon Z \to X$ and $g_Y\colon Z \to Y$ such that $f_X \circ g_X = f_Y \circ g_Y$, then the required unique map $g\colon Z \to X \times_S Y$ is simply given by $g(z) = (g_X(z), g_Y(z))$.

**Example 7.1.** If $S = \{*\}$ is a singleton, $X \times_S Y$ is the usual Cartesian product $X \times Y$. △

**Example 7.2.** For subsets $X, Y \subset S$ with their inclusion maps, the fiber product is equal to the intersection: $X \times_S Y = X \cap Y$. △

**Example 7.3.** For a map $f\colon X \to S$ and $s \in S$, the fiber product with the inclusion $\{s\} \hookrightarrow S$ is equal to the preimage: $X \times_S \{s\} = f^{-1}(s)$.

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In fact, a general fiber product as a disjoint union

$$X \times_S Y = \bigsqcup_{s \in S} f_X^{-1}(s) \times f_Y^{-1}(s).$$

This is the reason for the name 'fiber product'; the fibers of the map $X \times_S Y \to S$ are the Cartesian products of the fibers of the two maps $f_X$ and $f_Y$. △

## 7.2 Fiber products of schemes

The following existence theorem is the main result of this chapter.

**Theorem 7.4 (Existence of fiber products).** For any schemes $X \to S$ and $Y \to S$ over a base scheme $S$, there is a scheme $X \times_S Y$ satisfying the universal property of the fiber product.

When the base scheme $S = \operatorname{Spec} A$ is affine, the fiber product $X \times_S Y$ is often denoted by $X \times_A Y$.

The proof of the theorem consists of a series of reductions to the affine case. For the affine case the fiber product is defined using the tensor product of rings. The reductions rely heavily on the gluing techniques developed in Chapter **??**.

A remark before we begin: one cannot construct the fiber product $X \times_S Y$ by defining a structure sheaf on the fiber product of the sets. In fact, the underlying set of a product of schemes can be very different from the product of the underlying sets of $X$ and $Y$. This may seem counterintuitive at first, but is in fact a typical feature of the fiber products of schemes (see the examples in Section 7.3). It is important to keep in mind that we are taking the fiber product of the *morphisms* $X \to S$ and $Y \to S$, not merely of the schemes $X$ and $Y$ themselves.

The points of the fiber product $X \times_S Y$ become easier to understand if we consider the $R$-valued points. For a ring $R$, we have

$$(X \times_S Y)(R) = X(R) \times_{S(R)} Y(R) \tag{7.4}$$

where the right-hand side is the fiber product of sets. This is just a rephrasing of the universal property of the fiber product: to give a morphism $\operatorname{Spec} R \to X \times_S Y$ is equivalent to giving two morphisms, $\operatorname{Spec} R \to X$ and $\operatorname{Spec} R \to Y$, so that the compositions $\operatorname{Spec} R \to X \to S$ and $\operatorname{Spec} R \to Y \to S$ are equal.

### *Products of affine schemes*

We being by the constructing fiber products of affine schemes. The main observation is that the category of affine schemes is equivalent to the opposite category of rings $\mathsf{Rings}^{\mathrm{op}}$, and that the tensor product of algebras enjoys a universal property *dual* to the one of the fiber product.

Consider $A$-algebras $B_1$ and $B_2$ with structure maps $\alpha_i \colon A \to B_i$ for $i = 1, 2$. Then the tensor product $B_1 \otimes_A B_2$ is also an $A$-algebra, via $(b_1 \otimes b_2)(c_1 \otimes c_2) = (b_1 c_1) \otimes (b_2 c_2)$.

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Moreover, there are maps of $A$-algebras

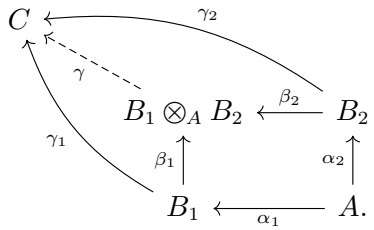
$$\beta_1 \colon B_1 \to B_1 \otimes_A B_2, \quad b_1 \mapsto b_1 \otimes 1$$
$$\beta_2 \colon B_2 \to B_1 \otimes_A B_2, \quad b_2 \mapsto 1 \otimes b_2$$

They fit into the commutative diagram

$$\begin{array}{ccc} B_1 \otimes_A B_2 & \xleftarrow{\beta_2} & B_2 \\ {\scriptstyle\beta_1}\uparrow & & \uparrow{\scriptstyle\alpha_2} \\ B_1 & \xleftarrow[\alpha_1]{} & A \end{array} \tag{7.5}$$

where commutativity holds because $\alpha_1(a) \otimes 1 = 1 \otimes \alpha_2(a)$ by definition of the tensor product $B_1 \otimes_A B_2$ (this is the significance of the tensor product being taken over $A$; one can move elements from $A$ between the components of the tensor).

Moreover, the tensor product is *universal* among diagrams such as (7.5). More presisely, suppose that $\gamma_i \colon B_i \to C$ are maps of $A$-algebras, i.e. $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$ (i.e., they fit into a commutative diagram analogous to (7.5), but with the $\beta_i$'s replaced by the $\gamma_i$'s). The assignment $b_1 \otimes b_2 \to \gamma_1(b_1)\gamma(b_2)$ is $A$-bilinear and hence extends uniquely to an $A$-algebra homomorphism $\gamma \colon B_1 \otimes_A B_2 \to C$, which obviously has the property $\gamma \circ \beta_i = \gamma_i$, as expressed in the following commutative diagram:




(7.6)

Applying $\operatorname{Spec}$ to (7.5), we arrive at the diagram

$$\begin{array}{ccc} \operatorname{Spec}(B_1 \otimes_A B_2) & \xrightarrow{p_2} & \operatorname{Spec} B_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ \operatorname{Spec} B_1 & \longrightarrow & \operatorname{Spec} A, \end{array} \tag{7.7}$$

and $\operatorname{Spec}(B_1 \otimes_A B_2)$ is universal among affine schemes sitting in a diagram like (7.7). Hence $\operatorname{Spec}(B_1 \otimes_A B_2)$ serves as the fiber product in the category $\mathsf{AffSch}$ of affine schemes. In fact, it is the fiber product in the larger category $\mathsf{Sch}$ of schemes:

**Proposition 7.5.** For morphisms $f_i \colon \operatorname{Spec} B_i \to \operatorname{Spec} A$ for $i = 1, 2$, the scheme $\operatorname{Spec}(B_1 \otimes_A B_2)$ with the two projection $p_1$ and $p_2$ is the fiber product of the $\operatorname{Spec} B_i$'s in the category of schemes $\mathsf{Sch}$.

Unravelled, this means: if $Z$ is any scheme and $g_i \colon Z \to \operatorname{Spec} B_i$ are morphisms with $f_1 \circ g_1 = f_2 \circ g_2$, there exists a unique morphism $g \colon Z \to \operatorname{Spec}(B_1 \otimes_A B_2)$ such that $p_i \circ g = g_i$ for $i = 1, 2$.

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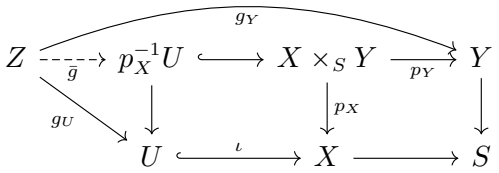
*Proof* To check the universal property, we apply Theorem 4.6 about maps into affine schemes. The morphisms $g_i$ give maps of $A$-algebras $B_i \to \mathcal{O}_Z(Z)$. By the universal property of the tensor product, these induce a unique map of $A$-algebras $B_1 \otimes_A B_2 \to \mathcal{O}_Z(Z)$, which in turn gives the desired map $g\colon Z \to \operatorname{Spec}(B_1 \otimes_A B_2)$ of schemes over $\operatorname{Spec} A$ by Theorem 4.6. By construction, this map satisfies $p_i \circ g = g_i$ for $i = 1, 2$. Finally, $g$ is unique by the uniqueness part of Theorem 4.6 and the universal property of the tensor product. □

### *Products of general schemes*

Recall that any open subset $U$ of a scheme $X$ has a canonically defined scheme structure as an open subscheme with the structure sheaf equal to the restriction $\mathcal{O}_X|_U$. Hence, if $f\colon X \to Y$ is any morphism and $V \subset Y$ is an open subscheme, the inverse image $f^{-1}V$ is an open subscheme of $X$, and any morphism $g\colon Z \to X$ such that $f \circ g$ factors through $V$, will factor through $f^{-1}V$.

**Lemma 7.6.** If $X \times_S Y$ exists and $U \subset X$ is an open subscheme, then $U \times_S Y$ exists and is canonically isomorphic to the open subscheme $p_X^{-1}U$ with the two restrictions $p_X|_{p_X^{-1}U}$ and $p_Y|_{p_X^{-1}U}$ as projections.

*Proof* Write $\iota\colon U \to X$ for the open embedding. We need to verify that $p_X^{-1}U$ together with the restriction of the two projections satisfies the universal property. Suppose $Z$ is a scheme and $g_U\colon Z \to U$ and $g_Y\colon Z \to Y$ are two morphisms over $S$. The situation is displayed in the diagram below




The composition $g_X = \iota \circ g_U$ is a map into $X$, and $g_X$ and $g_Y$ induce a unique map of schemes $g\colon Z \to X \times_S Y$ with $g_X = p_X \circ g$ and $g_Y = p_Y \circ g$. Clearly $p_X \circ g = \iota \circ g_U$ takes values in $U$. Therefore $g$ takes values in $p_X^{-1}U$, and we get an induced morphism $\overline{g}\colon Z \to p_X^{-1}U$. Using Exercise 7.7.1, we see that $\overline{g}$ is unique because $g$ is, and hence $p_X^{-1}U$ satisfies the universal property of the fiber product. □

The following lemma will allow us to construct fiber products in general by gluing.

**Lemma 7.7.** Let $f_X\colon X \to S$ and $f_Y\colon Y \to S$ be two morphisms and assume that there is an open cover $\{U_i\}_{i\in I}$ of $X$ such that $U_i \times_S Y$ exists for all $i \in I$. Then $X \times_S Y$ exists. The products $U_i \times_S Y$ form an open cover of $X \times_S Y$ and the projections restrict to projections.

*Proof* The proof involves gluing together the different schemes $U_i \times_S Y$ and verifying that the result indeed is a product $X \times_S Y$. Write $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$ for the intersections and let $p_i\colon U_i \times_S Y \to U_i$ denote the projections.

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By Lemma 7.6 the inverse images $p_i^{-1}(U_{ij})$ serve as fiber products $U_{ij} \times_S Y$ with the restrictions of $p_i$ and $p_Y$ as projections. Hence there are *unique* isomorphisms $\theta_{ji} \colon p_i^{-1}(U_{ij}) \to p_j^{-1}(U_{ij})$ making the following triangles commute

$$\begin{array}{ccc} p_i^{-1}(U_{ij}) & \xrightarrow[\simeq]{\theta_{ji}} & p_j^{-1}(U_{ij}) \\ & {\scriptstyle p_i} \searrow \quad \swarrow {\scriptstyle p_j} & \\ & U_{ij} & \end{array} \tag{7.8}$$

To be able to glue together the $p_i^{-1}(U_i)$'s using the $\theta_{ij}$'s, we need to verify the three conditions of Theorem 4.35 on page 81. The two items (i) and (ii) follow readily. For (iii), note that by Lemma 7.6 the preimages $p_i^{-1}(U_{ijk})$ serve as products $U_{ijk} \times_S Y$ with the restrictions of $p_i$ and $p_Y$ as projections. Moreover, the restrictions of the $\theta_{ij}$'s fit into a commutative diagram

$$\begin{array}{ccccc} p_i^{-1}(U_{ijk}) & \xrightarrow[\simeq]{\theta_{ji}} & p_j^{-1}(U_{ijk}) & \xrightarrow[\simeq]{\theta_{kj}} & p_k^{-1}(U_{ijk}) \\ & {\scriptstyle p_i} \searrow & \downarrow {\scriptstyle p_j} & \swarrow {\scriptstyle p_k} & \\ & & U_{ijk}. & & \end{array}$$

The two small triangles commute, so the big one commutes as well, and it follows by uniqueness of the isomorphisms that $\theta_{ki} = \theta_{kj} \circ \theta_{ji}$. The third gluing condition is therefore fulfilled, and we can glue the $p_i^{-1}(U_i)$'s together to a scheme $X \times_S Y$. Moreover, in view of the commutative diagram (7.8) and Proposition 4.36 on page 82, the $p_i$'s patch together to a map $p_X \colon X \times_S Y \to X$. The projections $U_i \times_S Y \to Y$ are basically unaffected by the gluing process and glue together to a morphism $p_Y \colon X \times_S Y \to Y$.

Finally, we check that $X \times_S Y$ together with $p_X$ and $p_Y$ satisfy the universal property. Let $Z$ be any scheme with morphisms $\phi_X \colon Z \to X$ and $\phi_Y \colon Z \to Y$ such that $f_X \circ \phi_X = f_Y \circ \phi_Y$. For each $i \in I$, the restrictions $\phi_X^{-1}(U_i) \to U_i$ and $\phi_Y : Z \to Y$ induce a unique morphism $\phi_i : \phi_X^{-1}(U_i) \to U_i \times_S Y$ by the universal property of $U_i \times_S Y$. On the overlaps $\phi_X^{-1}(U_{ij})$, the morphisms $\phi_i$ and $\phi_j$ agree by the uniqueness in the universal property. Therefore, the morphisms $\phi_i$ glue together to a (unique) morphism $\phi : Z \to X \times_S Y$ such that $p_X \circ \phi = \phi_X$ and $p_Y \circ \phi = \phi_Y$. □

An immediate consequence is that fiber products exist when the base $S$ is affine.

**Lemma 7.8.** Assume that $S$ is affine, then $X \times_S Y$ exists.

*Proof* First, if $Y$ is affine as well, we are done: cover $X$ by open affine subschemes $U_i$; then each $U_i \times_S Y$ exists by the affine case, and we may apply Lemma 7.7 above. In general, cover $Y$ by affine open subschemes $V_i$. As we just verified, the products $X \times_S V_i$ all exist, and applying Lemma 7.7 again, we conclude that $X \times_S Y$ exists. □

For the final reduction, we will need the following lemma.

**Lemma 7.9.** If $T \subset S$ is an open subscheme and the fiber product $X \times_T Y$ exists, then the fiber product $X \times_S Y$ exists and is isomorphic to $X \times_T Y$.

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*Proof* Let $\iota\colon T \to S$ be the inclusion. Let $p_X$ and $p_Y$ be the projection morphisms from $X \times_T Y$. For any scheme $Z$ with morphisms $\phi_X\colon Z \to X$ and $\phi_Y\colon Z \to Y$ such that $\iota \circ f \circ \phi_X = \iota \circ g \circ \phi_Y$, the condition $f \circ \phi_X = g \circ \phi_Y$ holds because $\iota$ is an inclusion (Exercise 7.7.1). By the universal property of $X \times_T Y$, there exists a unique morphism $Z \to X \times_T Y$ making the diagram commute. Hence, $X \times_T Y$ satisfies the universal property of $X \times_S Y$. □

Finally, with Lemmas 7.8 and Lemma 7.9, we can finish the proof of Theorem 7.4.

*Proof of Theorem 7.4* Let $\{S_i\}_{i\in I}$ be an open affine cover of $S$ and let $U_i = f_X^{-1}(S_i)$ and $V_i = f_Y^{-1}(S_i)$. By Lemma 7.8 the products $U_i \times_{S_i} V_i$ all exist. By Lemma 7.9 these serve as products $U_i \times_S V_i$. Then applying Lemma 7.7 again, we see that the schemes $U_i \times_S V_i$ glue to the fiber product $X \times_S Y$ and the proof is complete. □

Here are some of the basic properties of the fiber product. It is possible to deduce them directly using gluing arguments, but with the so-called 'functor of points', which we will introduce in Section 8.1, the proofs will become simple and natural.

**Proposition 7.10 (Basic formulas).** Let $X$, $Y$, $Z$ and $T$ be schemes over $S$. There are unique canonical isomorphisms over $S$, all compatible with projections:

(i) (Reflectivity) $X \times_S S \simeq X$.
(ii) (Symmetry) $X \times_S Y \simeq Y \times_S X$.
(iii) (Associativity) $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$.
(iv) (Transitivity) $(X \times_S T) \times_T Y \simeq X \times_S Y$.

In the last claim $Y$ is supposed to be a scheme over $T$, and $X \times_S T$ is considered a scheme over $T$ via the projection onto $T$.

## 7.3 Examples

As noted in the introduction, the fiber product of schemes can exihibt unexpected behaviour in some situations, differing from what we are used to in set theory or topology. The main difference is that the underlying set is almost never the product of the underlying sets of the factors. The next few examples illustrate this.

**Example 7.11.** For a ring $R$ and non-negative integers $m, n$, we have

$$R[x_1, \dots, x_m] \otimes_R R[y_1, \dots, y_n] \simeq R[x_1, \dots, x_m, y_1, \dots, y_n],$$

and so

$$\mathbb{A}^m_R \times_R \mathbb{A}^n_R \simeq \mathbb{A}^{m+n}_R.$$

Even when $R = \mathbb{C}$, the affine space $\mathbb{A}^{m+n}_{\mathbb{C}}$ has an underlying set which is different from the Cartesian product $\mathbb{A}^m_{\mathbb{C}} \times \mathbb{A}^n_{\mathbb{C}}$, and the topology is not equal to the product topology (see Example 2.17). △

**Example 7.12.** Let $X \subset \mathbb{A}^m_R = \operatorname{Spec} R[x_1, \dots, x_m]$ and $Y \subset \mathbb{A}^m_R = \operatorname{Spec} R[y_1, \dots, y_n]$ be two closed subschemes, defined by the ideals $\mathfrak{a} = (f_1, \dots, f_r) \subset k[x_1, \dots, x_m]$ and

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$\mathfrak{b} = (g_1, \ldots, g_s)$ respectively. Then $X \times_R Y$ is isomorphic to the closed subscheme of $\mathbb{A}_R^{m+n}$ given by

$$\operatorname{Spec} R[x_1, \ldots, x_m, y_1, \ldots, y_n]/(f_1, \ldots, f_r, g_1, \ldots, g_s)$$

This is the scheme version of the product in Example 1.36. △

Here are three examples of fiber products of spectra of fields:

**Example 7.13.** Consider the fiber product $\operatorname{Spec} \mathbb{C} \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C}$. This scheme has *two* distinct closed points, even if both factors are singletons.

To see this, we compute the tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$, by writing $\mathbb{C} = \mathbb{R}[t]/(t^2+1)$:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}[t]/(t^2+1) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[t]/(t^2+1) = \mathbb{C}[t]/(t-i)(t+i) = \mathbb{C} \times \mathbb{C}.$$

Here the last equality follows from the Chinese Remainder Theorem. △

**Example 7.14.** The fiber product $\operatorname{Spec} \mathbb{F}_2 \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_3$ is empty. Indeed, it is the spectrum of the ring

$$\mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/3 \simeq \mathbb{Z}/(2,3) = 0$$

See Exercise 7.7.2 for a generalization. △

**Example 7.15.** The fiber product $\operatorname{Spec} \mathbb{C}(x) \times_{\mathbb{C}} \operatorname{Spec} \mathbb{C}(y)$ is a bit surprising – it contains infinitely many points. This is because the tensor product $A = \mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$ is a ring of Krull dimension 1, and it contains infinitely many maximal ideals.

To see this, we observe that $A$ can be written as the localization $S^{-1}\mathbb{C}[x,y]$, where

$$S = \{\, p(x)q(y) \mid p(x)q(y) \neq 0 \,\}.$$

Every prime ideal $\mathfrak{p} \subset A$ has the form $\mathfrak{p} = S^{-1}\mathfrak{q}$ for some prime ideal $\mathfrak{q} \subset \mathbb{C}[x,y]$ disjoint from $S$. We observe that $\mathfrak{q}$ cannot be a maximal ideal. Maximal ideals in $\mathbb{C}[x,y]$ are of the form $(x-a, y-b)$ with $a, b \in \mathbb{C}$, but these intersect $S$ non-trivially (e.g., $(x-a)(y-b) \in S$). Hence, $\mathfrak{q}$ must have height at most 1. Since $S$ contains all non-zero polynomials in $\mathbb{C}[x]$ and $\mathbb{C}[y]$, $\mathfrak{q}$ must avoid both $\mathbb{C}[x]$ and $\mathbb{C}[y]$. Hence either $\mathfrak{q} = (0)$ or $\mathfrak{q} = (f(x,y))$, where $f(x,y)$ is an irreducible polynomial which does not lie in $\mathbb{C}[x] \cup \mathbb{C}[y]$.

In conclusion, all non-zero primes in $A$ are therefore maximal, and so $A$ has dimension 1, and $\operatorname{Spec} A$ contains infinitely many points. △

## 7.4 Base change

The fiber product leads to one of the most powerful techniques in algebraic geometry - the ability to change base schemes. This process generalizes the familiar idea of considering polynomial equations, such as $x^2 + 1$, over different rings, but in the geometric framework of schemes.

Consider a scheme $X$ over $S$ with structure morphism $p\colon X \to S$. If $T \to S$ is any morphism, we can form the fiber product $X_T = X \times_S T$, which is naturally a scheme over

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$T$. This fits into the Cartesian diagram

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow{\scriptstyle p_T} & & \downarrow{\scriptstyle p} \\ T & \longrightarrow & S. \end{array}$$

One says that $X_T$ is obtained from $X$ by *base change* along $T \to S$.

While the 'base change' is essentially a fiber product, the novelty lies in the interpretation: We start with a scheme over $S$, and obtain new schemes $X_T$ by changing the base schemes $T$. This philosophy turns out to be very powerful for studying schemes and proving properties about schemes and morphisms, as we will see shortly.

**Example 7.16** (Field extensions)**.** Let $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ be an affine scheme over $k$. For any field extension $k \subset K$ the base change $X_K$ is given by

$$X_K = X \times_k K = \operatorname{Spec} K[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

So $X_K$ is the scheme defined by the same equations as $X$, but viewed over $K$ instead of $k$.

For instance, if $X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2 + 1)$, then the base change $X_{\mathbb{C}}$ to $\mathbb{C}$ is given by

$$X_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]/(x^2 + y^2 + 1)$$

There is a significant change in the geometry: $X$ has no $\mathbb{R}$-points, but $X_{\mathbb{C}}$ has infinitely many $\mathbb{C}$-points. △

**Example 7.17.** Suppose that $X$ is a scheme over a field $k$ and that $\sigma\colon k \to k$ is a field automorphism. The automorphism $\sigma$ induces a morphism $\iota\colon \operatorname{Spec} k \to \operatorname{Spec} k$, and we may form the base change $\sigma X$ which fits into the diagram

$$\begin{array}{ccc} \sigma X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\iota} & \operatorname{Spec} k \end{array}$$

Then that $X$ and $\sigma X$ are isomorphic as abstract schemes, but they need not be isomorphic as $k$-schemes.

For a concrete example, consider $k = \mathbb{Q}(\sqrt{2})$ and the automorphism $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$. Then we get two schemes

$$X = \operatorname{Spec} k[x, y]/(x^2 + \sqrt{2}y^2 + 1) \quad \sigma X = \operatorname{Spec} \mathbb{Q}(\sqrt{2})[x, y]/(x^2 - \sqrt{2}y^2 + 1).$$

The first scheme has no $\mathbb{Q}(\sqrt{2})$-valued points, whereas the latter has infinitely many. △

**Example 7.18.** Base change also appears naturally in number theory when analyzing solutions to Diophantine equations. Here one can often simplify the analysis by working in some larger ring or modulo a suitable ideal. Consider, for instance, the Fermat equation $x^3 + y^3 = z^3$ and its associated scheme $X = \operatorname{Spec} \mathbb{Z}[x, y, z]/(x^3 + y^3 - z^3)$. The $\mathbb{Z}$-valued points of $X$ correspond exactly to the integer solutions to the equation. To study these solutions, it is natural to consider the base change to the ring $R = \mathbb{Z}[\omega]$ where $\omega = e^{2\pi i/3}$ is a

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primitive cube root of unity. Over $R$, the equation factors as:

$$x^3 = (z + y)(z + \omega y)(z + \omega^2 y)$$

Using this factorization, and the fact that $R$ is a UFD, one can prove that $X(R)$, and hence $X(\mathbb{Z})$, contains only the trivial solutions where $xyz = 0$. △

### *Properties stable under base change*

When studying morphisms of schemes, it is natural to ask how their properties behave under base change. If $\mathcal{P}$ is a property of morphisms, one says that $\mathcal{P}$ is *stable under base change* if for any $T$ over $S$, the map $f_T$ has the property $\mathcal{P}$ whenever the original morphism $: X \to Y f$ does. The same convention applies to properties of schemes.

Examples 7.20 and 7.21 below show that neither being irreducible nor being reduced are properties stable under base change.

On the other hand, many imporant geometric properties are preserved by base change. For instance, being a closed or open embedding:

**Proposition 7.19 (Embeddings and base change).** Let $f\colon Z \to X$ be a morphism and consider its base change along $Y \to X$:

$$\begin{array}{ccc} Z_Y & \longrightarrow & Z \\ {\scriptstyle f_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \longrightarrow & X. \end{array}$$

If the morphism $f$ is a closed, open or locally closed embedding, then the morphism $f_Y\colon Z_Y \to Y$ is as well.

*Proof* The case of an open embedding is a consequence of Lemma 7.6 on page 148. The case of locally closed embeddings follow directly from the two others, so it suffices to show that closed embeddings are stable under base change.

Assume first that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine. Then a closed subscheme $f\colon Z \to \operatorname{Spec} A$ is of the form $Z = \operatorname{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a}$ (Proposition 4.26 on page 74), and therefore

$$Z_Y = Z \times_X Y = \operatorname{Spec}(A/\mathfrak{a} \otimes_A B) = \operatorname{Spec}(B/\mathfrak{a}B).$$

and $f_Y$ corresponds to the closed subscheme $\operatorname{Spec}(B/\mathfrak{a}B) \to \operatorname{Spec} B$.

For the general case, the statement is local on $Y$ (Exercise 4.11.36 on page 88). For any open affine $U \subset Y$ mapping into an open affine $V \subset X$, Lemma 7.9 on page 149, implies that $f^{-1}V \times_X Y = f^{-1}V \times_V U$, and by the affine case this is a closed subscheme of $U$. Finally, one may cover $Y$ by such $U$'s by first covering $X$ by affine opens and subsequently cover each of their inverse images in $Y$ by affine opens. □

**Example 7.20** (Being irreducible is not stable under base change)**.** Consider the $\mathbb{R}$-algebra $A = \mathbb{R}[x, y]/(x^2 + y^2)$. Over $\mathbb{R}$, the polynomial $x^2 + y^2$ is irreducible, so $X = \operatorname{Spec} A$ is

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an irreducible $\mathbb{R}$-scheme. The base change to $\mathbb{C}$ however is not irreducible, because

$$A \otimes_{\mathbb{C}} \mathbb{C} \simeq \mathbb{C}[x,y]/(x-iy) \times \mathbb{C}[x,y]/(x+iy)$$

and so $X_{\mathbb{C}}$ is the union of two conjugate lines in $\operatorname{Spec} \mathbb{C}[x,y]$ △

**Example 7.21** (Being reduced is not stable under base change)**.** An example was already given in Exercise 7.7.4. Consider the scheme $X = \operatorname{Spec} \mathbb{Z}[x,y]/(x^2-2y^2)$, viewed as a scheme over $\operatorname{Spec} \mathbb{Z}$. Clearly $X$ is integral, as $x^2-2y^2$ is irreducible. However, if we take the base change via the morphism $\operatorname{Spec} \mathbb{F}_2 \to \operatorname{Spec} \mathbb{Z}$, the resulting scheme is non-reduced:

$$X_{\mathbb{F}_2} = \operatorname{Spec}(\mathbb{F}_2[x,y]/(x^2))$$

△

## 7.5 Scheme-theoretic fibers

One of the first applications of the fiber product is to define a scheme structure on the preimage of a point under a morphism.

Let $f\colon X \to Y$ be a morphism of schemes and $y \in Y$ be a point. Inspired by Example 7.3, we define the *scheme theoretic fiber* as the fiber product

$$X_y = \operatorname{Spec} \kappa(y) \times_Y X.$$

This scheme fits into the Cartesian diagram

$$\begin{array}{ccc} X_y = X \times_Y \operatorname{Spec} \kappa(y) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} \kappa(y) & \longrightarrow & Y, \end{array}$$

Here the morphism $\operatorname{Spec} \kappa(y) \to Y$ is the map corresponding to the point $y$. Recall that the residue field $\kappa(y)$ is defined by $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$, and that the 'point-map' $\operatorname{Spec} \kappa(y) \to Y$ is the composition $\operatorname{Spec} \kappa(y) \to \operatorname{Spec} \mathcal{O}_{Y,y} \to Y$.

By definition, the fiber $X_y$ satisfies the following universal property: a morphism $g\colon Z \to X$ factors through $X_y$ if and only if $f \circ g$ factors through $\operatorname{Spec} \kappa(y) \to Y$ (topologically this means it maps $Z$ to $y \in Y$ ).

It is common to write $X_y$ for the scheme-theoretic fiber and reserve the notation $f^{-1}(y)$ for the preimage as a topological space. The next proposition shows that the underlying topological space of $X_y$ is indeed equal to $f^{-1}(y)$.

**Proposition 7.22.** Let $f\colon X \to Y$ be a morphism and let $y \in Y$ be a point. Then $X_y \to X$ is a homeomorphism onto the topological fiber $f^{-1}(y)$.

*Proof* We may assume that $Y$ is affine, say $Y = \operatorname{Spec} A$.

We first treat the case where $X$ is also affine, say $X = \operatorname{Spec} B$ and $f\colon X \to Y$ is induced by a ring map $\phi\colon A \to B$. In this situation, Proposition 2.36 states that the fiber $f^{-1}(\mathfrak{p})$ over a point $\mathfrak{p} \in \operatorname{Spec} A$ is homeomorphic to the spectrum of the ring $(B/\mathfrak{p}B)_{\mathfrak{p}}$ via

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the composition $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \to \operatorname{Spec}(B_{\mathfrak{p}}) \to \operatorname{Spec}(B)$. On the other hand, standard formulas for tensor products give

$$(B/\mathfrak{p}B)_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} = B \otimes_A \kappa(\mathfrak{p}).$$

Here the Zariski topology on $\operatorname{Spec}(B/\mathfrak{p}B)_{\mathfrak{p}}$ (i.e. the induced topology on $f^{-1}(\mathfrak{p})$) coincides with the Zariski topology on $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ (i.e. the topology on the scheme $X_y$), and so the proposition holds when $X$ is affine.

For the general case, let $U$ be an open and affine subset of $X$. Let $\iota\colon X_y \to X$ denote the natural map $X \times_Y \operatorname{Spec}\kappa(y) \to X$. By Lemma 7.6 on page 148, the fiber product $U_y = U \times_Y \operatorname{Spec}\kappa(y)$, coincides with the open subscheme $\iota^{-1}(U)$ of $X_y$. By the affine case, $U_y$ maps homeomorphically onto $(f|_U^{-1})(y) = f^{-1}(y) \cap U$ inside $U$. As being a homeomorphism is local on the target, and this holds for any affine open, $\iota$ is a homeomorphism onto $f^{-1}(y)$. □

**Example 7.23.** In Example **??**, we showed that any scheme $X$ admits a unique morphism $f\colon X \to \operatorname{Spec}\mathbb{Z}$. This means that $X$ can be viewed as a sort of 'fibered object' over $\operatorname{Spec}\mathbb{Z}$; $X$ is the union of the schemes $X \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{F}_p)$ where $p$ is a prime number and $X \times_{\mathbb{Z}} \operatorname{Spec}(\mathbb{Q})$. Compare this with Figure **??**. △

## 7.6 The Segre embedding

In this section, we describe the *Segre embedding* of a product of two projective spaces.

At the level of $k$-points, the Segre embedding is defined by taking the products of homogeneous coordinates:

$$\begin{aligned} \sigma\colon \mathbb{P}^m(k) \times \mathbb{P}^n(k) &\longrightarrow \mathbb{P}^{(m+1)(n+1)-1}(k) \\ (a_0 : \cdots : a_m) \times (b_0 : \cdots : b_n) &\longmapsto (a_0b_0 : a_0b_1 : \cdots : a_ib_j : \cdots : a_mb_n) \end{aligned} \tag{7.9}$$

Here, there are $(m+1)(n+1)$ products $c_{ij} = a_ib_j$. This map is well-defined because scaling the $a_i$'s or the $b_j$'s will scale the products $c_{ij}$ accordingly. Moreover, if at least one of the $a_i$'s and one of the $b_j$'s are non-zero, one of the products will be non-zero as well.

The map $\sigma$ is also injective. Indeed, if $c_{pq} = a_pb_q \neq 0$, then the original coordinates can be recovered up to scaling via the ratios $a_i/a_p = c_{iq}/c_{pq}$ and $b_j/b_q = c_{pj}/c_{pq}$.

This construction has a natural generalization to schemes. For any ring $A$, there is a closed embedding

$$\sigma\colon \mathbb{P}^m_A \times_A \mathbb{P}^n_A \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}_A. \tag{7.10}$$

To define it, we use the standard affine covers of the projective spaces involved. Let

$$R_i = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_m}{x_i}\right],\ S_j = A\left[\frac{y_0}{y_j}, \ldots, \frac{y_n}{y_j}\right],\ T_{ij} = A\left[\frac{w_{00}}{w_{ij}}, \ldots, \frac{w_{mn}}{w_{ij}}\right]$$

be the rings defining the standard affine opens in $\mathbb{P}^m_A$, $\mathbb{P}^n_A$, and $\mathbb{P}^{(m+1)(n+1)-1}_A$, respectively.

The affine schemes $U_{ij} = \operatorname{Spec}(R_i \otimes_A S_j)$ give an open cover of $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$. Over $U_{ij}$, we consider the morphism $U_{ij} \to \operatorname{Spec}(T_{ij})$ induced by the ring map

$$\phi_{ij}\colon T_{ij} \longrightarrow R_i \otimes_A S_j, \quad \frac{w_{kl}}{w_{ij}} \mapsto \frac{x_k}{x_i} \otimes \frac{y_l}{y_j}, \tag{7.11}$$

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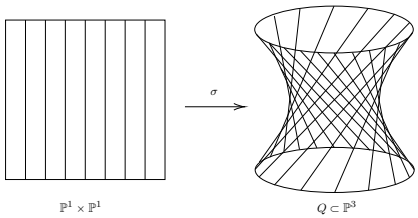


**Figure 7.1** The Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$

This ring map is surjective, as the tensor product $R_i \otimes_A S_j$ is generated by the products $x_k/x_i \otimes y_l/y_j$ as an $A$-algebra. Therefore, the induced morphism $U_{ij} \to \operatorname{Spec}(T_{ij})$ is a closed embedding. Moreover, it is not difficult to check that these local embeddings glue to a global closed embedding (7.10). Therefore, we get

**Proposition 7.24.** The scheme $\mathbb{P}^m_A \times_A \mathbb{P}^n_A$ is projective over $A$.

The ideal of the image of $\sigma$ is defined by the $2 \times 2$ minors of the matrix $(w_{ij})$ (see **?**).

**Example 7.25.** The product $\mathbb{P}^1_A \times_A \mathbb{P}^1_A$ embeds as the quadric

$$Q = V_+(w_{00}w_{11} - w_{01}w_{10}) \subset \mathbb{P}^2_A.$$

△

Proposition 7.24 also implies that the product of two projective $A$-schemes is again projective. If $X \subset \mathbb{P}^m_A$, and $Y \subset \mathbb{P}^n_A$, then $X \times_A Y$ embeds into projective space via the composition

$$X \times_A Y \longrightarrow \mathbb{P}^m \times_A \mathbb{P}^n_A \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}_A.$$

## 7.7 Exercises

**Exercise 7.7.1.** Assume that $U \subset X$ is an open subscheme and let $\iota\colon U \to X$ be the inclusion map. Let $f$ and $g$ be two maps from a scheme $Z$ to $U$ and assume that $\iota \circ f = \iota \circ g$. Show that $f = g$.

**Exercise 7.7.2.** Let $p$ and $q$ be different prime numbers. Show the following identities:

a) $\operatorname{Spec} \mathbb{F}_p \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{F}_q = \varnothing$.
b) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(p)} = \operatorname{Spec} \mathbb{Z}_{(p)}$.
c) $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(q)} = \operatorname{Spec} \mathbb{Q}$.

**Exercise 7.7.3.** Example 7.13 can be generalized as follows. Let $K/k$ be a finite Galois extension of fields with Galois group $G$. Show that the map $x \otimes y \mapsto (xg(y))_{g \in G}$ defines an isomorphism

$$K \otimes_k K \to \prod_{g \in G} K.$$

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Hint: Write $K = k[x]/(f(x))$ for a minimal polynomial $f(x)$ and compute $K \otimes_k K$ using the Chinese Remainder Theorem and the fact that $f$ factors in $K$.

Deduce that $\operatorname{Spec} K \times_k \operatorname{Spec} K$ has an underlying set with $|G|$ points.

**Exercise 7.7.4.** This exercise goes along the same lines as Exercise 7.7.3 and gives an example that a fiber product $X \times_k \operatorname{Spec} L$ may not be reduced even if $X$ is.

Let $k = \mathbb{F}_p(a)$ for a prime number $p$ and let $L = k[x]/(x^p - a)$. Show that

$$L \times_k L \simeq L[t]/(t^p - a) \simeq L[t]/(t - x)^p.$$

Conclude that $\operatorname{Spec} L \times_{\operatorname{Spec} k} \operatorname{Spec} L$ is not reduced.

**Exercise 7.7.5.** Let $X$ and $Y$ be schemes over $S$ with open affine covers $\{U_i\}$ and $\{V_j\}$. Show that $U_i \times_S V_j$ is an open cover of $X \times_S Y$.

**Exercise 7.7.6.** Let $X = \operatorname{Spec} \mathbb{R}[x, y]/(x^2 + y^2)$.

a) Show that $X$ is irreducible.
b) Show that $X \times_{\mathbb{R}} \mathbb{C}$ is not irreducible, and describe the irreducible components.
c) Compute $X(\mathbb{R})$ and $X(\mathbb{C})$.

**Exercise 7.7.7.**

a) Show that the base change of a surjective morphism is surjective.
b) Show that the base change of an injective morphism need not be injective.
HINT: Consider $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$.

**Exercise 7.7.8.** Let $X = \operatorname{Spec}(\mathbb{Q}[x, y]/xy)$ and $Y = \operatorname{Spec}(\mathbb{Q}[x, y]/(x^2 + y^2))$.

a) Compute $\operatorname{Hom}_{Sch}(\operatorname{Spec} \mathbb{Q}, X)$ and $\operatorname{Hom}_{Sch}(\operatorname{Spec} \mathbb{Q}, Y)$.
b) Show that $X$ and $Y$ are not isomorphic.
c) Let $X' = \operatorname{Spec}(\mathbb{Q}(i)[x, y]/xy)$ and $Y' = \operatorname{Spec}(\mathbb{Q}(i)[x, y]/(x^2+y^2))$. Show that $X'$ and $Y'$ are isomorphic.

**Exercise 7.7.9.**

a) Show that the map sending $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $\mathfrak{p} = \operatorname{Ker}(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to \overline{\mathbb{Q}})$ defines a bijection

$$\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \operatorname{Spec}\left(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}\right)$$

b) Describe the topological space $X = \operatorname{Spec}(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$. Is it connected? Irreducible? Reduced? Noetherian?

**Exercise 7.7.10.** In the setting above, show that the scheme-theoretic intersection is naturally a closed subscheme of $X$, with underlying topological space equal to the intersection $i(Y) \cap j(Z)$ in $X$.

**Exercise 7.7.11.** Prove statements (i) and (iv) in Proposition 7.10.

**Exercise 7.7.12.** Let $A = \mathbb{R}[x, y]/(x^2 + y^2 + 1)$ and let $X = \operatorname{Spec} A$. Show that the base-change $X_{\mathbb{C}}$ is isomorphic to $\mathbb{A}^1_{\mathbb{C}} - V(t)$, but $X$ is not isomorphic to $\mathbb{A}^1_{\mathbb{R}} - V(t)$.

**Exercise 7.7.13.** Show that if $B$ is an $A$-algebra, then $\mathbb{A}^n_B \simeq \mathbb{A}^n_A \times_A \operatorname{Spec} B$ and that $\mathbb{P}^n_B \simeq \mathbb{P}^n_A \times_A \operatorname{Spec} B$.

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**Exercise 7.7.14.** Let $L_m \to \mathbb{P}^1_k$ be the line bundle constructed in Section B.3 on page 451, and let $f_n \colon \mathbb{P}^1 \to \mathbb{P}^1$ be the map $u \mapsto u^n$. Show that $L_m \times_{\mathbb{P}^1_k} \mathbb{P}^1_k = L_{nm}$.

**Exercise 7.7.15** (Finite type and base change)**.**

a) Show that the properties 'finite' and 'finite type' are stable under base change.
b) Show that the product of two morphisms of finite type is of finite type. Similarly for two finite morphisms.

**Exercise 7.7.16.** Let $\mathfrak{m} = (x_1, \ldots, x_n)$ the origin in $\mathbb{A}^n_k$.

(i) Show that the graded $k$-algebra $\bigoplus_{i \geqslant 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is isomorphic to the polynomial ring $k[t_1, \ldots, t_n]$, where $t_i$ denotes the class of $x_i$ in $\mathfrak{m}/\mathfrak{m}^2$.
(ii) Let $\pi \colon X \to \mathbb{A}^n$ be the blow up of the origin 0 in $\mathbb{A}^n_k$ (that is, of $\mathfrak{m}$). Show that $X_0 = \mathbb{P}^{n-1}_k$.

**Exercise 7.7.17** (Flat base change)**.** Let $A$ be a ring and let $B$ be a flat $A$-algebra. Let $X$ be a Noetherian scheme over $A$ and let $X_B$ denote the base change to $B$. Show that there is a natural isomorphism

$$\Gamma(X_B, \mathcal{O}_{X_B}) = \Gamma(X, \mathcal{O}_X) \otimes_A B \tag{7.12}$$

Find an example where (7.12) fails for a non-flat $A$-algebra.

**Exercise 7.7.18.** Let $X$ be a scheme over a field $k$ and let $k \to K$ be a field extension. If $X_K$ denotes the base change to $K$, show that there is a natural bijection $X(K) = X_K(K)$.

**Exercise 7.7.19.** Show that $\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}} \not\simeq \mathbb{P}^2_{\mathbb{C}}$.

**Exercise 7.7.20.** Consider the projective schemes $X = V_+(x^2 + y^2 + z^2)$ and $Y = V_+(x^2 + y^2 + 3z^2)$ in $\mathbb{P}^2$.

a) Show that $X(\mathbb{Q}) = Y(\mathbb{Q}) = \varnothing$.
b) Show that $X(\mathbb{Q}(\sqrt{3})) = \varnothing$, but $Y(\mathbb{Q}(\sqrt{3})) \neq \varnothing$. Deduce that $X$ and $Y$ are not isomorphic.
c) Show that $X \times_{\mathbb{Q}} \mathbb{C}$ and $Y \times_{\mathbb{Q}} \mathbb{C}$ are isomorphic.

**Exercise 7.7.21.** Let $P$ and $Q$ be two lattice polytopes. Show that the product polytope $P \times Q$ is a lattice polytope. Show that for a field $k$, there is a natural isomorphism of toric varieties

$$X_{P \times Q} \simeq X_P \times_k X_Q$$

**Exercise 7.7.22.** Let $k$ be a field with $\operatorname{char} k \neq 2$.

a) Show that $X = \operatorname{Spec} k[x, y]/(y^2 - x^2 - x^3)$ is an integral scheme.
b) Show that the base change to $k[[x, y]]$, $\operatorname{Spec} k[[x, y]]/(y^2 - x^2 - x^3)$ is reducible. What is the geometric intuition here?

**Exercise 7.7.23** (Scheme-theoretic intersections)**.** If $X$ is a scheme and $Y, Z$ are two closed subschemes we define their *scheme-theoretic intersection* as the fiber product $Y \times_X Z$ of the closed embeddings $i : Y \to X$ and $j : Z \to X$.

a) Show that when $X = \operatorname{Spec} A$ and $Y$ and $Z$ are closed subschemes given by ideals $I$ and $J$ respectively, then $Y \times_X Z$ is the closed subscheme associated to the ideal $I + J$.

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b) Show that in general, $Y \times_X Z$ is a closed subscheme of $X$ with underlying topological space is homeomorphic to $i(Y) \cap j(Z)$ in $X$

**Exercise 7.7.24.** Let $R$ and $R'$ be two graded rings with degree 0 part $A = R_0 = R'_0$. Show that the tensor product $R \otimes_A R'$ is naturally a graded ring with a decomposition

$$R \otimes_A R' = \bigoplus_{n \geqslant 0} \bigoplus_{i+j=n} R_i \otimes_A R'_j,$$

**Exercise 7.7.25** (Proj and base change)**.** Let $B$ be a $R_0$-algebra. Show that there is a canonical isomorphism

$$\operatorname{Proj} R \times_{R_0} \operatorname{Spec} B \simeq \operatorname{Proj}(R \otimes_{R_0} B).$$

HINT: There is an isomorphism $R_f \otimes_{R_0} B \simeq (R \otimes_{R_0} B)_{f \otimes 1}$, defined by sending $x/f^n \otimes b$ to $(x \otimes b)/(f \otimes 1)^n$.

**Exercise 7.7.26** (Proj and fiber products)**.** Let $R$ and $R'$ be graded rings with $R_0 = R'_0 = A$. Consider the following subring $S$ of $R \otimes_A R'$:

$$S = \bigoplus_{i \geqslant 0} R_i \otimes_A R'_i$$

a) Let $f$ and $g$ be homogeneous elements of positive degree in $R$ and $R'$ respectively. Show that there is a canonical isomorphism of $A$-algebras

$$\theta_{f,g} \colon (R_f)_0 \otimes_A (R_g)_0 \longrightarrow (S_{f \otimes g})_0,$$

defined by $xf^{-a} \otimes yg^{-b} \mapsto (xf^b \otimes yg^a)(f \otimes g)^{-(a+b)}$.

b) Show that there is a canonical isomorphism between

$$\tau_{f,g} \colon D_+(f \otimes g) \xrightarrow{\simeq} D_+(f) \times_A D_+(g).$$

where $D_+(f \otimes g) \subset \operatorname{Proj} S$.

c) Show that there is a natural isomorphism

$$\operatorname{Proj} S \xrightarrow{\simeq} \operatorname{Proj} R \times_A \operatorname{Proj} R'.$$

d) In the case $R$ and $R'$ are polynomial rings with the standard grading, show that the isomorphism in (*c*)) identifies with the Segre embedding as defined in Section 7.6.

**Exercise 7.7.27** (Fiber product and products of fibers)**.** Let $f \colon X \to S$ and $g \colon Y \to S$ be two morphisms and let $s \in S$ be a point with 'point map' $\iota \colon \operatorname{Spec} \kappa(s) \to S$. Denote by $h \colon X \times_S Y \to S$ the structure map, i.e. $h = f \circ p_X = g \circ p_Y$. Show that the scheme-theoretic fiber of $h$ is the fiber product of the scheme-theoretic fibers of $f$ and $g$:

$$(X \times_S Y)_s = X_s \times_{\kappa(s)} Y_s.$$

**Exercise 7.7.28.** Check that the morphisms $U_{ij} \to \operatorname{Spec}(T_{ij})$ defined in (7.11) glue to a morphism $\sigma \colon \mathbb{P}^m \times_A \mathbb{P}^n_A \to \mathbb{P}^{(m+1)(n+1)-1}_A$.

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**Exercise 7.7.29** (Base changes are functorial)**.** If $f\colon X \to Y$ is a morphism over $S$, and $T \to S$ is a morphism, show that there is an induced morphism $f_T = X_T \to Y_T$ over $T$. Show that $p_T \circ f_T$ coincides with the natural projection map $X_T \to T$and is Cartesian:

$$\begin{array}{ccccc} X_T & \xrightarrow[f_T]{} & Y_T & \xrightarrow[p_T]{} & T \\ \downarrow & & \downarrow{\scriptstyle p_Y} & & \downarrow \\ X & \xrightarrow{f} & Y & \longrightarrow & S. \end{array}$$

Given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ over $T$, show that $(g \circ f)_T = g_T \circ f_T$.

**Exercise 7.7.30** (Subschemes of $\mathbb{P}^m \times \mathbb{P}^n$)**.** Let $R = A[x_0, \ldots, x_m, y_0, \ldots, y_n]$ with the *bigrading* defined by $\deg x_i = (1,0)$ and $\deg y_i = (0,1)$. An element $F \in R$ is called *bihomogeneous* of bidegree $(d,e)$ if all the monomials in $F$ have bidegree $(d,e)$. Given a bihomogeneous polynomial $F$ of bidegree $(d,e)$, define its *dehomogenization* with respect to $x_p$ and $y_q$ as:

$$F_{(p,q)} = x_p^{-d} y_q^{-e} F \in A\left[\frac{x_i}{x_p}, \frac{y_j}{y_q}\right].$$

For a bihomogeneous ideal $\mathfrak{a}$, define the dehomogenized ideal as

$$\mathfrak{a}_{(p,q)} = \left\{F_{(p,q)} \mid F \in \mathfrak{a} \text{ bihomogeneous}\right\} \subset A\left[\frac{x_i}{x_p}, \frac{y_j}{y_q}\right].$$

a) Show that the closed subschemes
$$V_+(\mathfrak{a}_{(p,q)}) \subset D_+(x_p) \times D_+(y_q) = \mathbb{A}^m_A \times_A \mathbb{A}^n_A$$
glue to a closed subscheme $V_+(\mathfrak{a}) \subset \mathbb{P}^m_A \times_A \mathbb{P}^n_A$.

b) Show that there is a bijection between closed subschemes of $\mathbb{P}^m_A \times \mathbb{P}^n_A$ and bihomogeneous ideals in $A[x_0, \ldots, x_m, y_0, \ldots, y_n]$ which are saturated with respect to the 'irrelevant ideal' $(x_0, \ldots, x_m) \cap (y_0, \ldots, y_n)$

**Exercise 7.7.31.** Show that $\mathbb{C} \otimes_{\mathbb{Q}} \mathbb{C}$ is not Noetherian.

**Exercise 7.7.32.** Show that if $X$ and $Y$ are of finite type over a scheme $S$, then $X \times_S Y$ is also of finite type over $S$.

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# 8

# The functor of points*

What is the best way to specify a scheme? By definition, a scheme $X$ can be described by a collection of rings $\{A_i\}_{i\in I}$, and for each pair $i, j \in I$, a ring isomorphism $\phi_{ij}\colon (A_i)_{g_{ij}} \simeq (A_{ji})_{g_{ij}}$ between the localizations. This is certainly an efficient way of presenting a scheme, but not the most conceptual. Often, schemes are better understood in terms of the problems they are designed solve. The fiber product of two schemes is a good example of this; the explicit construction of $X \times_S Y$ is mainly used to show that the product exists, most arguments use only the formal properties of the product. Likewise, projective space $\mathbb{P}^n$ is a scheme whose $k$-points parameterize 1-dimensional subspaces of $k^{n+1}$. This geometric description is much more intuitive and useful than the description in terms of gluing together $n+1$ affine spaces. However, a priori, there could be other schemes with the same property, so we have to be more precise about the 'universal property' that characterizes $\mathbb{P}^n$.

The precise way of specifying such universal properties is through the *functor of points*. This is an important concept in algebraic geometry, which provides a different way of thinking about schemes. Instead of focusing on the internal structure of a scheme, the functor of points focuses how a scheme interacts with other schemes.

Specifically, the functor of points of a scheme $X$ assigns to each scheme $T$ the set of morphisms $T \to X$. We have already seen that it is natural to study maps from a fixed scheme into $X$: we think of the points of $X$ as morphisms $\operatorname{Spec} k \to X$ and we think of tangent vectors as morphisms $\operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X$. By looking at all possible morphisms $T \to X$ at once, we get a functor from schemes to sets which contains all the information about $X$; in particular $X$ can be recovered by its functor.

The functor of points perspective is particularly useful for proving formal properties of schemes, such as Proposition 7.10.

## 8.1 The functor of points

Recall from Section 4.7, that for a scheme $X$ and a ring $R$, the set of $R$-valued points $X(R)$ is the set of all scheme maps $\operatorname{Spec} R \to X$. There is a generalization of this, where we consider the set of all morphisms $T \to X$ from a fixed scheme $T$ into $X$. Formally, we define the *functor of points* associated with a scheme $X$ to be the functor $h_X\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ defined by

$$h_X(T) = \operatorname{Hom}_{\mathsf{Sch}}(T, X).$$



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This functor sends a morphism $f\colon S \to T$ to the map of sets

$$\begin{aligned} h_X(f)\colon h_X(T) &\longrightarrow h_X(S) \\ g &\longmapsto g \circ f. \end{aligned}$$

If $f\colon X \to Y$ is a morphism of schemes, there is for each $T$ an induced map $h_f(T)\colon h_X(T) \to h_Y(T)$ defined by sending $g\colon T \to X$ to $f \circ g$.

A natural question is whether the scheme $X$ is determined by the functor $h_X$. The answer is 'yes', and this is essentially the content of Yoneda's lemma.

Let us formulate Yoneda's Lemma in its natural generality. If $\mathsf{C}$ is a category, and $X$ is an object in $\mathsf{C}$, we get a contravariant functor from $\mathsf{C}$ to $\mathsf{Sets}$ by

$$\mathrm{Hom}_{\mathsf{C}}(-, X)$$

A *natural transformation* $\eta$ between two such functors is a collection of morphisms

$$\eta_T\colon \mathrm{Hom}_{\mathsf{C}}(T, X) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(T, Y), \tag{8.1}$$

one for each object $T$ in $\mathsf{C}$ such that whenever $h\colon S \to T$ is a morphism in $\mathsf{C}$ there is a commutative diagram (of sets)

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}}(T, X) & \xrightarrow{\eta_T} & \mathrm{Hom}_{\mathsf{C}}(T, Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathsf{C}}(S, X) & \xrightarrow{\eta_S} & \mathrm{Hom}_{\mathsf{C}}(S, Y) \end{array}$$

and where the vertical arrows are given by $\sigma \mapsto h \circ \sigma$.

**Lemma 8.1 (Yoneda's Lemma).** Let $\mathsf{C}$ be a category and let $X$ and $Y$ be objects of $\mathsf{C}$. Given a natural transformation

$$\eta\colon \mathrm{Hom}_{\mathsf{C}}(-, X) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(-, Y), \tag{8.2}$$

there is a unique morphism $f\colon X \to Y$ that induces $\eta$, that is, for every object $T$ in $\mathsf{C}$, the map of sets

$$\eta_T\colon \mathrm{Hom}_{\mathsf{C}}(T, X) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(T, Y), \tag{8.3}$$

is given by $\eta_T(\sigma) = f \circ \sigma$.
Furthermore, $\eta$ is an isomorphism if and only if $f$ is an isomorphism.

*Proof* Applying $\eta$ to the object $X$, we have a map of sets

$$\eta_X\colon \mathrm{Hom}_{\mathsf{C}}(X, X) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(X, Y).$$

If there is a morphism $f\colon X \to Y$ that induces $\eta$, then we must have

$$\eta_X(\mathrm{id}_X) = f \circ \mathrm{id}_X = f.$$

Therefore $f$ is determined by $\eta$, so it is unique if it exists.

For the existence, we define $f = \eta_X(\mathrm{id}_X)$ and will check that $f$ induces $\eta$. This means that for any object $T$, the map of sets

$$\eta_T\colon \mathrm{Hom}_{\mathsf{C}}(T, X) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(T, Y)$$

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is equal to the map of sets that sends $\sigma\colon T \to X$ to $f \circ \sigma$.

Since $\eta$ is a natural transformation, we have for any $\sigma\colon T \to X$, a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}}(X,X) & \xrightarrow{\eta_X} & \mathrm{Hom}_{\mathsf{C}}(X,Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathsf{C}}(T,X) & \xrightarrow{\eta_T} & \mathrm{Hom}_{\mathsf{C}}(T,Y) \end{array}$$

Going through the diagram clockwise, we see that $\mathrm{id}_X$ gets sent to $\sigma \circ f$, while going counterclockwise, $\mathrm{id}_X$ gets sent to $\eta_T(\sigma)$. Hence

$$\eta_T(\sigma) = \sigma \circ f.$$

and so $\eta$ is induced by $f$. □

In particular, we have the following consequences:

**Corollary 8.2.** For two schemes $X$ and $Y$, we have:

(i) $h_X$ and $h_Y$ are isomorphic (as functors from $\mathsf{Sch}^{\mathrm{op}}$ to $\mathsf{Sets}$), if and only if $X \simeq Y$:

(ii) If a functor $F$ is isomorphic to $h_X$ for some scheme $X$, then $X$ is determined up to isomorphism.

Replacing the scheme $X$ with its associated functor of points $h_X$, may at this point seem like just yet another jump in abstraction, but the nice thing is that you can work with functors whose values are sets. For instance, by the Yoneda lemma, we see that giving a morphism $f\colon X \to Y$ of schemes, is the same thing as for each scheme $Y$ giving a map of sets $f(T)\colon X(T) \to Y(T)$ which is functorial in $T$ (i.e. a natural transformation). In fact, using that schemes are locally affine, and that morphisms of schemes glue together, it is even sufficient to test this condition on affine schemes $T = \operatorname{Spec} B$.

Another important consequence of this is that instead of specifying a scheme explictly, say by giving a projective embedding and a homogeneous ideal, we can simply specify a functor equivalent to $h_X$, and this will precisely pin down what scheme we are talking about. Many schemes are in the first place defined as solutions to universal problems (e.g. fiber products), so computations involving the functors is both more natural and simpler than say, explicit equations.

We say that a functor $F : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ is *representable* if there exists a scheme $X$ so that $F \simeq h_X$.

**Example 8.3** (The functor of points of $\mathbb{A}^1$)**.** The functor

$$F(T) = \mathcal{O}_T(T)$$

is representable by the affine line $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[t]$. This follows by Theorem **??** on page ??, which says that to give a morphism $T \to \mathbb{A}^1_{\mathbb{Z}}$ is the same thing as specifying an element of $\mathcal{O}_T(T)$.

More generally, $\mathbb{A}^n$ represents the functor $F(T) = \Gamma(T, \mathcal{O}_T)^n$. This is just a fancy way of saying that a morphism $X \to \mathbb{A}^n$ is the same thing as an $n$-tuple of regular functions. △

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**Example 8.4** (The functor of points of $\operatorname{Spec} A$)**.** More generally still, let $A$ be a ring and consider the functor $F\colon \mathsf{Sch}^{op} \to \mathsf{Sets}$ given by

$$F(T) = \operatorname{Hom}_{\mathsf{Rings}}(A, \mathcal{O}_T(T)).$$

Then Theorem 4.6 implies that $F$ is represented by $\operatorname{Spec} A$. △

**Example 8.5** (Group schemes)**.** A *group scheme* is a scheme $G$ so that $h_G\colon \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$ takes values in the category of groups (viewed as a subcategory of $\mathsf{Sets}$). By Yoneda's Lemma, being a group scheme implies that there are scheme morphisms $m\colon G \times G \to G$ and $i\colon G \to G$ satisfying the usual group axioms (associativity, identity, and inverses). △

**Example 8.6.** The scheme $\mathrm{GL}_n = \operatorname{Spec} \mathbb{Z}[x_{11}, \ldots, x_{nn}, \frac{1}{\det(x_{ij})}]$ is a group scheme. It represents the functor

$$\mathbf{GL}_n(T) = \{\, M \in \mathsf{Mat}_{n\times n}(\mathcal{O}_T(T)) \mid \exists N \in \mathsf{Mat}_{n\times n}(T) \text{ such that } MN = NM = \mathrm{id} \,\}.$$

If $A$ and $B$ are $n$-matrices with entries in $\mathcal{O}_T(T)$, then $(A, B) \mapsto AB$ defines a natural transformation $\mathbf{GL}_n(T) \times \mathbf{GL}_n(T) \to \mathbf{GL}_n(T)$, and so we get a morphism of schemes

$$m\colon \mathrm{GL}_n \times \mathrm{GL}_n \longrightarrow \mathrm{GL}_n\,.$$

which on $R$-points corresponds to multiplying the corresponding matrices. Likewise, the assignment $A \mapsto A^{-1}$ defines a natural transformation, and hence a morphism of schemes

$$i\colon \mathrm{GL}_n \longrightarrow \mathrm{GL}_n$$

It would be possible to write down formulas for the ring homomorphisms that define $m$ and $i$, but even checking associativity by composing the relevant ring maps would be quite cumbersome. However, as we know that matrix multiplication is associative, Yoneda's Lemma ensures that the corresponding scheme morphisms satisfy these properties. △

**Example 8.7** (Group actions)**.** If $G$ is a group scheme, and $X$ is a scheme, it is clear what it should mean to have an action of $G$ on $X$; for each $T$, the group $G(T)$ should act on $X(T)$ in a natural way. This translates into existence of a scheme morphism $\sigma\colon G \times X \to X$ satisfying the usual axioms for a group action. For instance, $\sigma(gh)(x) = g(h(x))$ for every $x \in X(T)$ and $g, h \in G(T)$.

As a concrete example, we can consider the group $\mathbb{G}_m$ acting on the scheme $U = \mathbb{A}^{n+1} - V(x_0, \ldots, x_n)$. Then the scheme morphism $\pi\colon U \to \mathbb{P}^n$ is $\mathbb{G}_m$ is $\mathbb{G}_m$-equivariant. In fact, $\mathbb{P}^n$ is universal among such $\mathbb{G}_m$-invariant morphisms, and so it can really be considered as a quotient of $U$ by $\mathbb{G}_m$ in the category of schemes. △

## 8.2 The fiber product in terms of the functor of points

There is a nice way to explain the universal property of fiber products of two $S$-schemes $X$ and $Y$ in terms of the functors of points $h_X$, $h_Y$ and $h_S$. For a scheme $T$, it translates into the following: the set $\operatorname{Hom}_{\mathsf{Sch}}(T, X \times_S Y)$ is the fiber product of the two sets $\operatorname{Hom}_{\mathsf{Sch}}(T, X)$ and $\operatorname{Hom}_{\mathsf{Sch}}(T, Y)$ over $\operatorname{Hom}_{\mathsf{Sch}}(T, S)$. In other words, there is a natural bijection of sets (!)

$$h_{X\times_S Y}(T) \longrightarrow h_X(T) \times_{h_S(T)} h_Y(T). \tag{8.4}$$

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By uniqueness, these bijections are functorial in $T$, and we conclude that the functor of points of the fiber product $X \times_S Y$ is isomorphic to the fiber product functor $h_X \times_{h_S} h_Y$, which assigns the set $h_X(T) \times_{h_S(T)} h_Y(T)$ to a scheme $T$. Thus the fiber product of schemes is not so mysterious after all – it is essentially forced upon us by the universal property of fiber products of sets.

Once we know the functor of points of $X \times_S Y$, Yoneda's Lemma implies that many computations involving fiber products reduce to ones involving sets only. To illustrate this, we give a proof of Proposition 7.10

*Proof of Proposition 7.10* By Yoneda's Lemma, it suffices to verify the corresponding statements for sets, and this is elementary: note that the assignments $(b, a) \mapsto b$; $(b, c) \mapsto (c, b)$; and $((b, c), d) \mapsto (b, (c, d))$ give natural bijections of sets

$$\begin{aligned} B \times_A A &\simeq B & (b, a) &\mapsto b \\ B \times_A C &\simeq C \times_A B & (b, c) &\mapsto (c, b) \\ (B \times_A C) \times_C D &\simeq B \times_A (C \times_C D) & ((b, c), d) &\mapsto (b, (c, d)). \end{aligned}$$

These translate into natural isomorphisms of functors

$$\begin{aligned} h_{X \times_S S} &\simeq h_X \\ h_{X \times_S Y} &\simeq h_{X \times_S Y} \\ h_{(X \times_S Y) \times_S Z} &\simeq h_{X \times_S (Y \times_S Z)} \end{aligned}$$

and by Yoneda's lemma, we have the isomorphisms between the corresponding fiber products as well. □

**Example 8.8.** To show $\mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}^m_{\mathbb{Z}} \simeq \mathbb{A}^{n+m}_{\mathbb{Z}}$, we can show that the two functors of points are isomorphic. Note for a scheme $T$, there is a bijection of sets

$$(\mathbb{A}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{A}^m_{\mathbb{Z}})(T) \simeq \mathbb{A}^{n+m}(T)$$

defined by

$$(a_1, \ldots, a_n) \times (b_1, \ldots, b_m) \mapsto (a_1, \ldots, a_n, b_1, \ldots, b_m).$$

for $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{O}_T(T)$. This assignment is natural in $T$, so we get an isomorphism between the functors of points and hence an isomorphism of schemes. △

## 8.3 Exercises

**Exercise 8.3.1.** Let $A$, $B$ and $C$ be sets.

a) Suppose $f_B \colon B \to A$, $f_{A'} \colon A' \to A$ and $f_C \colon C \to A$ are maps with $f_C = g \circ f_{A'}$ for $g \colon A' \to A$. Show that there is a bijection

$$B \times_A C \simeq (B \times_A A') \times_{A'} C$$

induced by $(b, c) \mapsto ((b, g(c)), c)$.

b) Deduce claim (iv) in Proposition 7.10.

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**Exercise 8.3.2.** Show that functor of points of $\mathbb{A}^1 - \{0\} = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ is isomorphic to

$$F(T) = \Gamma(T, \mathcal{O}_T)^\times.$$

**Exercise 8.3.3.** Show that the functor $T \mapsto \mathrm{GL}_n(\mathcal{O}_T(T))$ is represented by the scheme $\mathbf{GL}_n = \operatorname{Spec} \mathbb{Z}[t_{ij}, \det(t_{ij})^{-1}]$.

**Exercise 8.3.4** (A scheme is a sheaf)**.** Let $X$ and $T$ be two schemes.

a) Show that if $S \subset T$ is an open subscheme, then there is a natural restriction map

$$\rho_{T,S} \colon h_X(T) \to h_X(S). \tag{8.5}$$

b) Let $\{T_i\}_{i \in I}$ be an open over of $T$ and let $\phi, \psi \in h_X(T)$ be two elements so that $\rho_{T,T_i}(\phi) = \rho_{T,T_i}(\psi)$ for every $i$. Show that $\phi = \psi$.

c) Let $\{T_i\}_{i \in I}$ be an open over of $T$ and let $\phi_i \in h_X(T_i)$ be elements so that

$$\rho_{T_i, T_i \cap T_j}(\phi) = \rho_{T_i, T_i \cap T_j}(\psi)$$

for every $i, j$. Show that the $\phi_i$ glue to an element $\phi \in h_X(T)$.

Thus $X$ determines a sheaf of sets on every scheme $T$!

**Exercise 8.3.5.** Let $X$ be a scheme and let $h_X$ be its functor of points.

a) Show that $h_X$ is determined by its restriction to the subcategory of affine schemes $\mathsf{AffSch}$.

b) Deduce that $X$ is completely determined by the functor $R \mapsto X(R)$ from $\mathsf{Rings}$ to $\mathsf{Sets}$.

**Exercise 8.3.6.** Let $X$ be a scheme and let $h_X$ be its functor of points.

a) Show that $h_X$ is determined by its restriction to the subcategory of affine schemes $\mathsf{AffSch}$.

b) Deduce that $X$ is completely determined by the functor $R \mapsto X(R)$ from $\mathsf{Rings}$ to $\mathsf{Sets}$.

**Exercise 8.3.7.** For each $n \in \mathbb{N}$, the *group of $n$-th roots of 1* is defined by

$$\mu_n = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]/(t^n - 1).$$

Show that $\mu_n$ represents the functor $F(T) = \{\, a \in \mathcal{O}_T(T) \mid t^n = 1 \,\}$.

**Exercise 8.3.8.**

a) Let $n > 0$ be an integer. Find a scheme which represents the functor

$$F(X) = \{\, s \in \mathcal{O}_X(X) \mid s^n = 0 \,\}.$$

b) Show that the functor defined by

$$F(X) = \{s \in \mathcal{O}_X(X) \;\mid\; s \text{ is nilpotent}\}$$

is not representable.

**Exercise 8.3.9** (Surjectivity of maps of schemes)**.** A morphism of schemes $f \colon X \to Y$ is said to be surjective if it is surjective on the level of topological spaces.

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a) Show that $f$ is surjective if and only if the following condition holds: for every field $K$ and each $K$-valued point $y \in Y(K)$ there exist a field extension $L$ of $K$ and $x \in X(L)$ such that $f(L)(x) \in X$ is the image of $y$ under $Y(K) \to Y(L)$.
b) Show that surjective morphisms are stable under composition and base change.
c) Suppose $f\colon X \to Y$ is of finite type. Show that $f$ is surjective if and only if the induced map $X(k) \to Y(k)$ is surjective for every algebraically closed field $k$.

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# 9

# Separated and proper morphisms

## 9.1 Separated schemes

We have seen several examples showing that the topology on schemes behaves very differently from the usual Euclidean topology. In particular, schemes are essentially never Hausdorff – the open sets in the Zariski topology are simply too large. Still we would like to find an analogous property that can serve as a satisfactory substitute, so that we have good properties such as 'uniqueness of limits'. This leads to the notion of 'separatedness'.

The route we take to defining separatedness involves the *diagonal morphism*. The motivation comes from the following basic fact from basic topology.

**Proposition 9.1.** A topological space $X$ is Hausdorff if and only if the diagonal $\Delta = \{ (x, x) \mid x \in X \}$ is a closed subset of $X \times X$ (in the product topology).

*Proof* The diagonal $\Delta \subset X \times X$ is closed if and only if the complement $X \times X - \Delta$ is open, and with the product topology, this is equivalent to any point $(x, y) \in X \times X$ with $x \neq y$ being contained in $U \times V$ where $U, V \subset X$ are open and $U \times V \subset X \times X - \Delta$. But this is equivalent to the existence of $U, V$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. $\square$

Even for the affine line $X = \mathbb{A}^1_k$ over a field, the usual Hausdorff condition does not hold: any open set will contain the generic point $(0)$ (or even in the context of varieties, two non-open subsets intersect). On the other hand, the Zariski topology on a product is typically much finer than the product topology on the underlying sets. For instance, for $\mathbb{A}^1_k$, we have $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$, and it makes perfect sense to talk about the subset $V(x - y) \subset \mathbb{A}^2_k$ of points on the 'diagonal', and this is indeed a Zariski closed subset.

It turns out that the 'diagonal perspective' gives a completely satisfactory notion of 'Hausdorffness' for schemes. In fact, it works for relative schemes $X/S$ as well, and we will speak of a *morphism* $X \to S$ being separated, rather than the scheme itself.

The freedom to glue schemes together leads to many examples of non-separated schemes, but they are not commonly encountered in practice. For instance, all affine schemes and all projective schemes are separated. More importantly, some very nice and advantageous properties hold only for separated schemes, and this legitimates the notion. For instance, in a separated scheme, the intersection of two affine subsets is again affine (this is a property which will be important later on).



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### *Separated schemes*

Let $X/S$ be a scheme over $S$. The *diagonal morphism* is the morphism

$$\Delta_{X/S}\colon X \longrightarrow X \times_S X$$

defined by the two identity maps $X \to X$ as components. In other words, the defining property of $\Delta_{X/S}$ is that $p_i \circ \Delta_{X/S} = \mathrm{id}_X$ for $i = 1, 2$ where the $p_i$'s denote the two projections.

The following little lemma gives intuition for the diagonal morphism. In particular, it tells us that if $K$ is a field and $x_1, x_2 \in X(K)$ are two $K$-points, the induced $K$-point $x_1 \times x_2\colon \operatorname{Spec} K \to X \times_S X$ factors via the diagonal precisely whenever $x_1 = x_2$.

**Lemma 9.2.** A morphism $f\colon Z \to X \times_S X$ factors via the diagonal if and only if $p_1 \circ f = p_2 \circ f$.

*Proof* If $f$ factors, the equality holds by definition of the diagonal. If the equality holds, we put $g = p_1 \circ f\colon Z \to X$, and the uniqueness part of the universal property gives that $\Delta_{X/S} \circ g = f$. □

In the case that $X$ and $S$ are affine schemes, say $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$, the diagonal has a simple and natural interpretation in terms of algebras; it corresponds to the most natural map, namely the multiplication map:

$$\mu\colon B \otimes_A B \longrightarrow B.$$

Here $\mu$ sends $b \otimes b'$ to the product $bb'$, and we extend it to $B \otimes_A B$ by $A$-linearity. The projections correspond to the two maps of $A$-algebra $\beta_i\colon B \to B \otimes_A B$ that send $B$ to $b \otimes 1$ respectively to $1 \otimes b$. Clearly it holds that $\mu \circ \beta_i = \mathrm{id}_B$ for $i = 1, 2$. On the level of schemes this translates into the defining property of the diagonal map. Moreover, $\mu$ is clearly surjective so it induces a closed embedding. We have shown the following:

**Proposition 9.3.** If $X$ is an affine scheme over the affine scheme $S$, then the diagonal morphism $\Delta_{X/S}\colon X \to X \times_S X$ is a closed embedding.

The conclusion here is not generally true for schemes, and we will see simple counterexamples shortly. However, from the proposition we just proved, it follows readily that the image $\Delta_{X/S}(X)$ is always *locally closed*, i.e. the diagonal is locally a closed embedding:

**Proposition 9.4.** The diagonal $\Delta_{X/S}$ is locally a closed embedding.

*Proof* Begin with covering $S$ by open affine subsets and subsequently cover each of their inverse images in $X$ by open affines as well. In this way one obtains a cover of $X$ by affine open subsets $U_i$ whose images in $S$ are contained in affine open subsets $S_i$. The products $U_i \times_{S_i} U_i = U_i \times_S U_i$ are open and affine, and their union is an open subset containing the image of the diagonal. By Proposition 9.3 above the diagonal restricts to a closed embedding of $U_i$ in $U_i \times_{S_i} U_i$. □

With this in place, we are ready to give the general definition of separatedness:

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**Definition 9.5.** One says that the scheme $X/S$ is *separated* over $S$, or that the structure map $X \to S$ is separated, if the diagonal map $\Delta_{X/S}\colon X \to X \times_S X$ is a closed embedding. One says for short that $X$ is separated if it is separated over $\operatorname{Spec}\mathbb{Z}$.

Recall that being a closed embedding is a local property on the target. Translating this to the case of $\Delta_{X/S}$, a morphism $f\colon X \to S$ is separated if and only if for some open cover $\{S_i\}$ of $S$ it holds that all the restrictions $f^{-1}(S_i) \to S_i$ are separated.

Since $\Delta_{X/S}$ is a locally closed embedding, it suffices to check that the image $\Delta_{X/S}(X)$ is a closed subset of $X \times_S X$. In particular, this means that being separated is a condition that only involves the underlying topological part of the morphism $f\colon X \to S$.

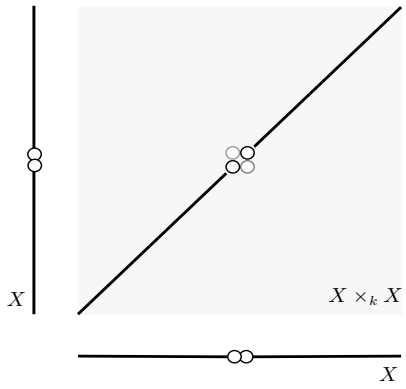
**Example 9.6.** Any morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes is separated, by Proposition 9.3. This, together with the above paragraph, shows that any affine morphism $f\colon X \to Y$ is separated (Exercise 9.4.4). △

**Example 9.7.** Open embeddings are separated: if $U \subset X$ is an open set, then $U \times_X U \simeq U$ and under this identification, the diagonal map $\Delta_{U/X}\colon U \to U$ is simply the identity map. △

**Example 9.8.** The affine line $X$ with two origins constructed in Section **??** on page ?? is not separated over $\operatorname{Spec} k$. Recall that $X$ is constructed by gluing two copies $U_1, U_2$ of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[u]$ along their common open subset $\operatorname{Spec} k[u, u^{-1}]$. Let $g_1\colon U_1 \to X$ and $g_2\colon U_2 \to X$ denote the two open embeddings. The scheme $X$ has two 'origins', which are the images $0_1$ and $0_2$ of the origin $0 \in \mathbb{A}^1_k$ under respectively $g_1$ and $g_2$.

To see that $X$ is not separated, it is instructive to study what happens with the diagonal. The scheme $X \times_k X$ is glued together by four affine charts $U_i \times_k U_j$ for $i, j \in \{1, 2\}$, each isomorphic to $\mathbb{A}^2_k$. Each $\mathbb{A}^2_k$ contains an origin, so there are four origins in total. These are the images $0_i \times 0_j$ of $(0) \in \mathbb{A}^1_k$ under the four embeddings $g_{ij}\colon \mathbb{A}^1_k \to X \times_k X$ with components $g_i$ and $g_j$. Away from the origin, these maps coincide and agree with the diagonal map.

By Lemma 9.2, only $0_1 \times 0_1$ and $0_2 \times 0_2$ lie on the diagonal. However, all four lie in the closure of the diagonal. Consider $0_1 \times 0_2$, for instance, which lies in the image of the map $g_{12}$. If $V$ is an open subset containing $0_1 \times 0_2$, the inverse image $g_{12}^{-1}V$ will be a non-empty open subset of $\mathbb{A}^1_k$, and hence must intersect $\mathbb{A}^1_k - \{0\}$. This means that $V$ must intersect $g_{12}(\mathbb{A}^1_k - \{0\})$, which is open in the diagonal. Thus, $0_1 \times 0_2$ lies in the closure of the diagonal.

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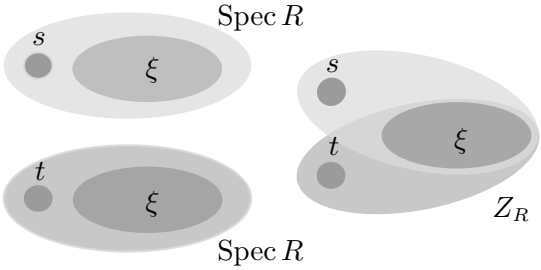
Heuristically, the maps $g_{ij}$ agree on $\mathbb{A}^1 - \{0\}$, but they differ at the origin, bridging the gap differently by passing through distinct points $0_i \times 0_j$. All four lie in the closure, but the diagonal itself only passes through $0_1 \times 0_1$ and $0_2 \times 0_2$. △

**Example 9.9.** An even more basic example of a scheme that is not separated is obtained by gluing the prime spectrum of a discrete valuation ring to itself along the generic point.

To give more details, let $R$ be a DVR with fraction field $K$. Then $\operatorname{Spec} R = \{s, \xi\}$ where $s$ is the closed point and $\xi$ is the generic and open point. By the Gluing Lemma for schemes (Proposition 4.35 on page 81), we may glue two copies of $\operatorname{Spec} R$ together by identifying the generic points; that is, the open subschemes $\operatorname{Spec} K$ in the two copies.

In this manner we construct a scheme $Z_R$ together with two open embeddings $g_i\colon \operatorname{Spec} R \to Z_R$. They send the generic point $\xi$ to the same point, which is an open point in $Z_R$, but they differ on the closed point $s$. It follows $Z_R$ is not separated.

The similar-looking examples of Examples 4.37 and 4.38 are separated however, because they are affine.




△

## 9.2 Properties of separated schemes

We introduce separatedness mostly because they give good formal properties. In some sense the schemes category is still a little bit 'too large', and separated schemes have properties that make them closer to varieties. In this section we survey a few of these properties.

### *Intersections of affine open subsets*

**Proposition 9.10.** Let $X$ be an $A$-scheme. The following are equivalent:

(i) $X$ is separated over $\operatorname{Spec} A$.

(ii) For any two affine open subschemes $U$ and $V$, the intersection $U \cap V$ is also affine, and the natural multiplication map

$$\mathcal{O}_X(U) \otimes_A \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V)$$

is surjective.

(iii) There is an affine cover $\{U_i\}_{i\in I}$ such that all intersections $U_i \cap U_j$ are affine, and $\mathcal{O}_X(U_i) \otimes_A \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_i \cap U_j)$ is surjective for every $i, j \in I$.

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*Proof* (i)⇒ (ii): The product $U \times_S V$ is an affine open subset of $X \times_S X$, and $U \cap V = \Delta_{X/S}(X) \cap (U \times_S V)$. If $X$ is separated, the diagonal is closed, $U \cap V$ is a closed subscheme of the affine scheme $U \times_S V$, hence affine (Proposition 4.26). By the construction of the fiber product of affine schemes, we have

$$\Gamma(U \times_S V, \mathcal{O}_{U \times_S V}) = \Gamma(U, \mathcal{O}_U) \otimes_A \Gamma(V, \mathcal{O}_V),$$

and as $U \cap V$ is a closed subscheme of $U \times_S V$, the restriction map

$$\Gamma(U \times_S V, \mathcal{O}_{U \times_S V}) \longrightarrow \Gamma(U \cap V, \mathcal{O}_{U \cap V})$$

is surjective, which gives (ii).

The implication (ii)⇒(iii) is clear, so let us show the last implication (iii)⇒(i). Let $p_1, p_2 \colon X \times_A X \to X$ denote the two projections and let $\Delta \colon X \to X \times_A X$ denote the diagonal morphism over $\operatorname{Spec} A$. Let $U_i = \operatorname{Spec} B_i$ and $U_j = \operatorname{Spec} B_j$ be two affines in the cover $\{U_i\}$. Then we have

$$\Delta^{-1}(p_1^{-1}(U_i) \cap p_2^{-1}(U_j)) = \Delta^{-1}(p_1^{-1}(U_i)) \cap \Delta^{-1}(p_2^{-1}(U_j)) = U_i \cap U_j, \tag{9.1}$$

Also, by the universal property of the fiber product it follows that $p_1^{-1}(U_i) \cap p^{-1}(U_j) = U_i \times_A U_j \subset X \times_A X$, and from this we deduce that $\Delta$ is a closed embedding if each restriction

$$\Delta_{ij} \colon U_i \cap U_j \longrightarrow U_i \times_A U_j$$

of $\Delta$ is a closed embedding. But this follows from the assumptions: the intersection $U_i \cap U_j$ is affine, say $U_i \cap U_j = \operatorname{Spec} C_{ij}$, and the ring homomorphism $B_i \otimes_A B_j \to C_{ij}$ is surjective. This completes the proof. □

**Example 9.11.** The above proposition provides us with a convenient criterion to check that a scheme is separated. For instance, projective space $\mathbb{P}^n_A$ is separated over $A$, because it is covered by the standard affine covering, and the multiplication map

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \otimes_A A\left[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}\right] \to A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right]$$

is surjective (all monomials on the right hand side are products of monomials on the left).

In fact, one can argue directly that the diagonal in $\mathbb{P}^n_A \times_A \mathbb{P}^n_A$ is a closed embedding. In the notation of Section 7.6, the diagonal is the closed subscheme defined by the bihomogeneous ideal generated by the $2 \times 2$-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \dots & x_n \\ y_0 & y_1 & \dots & y_n \end{pmatrix} \tag{9.2}$$

△

**Example 9.12.** More generally, $\operatorname{Proj} R$ is a separated scheme over $R_0$. The argument is similar: $\operatorname{Proj} R$ is covered by the affine open sets $D_+(f)$ where $f$ runs over the homogeneous elements of $R^+$, and each intersection $D_+(f) \cap D_+(g) = D_+(fg)$ is also affine. To prove that it is separated, it suffices to check that the multiplication map $(R_f)_0 \otimes_{R_0} (R_g)_0 \to (R_{fg})_0$ is surjective for any $f, g \in R^+$, which is the case. △

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**Example 9.13** (The affine plane with two origins)**.** Here is a non-separated scheme where two affine open subsets have non-affine intersection. We glue two copies of the affine plane $\mathbb{A}^2_k$ together along the complement $U_{12} = \mathbb{A}^2_k - V(x, y)$ of the origin. If $U_1$ and $U_2$ denote the two open embeddings of the affine plane, then $U_1 \cap U_2 = U_{12}$, but the open set $U_{12}$ is not affine (see the example in Section 4.22 on page 72). In this example, the multiplication map in the proposition coincides with $k[x, y] \otimes k[x, y] \to \Gamma(U_{12}, \mathcal{O}_{U_{12}})$, which *is* surjective. △

### *Extensions of rational maps*

A very useful property of separated schemes is that morphisms into separated schemes are determined on open dense subschemes, at least when the source is reduced. This can be viewed as an analogue of the fact that 'limits are unique' in Hausdorff spaces.

**Proposition 9.14.** Let $X$ and $Y$ be two schemes over $S$ and let $f, g\colon X \to Y$ with be two morphisms over $S$. Assume that

(i) $X$ is reduced, and
(ii) $Y$ is separated over $S$.

Then if there is a dense open subscheme $U \subset X$ such that $f|_U = g|_U$, then $f = g$.

*Proof* In order to prove that $f = g$, we may assume that $X$ is affine, say $X = \operatorname{Spec} A$. The two morphisms $f$ and $g$ induce a morphism $(f, g)\colon X \to Y \times_S Y$. We want to show that $H$ factors through the diagonal $Y \to Y \times_S Y$, as this will imply that $f = g$ by Lemme 9.2.

Consider the pullback of the diagonal $\Delta_{Y/S}$ via $(f, g)$. This fits into the following Cartesian diagram:

$$\begin{array}{ccccc} & & E & \longrightarrow & Y \\ & \nearrow & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \Delta_{Y/S}} \\ U & \xrightarrow{\iota} & X & \xrightarrow[(f,g)]{} & Y \times_S Y \end{array}$$

The scheme $E$ is the 'equalizer' of the two morphisms, and $j\colon E \to X$ represents the subscheme of points in $X$ where the morphisms coincide. Since pullbacks of closed embeddings are closed embeddings, $j\colon E \to X$ is a closed subscheme of $X$. Since $X$ is assumed to be affine, Proposition 4.26 on page 74 implies that $E$ is isomorphic to a subscheme of the form $\operatorname{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a}$.

The assumption $f|_U = g|_U$ implies that there exists a lift $U \to E$ of $\iota$. Hence the image $j(E)$ contains the dense set $U$ and therefore $j(E) = X$. By Lemma 2.4, $\mathfrak{a}$ is contained in the nilradical of $A$, which is zero as $A$ is reduced. Consequently, $j$ is an isomorphism, $H$ factors through the diagonal, and it follows that $f = g$. □

Here are two examples demonstrating that the hypotheses in the proposition are necessary.

**Example 9.15.** For the affine line with two origins, $X$, the two embeddings $g_1 : \mathbb{A}^1_k \to X$ and $g_2 : \mathbb{A}^1_k \to X$ agree over a dense open set, but they are not equal. △

**Example 9.16.** Consider the non-reduced scheme $X = \operatorname{Spec} k[x, y]/(y^2, xy)$ and the two

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morphisms $f, g\colon X \to \operatorname{Spec} k[u]$, defined by $u \mapsto x$ and $u \mapsto x + y$ respectively. These agree over the distinguished open set $D(x) \subset X$, but they are not equal. △

### *Formal properties*

**Proposition 9.17.**

(i) (Embeddings) Closed embeddings and open embeddings are separated.

(ii) (Composition) Let $f\colon X \to S$ and $g\colon Y \to X$ be morphisms. If both $f$ and $g$ are separated, the composition $f \circ g$ is separated as well. Moreover, if $Y$ is separated over $S$, it is separated over $X$.

(iii) (Base change) Being separated is a property stable under base change: if $f\colon X \to S$ is separated and $T \to S$ is any morphism, then $f_T\colon X_T \to T$ is separated.

(iv) If $f\colon X \to Y$ is a morphism and $\{U_i\}_{i\in I}$ is a cover of $Y$ so that each $f^{-1}(U_i) \to U_i$ is separated, then $f$ is separated.

*Proof* (i): Closed embeddings are affine, hence separated. Open embeddings are separated by Example 9.7.

(ii): Consider the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{\Delta_{Y/X}} & Y \times_X Y & \longrightarrow & Y \times_S Y \\ & & \downarrow & & \downarrow \\ & & X & \xrightarrow{\Delta_{X/S}} & X \times_S X \end{array}$$

The square in the diagram is Cartesian. As the morphism $Y \times_X Y \to Y \times_S Y$ is the base change of $\Delta_{X/S}$, which is a closed embedding, it is itself a closed embedding. Therefore, $\Delta_{X/S}$, which equals the composition $Y \to Y \times_X Y \to Y \times_S Y$, is also a closed embedding.

For the second part of the claim, if the composition $Y \to Y \times_X Y \to Y \times_S Y$ is a closed embedding, then also the first map, $\Delta_{Y/X}$, must be a closed embedding, by Exercise 9.4.12.

(iii): There is a natural isomorphism

$$X_T \times_T X_T = (X \times_S T) \times_T (X \times_S T) = (X \times_S X) \times_S T$$

Under this identification, the diagonal morphism $\Delta_{X_T/T}$ is given by the map $X \times_S T \to (X \times_S X) \times_S T$, that is, the pullback of $\Delta_{X/S}$ along $T \to S$. As $\Delta_{X/S}$ is a closed embedding, and closed embeddings are stable under base change, we find that $\Delta_{X_T/T}$ is a closed embedding as well.

(iv): This was proved in the paragraph following Definition 9.5. □

**Example 9.18.** Combining Proposition 9.17 with Example 9.11 we see that any projective subscheme of $\mathbb{P}^n_A$ is separated over $A$. △

## 9.3 Proper morphisms

Properness is an important property of morphisms serving as a replacement for 'proper maps' in topology. In topology, a map $f$ is called *proper* if $f^{-1}(K)$ is compact for every compact

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subset $K \subset Y$. In the setting of Hausdorff spaces $X$, one can show that $X$ is compact if and only if for any other topological space $Z$ the projection map $X \times Z \to Z$ is closed. (Here we are using the topological fiber product.) More generally, a continuous map $f\colon X \to Y$ is proper if and only if for any map $g : Z \to Y$, the induced map $X \times_Y Z \to Z$ is closed. In scheme theory, the latter property, along with a finiteness condition, is adopted as the definition of proper maps.

**Definition 9.19.** A morphism $f\colon X \to S$ is *universally closed* if it is closed and stays closed when pulled back; that is, given any morphism $T \to S$, the pulled back map $f_T$ in the Cartesian diagram below is closed.

$$\begin{array}{ccc} X \times_S T & \longrightarrow & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array}$$

The morphism $f$ is said to be *proper* if it is separated, of finite type and universally closed. When $S = \operatorname{Spec} A$, we say that $X$ is *proper over* $A$.

In Section 13.6, we will show that any morphisms $f\colon X \to Y$ be between projective schemes over a field $k$ are proper. In particular, any projective scheme over $k$ is proper over $\operatorname{Spec} k$. Admittedly, the main class of proper morphisms one meets in practice are the projective ones[1]. However, properness is a condition which is a bit more flexible and easier than projectivity to check.

**Example 9.20.** A simple example of a morphism that is not proper, is the structure map $\pi\colon \mathbb{A}^1_k \to \operatorname{Spec} k$. If we pull back $\pi$ along itself, we get the projection map onto the first factor $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$. In $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ there are lots of closed sets which project to non-closed sets, for instance the 'hyperbola' $V(xy-1)$, which maps to the non-closed set $\mathbb{A}^1_k - \{0\}$. △

**Example 9.21.** The standard example of a proper morphism is $\pi\colon \mathbb{P}^n_A \to \operatorname{Spec} A$. While this aligns with our intuition that 'projective varieties compact, whereas affine varieties are not', this statement is a non-trivial fact, which we will prove in Theorem 13.20. △

**Example 9.22.** Closed embeddings are proper. Indeed they are universally closed by Proposition 7.19 on page 153, separated by (i) of Proposition **??** on page ?? and of finite type by definition. △

[1] There are non-projective varieties which are proper. Over $\mathbb{C}$, the first examples were constructed by Hironaka in 1960.

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**Proposition 9.23 (Properties of properness).**

(i) Pullbacks of proper maps are proper.
(ii) Compositions of composable proper maps are proper.
(iii) The product $f \times g$ of two proper maps is proper.
(iv) If $f\colon X \to Y$ is a morphism, and $\{U_i\}_{i\in I}$ is a cover of $Y$ so that each $f^{-1}(U_i) \to U_i$ is proper, then $f$ is proper.
(v) If $X$ is proper over a field $k$, and $Y$ is separated over $k$, then any morphism $X \to Y$ over $k$ is proper.

The proposition also holds if 'proper' is replaced by 'universally closed'. In fact, we know the corresponding statements hold for 'separated' and 'finite type', so we only need to check it for 'universally closed'.

*Proof* (i): Suppose that $f\colon X \to Y$ is proper. Let $g\colon Z \to Y$ be a morphism and considere the base change morphism $f_Z\colon X \times_Y Z \to Z$. If $h : T \to Z$ is any morphism, we consider the following commutative diagram:

$$\begin{array}{ccccc} X \times_Y Z \times_Z T & \longrightarrow & X \times_Y Z & \longrightarrow & X \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f_Z} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{\quad h \quad} & Z & \xrightarrow{\quad g \quad} & Y \end{array}$$

Each square is Cartesian, and hence the outer rectangle is, as well. As $f$ is universally it closed, follows that $p$ is closed. This shows that $f_Z$ is universally closed as well.

(ii): Let $f\colon X \to Y$ and $g\colon Y \to Z$ be universally closed. Given a morphism $h : T \to Z$, consider the following commutative diagram:

$$\begin{array}{ccccc} X \times_Y Y \times_Z T & \xrightarrow{\quad p \quad} & Y \times_Z T & \xrightarrow{\quad q \quad} & T \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad g \quad} & Z \end{array}$$

The squares are Cartesian, and hence the outer rectangle is Cartesian as well. As $f$ and $g$ are universally closed, $p$ and $q$ are closed, and hence $q \circ p$ is closed as well.

(iii): If $f\colon X \to Y$ and $g\colon X' \to Y'$ are both proper, then it follows directly from (i) and (ii) that $f \times g = (f \times \mathrm{id}_{Y'}) \circ (\mathrm{id}_X \times g)$ is proper.

(iv): Let $g : T \to Y$ be a morphism and consider the induced morphism $p\colon X \times_Y T \to T$. Note that $T$ is covered by the open sets $g^{-1}(U_i)$ and the fiber product is covered by the open sets $p^{-1}(g^{-1}(U_i)) = f^{-1}(U_i) \times_{U_i} g^{-1}(U_i)$. As $f^{-1}(U_i) \to U_i$ is universally closed, we see that

$$f^{-1}(U_i) \times_{U_i} g^{-1}(U_i) \longrightarrow g^{-1}(U_i)$$

is closed. This implies that $p$ is closed as well.

(v): We can factor any morphism $f\colon X \to Y$ as $\pi_Y \circ \Gamma_f$ where $\Gamma_f\colon X \to X \times_k Y$ is the graph morphism $X \to X \times_k Y$, and $\pi_Y\colon X \times_k Y \to Y$ is the second projection. Note that $\Gamma_f$ is the pullback of $\Delta_Y$ via the morphism $(f, \mathrm{id}_Y)\colon X \times_k Y \to Y \times_k Y$. As $\Delta_Y$ is a closed embedding, and closed embeddings are stable under base change, $\Gamma_f$ is also a closed

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embedding, hence proper. Also $\pi_Y$ is proper because it is a base change of $X \to \operatorname{Spec} k$. Therefore $f$ is proper by (ii).

□

**Proposition 9.24.** Any finite morphism is proper.

*Proof* Finite morphisms are by definition affine, hence separated, and of finite type. Moreover, as finite morphisms are closed and being finite is stable under base change, finite morphisms are universally closed. □

## 9.4 Exercises

**Exercise 9.4.1.** Let $X$ be the gluing of two copies of $\operatorname{Spec}\mathbb{Z}$ along the open set $U = \operatorname{Spec}\mathbb{Z} - \{(2)\}$. Show that $X$ is not separated and describe the diagonal map.

**Exercise 9.4.2.** Let $X/S$ be a scheme over $S$ and let $U \subset X$ be an open set. Show that $\Delta_{X/S}|_U = \Delta_{U/S}$.

**Exercise 9.4.3.** Let $X = \operatorname{Spec}\mathbb{C}$ and $S = \operatorname{Spec}\mathbb{R}$. Recall that the product $X \times_S X$ consists of two (closed) points. Which one corresponds to the diagonal? Can you find another $\mathbb{R}$-algebra $A$ so that if $Y = \operatorname{Spec} A$ it holds that $Y \times_S Y \simeq X \times_S X$ and the diagonal is the other point?

**Exercise 9.4.4.** Show that affine morphisms are separated.

**Exercise 9.4.5.** Let $X$ and $Y$ be schemes separated over a scheme $S$. Show that their product $X \times_S Y$ is separated over $S$.

**Exercise 9.4.6.** Let $X$ be a separated scheme. Show that, for any affine opens $U_1, \ldots, U_m \subseteq X$, $U_1 \cap \cdots \cap U_m$ is affine.

**Exercise 9.4.7.** Show that if a scheme $X$ is separated (over $\mathbb{Z}$), then for every scheme $Y$ and every morphism $f\colon X \to Y$, the morphism $f$ is separated.

**Exercise 9.4.8.** Let $T \to S$ be a morphism and let $X$ and $Y$ be two schemes over $T$. Show that there is a Cartesian diagram

$$\begin{array}{ccc} X \times_T X & \xrightarrow{\iota} & X \times_S X \\ \downarrow & & \downarrow{\scriptstyle f\times f} \\ T & \xrightarrow{\Delta_{T/S}} & T \times_S T, \end{array}$$

and conclude that the natural map $\iota\colon X \times_T Y \to X \times_S Y$ is a locally closed embedding. Hint: Use the functor of points to reduce to a statement of sets.

**Exercise 9.4.9** (Pullback of diagonals)**.** Let $X \to S$ and $T \to S$ be morphisms between schemes, and let $X_T = X \times_S T$. Show that the diagonal $\Delta_{X/S}$ pulls back to the diagonal

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$\Delta_{X_T/T}$; in other words, that there is a canonical Cartesian square

$$\begin{array}{ccc} X_T & \xrightarrow{\Delta_{X_T/T}} & X_T \times_T X_T \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X. \end{array}$$

**Exercise 9.4.10.** Let $X/S$ be a scheme and let $\iota\colon W \to X$ be an open subscheme or a closed subscheme (over $S$). Show that the diagram below is Cartesian

$$\begin{array}{ccc} W & \hookrightarrow & X \\ \downarrow{\scriptstyle \Delta_{W/S}} & & \downarrow{\scriptstyle \Delta_{X/S}} \\ W \times_S W & \longrightarrow & X \times_S X \end{array}$$

Conclude that $W/S$ is separated if $X/S$ is.

**Exercise 9.4.11** (The graph of a morphism)**.** Let $S$ be a scheme and let $f\colon X \to Y$ be a morphism over $S$. Assume that $Y$ is separated over $S$. We define the *graph* $\Gamma_f$ as the map $X \to X \times_S Y$ with components $\mathrm{id}_X$ and $f$.

a) Show that $\Gamma_f$ is the pullback of the diagonal $\Delta_{Y/S}$ under the morphism $\mathrm{id}_X \times f\colon X \times_S Y \to X \times_S Y$.
b) Show that $\Gamma_f$ is a closed embedding.
c) Suppose that $X$ and $Y$ are integral. Show that $\Gamma_f$ (with the reduced scheme structure) is isomorphic to $X$.

**Exercise 9.4.12** (Closed embeddings)**.** Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of schemes.

a) Assume that $g$ is separated. Show that if the composition $g \circ f$ is a closed embedding, then $f$ is a closed embedding. HINT: Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\Gamma_f} & X \times_Z Y & \longrightarrow & Y \\ & & \downarrow & & \downarrow{\scriptstyle g} \\ & & X & \xrightarrow[g\circ f]{} & Z \end{array}$$

where the square is Cartesian and $\Gamma_f$ is the graph of $f$.
b) Show by an example that in general $f$ is not necessarily a closed embedding even if $g \circ f$ is. HINT: Let $X$ be the affine line with two origins and consider a morphism a morphism $X \to \mathbb{A}^1$.

**Exercise 9.4.13** (Equalizers)**.** Let $X$ and $Y$ be schemes over $S$ and $f_1, f_2\colon Y \to X$ two morphisms over $S$. Define $f\colon Y \to X \times_S X$ be the morphism whose components are the $f_i$'s; that is, $f_i = \pi_i \circ f$ (as usual, the $\pi_i$'s are the two projections). The *equalizer* of the $f_i$ is defined as the pullback of the diagonal $\Delta_{X/S}$ along $f$. In other words, the diagram below is

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Cartesian:

$$\begin{array}{ccc} E & \xrightarrow{\eta} & Y \\ \downarrow & & \downarrow f \\ X & \xrightarrow{\Delta_{X/S}} & X \times_S X. \end{array}$$

a) Show that a morphism $g\colon Z \to Y$ satisfies $f_1 \circ g = f_2 \circ g$ if and only if $g$ factors via $\eta$.
b) Show that $X$ is separated if and only if for every pair of morphisms $f_1, f_2\colon Y \to X$, the equalizer $\eta\colon E \to Y$ is a closed embedding.
c) Describe the equalizer of the two inlusions $\iota_1, \iota_2\colon \mathbb{A}^1 \to X$ into the affine line with two origins.

**Exercise 9.4.14** (Monomorphisms)**.** Let $f\colon X \to Y$ be a morphism of schemes. We say that $f$ is a *monomorphism* if the composition function

$$\operatorname{Hom}(T, X) \longrightarrow \operatorname{Hom}(T, Y)$$

is injective defined by $\alpha \mapsto f \circ \alpha$ is injective for any scheme $T$. That is, if $g_1 : T \to X$ and $g_2\colon T \to X$ are morphisms such that $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

a) Prove that if $f$ is a closed embedding or an open embedding then it is a monomorphism.
b) Prove that $f$ is a monomorphism if and only if the diagonal $\Delta\colon X \to X \times_Y X$ of the morphism $f$ is an isomorphism.
c) Prove that if $f$ is a monomorphism, then $f$ is separated.

**Exercise 9.4.15.** Let X be scheme. We say that a scheme X has *affine diagonal* if for every affine opens $U$ and $V$ of $X$ the intersection $U \cap V$ is affine.

a) Prove that $X$ has affine diagonal if and only if the diagonal morphism $\Delta\colon X \to X \times_{\operatorname{Spec}\mathbb{Z}} X$ is an affine morphism. In particular, separated schemes (over $\operatorname{Spec}\mathbb{Z}$) have affine diagonal.
b) Find a scheme with affine diagonal which is not separated.

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# 10

# Local properties of schemes

This chapter explores the local properties of schemes. These are properties of schemes which take place at and around specific points.

The main definition is that of a nonsingular scheme. The intuition behind a nonsingular scheme is that it locally looks like an affine space. Taking the definition of a manifolds as a guide, one could naively try to define a scheme over a field $k$ to be nonsingular of dimension $n$ if it is covered by open subsets which are isomorphic to an open subscheme of $\mathbb{A}^n_k$. However, this would be incorrect, as only very special schemes satisfy this condition. The issue is that the Zariski open subsets are too large: over $\mathbb{C}$, the curve $x^3 + y^3 - 1 = 0$ should by all accounts be considered 'nonsingular', but there is no open set which is isomorphic to affine space, as $X$ is not rational.

There is however another idea from differential geometry which generalizes to algebraic geometry. A subset $X \subset \mathbb{R}^n$ of dimension $d$ is a manifold if and only if locally around each point $P \in X$, $X$ can be defined by $r$ equations $f_1 = \cdots = f_r$, where $r = n - d$, and so that for each the Jacobian matrix $J = \left(\frac{\partial f_i}{\partial x_j}\right)$ has rank $r$ at $P$. Indeed, if this condition holds, the Implicit Function Theorem tells us that $f_1, \ldots, f_r$ can be extended to a coordinate system in a neighbourhood of $P$, and one proves that $X$ is locally diffeomorphic to $\mathbb{R}^d$ there. We will see that this generalizes well to schemes over a field in Section 10.3.

We begin by discussing tangent spaces and cotangent spaces of schemes. We then introduce ingular points of schemes. Nonsingular schemes have local rings which are regular, i.e., where the maximal ideal can be generated by $n$ elements, where $n$ is the dimension of $X$. Thus, they admit "systems of parameters" that resemble local coordinates on manifolds.

Towards the end of the chapter, we discuss normal schemes and the process of normalization. Like nonsingular schemes, normal schemes also have algebraic properties that make them easier to study than most schemes. For instance, their singular loci are of codimension at least 2. This property will prove particularly useful in Chapter 16, were we study divisors.

## 10.1 Tangent spaces

In this section, we define the *tangent space* of a scheme at a point. As a warm-up example, let us consider a plane curve $C \subset \mathbb{R}^2$ defined by a real polynomial $f(x, y) = 0$ at the point $P = (0, 0)$. From basic calculus, the tangent space of the curve $f(x, y) = 0$ at $P$ is defined by the linear equation

$$\frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y = 0 \tag{10.1}$$



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This linear form is the first-order Taylor expansion of $f(x, y)$ at $P$. It is the linear form which best approximates $f$ near $P$.

The *cotangent space* at $P$ is defined as the dual of this vector space: it consists of the linear forms in $x$ and $y$ modulo the linear form (10.1). Equivalently, the cotangent space can be viewed as the $\mathbb{R}$-vector space of smooth functions vanishing at $P$, up to second order terms.

If $X$ is a scheme and $P \in X$, we would like to define the tangent space $T_P X$ of $X$ at $P$. While we want to keep the above geometric picture, it turns out to be more convenient to define things by defining the cotangent space first and then define the tangent space as its dual. The definitions also work best when $X$ is a scheme over a field $k$, and $P$ is a $k$-point.

The definition of the cotangent space uses the local ring $\mathcal{O}_{X,P}$ and the maximal ideal $\mathfrak{m}_P$. This is a natural choice, as $\mathcal{O}_{X,P}$ captures the main geometric properties of the scheme locally near $P$. If $X$ is integral, the ring $\mathcal{O}_{X,P}$ consists of the rational functions $f \in K(X)$ which are defined at $P$ and $\mathfrak{m}_P$ consists of the functions that vanish at $P$. The quotient $\mathfrak{m}_P/\mathfrak{m}_P^2$ intuitively captures the space of functions vanishing at $P$ up to 'second order terms'. Note that $\mathfrak{m}_P/\mathfrak{m}_P^2 = \mathfrak{m}_P \otimes_{\mathcal{O}_{X,P}} \mathcal{O}_{X,P}/\mathfrak{m}_P$ is naturally an $\mathcal{O}_{X,P}/\mathfrak{m}_P$-module; in other words it is a $\kappa(P)$-vector space. For example, if $P$ is the origin in $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$ then $\mathfrak{m}_P = (x_1, \dots, x_n)$ and $\mathfrak{m}_P/\mathfrak{m}_P^2$ is the $k$-vector space generated by the linear forms $x_1, \dots, x_n$.

With this in mind, we make the following definition.

**Definition 10.1.** Let $X$ be a scheme and let $P \in X$ be a point.

(i) The *cotangent space* of $X$ at $P$ is defined the $\kappa(P)$-vector space $\mathfrak{m}_p/\mathfrak{m}_P^2$, where $\mathfrak{m}_P$ is the maximal ideal in the local ring $\mathcal{O}_{X,P}$.

(ii) The *tangent space* of $X$ at $P$ is defined as the dual $\kappa(P)$-vector space

$$T_P X = \operatorname{Hom}_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2, \kappa(P))$$

For a morphism $\phi\colon X \to Y$, there is an induced map of cotangent spaces. Let $P \in X$, and let $Q = \phi(P)$. Then the map of local rings $\phi_P^\sharp\colon \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$ maps the maximal ideal into the maximal ideal, and being a ring map, it sends $\mathfrak{m}_Q^2$ into $\mathfrak{m}_P^2$. Therefore, it induces a map of $\kappa(y)$-vector spaces:

$$\phi_P^*\colon \mathfrak{m}_Q/\mathfrak{m}_Q^2 \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2. \tag{10.2}$$

For a second morphism $\psi\colon Y \to Z$, we have $(\psi \circ \phi)_P^* = \phi_P^* \circ \psi_{\phi(P)}^*$.

The tangent space, however, does not satisfy the same functorial property in general. The issue is that the required duals will be with respect to different fields. More precisely, $\phi_P^*$ is only a map of $\kappa(Q)$-vector spaces, and there is no natural way to make $\mathfrak{m}_Q/\mathfrak{m}_Q^2$ into a $\kappa(P)$-vector space.

An exception occurs when $X$ and $Y$ are schemes of finite type over a field $k$, and both $P \in X$ and $Q \in Y$ are $k$-points, so that $\kappa(P) = \kappa(Q) = k$. In this case, each quotient $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional $k$-vector space. We can then take duals to obtain a map

$$d\phi_P\colon T_P X \longrightarrow T_Q Y. \tag{10.3}$$

In this setting, the tangent maps satisfy $d(\psi \circ \phi)_P = d\psi_{\phi(P)} \circ d\phi_P$ for any morphisms of $k$-schemes $\phi\colon X \to Y$ and $\psi\colon Y \to Z$.

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### *Classical tangent spaces*

To understand how the previous definition relates to the classical situation, let us consider an affine variety $X$ in $\mathbb{A}^n_k$. Before proceeding, we note that one does not have to go all the way to the local ring $\mathcal{O}_{X,P}$ to compute the cotangent space. If $A$ is a ring, and $\mathfrak{m}$ is a maximal ideal, then there is an isomorphism of $A/\mathfrak{m}$-vector spaces

$$\mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{m}A_{\mathfrak{m}}/\mathfrak{m}^2A_{\mathfrak{m}}.$$

This follows from the canonical isomorphism of $A$-modules $A/\mathfrak{m} = A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$, and

$$\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{m} \otimes_A A/\mathfrak{m} = \mathfrak{m} \otimes_A A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} = \mathfrak{m}A_{\mathfrak{m}}/\mathfrak{m}^2A_{\mathfrak{m}}.$$

Let us apply this to compute the tangent space of affine space $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ at a $k$-point $P$. For simplicity, we assume that $p = (0, \ldots, 0)$ is the origin, corresponding to the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$. (We may always achieve this by a linear change of coordinates.) The cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is then a $k$-vector space of dimension $n$, with basis $x_1, \ldots, x_n$.

More generally, consider an affine scheme $X = \operatorname{Spec} A$, where $A = k[x_1, \ldots, x_n]/\mathfrak{a}$ for an ideal $\mathfrak{a} \subset k[x_1, \ldots, x_n]$, and assume that $X$ contains $P$. Writing $\mathfrak{n} \subset A$ for the maximal ideal of the origin in $A$, the cotangent space at $P$ is isomorphic to the $k$-vector space

$$\mathfrak{n}/\mathfrak{n}^2 \simeq \frac{\mathfrak{m}}{\mathfrak{m}^2 + \mathfrak{a}}. \tag{10.4}$$

**Example 10.2.** Say $\mathfrak{a} = (f)$ is generated by $f = y^2 + x^3 + 3x + 4y$ in $\mathbb{A}^2_{\mathbb{C}}$. Then

$$\frac{\mathfrak{m}}{\mathfrak{m}^2 + (f)} = \frac{(x,y)}{(x^2, xy, y^2, y^2 + x^3 + 3x + 4y)} = \frac{(x,y)}{(x^2, xy, y^2, 3x + 4y)} = \mathbb{C}x.$$

If $f = y^2 + x^3 + x^2$, then

$$\frac{\mathfrak{m}}{\mathfrak{m}^2 + (f)} = \frac{(x,y)}{(x^2, xy, y^2, y^2 + x^3 + x^2)} = \frac{(x,y)}{(x^2, xy, y^2)} = \mathbb{C}x \oplus \mathbb{C}y.$$

△

It is clear that the quotient (10.4) only depends on the 'linear part' of the Taylor expansion of the elements of $\mathfrak{a}$ at $P$. More precisely, let us define, for $f \in k[x_1, \ldots, x_n]$,

$$d_P f = \frac{\partial f}{\partial x_1}(P) \cdot x_1 + \cdots + \frac{\partial f}{\partial x_n}(P) \cdot x_n \tag{10.5}$$

Here the $\frac{\partial f}{\partial x_i}$ denote the *formal derivatives* of $f$. They are defined using the usual formulas for differenting monomials, that is, if $f = x_1^{e_1} \ldots x_n^{e_n}$, then $\frac{\partial f}{\partial x_1} = e_1 x_1^{e_1 - 1} \ldots x_n^{e_n}$, and so on. All the usual rules for derivatives still hold, such as the chain rule, quotient rule etc. However, one should keep in mind that there may be unusual behaviour when $k$ has characteristic $p$. For instance, if $f = x^p$ then $\frac{\partial f}{\partial x} = 0$, even though it is not constant.

In terms of (10.5), the cotangent space of $X$ at $P$ is given by

$$\mathfrak{m}/(\mathfrak{m}^2 + d_P(\mathfrak{a})),$$

where $d_P(\mathfrak{a}) \subset \mathfrak{m}/\mathfrak{m}^2$ is the subspace spanned by all the linear parts $d_P f$ for $f \in \mathfrak{a}$. In

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other words, the cotangent space is obtained from $\mathfrak{m}/\mathfrak{m}^2$ by modding out by the linearized equations defining $X$ at $P$.

To compute $T_P X$, we are interested in the dual of this vector space. For this, note that $\mathfrak{m}/\mathfrak{m}^2$ is generated by $x_1, \ldots, x_n$, so we may identify $\mathfrak{m}/\mathfrak{m}^2 = (k^n)^\vee$. Under this identification, $d_P f$ corresponds to a linear functional on $k^n$. Taking duals, we get an isomorphism

$$T_P \mathbb{A}^n_k \simeq k^n.$$

where the $i$-th basis vector of $k^n$ corresponds to the functional $\mathfrak{m}/\mathfrak{m}^2 \to k$ which sends $x_i$ to 1 and the other $x_j$'s to 0.

To compute $T_P X$, we recall the following fact from linear algebra. Let $V$ be a finite dimensional $k$-vector space and let $S \subset V^\vee$ be a linear subspace of the dual space of $V$. Consider the subspace of $V$ defined by

$$W = \big\{ v \in V \,\big|\, f(v) = 0 \text{ for all } f \in S \big\}$$

Then restrictions of linear functionals induces an isomorphism $W^\vee \simeq V^\vee / S$. In particular, since $W$ is isomorphic to $(W^\vee)^\vee$, there is an isomorphism $W = (V^\vee/S)^\vee$.

Applying this to $V = k^n$ and $S = d_P(\mathfrak{a}) \subset \mathfrak{m}/\mathfrak{m}^2 = (k^n)^\vee$, we may identify the tangent space $T_P X$ with

$$T_P X = \left\{ (v_1, \ldots, v_n) \in k^n \mid v_1 \frac{\partial f}{\partial x_1}(P) + \cdots + v_n \frac{\partial f}{\partial x_n}(P) = 0 \quad \forall f \in \mathfrak{a} \right\}. \tag{10.6}$$

This is the 'classical tangent space' of $X$ at $P$. The description (10.6) is valid for any point $P = (a_1, \ldots, a_n)$, not just the origin.

If $\mathfrak{a} = (f_1, \ldots, f_r)$, then $T_P X$ is the subspace of $k^n$ defined by the $r$ linear equations in $k^n$:

$$d_P(f_1) = \cdots = d_P(f_r) = 0$$

One sometimes also considers the *affine tangent space* of $X$ at the point $P$. This is the subvariety of $\mathbb{A}^n_k$ defined by the $r$ linear equations

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(P) \cdot (x_i - a_i) = 0 \quad \text{for } j = 1, \ldots, r. \tag{10.7}$$

One can view this as the tangent space $T_P X$, but translated so that $P$ is the origin. Another

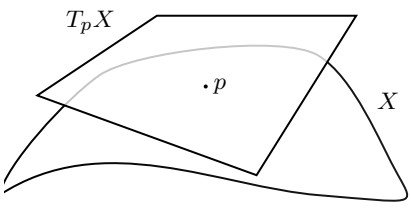


**Figure 10.1** The affine tangent space

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way to view the tangent space is in terms of the *Jacobian matrix*

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}. \tag{10.8}$$

Then the matrix $J(P) = J \otimes_A \kappa(P)$ is an $r \times n$-matrix with entries in $\kappa(P) = k$, which can be identified with the matrix $\left(\frac{\partial f_i}{\partial x_j}(P)\right)$. From (10.6), we see that $T_P X$ can be identified with the null space of $J(P)$. In particular, the dimension of $T_P X$ is given by

$$\dim_k T_P X = n - \text{rank } J(P). \tag{10.9}$$

**Example 10.3.** Consider the quadric surface $X = V(x^2 + yz - z)$ in $\mathbb{A}^3_{\mathbb{C}}$. Then the Jacobian at a closed point $P = (a, b, c)$ is equal to

$$J = \begin{pmatrix} 2a & c-1 & b \end{pmatrix}.$$

Therefore, $T_P X$ has dimension 3 at the $k$-point $P = (0, 0, 1)$ and dimension 2 otherwise. △

### *Zariski tangent spaces and the ring of dual numbers*

When $X$ is a scheme over a field $k$, there is an interesting relation between the Zariski tangent space at $k$-points and the ring $k[\epsilon]/\epsilon^2$. This ring is called the *ring of dual numbers* over $k$. The spectrum of $k[\epsilon]/\epsilon^2$ is a very simple scheme: its underlying topological space is a single point. However, the non-reduced structure on $\operatorname{Spec} k[\epsilon]/\epsilon^2$ shows that it is more interesting than $\operatorname{Spec} k$. It is common to picture it as a point $\varepsilon$ with a vector 'sticking out of it'.

The following proposition generalizes the perspective in differential geometry where tangent vectors are defined as equivalence classes of curves passing through $P$.

**Proposition 10.4.** Let $X$ be a scheme over a field $k$ and let $P \in X$ be a $k$-point. Then there are natural bijections between:

(i) The elements of $T_P X$, i.e., $k$-linear maps $\mathfrak{m}_P/\mathfrak{m}_P^2 \to k$
(ii) Morphisms of schemes over $k$,

$$\operatorname{Spec} k[\epsilon]/\epsilon^2 \longrightarrow X$$

with image $P$.

*Proof* We may assume that $X$ is affine, say $X = \operatorname{Spec} A$, where $A$ is a $k$-algebra. Let $\mathfrak{m} = \mathfrak{m}_P$ be the maximal ideal, and let $\rho\colon A \to k = A/\mathfrak{m}$ be the quotient map corresponding to $P$. As $P$ is a $k$-point, the composition of the structure map $k \to A$ and $\rho\colon A \to k$ is the identity map.

First of all, any map of $k$-algebras $A \to k[\epsilon]/\epsilon^2$ factors uniquely via $A/\mathfrak{m}^2 \to k[\epsilon]/\epsilon^2$. Hence there is a natural bijection

$$\operatorname{Hom}_{\mathsf{Alg}/k}(A, k[\epsilon]/\epsilon^2) = \operatorname{Hom}_{\mathsf{Alg}/k}(A/\mathfrak{m}^2, k[\epsilon]/\epsilon^2)$$

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Next, consider the exact sequence

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow A/\mathfrak{m}^2 \longrightarrow A/\mathfrak{m} \longrightarrow 0$$

This sequence splits, as the structure map $k \to A/\mathfrak{m}^2$ provides a section, so we may write $A/\mathfrak{m}^2 = k \oplus \mathfrak{m}/\mathfrak{m}^2$. Given a map of $k$-algebras $h\colon A/\mathfrak{m}^2 \to k[\epsilon]/\epsilon^2$, the restriction of $h$ defines a $k$-linear map $\mathfrak{m}/\mathfrak{m}^2 \to k\epsilon \simeq k$. This defines a map of $k$-vector spaces

$$\mathrm{Hom}_{\mathsf{Alg}/k}(A/\mathfrak{m}^2, k[\epsilon]/\epsilon^2) \longrightarrow \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

The inverse is constructed as follows: given $\alpha\colon \mathfrak{m}/\mathfrak{m}^2 \to k$, define $h$ by $h(a+t) = a + \alpha(t)$ where $a \in k$ and $t \in \mathfrak{m}/\mathfrak{m}^2$. One checks that this is a map of $k$-algebras. □

There is also a description of $T_P X$ as the vector space of derivations of $\mathcal{O}_{X,P}$ into $k$, i.e., $k$-linear maps $D\colon \mathcal{O}_{X,P} \to k$ satisfying the Leibniz rule (see Exericise 17.11.2).

## 10.2 Nonsingular schemes

We say that a Noetherian scheme $X$ *nonsingular* or *regular*, at a point $P$ if $\mathcal{O}_{X,P}$ is a regular local ring. Here we recall that a Noetherian local ring $A$ with maximal ideal $\mathfrak{m}$ is defined to be regular if the maximal ideal can be generated by $n$ elements, where $n = \dim A$ is the Krull dimension of $A$.

For instance, if $P \in \mathbb{A}^n_k$ is the origin, corresponding to the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$, then the local ring is isomorphic to

$$\mathcal{O}_{\mathbb{A}^n_k, p} = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$$

which is regular, as it has dimension $n$ and $\mathfrak{m}$ is generated by $n$ elements. This is the intuition of regularity: the $n$ generators for $\mathfrak{m}$ define $n$ 'coordinate functions' on $X$ near the point.

It is can certainly happen that $\mathfrak{m}_P$ requires more generators than the dimension of $\mathcal{O}_{X,P}$. For instance, if $X = \operatorname{Spec} k[x,y]/(x^2 - y^3)$, and $P$ is the origin, then $\mathcal{O}_{X,P}$ has dimension 1, but $\mathfrak{m}_P = (x, y)$ cannot be generated by a single element. If this happens, we say that $P$ is a *singular point*. The set of singular points of $X$ is denoted by $\mathrm{sing}(X)$. The scheme $X$ is said to be nonsingular if every point is nonsingular.

As it turns out, for a Noetherian local ring $A$, the condition of regularity can be formulated in terms of the quotient $\mathfrak{m}/\mathfrak{m}^2$, which is naturally a vector space over the residue field $A/\mathfrak{m}$. As such, it is finite-dimensional, because $\mathfrak{m}$ is finitely generated. If $x_1, \ldots, x_r$ are elements from $A$ that generate $\mathfrak{m}/\mathfrak{m}^2$ as a $A/\mathfrak{m}$-vector space, then Nakayama's lemma implies that the $x_1, \ldots, x_r$ generate $\mathfrak{m}$. Furthermore, by Krull's Principal ideal theorem, we have

$$\dim A = \mathrm{ht}\, \mathfrak{m} \leqslant r = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 < \infty \tag{10.10}$$

In other words, $A$ is regular if and only if equality holds in the inequality (10.10).

On the geometric side, this means that a scheme $X$ is nonsingular at a point $P$ if and only if the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ has dimension equal to the Krull dimension of $\mathcal{O}_{X,P}$.

Here are three important facts about regular local rings.

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**Theorem 10.5.** Let $A$ be a regular local ring. Then
(i) For each prime ideal $\mathfrak{p} \subset A$, the ring $A_{\mathfrak{p}}$ is also a regular local ring.
(ii) $A$ is an integral domain.
(iii) $A$ is a UFD.

*Proof* For (i), see [Matsumura, Chapter 7, Theorem 45]. For (ii), see (Atiyah, 2018, p. 123). □

The first item (i) is a rather non-trivial result; it is not even obvious that a localization of a polynomial ring is regular.

The second item tells us that in a nonsingular scheme, any point is contained in a unique irreducible component, and this component is an integral scheme. Hence the scheme is a disjoint union of integral schemes. In particular, any connected nonsingular scheme is integral. This observation is sometimes useful when proving that a ring is an integral domain (see Exercise 10.6.17).

The third item (iii) is known as the *Auslander–Buchsbaum theorem*. This will be used several times in this book, in particular when we discuss divisors in Chapter 16.

Another consequence of (i) is the following:

**Corollary 10.6.** A Noetherian scheme is nonsingular if and only if it is nonsingular at every closed point.

**Example 10.7.** The schemes $\mathbb{A}^n_{\mathbb{Z}}$ and $\mathbb{P}^n_{\mathbb{Z}}$ are nonsingular. This is because the local rings are localizations of the polynomial ring $\mathbb{Z}[t_1, \ldots, t_n]$, and these are regular. More generally, $\mathbb{A}^n_A$ and $\mathbb{P}^n_A$ is nonsingular if $\operatorname{Spec} A$ is. △

**Example 10.8.** The $\mathbb{C}$-scheme $X = \operatorname{Spec}(\mathbb{C}[x]/(x + x^2))$ is nonsingular, because it is isomorphic to the two-point scheme $\operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}$.

On the other hand, the scheme $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ is singular, because $\mathbb{C}[x]/(x^2)$ is a local ring of dimension 0, and the maximal ideal $\mathfrak{m} = (x)$ is nonzero. Alternatively, $\mathbb{C}[x]/(x^2)$ is clearly not an integral domain. △

**Example 10.9.** More generally, any regular local ring of dimension 0 must be a field (because it is both Artinian and an integral domain.) Hence, $X = \operatorname{Spec} A$ is a nonsingular scheme of dimension 0, then $A$ is a product of fields. △

**Example 10.10** (Complete intersections)**.** If $A$ is a regular local ring of dimension $n$, and $f \in \mathfrak{m}$ is a a nonzero element, then the quotient $A/(f)$ is regular if and only if $f \notin \mathfrak{m}^2$. Indeed, $A$ is an integral domain, so $A/(f)$ has dimension $n - 1$. If $\mathfrak{n} = \mathfrak{m}/(f)$ denotes the maximal in $A/(f)$, then

$$\frac{\mathfrak{n}}{\mathfrak{n}^2} \simeq \frac{\mathfrak{m}}{\mathfrak{m}^2 + (f)}$$

which has dimension $n - 1$ if and only if $f$ is non-zero in $\mathfrak{m}/\mathfrak{m}^2$, i.e., $f \notin \mathfrak{m}^2$.

More generally, an inductive argument shows that if $f_1, \ldots, f_r \in \mathfrak{m}$ are elements of $A$ such that the classes $\overline{f_1}, \ldots, \overline{f_r}$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$, then $A/(f_1, \ldots, f_r)$ is a regular local ring. △

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**Example 10.11.** Consider the ring $A = \mathbb{Z}[x]/(x^2+4)$ and the maximal ideal $\mathfrak{m} = (2, x)$. The scheme $X = \operatorname{Spec} A$ is singular at the point $x$ corresponding to $\mathfrak{m}$. Indeed, here the cotangent space is given by

$$\mathfrak{m}/\mathfrak{m}^2 = (2,x)/(4,2x,x^2) = (2,x)/(4,2x,-4)$$

which is a $\mathbb{F}_2$-vector space of dimension 2 (the elements are: $0, 2, x, x+2$). △

## 10.3 The Jacobian criterion and smoothness

Using the description of the tangent space in terms of the Jacobian matrix (10.8), we get the following very useful criterion to check if a scheme is singular at a point.

**Proposition 10.12 (Jacobian criterion).** Let $k$ be a field and let $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Let $P \in X$ be a $k$-point and let $J(P) = J \otimes_A \kappa(P)$, where $J$ is the Jacobian matrix (10.8). Then

$$\operatorname{rank} J(P) \leqslant n - \dim X. \tag{10.11}$$

and $X$ is nonsingular at $P$ if and only if equality holds.

In light of Corollary 10.6, this means that if $X$ is a scheme of finite type over an algebraically closed field $k$, then we can determine whether $X$ is singular by computing the Jacobian matrices at $k$-points in local affine charts. Here are a few examples of this.

**Example 10.13.** Let $X = \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^3)$. The Jacobian matrix is given by

$$J = \begin{pmatrix} -3x^2 & 2y \end{pmatrix}$$

which has rank 1 at every $\mathbb{C}$-point except the origin. At the origin, the local ring is given by

$$\mathcal{O}_{X,P} = \big(\mathbb{C}[x,y]/(y^2-x^3)\big)_{(x,y)}$$

and the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2 \simeq \mathbb{C}\bar{x} \oplus \mathbb{C}\bar{y}$ has dimension 2. △

**Example 10.14.** Consider the $\mathbb{C}$-scheme $X = \operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - z^2))$. The Jacobian is given by the matrix

$$J = \begin{pmatrix} y & x & -2z \end{pmatrix}$$

which has rank 1 at every point, outside the origin $p = (0,0,0)$, so the tangent space $T_P X$ has dimension 2 at these points. At $P$, the cotangent space has dimension 3: it is given by

$$\mathfrak{m}_P/\mathfrak{m}_P^2 = (x,y,z)/(x^2, xy, y^2, xz, z^2) \simeq k\,x \oplus k\,y \oplus k\,z.$$

△

**Example 10.15.** Let $X \subset \mathbb{A}^3_k$ be the curve defined by the ideal $I = (xz + y^2, x^2 + y^2 + 2z)$. The Jacobian is given by

$$J = \begin{pmatrix} y & x & 2z \\ 2x & 2y & 2 \end{pmatrix}$$

The singular locus $\operatorname{sing}(X)$ is defined by the ideal generated by the $2 \times 2$-minors of $J$ and



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$I$. If char $k \neq 2$, we find that there is a unique singular point the origin $p = (0,0,0)$. If char $k = 2$, then $X$ is actually non-reduced, because $x^2+y^2+2z = (x+y)^2$, but $x+y \notin I$. In fact, in this case, $\sqrt{I} = (x+z, y+z)$. △

Unfortunately, the Jacobian criterion is not sufficient to check nonsingularity in general. We will see simple counterexamples shortly. In general, if $X$ is scheme of finite type over a field $k$ and $P \in X$ is a point, then the property that the Jacobian matrix at $P$ has maximal rank at $P$ is called 'smoothness'. Smoothness and nonsingularity are equivalent notions over perfect fields, but not in general.

**Definition 10.16** (Smoothness over a field)**.** Let $X$ be a scheme of finite type over a field $k$. We say that a point $P \in X$ is *smooth* at $P$ if there is an affine neighbourhood $U = \operatorname{Spec} A$ of $P$, where $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ and the Jacobian $J$ of the $f_1, \ldots, f_r$ satisfies

$$\operatorname{rank} J(P) = n - \dim_P X.$$

where $\dim_P X$ is the dimension of $X$ at $P$.

One key difference between 'smoothness' and 'nonsingularity' is that smoothness should really be considered as a property of the *morphism* $X \to \operatorname{Spec} k$, whereas nonsingularity is an absolute property of $X$ (this will be explained in Exercise 17.11.14). Smoothness is furthermore a property which is stable under base change, wheras nonsingularity is not. The next result tells us that we can relate the two notions by going to the algebraic closure of the base field.

**Proposition 10.17.** Let $X$ be a scheme of finite type over a field $k$ and let $P \in X$ be a closed point. Then the following are equivalent:

(i) $X$ is smooth at $P$.
(ii) For each closed point $\bar{P} \in X_{\bar{k}}$ which maps to $P$, $X_{\bar{k}}$ is smooth at $\bar{P}$.
(iii) For each closed point $\bar{P} \in X_{\bar{k}}$ which maps to $P$, $X_{\bar{k}}$ is nonsingular at $\bar{P}$.

*Proof* Over an algebraically closed field, the conditions of nonsingularity and smoothness are equivalent, by the Jacobian criterion, hence (ii) and (iii) are equivalent.

To prove (i)⇔(ii), note that the statement is local around $P$, so we may assume that $X = \operatorname{Spec} A$ is affine, where $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$.

The base change morphism $X_{\bar{k}} \to X$ is surjective (by the Lying Over Theorem). Let $\bar{P} \in X_{\bar{k}}$ be a point which maps to $P$. Then as $\bar{k}$ is algebraically closed, $X_{\bar{k}}$ is nonsingular at $\bar{P}$ if and only if $X_{\bar{k}}$ is smooth at $\bar{P}$, and this happens if and only if $\operatorname{rank} J(\bar{P}) = n - \dim X$. But as $\operatorname{rank} J(P) = \operatorname{rank} J(\bar{P})$, this happens if and only if $X$ is smooth at $P$. □

**Proposition 10.18.** Let $X$ be a scheme of finite type over a field $k$ and let $P \in X$ be a closed point. If $X$ is smooth at $P$, then $X$ is nonsingular at $P$. The converse holds if $\kappa(P)$ is separable over $k$.

*Proof* The statement is local on $X$, so we may assume that $X = \operatorname{Spec} A$ where $A =$

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$k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. We may also assume that $X$ is irreducible, so $\dim_P X = \dim X$.

The statement is clear if $\kappa(P) = k$ and $P$ is a $k$-point, by the Jacobian Criterion. For a general closed point $P$, we reduce to this case as follows. Consider the base change $X_{\kappa(P)} = \operatorname{Spec} B$, where

$$B = A \otimes_k \kappa(P) = \kappa(P)[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

As $k \to \kappa(P)$ is a finite field extension, hence integral, also $A \to B$ is integral, and so $X_{\kappa(P)} \to X$ is surjective by the Lying Over Theorem. Let $\bar{P} \in X_{\kappa(P)}$ be a $\kappa(P)$-point which maps to $P$.

The Jacobian matrices $J(P)$ and $J(\bar{P})$ of $f_1, \ldots, f_r$ at the points $P$ and $\bar{P}$ have the same rank (we may compute the rank by going to $\bar{k}$). Therefore, $X_{\kappa(P)}$ is smooth at $\bar{P}$ if and only if $X$ is smooth at $P$. Since $\bar{P}$ is a $\kappa(P)$-point, we see that $X_{\kappa(P)}$ is nonsingular at $\bar{P}$ by the Jacobian Criterion. This means that the classes of $f_1, \ldots, f_r$ are linearly independent in $\mathfrak{m}_{\bar{P}}/\mathfrak{m}_{\bar{P}}^2$. Considering the map $\mathfrak{m}_P/\mathfrak{m}_P^2 \to \mathfrak{m}_{\bar{P}}/\mathfrak{m}_{\bar{P}}^2$, we see that $f_1, \ldots, f_r$ must be linearly independent in $\mathfrak{m}_P/\mathfrak{m}_P^2$ as well, and hence $\mathcal{O}_{X,P}$ is a regular local ring by Example 10.10.

For the converse, we will consider the base change $X_{\bar{k}}$ to the algebraic closure $\bar{k}$. Suppose that $k \subset \kappa(P)$ is separable of degree $d$. By the Primitive Element Theorem, we have $\kappa(P) = k(\xi) = k[t]/(h(t))$ for some polynomial $h$ with $h'(\xi) \neq 0$ in $\kappa(P)$. As $h$ is separable, we have, as rings

$$\kappa(P) \otimes_k \bar{k} = \bar{k} \times \cdots \times \bar{k}.$$

This implies that the scheme-theoretic fiber of $X_{\kappa(P)} \to X$ over $P$ consists of $d$ *reduced* points.

Let $\mathfrak{n} = \mathfrak{m} \otimes_k \bar{k} \subset A \otimes_k \bar{k}$. Then $\mathfrak{n}$ is the ideal describing $P_1, \ldots, P_d$ as a closed subscheme of $X_{\bar{k}}$. We have

$$\frac{\mathfrak{n}}{\mathfrak{n}^2} = \frac{\mathfrak{m}_{P_1}}{\mathfrak{m}_{P_1}^2} \oplus \cdots \oplus \frac{\mathfrak{m}_{P_d}}{\mathfrak{m}_{P_d}^2}$$

Note that $\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{m}/\mathfrak{m}^2 \otimes_k \bar{k}$ has dimension $d \cdot \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ as a $\bar{k}$-vector space. Therefore, we have

$$d \cdot \dim_k(\mathfrak{m}/\mathfrak{m}^2) = \sum_{i=1}^{d} \dim_{\bar{k}}\left(\frac{\mathfrak{m}_{P_i}}{\mathfrak{m}_{P_i}^2}\right) \tag{10.12}$$

Recall that $\dim_{\bar{k}}(\mathfrak{m}_{P_i}/\mathfrak{m}_{P_i}^2) \geqslant \dim X$ for every $i$. If this is an equality for every $i$, then (10.12) implies that $P$ is nonsingular. Conversely, some inequatlity is strict, then the sum on the right-hand side is $> d \dim X$ and hence $\dim_k(\mathfrak{m}/\mathfrak{m}^2) > \dim X$. In other words, $P$ is nonsingular if and only if $P_i$ is nonsingular for every $i$. Therefore, assumping $P$ is nonsingular, the $P_i$ are nonsingular, hence smooth, and hence $P$ is also smooth by Proposition 10.17.

□

Here is a concrete example where the converse fails:

**Example 10.19** (A nonsingular, but non-smooth curve)**.** Let $k = \mathbb{F}_p(t)$ where $p > 2$ is a

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prime number and let $X$ be the plane curve defined by the polynomial $f = y^2 + x^p + t$. Note that $X$ is integral. In fact, $f$ is irreducible even in $\bar{k}[x, y]$, so also $X_{\bar{k}}$ is integral.

The Jacobian of $f$ is given by

$$J = \begin{pmatrix} px^{p-1} & 2y \end{pmatrix} = \begin{pmatrix} 0 & 2y \end{pmatrix}.$$

The closed subset of $X$ defined by the vanishing of the Jacobian matrix is given by $\{P\}$, where $P \in X$ is the point corresponding to the maximal ideal $\mathfrak{m} = (x^p + t, y)$ (this is a prime ideal, because $x^p + t$ is irreducible over $k$). This means that $X$ is nonsingular at every point in $X - \{P\}$. However, $X$ is not smooth at $P$, as the Jacobian is identically 0 there.

To show that $X$ is still nonsingular at $P$, note that $f \in k[x, y]$ is not contained in $\mathfrak{m}^2 = (x^{2p} + 2t + t^2, y(x^p + t), y^2)$, and so $\mathcal{O}_{X,P}$ is regular by Example 10.10. Hence $X$ is a nonsingular scheme.

The point of this example is that 'smoothness' behaves well under base change, whereas nonsingularity does not. If we go to the algebraic closure $\bar{k}$, then $t$ has a $p$-th root $s$, and then $\operatorname{Spec} \bar{k}[x, y]/(y^2 + x^p + s^p)$ is no longer nonsingular (e.g., at the point $(x, y) = (-s, 0)$). △

**Example 10.20.** Let $k$ be a field of characteristic 2 containing an element $\alpha \in k$ which is not a square. Let $P \in \mathbb{A}^2_k$ be the closed point of $\mathbb{A}^2_k$ corresponding to the maximal ideal $\mathfrak{m} = (v, u^2 + \alpha)$ in $k[u, v]$.

The plane affine curve $X \subset \mathbb{A}^2_k$ defined by $f = v^2 - u(u^2 + \alpha)$ has a Jacobian given by $J = (u^2 + \alpha, 0)$. This has rank 1 except at the point $P$ corresponding to the maximal ideal $\mathfrak{m} = (u^2 + \alpha, v)$. Therefore, $X$ is smooth at all points of $X - \{P\}$, but not at $P$. $X$ is however nonsingular at $P$, as $f$ does not belong to $\mathfrak{m}^2 = (v^2, v(u^2 + \alpha), u^4 + \alpha^2)$.

If $K = k(\beta)$ is the extension of $k$ obtained by adjoining a square root $\beta$ of $\alpha$, then the curve $X_K = X \times_k K$ acquires a singular point. In terms of $\beta$, $f$ takes the form $f = v^2 - u(u^2 + \beta^2) = v^2 - u(u + \beta)^2$, and $X_K$ has a singular point at $(u, v) = (-\beta, 0)$. △

**Proposition 10.21.** Let $X$ be a scheme of finite type over field $k$. Then the *smooth locus* $X^{\mathrm{sm}} = \{\, P \in X \mid f \text{ is smooth at } P \,\}$ is an open subset of $X$.

*Proof* Let $P \in X$ be a point and choose an open set $U = \operatorname{Spec} A$ as in the definition of a smooth point. If the Jacobian matrix $J$ of the defining polynomials has maximal rank $n - r$ at $x$, then some $(n-r) \times (n-r)$-minor of $J$ is non-zero, and it will continue to be non-zero in some open neighborhood $W$ of $P$. Therefore $W \subset X^{\mathrm{sm}}$ and hence $X^{\mathrm{sm}}$ is open. □

**Corollary 10.22.** For a reduced scheme of finite type over a perfect field $k$, the smooth locus $X^{\mathrm{sm}}$ is a dense open subset of $X$.

*Proof* We may assume that $X$ is irreducible, hence integral. By the previous proposition, it suffices to show that $X^{\mathrm{sm}}$ is nonempty, i.e., $X$ contains a single smooth point. To see this, it is simplest to use the fact that $X$ is birational to a hypersurface $Y \subset \mathbb{A}^n_k$ (Proposition **??**). Write $Y = V(f)$, where $f$ is an irreducible polynomial in $k[x_1, \ldots, x_n]$. For $Y$, the statement of the proposition is clear, because if the Jacobian matrix is zero everwhere on $Y$, then $\frac{\partial f}{\partial x_i} \in I(Y) = (f)$ for each $i$. But by degree reasons this means that the partial

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derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ must all be the zero polynomial. This can only happen if $k$ has characteristic $p$ and $f$ is a polynomial in $x_1^p, \ldots, x_n^p$. Over an algebraically closed field, this even implies that $f = g^p$ is the $p$-th power of a polynomial $g$. However, this is not possible, as we assumed $f$ to be irreducible.

Finally, let $U \subset X$ and $V \subset Y$ be open sets related by an isomorphism $\iota\colon U \to V$. By what we just showed, the smooth locus $Y^{\mathrm{sm}}$ is dense in $Y$, and hence intersects $V$. As $\iota$ is an isomorphism, it induces an isomorphism on tangent spaces, and hence $U$ contains smooth points as well. □

**Corollary 10.23.** Let $X$ be a scheme of finite type over a perfect field $k$. Then $X$ is nonsingular if and only if it is smooth over $k$.

*Proof* $X$ is nonsingular if and only if it is nonsingular at each closed point $X$, which happens if and only if it is smooth at every closed point. On the other hand, the smooth locus of $X$ is open, so if it contains all closed points of $X$, it must be equal to $X$, and hence $X$ is smooth. □

**Example 10.24.** For varieties over non-perfect fields, it can happen that the smooth locus is empty. For instance, if $k = \mathbb{F}_p(t)$ for an odd prime $p$ and $X$ is the affine scheme defined by the polynomial $f = x^p - ty^p$ in $\mathbb{A}^2_k$, then the Jacobian matrix is the zero matrix, so $X^{\mathrm{sm}} = \varnothing$. This is 'explained' by the fact that the base change $X_{\bar{k}}$ is non-reduced, hence singular at every point: if $s$ is a $p$-th root of $t$, then $f = (x - sy)^p$. △

## 10.4 Normal schemes

Recall that an integral domain $A$ is said to be *normal* if it is integrally closed in its fraction field $K = K(A)$. This means that any element $w \in K$ which satisfies a monic equation $w^n + a_{n-1}w^{n-1} + \cdots + a_0 = 0$ with coefficients in $A$, is already contained in $A$.

Here are a few examples of normal rings:

**Example 10.25.** Any unique factorization domain $A$ is normal. This includes rings such as $\mathbb{Z}$ or $k[x_1, \ldots, x_n]$ where $k$ is a field.

To prove this, suppose $\frac{u}{v} \in K$ is integral over $A$. Since $A$ is a UFD, we may assume without loss of generality that $u$ and $v$ are coprime (i.e., they share no common prime factors). By assumption, $\frac{u}{v}$ satisfies a monic polynomial equation:

$$\left(\frac{u}{v}\right)^n + a_{n-1}\left(\frac{u}{v}\right)^{n-1} + \cdots + a_0 = 0,$$

where $a_i \in A$. Multiplying by $v^n$ gives:

$$u^n + a_{n-1}u^{n-1}v + \cdots + a_0 v^n = 0.$$

This implies that $v$ divides $u^n$, which by our assumption means that $v$ must be a unit in $A$. Therefore, $\frac{u}{v} \in A$, and hence $A$ is normal. △

**Example 10.26.** Any localization $S^{-1}A$ of a normal integral domain $A$ is normal. In fact, there is a partial converse: an integral domain $A$ is normal if and only if $A_{\mathfrak{p}}$ is normal for all prime ideals $\mathfrak{p}$, if and only if $A_{\mathfrak{m}}$ is normal for all maximal ideals $\mathfrak{m}$. △

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**Example 10.27.** If $A$ is a normal, then so is $A[x]$. △

**Example 10.28.** The ring $A = \mathbb{Z}[\sqrt{-5}]$ is normal. To see this, let $\alpha = r + s\sqrt{-5} \in \mathbb{Q}(\sqrt{-5})$, with $r, s \in \mathbb{Q}$. If $s \neq 0$, then the minimal polynomial $\alpha$ is $q(x) = x^2 - 2rx + (r^2 + 5s^2)$. Since $\alpha$ is integral over $\mathbb{Z}[\sqrt{-5}]$, then it satisfies a monic polynomial equation $f(x) = 0$ with $f(x) \in \mathbb{Z}[x]$. As $q(x)$ is the minimal polynomial of $\alpha$ over $\mathbb{Q}$, it must divide $f(x)$. By Gauss' Lemma, $q(x)$ must have integer coefficients. For this to happen, we must have $2r \in \mathbb{Z}$ and $r^2 + 5s^2 \in \mathbb{Z}$. These two imply that $r$ and $s$ can be written as $r = m/2$ and $s = n/2$ for some $m, n \in \mathbb{Z}$. Then:

$$r^2 + 5s^2 = \left(\frac{m}{2}\right)^2 + 5\left(\frac{n}{2}\right)^2 = \frac{m^2 + 5n^2}{4} \in \mathbb{Z}.$$

This implies that $m^2 + 5n^2$ is divisible by 4. Considering the possiblities of $m$ and $n$ modulo 4, shows that $m$ and $n$ must be even, and hence $r, s \in \mathbb{Z}$. Hence $\alpha \in \mathbb{Z}[\sqrt{-5}]$. △

**Example 10.29.** The ring $\mathbb{Z}[\sqrt{5}]$ is not normal. The fraction field equals $K = \mathbb{Q}(\sqrt{5})$, and this contains the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. Note that $\phi$ satisfies the monic equation

$$\phi^2 - \phi - 1 = 0,$$

so $\phi$ is integral over $\mathbb{Z}$, hence over $\mathbb{Z}[\sqrt{5}]$, but of course $\phi \notin \mathbb{Z}[\sqrt{5}]$. △

Motivated by the many desirable properties of normal rings, we make the following definition for schemes.

**Definition 10.30.** We say that a scheme $X$ is *normal* if for each point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integrally closed integral domain.

The primary example of a normal scheme is $X = \operatorname{Spec} A$, where $A$ is a normal integral domain. Note however, that in the definition of a normal scheme we do not make the assumption that $X$ is integral. For instance, $X = \operatorname{Spec}(\mathbb{C} \times \mathbb{C})$ is a normal scheme. However, if $X$ is normal at a point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is an integral domain, which implies that there is a unique irreducible component of $X$ containing $x$. This component, with its scheme structure as an open subscheme, is integral. Hence any normal scheme is a disjoint union of integral normal schemes. In any case, a normal scheme is always reduced.

**Example 10.31.** $\mathbb{A}^n_{\mathbb{Z}}$ and $\mathbb{P}^n_{\mathbb{Z}}$ are normal schemes, because the local rings are isomorphic to localizations of the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$. △

**Lemma 10.32.** Let $X$ be an integral normal scheme. Then $\mathcal{O}_X(U)$ is normal for every open subscheme $U \subset X$.

*Proof* Let $A = \mathcal{O}_X(U)$. Suppose first that $U$ is affine. Then all of the localizations $A_{\mathfrak{p}}$ are normal, by assumption, and so $A$ is normal as well by Example 10.26.

If $U \subset X$ is a general open subset, take elements $u, v \in A$ and suppose that $u/v \in K$ satisfies a monic relation

$$(u/v)^n + a_{n-1}(u/v)^{n-1} + \cdots + a_0 = 0 \tag{10.13}$$

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where $a_i \in A$. If $V \subset U$ is an affine open, then restricting (10.13) to $V$ shows that $(u|_V)/(v|_V)$ satisfies a monic equation. As $\mathcal{O}_X(V)$ is normal, we have $u|_V/v|_V \in \mathcal{O}_X(V)$, i.e., there is a regular element $s_V \in \mathcal{O}_X(V)$ so that $v|_V s_V = u|_V$. If $W \subset V$ is another affine, then we similarly have an element $s_W$ so that $v|_W s_W = u|_W$, and clearly $s_V|_W = s_W$ as $\mathcal{O}_X(W)$ is an integral domain. Therefore the sections $s_V$ glue to a section $s \in \mathcal{O}_X(U)$, which has the property that $as = b$, i.e., $a/b = s \in \mathcal{O}_X(U)$. □

Although it is not obvious from the definition, the notion of normality is related to nonsingularity. This is because of the algebraic fact that a regular local ring is normal. In fact, by the *Auslander–Buchsbaum theorem* local regular rings are unique factorization domains, and hence they are normal by Example 10.25.

**Proposition 10.33.** Any nonsingular scheme is normal.

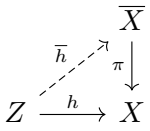
While normal schemes are more general than regular schemes, they still have several nice properties. For instance, if $X$ is normal and of finite type over a field, then:

(i) The singular locus of $X$ has codimension at least 2 in $X$ (Theorem **??**).
(ii) Any finite birational morphism $Y \to X$ is an isomorphism (Proposition 10.54).
(iii) Any rational function defined outside a closed set of codimension at least 2, extends to a regular function on all of $X$ (Theorem 10.42).

### *Normalization*

A useful fact is that any integral scheme has a *normalization*. More precisely, if $X$ is an integral scheme, then its *normalization* is a normal scheme $\overline{X}$ together with a dominant morphism $\pi\colon \overline{X} \to X$, which is universal among dominant morphisms from normal schemes.

**Theorem 10.34 (Normalization).** For an integral scheme $X$, there is a normal scheme $\overline{X}$, and a morphism $\pi\colon \overline{X} \to X$ satisfying the following universal property: For any dominant morphism $h\colon Y \to X$ from a normal scheme $Z$, there is a unique morphism $\overline{h}\colon Z \to \overline{X}$ such that $h = \pi \circ \overline{h}$.




*Proof* If $X = \operatorname{Spec} A$ is affine, define $\overline{X} = \operatorname{Spec} \overline{A}$ where $\overline{A} \supset A$ is the integral closure of $A$ in $K = K(X)$, and $\pi\colon \operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ is the morphism induced by the inclusion. Note that $\overline{A}$ is an integral domain, and that $\operatorname{Spec} \overline{A}$ is normal.

Next we verify the universal property. Let $h : Z \to X$ be a dominant morphism from an integral normal scheme $Z$. This means that the map $h^\sharp\colon A \to \mathcal{O}_Z(Z)$ is injective (by Example 6.17). As the ring $\mathcal{O}_Z(Z)$ is normal, the ring map $A \to \mathcal{O}_Z(Z)$ factors uniquely as $A \to \overline{A} \to \mathcal{O}_Z(Z)$. This means that $h$ factors via $\overline{X}$, as we want.

The proof for a general scheme involves gluing the morphisms $\overline{U_i} \to U_i$ for an affine cover $\{U_i\}$ of $X$. See Example B.32 for details.

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□

Passing from $X$ to its normalization $\overline{X}$ is a quite common operation in algebraic geometry. This is because $\overline{X}$ it has better properties than $X$, and we can hope to study $X$ by performing computations on $\overline{X}$.

**Proposition 10.35.** For a Noetherian integral scheme $X$, the normalization $\overline{X}$ has the following properties:

(i) $\pi\colon \overline{X} \to X$ is surjective.
(ii) $\overline{X}$ and $X$ have the same dimension.
(iii) There is a dense open subset $U \subset X$ so that $\pi$ restricted to $\pi^{-1}(U)$ is an isomorphism.
(iv) If $X$ is of finite type over a field or over $\mathbb{Z}$, then $\pi\colon \overline{X} \to X$ is a finite morphism.

*Proof* All of these properties are 'local on $X$', so by the gluing construction used in the construction of $\overline{X}$, we reduce to $X = \operatorname{Spec} A$ and $\overline{X} = \operatorname{Spec} \overline{A}$ and $\pi$ is induced by the inclusion $A \subset \overline{A}$. The first two statements (i) and (ii) follow from the Going-Up theorem. The statement (iii) holds because by construction, $X$ and $\overline{X}$ have the same fraction field $K$, and $\pi$ maps the generic point $\overline{\eta} = \operatorname{Spec} K$ of $\overline{X}$ maps to the generic point of $X$. Finally, the statement (iv) follows from Theorem A.20, which tell us that with our assumptions, $\overline{A}$ is *finite* as an $A$-module.

□

For a general scheme $X$, the normalization morphism $\pi\colon \overline{X} \to X$ may fail to be a finite. In fact, there are examples of Noetherian integral domians $A$ whose integral closure is not finite over $A$.

**Example 10.36** (The Cuspidal cubic)**.** Consider the cuspidal cubic curve $X = \operatorname{Spec} A$ where $A = k[x,y]/(y^2 - x^3)$. In Example **??**, we showed that there is an isomorphism of $k$-algebras $A \to k[t^2, t^3]$ defined the assignment $x \mapsto t^2$ and $y \mapsto t^3$. It is clear that $k[t^2,t^3]$ is an integral domain with fraction field $K = k(t)$. On the other hand, this ring is visibly not normal, because $t \notin k[t^2,t^3]$, but yet it satisfies the monic equation $T^2 - t^2 = 0$. As $k[t]$ is integrally closed in $K$ (being a UFD), any element in $K$ which is integral over $A$, can be written as a polynomial in $t$. Therefore the integral closure of $A$ is given by $\overline{A} = k[t]$. The corresponding normalization morphism $\operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ is exactly the morphism $\pi\colon \mathbb{A}^1_k \to \operatorname{Spec} A$ from Example **??**. △

**Example 10.37** (The Nodal cubic)**.** Let $X = \operatorname{Spec} A$, where $A = k[x,y]/(y^2 - x^3 - x^2)$, where $k$ now is a field of characteristic not equal to 2 (if the characteristic is 2, we are back in previous cuspidal case). This is the *nodal cubic curve* in $\mathbb{A}^2_k$. Here it is less obvious what the normalization should be, but it helps to think about it geometrically.

If we think of the corresponding affine algebraic set $\{\, (x,y) \mid y^2 = x^3 + x^2 \,\} \subset \mathbb{A}^2(k)$, we see that the origin $(0,0)$ is a special point: any line $l$ defined by an equation $y = tx$, intersects the curve in $(0,0) \in X$ and one more point (namely $x = t^2 - 1$ and $y = t(t^2-1)$). The polynomial map $t \mapsto (t^2 - 1, t^3 - t)$ gives a parameterization of the curve, which is one-to-one when $t \neq 0$.

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This geometric observation can be turned into algebra as follows. Introduce the parameter $t = y/x$ in the function field $K$ of $X$. The equation $y^2 = x^3 + x^2$ then reduces to $t^2 = 1 + x$, after dividing by $x^2$. Moreover, the element $t$ is integral, because it satisfies the monic equation $T^2 - x - 1 = 0$ (which has coefficients in $A$). As $x = t^2 - 1$ and $y = x \cdot y/x = t^3 - t$, we see that

$$A = k[t^2 - 1, t^3 - t] \subset k[t] \subset K = k(t).$$

Moreover, since $k[t]$ is integrally closed in $K$, we see that $\overline{A} = k[t]$. The normalization map $\pi\colon \operatorname{Spec} \overline{A} \to \operatorname{Spec} A$ is an isomorphism outside the origin $(0,0) \in X$. Geometrically, the map $\pi$ identifies two points $(t+1)$ and $(t-1)$ in $\mathbb{A}^1_k$ to the origin in $X$.

△

**Example 10.38.** Let $A = \mathbb{Z}[\sqrt{-7}]$. Then $A$ is not normal, because the element $\alpha = \frac{1}{2} + \frac{\sqrt{-7}}{2}$ belongs to the fraction field $\mathbb{Q}(\sqrt{-7})$, and satisfies the integral equation $x^2 - 2x + 8 = 0$, yet $\alpha \notin A$. In fact, we have $A = \mathbb{Z}[2\alpha]$ and the normalization is given by $\overline{A} = \mathbb{Z}[\alpha]$. This is because $\mathbb{Z}[\alpha] \subset \overline{A}$, and $\mathbb{Z}[\alpha]$ is integrally closed by Exercise 10.6.8.

Note that there are two ring maps $\mathbb{Z}[\alpha] \to \mathbb{Z}/2$; one which sends $\alpha$ to 0 and one that sends $\alpha$ to 1. Their kernels are the two prime ideals $\mathfrak{p}_1 = (2, \alpha)$ and $\mathfrak{p}_2 = (2, \alpha - 1)$. Note that

$$\mathfrak{p}_1 \cap \mathbb{Z}[2\alpha] = (2, 2\alpha) = (2, 2\alpha - 2) = \mathfrak{p}_2 \cap \mathbb{Z}[2\alpha]$$

In other words, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ both map to the same prime ideal, $\mathfrak{q} = (2, 2\alpha) = (2, 1 + \sqrt{-7})$ via the normalization map

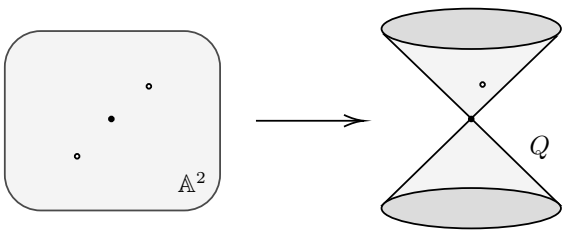
$$\pi\colon \operatorname{Spec} \mathbb{Z}[\alpha] \longrightarrow \operatorname{Spec} \mathbb{Z}[2\alpha].$$

This example shares many similarities with the nodal cubic example. △

**Example 10.39** (The quadratic cone)**.** Consider the affine scheme $X = \operatorname{Spec} A$ where $A = \mathbb{C}[x,y,z]/(xy - z^2)$. Note that $A$ is not a UFD as $xy = z^2$ and one easily checks that $x$, $y$ and $z$ all are irreducible elements, so we cannot immediately conclude that $A$ is normal. However, there is an isomorphism of rings

$$\phi\colon A \to \mathbb{C}[u^2, uv, v^2],$$

and the latter algebra is normal in its fraction field $K = \mathbb{C}(u^2, uv, v^2)$. Indeed, suppose that $T = p/q \in \mathbb{C}(u^2, uv, v^2)$ satisfies a monic equation with coefficients in $\mathbb{C}[u^2, uv, v^2]$, then in particular $T \in \mathbb{C}[u,v]$ is a polynomial in $u$ and $v$ (as $\mathbb{C}[u,v]$ is integrally closed). Therefore, $q$ divides $p$ and so $T \in \mathbb{C}[u^2, uv, v^2]$. (For another proof, see Exercise 10.6.9.)

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**Example 10.40** (Toric varieties)**.** Let $C \subset \mathbb{R}^n$ be a cone generated by finitely many lattice points in $\mathbb{Z}^n$, and let $X = \operatorname{Spec} k[C]$ be the associated toric variety. Then $X$ is normal.

To prove this, we begin by writing $C = H_1 \cap \cdots \cap H_s$ where the $H_i$ are half-spaces of the form

$$H_i = \{\, \mathbf{v} \in \mathbb{R}^n \mid \rho_i \cdot \mathbf{v} \geqslant 0 \,\},$$

where $\rho \in \mathbb{Z}^n$ is a primitive lattice vector, that is, $\rho_i$ is not a positive integer multiple of another vector in $\mathbb{Z}^n$.

Then, inside $K(X) = k(x_1, \ldots, x_n)$, we have

$$k[C] = \bigcap_{i=1}^{s} k[H_i]$$

The intersection of normal subrings is again normal, so it suffices to prove that each $k$-algebra $k[H_i]$ is normal. To see this, since $\rho$ is primitive, it can be extended to a basis of $\mathbb{Z}^n$. After a change of cooordinates, we may therefore assume that $\rho = e_1$. Then $k[H] = k[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, which is clearly normal, being the localization of a polynomial ring.

This implies that all projective toric varieties, as defined in Section B.1 are normal. △

## 10.5 Properties of normal schemes

The following is another nice property of normal schemes. It essentially says that rational functions can be extended over codimension 2 subsets, that is, any rational function which is regular over an open subset $U \subset X$ can be extended uniquely to a regular function on all of $X$, provided that $X - U$ has has no codimension 1 components. This is sometimes referred to as the *Algebraic Hartogs's Theorem*, even though it merely is an algebraic version of a much deeper result from complex function theory due to Friedrich Hartogs.

We will need the following algebraic fact, which says that a normal ring $A$ is equal to the intersection (inside the fraction field) of all its localizations at height 1 prime ideals.

**Proposition 10.41.** Let $A$ be a normal integral domain. Then

$$A = \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_{\mathfrak{p}} \tag{10.14}$$

This has the following geometric consequence, which holds for all Noetherian normal schemes.

**Theorem 10.42 ("Algebraic Hartogs's theorem").** Let $X$ be a Noetherian normal scheme, and let $U \subset X$ be an open subset with $\operatorname{codim}_X(X - U) \geqslant 2$. Then the restriction map

$$\mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U) \tag{10.15}$$

is an isomorphism.

*Proof* As $X$ is a disjoint of integral normal schemes, we may without loss of generality assume that $X$ is integral, so that we may view the rings of regular functions $\mathcal{O}_X(V)$ as

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subrings of the function field $K(X)$ and the restriction map (10.15) is simply the inclusion of subrings $\mathcal{O}_X(X) \subset \mathcal{O}_X(U)$.

For $f \in \mathcal{O}_X(U)$, we need to show that $f$ is in fact regular on all of $X$. In light of Proposition **??**, this is equivalent to saying that $f \in \mathcal{O}_{X,p}$ for every $p \in X$. As the latter condition only involves the local rings, we reduce to proving the proposition for $X$ affine, say $X = \operatorname{Spec} A$, where $A$ is a normal integral domain.

Now, as $X - U$ is assumed to be of codimension at least 2, $U$ contains all points $p$ corresponding to prime ideals $\mathfrak{p}$ of height 1. This means that $\mathcal{O}_X(U) \subset \mathcal{O}_{X,p} = A_{\mathfrak{p}}$ for every such $\mathfrak{p}$. Therefore, by (10.14), we conclude that

$$\mathcal{O}_X(U) \subset \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_{\mathfrak{p}} = A = \mathcal{O}_X(X).$$

□

In particular, if $X$ is integral, and a rational function $f \in K(X)$ is regular outside a closed subset $Z$ of codimension at least 2, then it is regular everywhere.

**Example 10.43.** The assumption that the codimension is at least 2 can not be removed: for the open set $D(t) \subset \mathbb{A}^1_k$ we have $\mathcal{O}_{\mathbb{A}^1_k}(D(t)) = k[t, t^{-1}]$, whereas $\mathcal{O}_{\mathbb{A}^1_k} = k[t]$.

By the way, the above proposition gives another way to see the why $\mathcal{O}_U(U) = k[u, v]$ for the non-affine open set $U = \mathbb{A}^2_k - V(u, v)$ in $\mathbb{A}^2_k$ from Chapter **??**. △

**Example 10.44.** Consider the subring of $k[x, y]$ given by

$$A = k[x^2, xy, y^2, x^3, x^2y, xy^2, y^3]$$

We can see directly that $A$ is not normal, as $x$ the monic equation $T^2 - x^2 = 0$ but $x \notin A$. Similarly, $y$ satisfies $T^3 - y^3 = 0$. In any case, these equations imply that the extension $A \subset k[x, y]$ is an integral extension.

Consider $X = \operatorname{Spec} A$ and let $\mathfrak{m} = (x, y) \cap A = (x^2, xy, \ldots, y^3)$. We claim that the morphism $\pi\colon \mathbb{A}^2_k \to X$ induced by $A \subset k[x, y]$ is an isomorphism outside $\{\mathfrak{m}\}$. To check this, we can work over the distinguished open sets $D(x^2)$ and $D(y^2)$. Localizing at $x^2$, we have $A_{x^2} = k[x, y]_{x^2} = k[x, y]_x$, so $\pi$ is an isomorphism over $D(x^2)$. Similarly, $\pi$ is an isomorphism over $D(y^2)$.

If $\mathfrak{p} \subset A$ is a prime ideal of height 1, we must have $\mathfrak{p} \subset X - \{\mathfrak{m}\}$. This means that $\pi$ gives a bijection between the points of codimension 1 in $X$ and the points of codimension 1 in $\mathbb{A}^2_k$. Therefore,

$$\bigcap_{\operatorname{ht}\mathfrak{p}=1} A_{\mathfrak{p}} = \bigcap_{\operatorname{codim}\overline{\{p\}}=1} \mathcal{O}_{X,p} = \bigcap_{\operatorname{codim}\overline{\{p\}}=1} \mathcal{O}_{\mathbb{A}^2,p} = k[x, y] \supsetneq A.$$

In this example, the singular locus of $X$ is given by $\{\mathfrak{m}\}$, which is of codimension 2. △

**Corollary 10.45.** Let $X$ and $Y$ be affine schemes of finite type over a field $k$, and let $\phi\colon X - Z \to Y$ be a morphism, where $Z$ has codimension at least 2. Then if $X$ is normal, then $\phi$ extends to a morphism $\overline{\phi}\colon X \to Y$.

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*Proof* Let $Y \subset \mathbb{A}^n_k$ be an embedding of $Y$ into affine space. The composition $X - Z \to Y \to \mathbb{A}^n_k$ is defined by $n$ regular functions defined on $X - Z$. By the proposition above, these extend to regular functions defined on all of $X$, and the extended map $X \to \mathbb{A}^n_k$ factors via $Y$. □

**Example 10.46.** The morphism $\pi\colon \mathbb{A}^2_k - \{0\} \to \mathbb{P}^1_k$ does not extend to a morphism $\pi\colon \mathbb{A}^2 \to \mathbb{P}^1_k$ (it does not even extend as a continuous map). △

We saw that a Noetherian ring of Krull dimension 0 is normal if and only if it is a product of fields. The next result gives a characterization of 1-dimensional normal rings. In short, it says that a ring of dimension 1 is normal if and only if its localizations are discrete valuation rings. Discrete valuation rings, in turn, have excellent properties: the structure of the ideals is particularly simple, as the only non-zero ideals are powers of the maximal ideal. This fact will be important later on when we study divisors on curves.

**Proposition 10.47.** Let $A$ be a Noetherian local domain of dimension 1 with maximal ideal $\mathfrak{m}$. The following statements are equivalent:

(i) $A$ is a DVR.
(ii) The maximal ideal $\mathfrak{m}$ is principal.
(iii) Every non-zero ideal is a power of $\mathfrak{m}$.
(iv) $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$.

*Proof* See Proposition A.49. □

**Corollary 10.48.** Let $X$ be a Noetherian integral scheme of dimension 1. Then $X$ is normal if and only if it is nonsingular.

**Proposition 10.49.** Let $X$ be an integral scheme of finite type over an algebraically closed field $k$. If $X$ is normal, then the singular locus $\operatorname{sing}(X)$ has codimension at least 2.

*Proof* It suffices to prove this in the case $X = \operatorname{Spec} A$ is affine, where say $A = k[x_1, \ldots, x_n]/\mathfrak{a}$ for some $\mathfrak{a} = (f_1, \ldots, f_r)$. By definition, the singular locus $\operatorname{sing}(X)$ consists of all the points $x \in X$ for which $\mathcal{O}_{X,x}$ is not a regular local ring. Suppose that $Z$ is a component of $\operatorname{sing}(X)$ of codimension 1 and let $\zeta$ be its generic point. Then as $X$ is normal, the local ring $\mathcal{O}_{X,\zeta}$ is a normal integral domain, of dimension 1, hence regular, by the above proposition. Therefore, $\zeta \notin \operatorname{sing}(X)$. However, with our assumptions, the singular locus $\operatorname{sing}(X)$ is a closed subset in $X$, being defined by the ideal $\mathfrak{b}$ generated by $\mathfrak{a}$ and the $r \times r$-minors of the Jacobian matrix of the $f_i$. Therefore $\operatorname{sing}(X)$ must contain all the generic points of its components, including $\zeta$. This is a contradiction, and so $\operatorname{sing}(X)$ cannot contain any codimension 1 components. □

There is a converse to this result, known as *Serre's Criterion*. It gives a more geometric characterisation of the property of 'normality' (which is fundamentally an algebraic notion).

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**Theorem 10.50 (Serre's Criterion).** Let $X$ be a Noetherian integral scheme. Then $X$ is normal if and only

(i) $X$ is nonsingular at all codimension 1 points, that is, $\mathcal{O}_{X,\zeta}$ is a regular local ring for every point $\zeta$ of codimension 1.

(ii) Whenever $U \subset X$ is an open set whose complement has codimension at least 2, the restriction map (10.15) is an isomorphism.

**Example 10.51** (Curves)**.** A curve $X$ is normal if and only it is nonsingular. △

**Example 10.52** (Hypersurfaces)**.** Let $X$ be a nonsingular affine variety and let $Y \subset X$ be a hypersurface defined by $f \in \mathcal{O}_X(X)$. Then the condition (ii) in Theorem 10.50 is automatically satisfied (this is a non-trivial fact; see Eisenbud (2013)). Hence $Y$ is normal if and only if $\operatorname{sing}(X)$ has codimension at least 2. △

**Example 10.53.** Let $X$ be the scheme obtained from $\mathbb{A}^2_k$ by identifying two points $p, q \in \mathbb{A}^2_k$ (see Example B.42 on page 475). Then $X$ is an integral scheme of dimension 2, and the singular locus consists of a single point $p$. However, consider now the complement $U = X - p$, which consists of two disjoint copies of $\mathbb{A}^2_k - p$. The regular function $f \in \mathcal{O}_X(U)$ which takes the value 0 on one component and 1 on the other clearly does not extend to all of $X$. △

The second important property of normal schemes concerns birational morphisms. Recall that a morphism $f\colon X \to Y$ is birational if it induces an isomorphism between open sets $U \subset X$ and $V \subset Y$. We have already seen several birational morphisms which are not isomorphisms. For instance, when $f$ is the blow-up of $\mathbb{A}^2_k$ at a point, there is a whole $\mathbb{P}^1_k$ which collapses to a point. Even when $f$ is finite, i.e., when it does not collapse a positive-dimensional subscheme to a point, $f$ can still fail to be an isomorphism.

$$f\colon \operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} k[x,y]/(y^2 - x^3)$$

described in Example **??** is both birational and bijective on points, but not an isomorphism in any neighborhood containing the origin. However, our next result shows that this type of phenomenon does not occur if the target is a normal scheme. This previous is not normal, and the failure of being an isomorphism is entirely concentrated at the singular point at the origin.

**Proposition 10.54.** Let $X$ and $Y$ be integral schemes, and let $f\colon X \to Y$ be a finite, birational morphism. If $Y$ is normal, then $f$ is an isomorphism.

*Proof* Since the property of being an isomorphism is local on the target, and finite morphisms are affine, we may reduce to the case where both $X$ and $Y$ are affine, say $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$, and $f$ is induced by an injective ring map $\phi\colon A \to B$, where $B$ is finite as an $A$-module. Now, as $f$ is birational, $\phi$ induces an isomorphism of the function fields $\phi_K : K(A) \to K(B)$. Since $A$ is integrally closed, the finite extension $A \to B$ must be an isomorphism. Therefore, $f$ is an isomorphism as well. □

If $X$ is a curve, being normal is the same thing as being nonsingular, so the normalization $\pi\colon \overline{X} \to X$ produces a nonsingular curve which is birational to $X$.

In light of the relationship between the properties of normality and nonsingularity, one

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can ask whether this holds in general i.e., whether the exists a 'resolution of singularities' $\pi\colon Y \to X$ where $Y$ is nonsingular, and $\pi$ is birational. This is true if the ground field has characteristic zero, by the following famous theorem of Hironaka:

**Theorem 10.55 (Hironaka).** Let $k$ be a field of characteristic zero, and let $X$ be a reduced scheme of finite type over $k$. Then there exists a projective morphism $\pi\colon Y \to X$, where $Y$ is nonsingular and an open dense set $U \subset X$ such that the restruction $\pi^{-1}(U) \to U$ is an isomorphism.

Hironaka's result is in fact more precise, saying that one can construct the morphism $\pi$ through a series of *blow-ups* along nonsingular closed subschemes, which are systematically chosen to eliminate singularities. While the proof is highly technical, this process ultimately yields a nonsingular scheme. The analogous statement in positive characteristic is unsolved, and remains one of the most important open problems in algebraic geometry.

## 10.6 Exercises

**Exercise 10.6.1** (Tangent space to a projective variety)**.** Continuing the previous exercise, let $X \subset \mathbb{P}^n_k$ be a closed subscheme given by homogeneous polynomials $F_1, \dots, F_r$. Show that for a $k$-point $p = (a_0 : \cdots : a_n) \in X$, the tangent space of $X$ at $p$ is given by

$$T_pX = (\operatorname{Ker} J(p))\,/k(a_0, \dots, a_n)$$

where $J$ is the Jacobian matrix of the $F_i$. HINT: Consider the affine cone $C(X) \subset \mathbb{A}^{n+1}$ and the map $C(X) - 0 \to X$.

**Exercise 10.6.2.** Compute the singular points of the scheme $\operatorname{Spec} \mathbb{C}[x, y]/(xy)$.

**Exercise 10.6.3.** Compute the singular points for the *Whitney umbrella* $A = k[x, y, z]/(y^2 + x^3 - x^2z^2)$.

**Exercise 10.6.4.** Let $k$ be a field of characteristic 2 or 3 and let $X \subset \mathbb{A}^2_k$ be a curve defined by the equation $y^2 = x^3 + ax + b$, where $a, b \in k$. Show that $X$ is nonsingular if and only if $4a^3 + 27b^2 \neq 0$. What happens if $k$ has characteristic 2 or 3?

**Exercise 10.6.5.** Let $k$ be a field and let $C$ be the image of the morphism

$$f\colon \mathbb{A}^1_k \longrightarrow \mathbb{A}^3_k; \qquad t \mapsto (t^3 - 1, t^4 - t, t^5 - t^2).$$

Explicitly, $C$ is defined by the ideal $I = (y^2 - xz, x^2y + xy - z^2, x^3 + x^2 - yz)$.

a) Show that the singular locus of $C$ consists of the origin $p = (0, 0, 0)$ and show that $T_pX$ is 3-dimensional.
b) Show that the curve $C$ does not lie on any nonsingular surface in $\mathbb{A}^3_k$.

**Exercise 10.6.6.** Let $A$ be an integral domain with fraction field $K$. Let $x \in K$ be an element. Show that the following are equivalent:

a) $x$ is integral over $A$
b) $A[x]$ is a finite $A$-module
c) There exists a subalgebra $A' \subset A$ such that $x \in A'$ and $R$ is a finite $A'$-module.

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**Exercise 10.6.7.** Let $K/\mathbb{Q}$ be a finite field extension of $\mathbb{Q}$ and let $\alpha \in K$. Show that $\alpha$ is integral over $\mathbb{Z}$ if and only if its minimal polynomial $f(x) \in \mathbb{Q}[x]$ has integer coefficients.

**Exercise 10.6.8.** Let $d$ be a square-free integer and let $K = \mathbb{Q}(\sqrt{d})$. Let $\mathcal{O}_K$ denote the ring of integers in $K$, that is, the integral closure of $\mathbb{Z}$ in $K$.

a) Show that the minimal polynomial of an element $w = a + b\sqrt{d}$, where $a, b \in \mathbb{Q}$ and $b \neq 0$ is given by
$$W^2 - 2aW + (a^2 - b^2 d) = 0$$
b) Show that if $w$ is integral over $\mathbb{Z}$, then $2a, a^2 - b^2 d \in \mathbb{Z}$. HINT: Use Exercise 10.6.7
c) Show that $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{d}}{2}]$ if $d \equiv 1 \mod 4$ and $\mathcal{O}_K = \mathbb{Z}[\sqrt{d}]$ otherwise.

**Exercise 10.6.9.** Prove directly that $A = \mathbb{C}[x, y, z]/(z^2 - xy)$ is normal as follows. Let $B = \mathbb{C}[x, y]$, so that $A = B[z]/(z^2 - xy)$.

a) Show that $A$ is a finite $B$-module of rank 2, with basis $1, z$.
b) Show that $K(B) = \mathbb{C}(x, y)$ and the field extension $K(B) \subset K(B)$ has degree 2.
c) Show that $w = u + vz \in A$ satisfies the monic polynomial
$$T^2 - 2uT + (u^2 - xyv^2) = 0.$$
d) Show that if $w$ is integral over $B$, then $u \in \mathbb{C}[x, y]$; $xyv^2 \in \mathbb{C}[x, y]$; and hence $v \in \mathbb{C}[x, y]$. Conclude that $w \in A$.

**Exercise 10.6.10.** Let $X$ be a Noetherian integral scheme. Show that $X$ is normal if and only if $\mathcal{O}_{X,x}$ is normal for every *closed* point $x \in X$. HINT: If $y \in X$ is any point, there is a closed point in $\overline{\{y\}}$.

**Exercise 10.6.11.** Show that the projective scheme $X = V_+(x_1^2 - 15x_0x_2) \subset \mathbb{P}^2_{\mathbb{Z}}$ is normal.

**Exercise 10.6.12** (The cone over a rational quartic curve)**.** Consider $X = \operatorname{Spec} A$, where $A$ is the $\mathbb{C}$-algebra
$$A = \mathbb{C}[u^4, u^3v, uv^3, v^4] \simeq \mathbb{C}[t_0, t_1, t_3, t_4]/(t_0t_4 - t_1t_3, t_1^3 - t_0^2t_3, t_3^3 - t_1t_4^3).$$

a) Show that $X$ is a variety of dimension 2.
b) Show that $X$ is nonsingular outside the origin $p = V(t_0, t_1, t_3, t_4)$.
c) Show that
$$\frac{t_1^2}{t_0} = u^2v^2 = \frac{t_3^2}{t_4}$$
defines a regular function on $X - p$, but it does not extend to all of $X$. Conclude that $X$ satisfies (i) but not (ii) of Serre's criterion.
d) Show that the ideal $(t_0)$ is not principal in $A$. HINT: A primary decomposition of $(t_0)$ is given by
$$(t_0) = (t_0, t_1^2) \cap (t_0, t_4)$$

**Exercise 10.6.13.** Show that the normalization of the scheme $X = \operatorname{Spec} \mathbb{Z}[6i]$ is given by $\operatorname{Spec} \mathbb{Z}[i]$.

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**Exercise 10.6.14.** Let $A$ be a local principal ideal domain which is not a field. Show that $A$ is a discrete valuation ring.

**Exercise 10.6.15.** Describe the normalization of the scheme $\operatorname{Spec} \mathbb{Z}[\sqrt{-3}]$.

**Exercise 10.6.16** (Tangent space to projective space)**.** Let $k$ be a field and consider the quotient map $\pi\colon \mathbb{A}_k^{n+1} - 0 \to \mathbb{P}_k^n$. Let $P = (a_0, \ldots, a_n) \in \mathbb{A}_k^n - 0$ and $p = (a_0 : \cdots : a_n) \in \mathbb{P}_k^n$ be two $k$-points. Show that $\pi$ induces a map $T_P\mathbb{A}_k^n \to T_p\mathbb{P}_k^n$, and show that the kernel can be naturally identified with $k \cdot (a_0, \ldots, a_n) \subset k^{n+1}$. Conclude that the tangent space of $\mathbb{P}_k^n$ at $p$ is given by $k^{n+1}/(k \cdot (a_0, \ldots, a_n))$.

**Exercise 10.6.17.** Let $X \subset \mathbb{P}_k^3$ be the twisted cubic curve, defined by the ideal $I = (y^2 - xy, xw - yz, z^2 - yw)$ in $k[x, y, z, w]$.

a) Show directly that $X$ is nonsingular.
b) Show that $X$ is connected. HINT: Look in the affine charts.
c) Deduce that $I$ is a prime ideal.

**Exercise 10.6.18** (Nonsingular Toric Varieties)**.**

a) Let $C \subset \mathbb{R}^n$ be a cone generated by finitely many vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in \mathbb{Z}^n$. Assume that the $\mathbf{v}_i$ form a basis for the lattice $\mathbb{Z}^n$. Show that the associated toric variety $X = \operatorname{Spec} k[C]$ is nonsingular.
b) Let $P \subset \mathbb{R}^n$ be a lattice polytope. Suppose that for each vertex $\mathbf{v} \in P$, the cone $C_{\mathbf{v}}$ at $\mathbf{v}$ can be generated by $n$ vectors that form a basis for $\mathbb{Z}^n$. Show that the associated toric variety $X$ is nonsingular.
c) * Prove that the converses of parts (a) and (b) hold: if $X$ is nonsingular, then, up to a change of coordinates, $C$ is generated by the standard basis vectors of $\mathbb{Z}^n$, and for each vertex $\mathbf{v} \in P$, the cone $C_{\mathbf{v}}$ can be generated by $n$ basis vectors for $\mathbb{Z}^n$. HINT: See (Cox et al., 2024, Theorem 1.3.12).

**Exercise 10.6.19** (Involutions)**.** Let $B$ be an integral domain and let $\sigma\colon B \to B$ be a ring map satisfying $\sigma^2 = \mathrm{id}_B$. The map $\sigma$ extends to an involution of the fraction field $K$ of $B$ by the assignment $\sigma(x/y) = \sigma(x)/\sigma(y)$. Define

$$A = B^\sigma = \{x \in B \mid \sigma(x) = x\}$$

be the ring of invariants and let $L = K(A)$ be its field of fractions.

a) Show that $L = K^\sigma = \{z \in K \mid \sigma(z) = z\}$.
b) Show that $B$ is integral over $A$ and if $B$ is normal, $A$ will be normal. HINT: Any $x \in B$ will satisfy a quadratic equation with coefficients in $A$.
c) Consider $B = k[x, y]$ and the involution $\sigma(x, y) = (-x, -y)$. Show that $B^\sigma = k[x^2, xy, y^2]$. Deduce that the ring $A = k[u, v, w]/(v^2 - uw)$ is normal.

**Exercise 10.6.20.** Let $A = \mathbb{Z}[\sqrt{8}]$. Show that the integral closure of $A$ is given by $\overline{A} = \mathbb{Z}[\sqrt{2}]$. Show that the normalization map

$$\pi\colon \operatorname{Spec} \mathbb{Z}[\sqrt{2}] \longrightarrow \operatorname{Spec} \mathbb{Z}[\sqrt{8}]$$

is bijective on points, but it is not an isomorphism. Compare this with the normalization of the cuspidal cubic.

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# Part II

## Quasi-coherent sheaves



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# 11

# More on sheaves

Over the next few sections, we will develop the theory of sheaves in greater detail. We will see that most of the familiar algebraic constructions, such as forming kernels, cokernels and images, direct sums and products, have analogues for sheaves. This means that for a scheme $X$, the category of sheaves on $X$ is a particularly nice category which behaves very much like the category of modules over a ring. In short, it is an abelian category with direct products and direct sums.

## 11.1 Kernels and images

For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, we define its *kernel* as follows:

**Definition 11.1.** The *kernel* $\operatorname{Ker}\phi$ of $\phi$ is the subsheaf of $\mathcal{F}$ defined by

$$(\operatorname{Ker}\phi)(U) = \operatorname{Ker}\phi_U$$

for each open $U \subset X$. In other words, $(\operatorname{Ker}\phi)(U)$ consists of the sections in $\mathcal{F}(U)$ that are mapped to zero by $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$.

The restriction maps of $\mathcal{F}$ induce restriction maps of $\operatorname{Ker}\phi$. Indeed, if $s \in \mathcal{F}(U)$ maps to zero by $\phi_U$, then for any $V \subset U$,

$$\phi_V(s|_V) = \phi_U(s)|_V = 0$$

and hence $s|_V \in (\operatorname{Ker}\phi)(V)$.

The Locality axiom for $\operatorname{Ker}\phi$ follows automatically from the Locality axiom for $\mathcal{F}$. For the Gluing axiom, let $\{U_i\}_{i\in I}$ be an open cover of an open set $U$ and let $s_i \in (\operatorname{Ker}\phi)(U_i)$ be sections that agree on the overlaps. As $\mathcal{F}$ is a sheaf, we may glue together the $s_i$'s to a section $s \in \mathcal{F}(U)$. Now we have $\phi(s)|_{U_i} = \phi(s|_{U_i}) = \phi(s_i) = 0$ for all $i$. By the Locality axiom for $\mathcal{G}$, it follows that $\phi(s) = 0$, and hence $s \in (\operatorname{Ker}\phi)(U)$.

**Lemma 11.2.** For each point $x \in X$, one has $(\operatorname{Ker}\phi)_x = \operatorname{Ker}\phi_x$.

*Proof* To see the inclusion $(\operatorname{Ker}\phi)_x \subset \operatorname{Ker}\phi_x$, let $s_x \in (\operatorname{Ker}\phi)_x$. By definition of the stalk, we may find an open set $V$ and $s \in \operatorname{Ker}\phi_V$ so that $s_x$ is the germ of $s$ at $x$. As $\phi_V(s) = 0$, we also have $\phi_x(s_x) = (\phi_V(s))_x = 0$ in $\mathcal{G}_x$, and hence $s_x \in \operatorname{Ker}\phi_x$.

Conversely, let $s_x \in \operatorname{Ker}\phi_x$ and let $s \in \mathcal{F}(V)$ be a section representing $s_x$. As $\phi_x(s_x) =$



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0, we have $(\phi_V(s))_x = 0$. This means that there exists some open set $W \subset V$ so that $\phi_U(s)|_W = 0$, and hence $s|_W \in (\operatorname{Ker}\phi)(W)$. But this means that $s_x \in (\operatorname{Ker}\phi)_x$. □

Defining the *image* of a map $\phi\colon \mathcal{F} \to \mathcal{G}$ between sheaves is more subtle than defining the kernel. One can define the *image presheaf* by setting

$$\operatorname{Im}(\phi_U) = \left\{ \phi_U(s) \in \mathcal{G}(U) \;\middle|\; s \in \mathcal{F}(U) \right\}. \tag{11.1}$$

The restriction maps $\mathcal{G}$ induce restriction maps of $\operatorname{Im}\phi$, as $\phi$ is compatible with restrictions, but it will not be a sheaf in general. The issue arises from the Gluing axiom: given an open cover $U_i$ of $U$ and sections of the form $t_i = \phi_{U_i}(s_i)$, we can glue the $t_i$ to a section $t \in \mathcal{G}(U)$. However, there is no guarantee that $t$ is of the form $\phi(s)$ for some $s \in \mathcal{F}(U)$. For that, the sections $s_i$ would need to agree on the overlaps $U_i \cap U_j$, and there is no reason for why they should do so. Example 11.4 below illustrates this.

To define the image sheaf, we need to add in all sections that can be obtained by gluing together local sections of the form $\phi_{U_i}(s_i)$ as above. In other words, we include the sections of $\mathcal{G}(U)$ which are 'locally images of $\phi$'. For a later applications, we allow $\mathcal{F}$ to be simply a presheaf.

**Definition 11.3.** For a map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, we define the *image sheaf* $\operatorname{Im}\phi$ by

$$(\operatorname{Im}\phi)(U) = \left\{ t \in \mathcal{G}(U) \;\middle|\; \begin{array}{l} \text{there is a cover } U_i \text{ of } U \text{ and sections} \\ s_i \in \mathcal{F}(U_i) \text{ such that } t|_{U_i} = \phi_{U_i}(s_i) \end{array} \right\}.$$

The restriction maps of $\mathcal{G}$ induce restriction maps of $\operatorname{Im}\phi$. The Locality axiom holds for free because $\mathcal{G}$ is a sheaf. As for the Gluing axiom, suppose we are given an open cover $\{U_i\}_{i\in I}$ of an open set $U$ and sections $t_i \in (\operatorname{Im}\phi)(U_i)$ that agree on the overlaps. Since $\mathcal{G}$ is a sheaf, the $t_i$'s glue together to a section $t \in \mathcal{G}(U)$, and $t$ is by construction locally an image because each $t_i$ is. The sheaf $\operatorname{Im}\phi$ is therefore a subsheaf of $\mathcal{G}$: it is the smallest subsheaf of $\mathcal{G}$ containing the images of $\phi$.

Unlike the situation for kernels, $(\operatorname{Im}\phi)(U)$ is not always equal to $\operatorname{Im}\phi_U$. We always have the inclusion $\operatorname{Im}\phi_U \subset (\operatorname{Im}\phi)(U)$, as any section of the form $\phi(s)$ clearly lies in $\operatorname{Im}\phi$, but there may also be additional elements of $(\operatorname{Im}\phi)(U)$ which are only locally images of $\phi$.

Here is a concrete example where this happens:

**Example 11.4.** Let $Z$ be the closed subscheme given by the '$x$-axis' in $\mathbb{A}^2_k$, that is, $Z = \operatorname{Spec} k[x]$ inside $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$. Let $\iota\colon Z \to \mathbb{A}^2_k$ denote the inclusion, and consider the associated map of sheaves

$$\iota^\sharp\colon \mathcal{O}_{\mathbb{A}^2_k} \longrightarrow \iota_*\mathcal{O}_Z.$$

We will show that the naive image presheaf $\mathcal{G}$ defined by $\mathcal{G}(W) = \operatorname{Im}(\iota^\sharp(W))$ is not a sheaf. To see this, let $U = D(x) = \operatorname{Spec} k[x,y]_x$ and $V = D(y) = \operatorname{Spec} k[x,y]_y$. Then

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$U \cap Z = Z - V(x)$ and $V \cap Z = \varnothing$. Over these open sets, the map $\iota^\sharp$ is given by

$$\begin{aligned}
\iota_U^\sharp : &\quad \mathcal{O}_{\mathbb{A}^2}(U) = k[x,y]_x \longrightarrow \mathcal{O}_Z(\iota^{-1}U) = k[x]_x \\
\iota_V^\sharp : &\quad \mathcal{O}_{\mathbb{A}^2}(V) = k[x,y]_x \longrightarrow \mathcal{O}_Z(\iota^{-1}V) = 0 \\
\iota_{U\cap V}^\sharp : &\quad \mathcal{O}_{\mathbb{A}^2}(U \cap V) = k[x,y]_{xy} \longrightarrow \mathcal{O}_Z(\iota^{-1}U \cap V) = 0. \\
\iota_{U\cup V}^\sharp : &\quad \mathcal{O}_{\mathbb{A}^2}(U \cup V) = k[x,y] \longrightarrow \mathcal{O}_Z(\iota^{-1}(U \cup V)) = k[x]_x.
\end{aligned}$$

Here we have used Example 4.22 for $\mathcal{O}_{\mathbb{A}^2}(U \cup V) = k[x,y]$.

Now note that the elements $x^{-1} \in \mathcal{G}(U)$ and $0 \in \mathcal{G}(V)$ both restrict to 0 in $\mathcal{G}(U \cap V) = 0$. However, they do not glue together to a section of $\mathcal{G}$ over $U \cup V$, because there is no element of $k[x,y]$ that maps to $x^{-1}$ in $k[x]_x$. In fact, the image sheaf is equal to all of $\iota_*\mathcal{O}_Z$ (see also Example 11.11). △

**Example 11.5.** There is one situation when the formula $(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U$ in fact holds for every $U$, namely when each $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective.

To see this, let $t \in (\operatorname{Im}\phi)(U)$ be a section and let $s_i \in \mathcal{F}(U_i)$ be elements so that $t|_{U_i} = \phi(s_i)$ for every $i$. Then $s_i|_{U_i\cap U_j}$ and $s_j|_{U_i\cap U_j}$ map to the same element in $\mathcal{G}(U_i \cap U_j)$ for every $i$ and $j$, so by injectivity, this forces $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Therefore, the $s_i$ glue to an element $s \in \mathcal{F}(U)$. We must have $\phi(s) = t$, because $\phi(s)|_{U_i} = \phi(s|_{U_i}) = \phi(s_i) = t|_{U_i}$ for all $i$. Hence $t = \phi(s)$ and so $t \in (\operatorname{Im}\phi)(U)$. △

In general, the image sheaf behaves well when it comes to stalks:

**Lemma 11.6.** For each $x \in X$, we have $(\operatorname{Im}\phi)_x = \operatorname{Im}\phi_x$.

*Proof* Let $t_x \in \operatorname{Im}\phi_x$ and pick an $s_x \in \mathcal{F}_x$ with $\phi_x(s_x) = t_x$. Choose sections $s \in \mathcal{F}(V)$ and $t \in \mathcal{G}(V)$ representing $s_x$ and $t_x$ respectively, for some neighbourhood $V$ of $x$. As $\phi_x(s_x) = t_x$, we have $\phi_V(s) = t$ after possibly shrinking $V$. This implies that $t \in (\operatorname{Im}\phi)(V)$, and hence $t_x \in (\operatorname{Im}\phi)_x$.

Conversely, let $t_x \in (\operatorname{Im}\phi)_x$. By definition, there exists an open set $U$ and a section $t \in (\operatorname{Im}\phi)(U)$ so that $t_x$ is the germ of $t$ at $x$. As $t \in (\operatorname{Im}\phi)(U)$, there exists an open cover $\{U_i\}_{i\in I}$ of $U$ and sections $s_i \in \mathcal{F}(U_i)$ such that $t|_{U_i} = \phi_{U_i}(s_i)$ for each $i$. Choosing an index $i$ such that $x \in U_i$, we have

$$t_x = (t|_{U_i})_x = (\phi_{U_i}(s_i))_x = \phi_x((s_i)_x)$$

This means that $t_x \in \phi_x$. □

## 11.2 Injective and surjective maps of sheaves

We say that a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ is *injective* if $\operatorname{Ker}\phi = 0$, and that $\phi$ is *surjective* if $\operatorname{Im}\phi = \mathcal{G}$.

The next two lemmas say that being injective or surjective are 'local in nature'; they can be checked on stalks.

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**Lemma 11.7.** For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, the following are equivalent:
(i) $\phi$ is injective.
(ii) $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all $U \subset X$.
(iii) $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$.

This is a consequence of the formula $(\operatorname{Ker}\phi)(U) = \operatorname{Ker}\phi_U$, and the fact that a sheaf is zero if and only if its stalks are all zero.

When it comes to surjectivity, the formula $(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U$ does not hold in general, and in fact the item (ii) above does not hold with 'injective' replaced by 'surjective'. Indeed, surjectivity of $\phi$ means that for every open set $U \subset X$, and $t \in \mathcal{G}(U)$, there exists an open cover $\{U_i\}$ of $U$ and elements $s_i \in \mathcal{F}(U)$ so that $\phi(s_i) = t|_{U_i}$. The best we can say is:

**Lemma 11.8.** For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, the following are equivalent:
(i) $\phi$ is surjective.
(ii) $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$.

This follows from the following small lemma, which will be useful later:

**Lemma 11.9.** Two subsheaves $\mathcal{H}, \mathcal{G}$ of a sheaf $\mathcal{F}$ are equal if and only if $\mathcal{H}_x = \mathcal{G}_x$ (as subgroups of $\mathcal{F}_x$) for all $x \in X$.

*Proof* The 'only if'-direction is trivial. For the 'if' direction, let $U \subset X$ be an open set and let $s \in \mathcal{G}(U)$ be a section. By assumption, for each $x \in X$, $s_x$ lies in $\mathcal{G}_x = \mathcal{H}_x$. This means that for each $x \in U$, there exists a neighbourhood $U_x$ of $x$ and a section $t^x \in \mathcal{H}(U_x)$ such that $t^x$ represents the germ $s_x$. As the $\{U_x\}_{x\in U}$ form a cover of $U$, and the sections $t^x$ and $t^y$ agree over $U_x \cap U_y$ (they are both germs of $s$), they glue together to a section $t \in \mathcal{H}(U)$. By construction $s$ and $t$ have the same germs at every point $x \in U$, so by the Locality axiom, $s = t$. This shows that $\mathcal{G}(U) \subset \mathcal{H}(U)$ for every open set $U$, and so $\mathcal{G}$ is a subsheaf of $\mathcal{H}$. By symmetry, we also have $\mathcal{H} \subset \mathcal{G}$ and hence $\mathcal{H} = \mathcal{G}$. □

**Example 11.10.** If $X = \operatorname{Spec} A$ is an affine scheme, and $\phi\colon \widetilde{M} \to \widetilde{N}$ is a map of sheaves of 'tilde-type', then the following are equivalent:
(i) $\phi\colon \widetilde{M} \to \widetilde{N}$ is injective (resp. surjective)
(ii) $\phi_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective (resp. surjective) for every $\mathfrak{p} \in \operatorname{Spec} A$.
(iii) $\phi_X\colon M \to N$ is injective (resp. surjective).
This follows because the stalk maps of $\phi$ are exactly the localized maps in item (ii). △

**Example 11.11.** The map $\iota^\sharp\colon \mathcal{O}_{\mathbb{A}^2_k} \to \iota_*\mathcal{O}_Z$ of Example 11.4 is not surjective when evaluated over $U = \mathbb{A}^2_k$. The map $\iota^\sharp$ is however surjective as a map of sheaves. To see this, note that $\mathbb{A}^2_k$ is covered by the two opens $U = D(x)$ and $U' = D(x-1)$. We already showed that $\iota^\sharp_U$ is surjective, as this is given by the quotient map $k[x,y]_x \to k[x]_x$. By Example 11.10, $\iota^\sharp_p$ is surjective for all $p \in U$. A similar argument applies to $U'$. Therefore $\iota^\sharp$ is surjective on every stalk, and so it is surjective. △

For a map $\phi\colon \mathcal{F} \to \mathcal{G}$ to be an *isomorphism*, i.e., it admits an inverse map $\psi\colon \mathcal{G} \to \mathcal{F}$, we have the following clean statement.

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**Proposition 11.12.** Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves. Then the following four conditions are equivalent.

(i) The map $\phi$ is an isomorphism.
(ii) For every $x \in X$, the map on stalks $\phi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism.
(iii) One has $\operatorname{Ker}\phi = 0$ and $\operatorname{Im}\phi = \mathcal{G}$.
(iv) For all open subsets $U \subset X$ the map on sections $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.

*Proof* (i) $\Rightarrow$ (ii). This implication is clear.

(ii) $\Rightarrow$ (iii). $\operatorname{Ker}\phi = 0$ follows by Lemma 11.2 and $\operatorname{Im}\phi = \mathcal{G}$ follows by Lemma 11.6 and Lemma 11.9.

(iii) $\Rightarrow$ (iv). As $\operatorname{Ker}\phi = 0$, $\phi$ is injective. In that case, taking images commutes with taking sections (Example 11.5), and so we have $\operatorname{Im}\phi_U = (\operatorname{Im}\phi)(U)$. But by assumption, $\operatorname{Im}\phi = \mathcal{G}$, so $\phi_U$ is also surjective.

(iv) $\Rightarrow$ (iii). If $\phi_U$ is an isomorphism for every $U$, the inverse maps $\psi_U = \phi_U^{-1}$ define an inverse morphism $\psi\colon \mathcal{G} \to \mathcal{F}$. □

## 11.3 Exact sequences

A sequence of maps of sheaves

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \tag{11.2}$$

is said to be *exact* if $\operatorname{Im}\phi = \operatorname{Ker}\psi$ as subsheaves of $\mathcal{G}$.

A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0 \tag{11.3}$$

where we have exactness at all stages. This is just a convenient way of simultaneously saying that $\phi$ is injective, that $\psi$ is surjective and that $\operatorname{Im}\phi = \operatorname{Ker}\psi$.

Exactness for a sequence of sheaves is a purely local condition; the sequence (11.2) is exact if and only if for each $x \in X$ the sequence induced on stalks

$$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \tag{11.4}$$

is exact. This follows from Lemma 11.9 applied to $\operatorname{Ker}\phi$ and $\operatorname{Im}\phi$.

The following proposition will be very important:

**Proposition 11.13 (Taking sections is left exact).** Given a short exact sequence as in (11.3), then for each open subset $U \subset X$, the sequence

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$$

is exact.

*Proof* As $\phi$ is injective, we have that $\phi_U$ is injective, and also that $(\operatorname{Im}\phi)(U) = \operatorname{Im}\phi_U$ by

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Example 11.5. By definition, we have $\operatorname{Ker}\phi_U = (\operatorname{Ker}\phi)(U)$. Combining these, we obtain

$$\operatorname{Im}\phi_U = (\operatorname{Im}\phi)(U) = (\operatorname{Ker}\psi)(U) = \operatorname{Ker}\psi_U,$$

and hence the above sequence is exact. □

One way of phrasing Proposition 11.13 is to say that taking sections over an open set $U$ is a *left exact functor*. This functor, however, is not right exact in general. This failure of exactness is a fundamental problem in algebraic geometry. We will explore this in greater detail in Chapter 15 where we discuss *cohomology*.

**Example 11.14.** Consider the two points $P = (0 : 1)$ and $Q = (1 : 0)$ in $\mathbb{P}^1_k$ and let $\iota\colon Z \to \mathbb{P}^1_k$ be the closed embedding given by their union. Let $\mathcal{I}$ be the kernel of the map $\iota^\sharp : \mathcal{O}_{\mathbb{P}^1_k} \to \iota_*\mathcal{O}_Z$. The sheaf $\mathcal{I}$ fits into the following sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \xrightarrow{\iota^\sharp} \iota_*\mathcal{O}_Z \longrightarrow 0. \tag{11.5}$$

We claim that this sequence is exact, i.e., that $\iota^\sharp$ is surjective. For this, it suffices to check that the map is surjective locally. If $U_0 = \mathbb{P}^1_k - P \simeq \operatorname{Spec} k[s]$, then $(\iota_*\mathcal{O}_Z)(U) = k[s]/s$ and the map $\iota^\sharp(U)$ is given by the quotient map $k[s] \to k[s]/s$, which is surjective. A similar argument shows that $\iota^\sharp$ is surjective over $U_1 = \mathbb{P}^1_k - Q$. Hence the sequence (11.5) is exact.

Now, consider the global sections of this sequence. We have $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_k}) = k$ by Proposition **??** and $\iota_*\mathcal{O}_Z(\mathbb{P}^1_k) = \mathcal{O}_Z(Z) = k \oplus k$ and the sequence becomes

$$0 \longrightarrow \Gamma(\mathbb{P}^1_k, \mathcal{I}) \longrightarrow k \longrightarrow k \oplus k.$$

Here the right-most map cannot be surjective, so the sequence is not exact. △

## 11.4 The sheaf associated to a presheaf

Essentially any construction for abelian groups, such as forming kernels, cokernels, tensor products, direct sums etc. has an analogue for sheaves. For these constructions, one typically starts by writing down a naive presheaf and then proceeds to show that it satisfies the two sheaf axioms. This works well in some cases (e.g., for the kernel sheaf), but in general, it can fail to be a sheaf (as for the image presheaf). To obtain an actual sheaf, we sometimes need to replace the naive presheaf with a sheaf which in some sense best approximates it; as one says, we *sheafify* it.

What prevents a presheaf $\mathcal{F}$ in being a sheaf is of course the failure of one of the sheaf axioms:

(i) There are non-zero sections which are locally zero, i.e., become zero when restricted to the open sets of a covering. These are sections $s \in \mathcal{F}(U)$ so that $s_x = 0$ for every $x \in U$.
(ii) There are collections of local sections which are compatible over the overlaps, but do not glue to a global section of $\mathcal{F}$.

In light of this, it is clear what the sheafifcation should do: mod out by the sections in the item (i) and add in all the missing sections in item (ii).

The sheafification should also satisfy some sort of uniqueness property in the form of a

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universal property. More precisely, to any presheaf $\mathcal{F}$, we will construct a sheaf $\mathcal{F}^+$ and a map of presheaves

$$\kappa_{\mathcal{F}} \colon \mathcal{F} \longrightarrow \mathcal{F}^+$$

which is universal among maps from $\mathcal{F}$ into a sheaf.

The main properties of $\mathcal{F}^+$ and $\kappa_{\mathcal{F}}$ are summarised in the following proposition.

**Proposition 11.15.** Given a presheaf of abelian groups $\mathcal{F}$ on $X$, there is a sheaf $\mathcal{F}^+$ and a map of presheaves $\kappa_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{F}^+$ satisfying the following properties:

(i) Any map of presheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ induces a map of sheaves $\phi^+ \colon \mathcal{F}^+ \to \mathcal{G}^+$ such that following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ \downarrow{\scriptstyle \kappa_{\mathcal{F}}} & & \downarrow{\scriptstyle \kappa_{\mathcal{G}}} \\ \mathcal{F}^+ & \xrightarrow{\phi^+} & \mathcal{G}^+ \end{array} \tag{11.6}$$

(ii) If $\mathcal{F}$ is a sheaf, then $\kappa_{\mathcal{F}}$ is an isomorphism.

(iii) $\kappa_{\mathcal{F}}$ and $\mathcal{F}^+$ satisfy the universal property that any map of presheaves $\mathcal{F} \to \mathcal{G}$ where $\mathcal{G}$ is a sheaf, factors through $\mathcal{F}^+$ in a unique way. In other words, if $\mathcal{G}$ is a sheaf, there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{PAb}(X)}(\mathcal{F}, \mathcal{G}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{Ab}(X)}(\mathcal{F}^+, \mathcal{G}), \tag{11.7}$$

where on the left-hand side $\mathcal{G}$ is considered as a presheaf. This property characterizes $\mathcal{F}^+$ up to a unique isomorphism.

(iv) $\kappa_{\mathcal{F}}$ induces an isomorphism on stalks: $\mathcal{F}_x \simeq \mathcal{F}^+_x$ for every $x \in X$.

As in the construction of a sheaf from a sheaf on a basis, the details of the explicit construction will not be terribly important. All arguments involving $\mathcal{F}^+$ in this book rely mainly on the four properties in the Proposition 11.15. This is a good illustration of the principle: *ask not what the thing is, but what it does*.

Explicitly, the sections of $\mathcal{F}^+$ are sequences $(s_x)_{x\in U}$ of germs of $\mathcal{F}$, i.e., elements of the product $\prod_{x\in U} \mathcal{F}_x$, but only those sequences that arise from local sections of $\mathcal{F}$ are allowed. More precisely, for an open set $U \subset X$, and we say that an element [1] $(s_x) \in \prod_{x\in U} \mathcal{F}_x$ is *locally induced by sections of* $\mathcal{F}$ if there is an open cover $\{U_i\}$ of $U$ and sections $t_i \in \mathcal{F}(U_i)$ so that $(t_i)_x = s_x$ for $x \in U_i$. We then define $\mathcal{F}^+$ by setting

$$\mathcal{F}^+(U) = \Big\{ (s_x)_{x\in U} \;\Big|\; (s_x) \text{ is locally induced by sections of } \mathcal{F} \Big\} \subset \prod_{x\in U} \mathcal{F}_x$$

The projection maps $\prod_{x\in U} \mathcal{F}_x \to \prod_{x\in V} \mathcal{F}_x$ give restriction maps $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$ for each $U \supset V$, making $\mathcal{F}^+$ into a presheaf.

**Lemma 11.16.** $\mathcal{F}^+$ is a sheaf.

[1] The notation is not ideal: $s_x$ is a germ at $x$, but at the same time, $x$ serves as an index.

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*Proof* Locality holds: if $\{U_i\}$ is an open cover of $U$, and $t = (s_x)_{x\in U}$ is a section of $\mathcal{F}^+(U)$ such that $t|_{U_i} = 0$ for each $i$, then $s_x = 0$ for every $x \in U_i$. Hence, if $t|_{U_i} = 0$ for all $i$, it follows that $t = 0$.

Gluing holds: Suppose we are given an open cover $\{U_i\}$ of $U$ and sections $t_i = (s_x^i)_{x\in U_i}$ of $\mathcal{F}^+$ over $U_i$ matching on the intersections $U_i \cap U_j$. Saying that the sections agree over the overlaps, means that for any $x \in U_i \cap U_j$, the components $s_x^i$ and $s_x^j$ are equal in $\mathcal{F}_x$. Therefore, we get a well-defined section $t \in \mathcal{F}^+(U)$ by using this common component as the component of $t$ at $x$. It is clear that $t|_{U_i} = t_i$. Moreover, $t$ is locally induced by sections of $\mathcal{F}$ because the $t_i$ are. □

There is a canonical map of presheaves

$$\kappa_{\mathcal{F}} \colon \mathcal{F} \longrightarrow \mathcal{F}^+$$

that sends a section $s \in \mathcal{F}(U)$ to the sequence of all its germs; that is, to the element $(s_x)_{x\in U}$. The kernel of this map consists exactly of the sections with all germs equal to zero, that is, the sections of $\mathcal{F}$ which are 'locally zero'.

For a map of presheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, we may define a map of sheaves $\phi^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ over an open set $U$ using the product of all the stalk maps $\mathcal{F}_x \to \mathcal{G}_x$ with $x \in U$. In other words, $\phi^+_U$ sends $(s_x)_{x\in U}$ to $(\phi_x(s_x))_{x\in U}$. With this definition, the diagram (11.6) is commutative. It is not hard to check that $\mathrm{id}^+_{\mathcal{F}} = \mathrm{id}_{\mathcal{F}^+}$ and that $(\psi \circ \phi)^+ = \psi^+ \circ \phi^+$ for two composable morphisms between presheaves on $X$, so that $\mathcal{F} \mapsto \mathcal{F}^+$ is a functor from the category of presheaves on $X$ to the category of sheaves on $X$.

*Proof of Proposition 11.15* The statement (i) was proved in the paragraph above.

As for (ii), let us assume that $\mathcal{F}$ is a sheaf. Then the Locality axiom implies that $\kappa_{\mathcal{F}}$ is injective. For surjectivity, let $t \in \mathcal{F}^+(U)$ be an element and let $s_i \in \mathcal{F}(U_i)$ be a collection of sections that locally induces $t$. Then since $s_i$ and $s_j$ have the same germs for every point in $U_i \cap U_j$, the restrictions $s_i|_{U_i\cap U_j}$ and $s_j|_{U_i\cap U_j}$ are equal. Hence the $s_i$ glue to an element $s \in \mathcal{F}(U)$, (because $\mathcal{F}$ is a sheaf). By construction, $s$ has the same germs as the $s_i$ over $U_i$, so we have $\kappa_{\mathcal{F}}(s) = t$.

For (iii), if $\mathcal{G}$ is a sheaf, then the map (11.7) sends $\phi\colon \mathcal{F} \to \mathcal{G}$ to the map $\kappa_{\mathcal{G}}^{-1} \circ \phi^+$, which is the composition $\mathcal{F}^+ \to \mathcal{G}^+ \to \mathcal{G}$. This defines a bijection, as the inverse sends a map $\psi : \mathcal{F}^+ \to \mathcal{G}$ to the map of presheaves $\mathcal{F} \to \mathcal{G}$ given by the composition $\psi \circ \kappa_{\mathcal{F}}$.

Finally, let us prove claim (iv), starting with injectivity. Let $t_x \in \mathcal{F}_x$, and represent it as the germ of a section $t \in \mathcal{F}(V)$ defined in some neighbourhood of $x$. Then over $V$, the map $\kappa_{\mathcal{F}}$ sends $t$ to the collection of germs $(t_y)_{y\in V}$. If $t$ maps to zero in $\mathcal{F}^+(V)$, then in particular, $t_x = 0$.

For surjectivity, take an element in $(\mathcal{F}^+)_x$ and represent it by a section $t \in \mathcal{F}^+(V)$ over some neighbourhood $V$ of $x$. Taking a smaller $V$ if necessary, we may assume that $t$ is induced from a section of $\mathcal{F}$, that is, there is a section $s \in \mathcal{F}(V)$ such that for each $y \in V$, the $y$-th component of $t$ is equal to the germ $s_y$. But then over $V$, $\kappa_{\mathcal{F}}$ sends $s$ to $t$, and so the germ $s_x \in \mathcal{F}_x$ maps to $t_x \in (\mathcal{F}^+)_x$.

□

**Example 11.17.** A presheaf $\mathcal{F}$ which is contained in a sheaf $\mathcal{G}$ is particularly easy to sheafify. The sheafification $\mathcal{F}^+$ equals the image sheaf of the inclusion map $\mathcal{F} \to \mathcal{G}$. In other words,

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the sections in $\mathcal{F}^+(U)$ are the sections in $\mathcal{G}(U)$ that locally lie in $\mathcal{F}$; that is, sections $s$ so that $s|_{U_i} \in \mathcal{F}(U_i)$ for some open cover $\{U_i\}$ of $U$. △

**Example 11.18** (The structure sheaf)**.** Let $X = \operatorname{Spec} A$ be the spectrum of a ring $A$. For an open set $U \subset X$, define $S(U) = \{\, s \in A \mid D(s) \supset U \,\}$ as the set of elements do not vanish anywhere on $U$. As in Section 3.4, we consider the assignment $U \mapsto S(U)^{-1}A$, which defines a presheaf on $X$. This presheaf is not always a sheaf (see Exercise 3.6.20). The associated sheaf is exactly the structure sheaf $\mathcal{O}_X$.

To see this, note that the natural maps $S(U)^{-1}A \to \mathcal{O}'_X(U)$, given by the universal property of sheafification, define a morphism of $\mathscr{B}$-sheaves (where as usual, $\mathscr{B}$ is the basis of distinguished opens), and hence a morphism of sheaves $\mathcal{O}_X \to \mathcal{O}'_X$. This is an isomorphism, because it induces an isomorphism on stalks. △

**Example 11.19.** Let $X = \mathbb{A}^1_{\mathbb{C}}$ and let $p, q$ be two distinct points. Consider the presheaf $\mathcal{F}$ defined by

$$\mathcal{F}(U) = \{\, f \in \mathcal{O}_X(U) \mid f(p) = f(q) \,\}.$$

Then $\mathcal{F}$ is not a sheaf. The reason is that $\mathcal{F}|_U = \mathcal{O}_U$ for any open set $U$ which contains $p$ but not $q$ (as the condition $f(p) = f(q)$ becomes redundant there). The same holds for open sets containing $q$ but not $p$. Therefore, if we take any $f \in \mathcal{O}_X(X)$ such that $f(p) \neq f(q)$, we can restrict it to the open sets $U = \mathbb{A}^1_{\mathbb{C}} - p$ and $V = \mathbb{A}^1_{\mathbb{C}} - q$, which form an open cover of $X$. The restrictions $f|_U$ and $f|_V$ give two sections of $\mathcal{F}(U) = \mathcal{O}_X(U)$ and $\mathcal{F}(V) = \mathcal{O}_X(V)$, which are equal to $f|_{U\cap V}$, over $U \cap V$, but the element they glue to, namely $f$, is not an element of $\mathcal{F}(X)$.

Note that $\mathcal{F}$ is a subpresheaf of $\mathcal{O}_X$. The above argument shows that every section of $\mathcal{O}_X$ locally lies in $\mathcal{F}$, so that when we sheafify, we get $\mathcal{F}^+ = \mathcal{O}_X$. △

## 11.5 Cokernels and quotients

The main reason to introduce sheafification is to be able to define cokernels and quotient sheaves. For a map of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$, we define the *cokernel* $\operatorname{Coker}\phi$ to be the sheaf associated to the presheaf

$$(\operatorname{Coker}'\phi)(U) = \mathcal{G}(U)/\operatorname{Im}\phi(U).$$

For a subsheaf $\mathcal{G} \subset \mathcal{F}$ of a sheaf $\mathcal{G}$, the *quotient sheaf* $\mathcal{F}/\mathcal{G}$ is the sheaf associated to the presheaf

$$(\mathcal{F}/\mathcal{G})'(U) = \mathcal{F}(U)/\mathcal{G}(U).$$

In other words, $\mathcal{F}/\mathcal{G}$ is the cokernel of the inclusion map $\mathcal{G} \to \mathcal{F}$.

Note that over an open set $U$, the cokernel presheaf is simply given by $\operatorname{Coker}\phi_U$. Composing $\phi$ with the canonical map $\operatorname{Coker}'\phi \to \operatorname{Coker}\phi$ we obtain a map $\mathcal{G} \to \operatorname{Coker}\phi$. It sits in the sequence

$$\mathcal{F} \xrightarrow{\ \phi\ } \mathcal{G} \longrightarrow \operatorname{Coker}\phi \longrightarrow 0. \tag{11.8}$$

**Example 11.20.** In the sequence (11.5) the subsheaf $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^1_k}$ identifies with the sections of $\mathcal{O}_{\mathbb{P}^1_l}$ vanishing along the subscheme $Z$. By the uniqueness of the cokernel, we get an

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isomorphism of sheaves $\mathcal{O}/\mathcal{I} \simeq \iota_*\mathcal{O}_Z$. Even in this example it is necessary to sheafify, as the 'naive' quotient sheaf on global sections satisfies $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1)/\mathcal{I}(\mathbb{P}^1) = k$, whereas $(\iota_*\mathcal{O}_Z)(\mathbb{P}^1) = \mathcal{O}_Z(Z) = k \oplus k$. △

## 11.6 The inverse image sheaf

For a morphism $f\colon X \to Y$ and a sheaf $\mathcal{G}$ on $Y$, we can define the *inverse image sheaf* $f^{-1}\mathcal{G}$, which is a sheaf on $X$. This sheaf has a universal property which is closely related to the pushforward functor $f_*$: for any sheaf $\mathcal{F}$ on $X$, the sheaf morphisms $f^{-1}\mathcal{G} \to \mathcal{F}$ are in one-to-one correspondence with sheaf morphisms $\mathcal{G} \to f_*\mathcal{F}$. The inverse image sheaf is trickier to define than the pushforward sheaf, but it has better properties. For instance, applying $f^{-1}$ preserves both stalks and exact sequences.

To define the sheaf $f^{-1}\mathcal{G}$ over an open set $U \subset X$, it may be tempting to consider something like $\mathcal{G}(f(U))$. This does not work in general, because the sheaf $\mathcal{G}$ only knows about the open sets in $Y$, and typically subsets of the form $f(U)$ will not be open in $Y$. The solution is to adapt the definition of the stalk, and consider "germ-like" equivalence classes of sections $\mathcal{G}(V)$ as $V$ runs over the collection of open sets containing $f(U)$. Precisely, we define the inverse image presheaf $f_p^{-1}\mathcal{G}$ by setting, for an open set $U \subset X$,

$$(f_p^{-1}\mathcal{G})(U) = \coprod_{V \supset f(U)} \mathcal{G}(V)/\sim . \tag{11.9}$$

The elements in the disjoint union are indexed as pairs $(s, V)$, where $V \subset Y$ is an open set in $Y$ containing $f(U)$ and $s \in \mathcal{G}(V)$. Two such sections $(s, V)$ and $(s', V')$ are defined to be equivalent if they agree over some smaller open set $W$ with $f(U) \subset W \subset V \cap V'$.

The restriction maps are defined as follows: if $U' \subset U$, then any open set $V$ containing $f(U)$ also contains $f(U')$. This implies that a pair $(s, V)$ representing an element in $(f_p^{-1}\mathcal{G})(U)$ naturally determines an element in $(f_p^{-1}\mathcal{G})(U')$.

A different way of expressing (11.9) is as a direct limit

$$f_p^{-1}(\mathcal{G})(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V), \tag{11.10}$$

where $V$ runs over open sets in $Y$ containing $f(U)$. The restriction maps then arise directly from the universal property of the direct limit, because if $U' \subset U$, the set of opens containing $f(U)$ is contained in the set of open sets containing $f(U')$.

The definition of $f_p^{-1}\mathcal{G}$ has many similarities with that of the stalk of a sheaf. If $V \subset Y$ is a subset which contains $f(U)$ and $s \in \mathcal{G}(V)$, then the equivalence class of $s$ defines an element in $f_p^{-1}(\mathcal{G})(U)$. Conversely, any section of $f_p^{-1}\mathcal{G}$ over $U$ arises in this way from a section of $\mathcal{G}(V)$ over some $V$ containing $f(U)$.

**Example 11.21.** Let $\iota\colon Y \to X$ be the inclusion of a closed subscheme in a scheme $X$. Then $\iota_p^{-1}\mathcal{O}_X$ is a presheaf of rings on $Y$. A section of $\iota_p^{-1}\mathcal{O}_X$ over $V \subset Y$, can be thought of as a regular function on $V$ obtained by restriction from some open neighbourhood in $X$ containing $V$. △

Unfortunately, the presheaf (11.9) is not a sheaf in general, so we take the associated sheaf:

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**Definition 11.22.** Let $f\colon X \to Y$ be a continuous map and $\mathcal{G}$ a presheaf on $Y$. The *inverse image* $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf (11.9).

Unraveling the definition, the sections of $f^{-1}\mathcal{G}$ can be explicitly described as follows. For an open set $U \subset X$, the group $(f^{-1}\mathcal{G})(U)$ is the set of sequences $(s_x)_{x\in U} \in \prod_{x\in U} \mathcal{G}_{f(x)}$ that satisfy the following property: for every $x \in U$, there is an open set $V$ containing $f(x)$, a section $t \in \mathcal{G}(V)$, and an open set $U' \subset U$ so that $f(W) \subset V$ and $s_x = t_{f(x)}$ for every $x \in U'$.

Given a morphism of sheaves on $Y$, $\phi\colon \mathcal{G} \to \mathcal{H}$, we obtain a collection of maps $\phi_V\colon \mathcal{G}(V) \to \mathcal{H}(V)$ for open sets $V$ containing $f(U)$. These maps are compatible with restrictions, so they pass to the direct limit and induce a map of presheaves $f_p^{-1}\mathcal{G} \to f_p^{-1}\mathcal{H}$ and consequently to a map of sheaves $f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$ . Therefore the inverse image defines a functor $f^{-1}\colon \mathsf{Ab}(Y) \to \mathsf{Ab}(X)$.

While the pushforward $f_*\mathcal{F}$ is easier to define and the sections are more intuitively understood, the inverse image $f^{-1}\mathcal{G}$ has better formal properties. For instance, the stalks are easy to compute:

**Proposition 11.23.** The stalk of $f^{-1}\mathcal{G}$ at a point $x \in X$ is isomorphic to $\mathcal{G}_{f(x)}$.

*Proof* As sheafification preserves stalks, it suffices to verify this on the level of presheaves:

$$(f_p^{-1}\mathcal{G})_x = \varinjlim_{U\ni x} f_p^{-1}\mathcal{G}(U) = \varinjlim_{U\ni x}\ \varinjlim_{V\supset f(U)} \mathcal{G}(V) = \varinjlim_{V\ni f(x)} \mathcal{G}(V) = \mathcal{G}_{f(x)}. \qquad \square$$

Note that there is no such description of the stalk of a pushforward. Also, unlike the pushfoward, the inverse image is an exact functor:

**Corollary 11.24.** If $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence on $Y$, then

$$0 \longrightarrow f^{-1}\mathcal{G}' \longrightarrow f^{-1}\mathcal{G} \longrightarrow f^{-1}\mathcal{G}'' \longrightarrow 0$$

is an exact sequence of sheaves on $X$.

*Proof* A sequence of sheaves is exact if and only if it is exact on stalks, so the lemma follows by Proposition 11.23. $\square$

**Example 11.25** (Restriction to an open set)**.** Let $\iota\colon U \to X$ be the inclusion of an open set in $X$. Then for a sheaf $\mathcal{F}$ on $X$, we have

$$\iota^{-1}\mathcal{F} = \mathcal{F}|_U.$$

Indeed, if $V \subset U$ is an open set, then $V$ is also open in $X$, and the direct limit $\varinjlim_{W\supset\iota(V)} \mathcal{F}(W)$ simply evaluates to $\mathcal{F}(V)$. Note that the inverse inverse presheaf $\iota_p^{-1}\mathcal{F}$ is a sheaf in this case. $\triangle$

**Example 11.26** (General restrictions)**.** If $Z \subset X$ is an arbitrary subset, then the naive restriction of sections does not directly give a sheaf on $Z$, because an open subset $V \subset Z$ will typically not be open in $X$. The sections of the inverse image sheaf are instead determined by sections of $\mathcal{F}(U)$ as $U$ runs over the open sets in $X$ containing $Z$.

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In particular, if $Z = \{x\}$, we recognize the definition of the stalk, and $\iota^{-1}\mathcal{F} = \mathcal{F}_x$, as a constant sheaf on $\{x\}$. △

**Example 11.27.** The presheaf defined by (11.9) is not in general a sheaf. For instance, if $f\colon X \to Y$ is the constant map with image $y \in Y$, then for each open set $U \subset X$, the direct limit in (11.9) will be the stalk $\mathcal{G}_y$. Therefore, (11.9) defines the constant presheaf with value $\mathcal{G}_y$, and as we observed in Example 3.9 on page 49, this is not always a sheaf.

For an explicit example, we can let $X = \operatorname{Spec} k \sqcup \operatorname{Spec} k$, $Y = \operatorname{Spec} k$ and $f\colon X \to \operatorname{Spec} k$. Then $f^{-1}\mathcal{O}_Y = \mathcal{O}_X$, which is the constant sheaf on $k$ on $X$, whereas $f_p^{-1}\mathcal{O}_Y$ is the constant presheaf on $k$.

More generally, if $f\colon X \to Y$ is any morphism, and $\mathcal{A}$ is a constant presheaf with value $A$ on $Y$, then $f_p^{-1}\mathcal{A}$ is the constant presheaf on $X$ and $f^{-1}\mathcal{A}$ is a constant sheaf on $X$ (both with value $A$.). △

**Example 11.28.** As $f_p^{-1}\mathcal{O}_Y$ is a presheaf of rings, $f^{-1}\mathcal{O}_Y$ is a sheaf of rings on $X$. To see this, we can either apply Exercise 11.7.11, or use the explicit description of the sections of $f^{-1}\mathcal{O}_Y$ in terms of germs. △

Here is the aforementioned universal property of the inverse image sheaf.

**Theorem 11.29.** Let $f\colon X \to Y$ be a map of schemes and let $\mathcal{G}$ be a sheaf on $Y$. Then there exist functorial bijections

$$\operatorname{Hom}_{\mathsf{Ab}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathsf{Ab}(Y)}(\mathcal{G}, f_*\mathcal{F}) \tag{11.11}$$

for each sheaf $\mathcal{F}$ on $X$.

*Proof* As $\mathcal{F}$ is a sheaf, the universal property of sheafification tells us that there is a one-to-one correspondence between sheaf maps $f^{-1}\mathcal{G} \to \mathcal{F}$ and maps of presheaves $f_p^{-1}\mathcal{G} \to \mathcal{F}$. Therefore, it suffices to establish a bijection between:

(i) The set of sheaf maps $\phi\colon \mathcal{G} \to f_*\mathcal{F}$
(ii) The set of presheaf maps $\psi\colon f_p^{-1}\mathcal{G} \to \mathcal{F}$.

We will show that both sets are in bijection with

(iii) The set of maps

$$\Lambda_{U,V}\colon \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

so that for each inclusion $U \supset U'$ in $X$ and $V \supset V'$ in $Y$ with $V \supset f(U)$ and $V' \supset f(U')$, then the following diagram commutes

$$\begin{array}{ccc} \mathcal{G}(V) & \xrightarrow{\Lambda_{U,V}} & \mathcal{F}(U) \\ \big\downarrow & & \big\downarrow \\ \mathcal{G}(V') & \xrightarrow{\Lambda_{U',V'}} & \mathcal{F}(U'). \end{array} \tag{11.12}$$

(ii) $\Leftrightarrow$ (iii): Let $U \subset X$ be an open set. As $(f_p^{-1}\mathcal{G})(U)$ is defined as a direct limit, a map $\psi_U\colon f_p^{-1}\mathcal{G}(U) \to \mathcal{F}(U)$ is specified by a collection of maps $\Lambda_{U,V}\colon \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$ so that whenever $V \supset V' \supset f(U)$, we have $\Lambda_{U',V} = \Lambda_{U,V} \circ \rho_{VV'}$. That $\psi$ is compatible with the restriction maps translates is exactly the condition that (11.12) commutes

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for all $U, U', V, V'$. Conversely, any such collection $\Lambda_{U,V}$ determines a map of presheaves $\psi\colon f_p^{-1}\mathcal{G} \to \mathcal{F}$ by letting $\psi_U$ be the map induced by $\Lambda_{U,V}$ in the direct limit.

(i) $\Leftrightarrow$ (iii): Given a collection $\Lambda_{U,V}$, we obtain a sheaf map $\phi\colon \mathcal{G} \to f_*\mathcal{G}$ by defining $\phi_V = \Lambda_{f^{-1}V,V}$. The diagrams (11.12) ensure that $\phi$ is a map of sheaves.

Conversely, we need to show that every map of sheaves $\phi\colon \mathcal{G} \to f_*\mathcal{G}$ arises from a unique collection $\Lambda_{U,V}$ in this way. Given $\phi$, we define $\Lambda_{U,V}$ to be the composition

$$\mathcal{G}(V) \xrightarrow{\phi_V} \mathcal{F}(f^{-1}V) \longrightarrow \mathcal{F}(U).$$

This assignment makes the diagrams (11.12) commute. Moreover, for an open set of the form $U = f^{-1}V$, we have $\Lambda_{U,V} = \phi_V$, so we see that the collection $\Lambda_{U,V}$ induces $\phi$.

The collection $\Lambda_{U,V}$ is in fact uniquely determined by the maps of the form $\Lambda_{f^{-1}V,V}$ for $V \subset Y$. Indeed, the diagram (11.12) implies that if $U \subset f^{-1}V$, then $\Lambda_{U,V}$ factors as $\Lambda_{f^{-1}V,V}\colon \mathcal{G}(V) \to \mathcal{F}(f^{-1}V)$ followed by the restriction $\mathcal{F}(f^{-1}V) \to \mathcal{F}(U)$. In particular, there is one and only one $\Lambda_{U,V}$ that induces $\phi$. □

Despite of the name, the functors $f_*$ and $f^{-1}$ are not inverses: it is not true that $f_*f^{-1}\mathcal{G} \simeq \mathcal{G}$ or $f^{-1}f_*\mathcal{F} \simeq \mathcal{F}$ in general. However, there is a relationship between them. Applying the bijection (11.11) to the identity maps $f^{-1}\mathcal{G} \to f^{-1}\mathcal{G}$ and $f_*\mathcal{F} \to f_*\mathcal{F}$, we obtain canonical sheaf maps

$$\eta\colon \mathcal{G} \longrightarrow f_*f^{-1}\mathcal{G} \qquad \text{and} \qquad \varepsilon\colon f^{-1}f_*\mathcal{F} \longrightarrow \mathcal{F}.$$

These can be understood a bit more explicitly as follows. For an open set in $X$ of the form $f^{-1}V$, the direct limit (11.10) is canonically isomorphic to $\mathcal{G}(V)$. Therefore, a section $\mathcal{G}(V)$ defines a section of $f_p^{-1}\mathcal{G}$ over $f^{-1}(V)$. Then $\eta$ is induced by the composition

$$\mathcal{G}(V) \longrightarrow f_*f_p^{-1}\mathcal{G}(V) \longrightarrow f_*f^{-1}\mathcal{G}(V).$$

In particular, there is a natural map

$$\Gamma(Y, \mathcal{G}) \longrightarrow \Gamma(X, f^{-1}\mathcal{G}). \tag{11.13}$$

Likewise, the map $\varepsilon$ can be defined as follows. For an open set $U \subset X$, taking the direct limit of the restriction maps $\mathcal{F}(f^{-1}V) \to \mathcal{F}(U)$ over open sets $V$ with $V \supset f(U)$, we get a map of presheaves $f_p^{-1}f_*\mathcal{F} \to \mathcal{F}$. Then sheafifying, we get $\varepsilon$.

**Example 11.30.** Let $f\colon \mathbb{A}^2_k = \operatorname{Spec} k[x,y] \to \operatorname{Spec} k$ be the structure morphism. Then $f^{-1}\mathcal{O}_{\operatorname{Spec} k}$ is the constant sheaf on $k$ on $\mathbb{A}^2_k$. The morphism $\mathcal{O}_{\operatorname{Spec} k} \to f_*f^{-1}\mathcal{O}_{\mathbb{A}^2_k}$ is the identity map. On the other hand, $f_*\mathcal{O}_{\mathbb{A}^2_k}$ is the constant sheaf with value $k[x,y]$ on $\operatorname{Spec} k$ and $f^{-1}f_*\mathcal{O}_{\mathbb{A}^2_k}$ is also the constant sheaf on $\mathbb{A}^2_k$ with value $k[x,y]$. The map $f^{-1}f_*\mathcal{O}_{\mathbb{A}^2_k} \to \mathcal{O}_{\mathbb{A}^2}$ is given by the inclusion $k[x,y] \to \mathcal{O}_{\mathbb{A}^2_k}(V)$ for every $V \subset \mathbb{A}^2_k$. △

**Example 11.31.** In contrast, for the structure morphism $f\colon \mathbb{P}^1_k \to \operatorname{Spec} k$ and $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1_k}(-1)$, we have $f_*\mathcal{F} = \Gamma(\mathbb{P}^1_k, \mathcal{F}) = 0$, and $f^{-1}f_*\mathcal{F} = 0$. Hence $f_*f^{-1}\mathcal{F} \neq \mathcal{F}$. △

The sheaf $f^{-1}\mathcal{G}$ is uniquely determined by the property (11.11). This follows from the following version of Yoneda's Lemma:

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**Lemma 11.32 (Yoneda's Lemma).** Let $\mathsf{C}$ be a category and let $X$ and $Y$ be objects of $\mathsf{C}$. Given a natural transformation

$$\eta\colon \operatorname{Hom}_{\mathsf{C}}(Y,-) \longrightarrow \operatorname{Hom}_{\mathsf{C}}(X,-), \tag{11.14}$$

there is a unique morphism $f\colon X \to Y$ that induces $\eta$, that is, for every object $W$ in $\mathsf{C}$, the map of sets

$$\eta_W\colon \operatorname{Hom}_{\mathsf{C}}(Y,W) \longrightarrow \operatorname{Hom}_{\mathsf{C}}(X,W), \tag{11.15}$$

is given by $\eta_W(\sigma) = \sigma \circ f$.

*Proof* The proof is very similar to Lemma 8.1. □

## 11.7 Exercises

**Exercise 11.7.1.** Let $\mathcal{F}$ be a presheaf on a scheme $X$ and let $\mathcal{F}^+$ be the sheaf associated to $\mathcal{F}$. Show that there is a natural map $\mathcal{F}(X) \to \mathcal{F}^+(X)$. Find examples where this fails to be injective or surjective.

**Exercise 11.7.2.** Let $T = \{x, y, z, v\}$ be a topological space with open subsets $\{x\}$, $\{y\}$, $O = \{x, y, z\}$, $P = \{x, y, v\}$, and $O \cap P = \{x, y\}$.

Define a presheaf $\mathcal{G}$ on $T$ as follows:

- $\mathcal{G}(\{x\}) = \mathcal{G}(\{y\}) = \mathbb{Z}/2$,
- $\mathcal{G}(O) = \mathcal{G}(P) = \mathcal{G}(O \cap P) = \mathcal{G}(T) = \mathbb{Z}$,

with the natural quotient maps as the restrictions.

a) If $\mathcal{G}^+$ denotes the sheafification, show that $\mathcal{G}^+(T) = \{(m, n) \in \mathbb{Z} \oplus \mathbb{Z} \mid m \equiv n \mod 2\}$.
b) Show that the element $(0, 2) \in \mathcal{G}^+(T)$ cannot come from a global section of $\mathcal{G}$ over $T$. Conclude that $\mathcal{G}$ is not a sheaf.

**Exercise 11.7.3.** Define a sheaf on a topological space $X$ by $\mathcal{F}(X) = \mathbb{Z}$ and $\mathcal{F}(U) = 0$ for all other open sets. Show that $\mathcal{F}$ is a presheaf and describe the associated sheaf $\mathcal{F}^+$.

**Exercise 11.7.4.** Suppose $\phi : \mathcal{F} \to \mathcal{G}$ is a map of sheaves with the property that there is a covering $\{U_i\}$ of $X$ so that each $\phi_{U_i}$ is injective. Must $\phi$ be injective? If so, prove it, if not, give a counterexample.

**Exercise 11.7.5.** Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves on $X$ and let $\mathscr{B}$ be a basis for the topology on $X$. Show that $\phi$ is an isomorphism if and only if $\phi_V$ is an isomorphism for every $V \in \mathscr{B}$.

**Exercise 11.7.6.** Show that the sheaf associated to the 'constant presheaf' $\mathcal{F}(U) = A$ of Example 3.9 is the sheaf $A_X$ described in (**??**).

**Exercise 11.7.7.** Prove that the sheafification is unique up to a unique isomorphism.

**Exercise 11.7.8.** Let $X$ be a topological space and let $j : U \to X$ be an open set. For a

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sheaf $\mathcal{G}$ on $U$, we define a sheaf $j_!\mathcal{G}$ on $X$, the *extension by zero* by the sheaf associated to the presheaf

$$j_{p!}(V) = \begin{cases} \mathcal{G}(V) & V \subset U \\ 0 & \text{otherwise} \end{cases}$$

a) Show that the stalk $(j_!\mathcal{G})_x$ is equal to $\mathcal{G}_x$ if $x \in U$, and 0 otherwise.
b) Show that $j_!$ is a left adjoint to $j^!$.
c) Assume $U = X - Z$, where $i : Z \to X$ is a closed subset. If $\mathcal{F}$ is a sheaf on $X$, show that there is an exact sequence

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0. \tag{11.16}$$

d) Study the exact sequence (11.16) for $X = \mathbb{A}^n_k$, $Z = (0, \ldots, 0)$ and $\mathcal{F} = \mathcal{O}_{\mathbb{A}^n_k}$.

**Exercise 11.7.9.** Let $f\colon X \to Y$ and $g : Y \to Z$ be two morphisms.
a) Show that $(g \circ f)_* = g_* \circ f_*$
b) Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. HINT: There are sheafifications involved. Define a map from one side to the other and show that it induces an isomorphism on stalks.

**Exercise 11.7.10.** Let $X$ be a scheme and let $\mathcal{F}$ be a presheaf on $X$. Show that sheafification commutes with restriction, i.e., $\mathcal{F}^+|_U = (\mathcal{F}|_U)^+$. HINT: Use the universal property and check the stalks.

**Exercise 11.7.11.** Let $\mathcal{F}$ be a presheaf of rings on $X$. Show that the associated sheaf $\mathcal{F}^+$ is a sheaf of rings.

**Exercise 11.7.12.** Let $X = \{x, y, z\}$ be a topological space with three elements and open sets $\varnothing, X, V = \{x\}, U_1 = \{x, y\}, U_2 = \{x, z\}$. Let $\mathcal{F}$ be the constant presheaf on $X$.
a) Show that $\mathcal{F}$ is a sheaf.
b) Let $\mathcal{G}$ be the presheaf defined by $\mathcal{G}(\varnothing) = 0, \mathcal{G}(X) = \mathbb{Z}\oplus\mathbb{Z}$, and $\mathcal{G}(V), \mathcal{G}(U_1), \mathcal{G}(U_2)$ all equal to $\mathbb{Z}$, and let the restriction maps be $\rho_{U_1,V} = \rho_{U_2,V} = \mathrm{id}_{\mathbb{Z}}$ and let $\mathcal{G}(X) \to \mathcal{G}(V_i)$ be the $i$-th projection, for $i = 1, 2$. Show that $\mathcal{G}$ is a sheaf.
c) Consider the map $\phi\colon \mathcal{F} \to \mathcal{G}$ defined by $\phi_X = \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, $x \mapsto (x, x)$, and all the other maps equal to the identity map. Show that $\phi$ is a map of sheaves.
d) Compute the values of the image presheaf and conclude that it is not a sheaf.

**Exercise 11.7.13.** Consider the adjunction between the inverse image functor $f^{-1}$ and the direct image functor $f_*$. Recall the maps $\eta\colon \mathcal{G} \to f_*f^{-1}\mathcal{G}$ and $\varepsilon\colon f^{-1}f_*\mathcal{F} \to \mathcal{F}$.
a) Given a morphism $\phi\colon f^{-1}\mathcal{G} \to \mathcal{F}$, show that the corresponding morphism on the right-hand side of (11.11) is obtained by applying $\eta\colon \mathcal{G} \to f_*f^{-1}\mathcal{G}$ and then composing with the morphism $f_*\phi\colon f_*f^{-1}\mathcal{G} \to f_*\mathcal{F}$.
b) Given a morphism $\psi\colon \mathcal{G} \to f_*\mathcal{F}$, show that the corresponding morphism on the left-hand side is obtained by first applying $f^{-1}\psi\colon f^{-1}\mathcal{G} \to f^{-1}f_*\mathcal{F}$ and then composing with the map $\varepsilon\colon f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

**Exercise 11.7.14.** Show that the map $\Gamma(Y, \mathcal{G}) \longrightarrow \Gamma(X, f^{-1}\mathcal{G})$ of (11.13) needs not be injective or surjective in general, even in the basic case in Example 11.25.

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**Exercise 11.7.15.** Let $\iota\colon \operatorname{Spec} k \to \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ be the inclusion of a $k$-point in $\mathbb{A}^2_k$. Compute $\iota^{-1}\mathcal{O}_{\mathbb{A}^2_k}$ and study the morphisms $\mathcal{O}_{\mathbb{A}^2_k} \to \iota_*\iota^{-1}\mathcal{O}_{\mathbb{A}^2_k}$ and $\iota^{-1}\iota_*\mathcal{O}_{\operatorname{Spec} k} \to \mathcal{O}_{\operatorname{Spec} k}$.

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# 12

# Quasi-coherent sheaves

## 12.1 Sheaves of modules

A module over a ring is an additive abelian group equipped with a multiplicative action of the ring. Loosely speaking, we can multiply elements of the module by elements from the ring.

Similarly, an *$\mathcal{O}_X$-module* is a sheaf $\mathcal{F}$ whose sections can be multiplied by sections of $\mathcal{O}_X$. That is, an $\mathcal{O}_X$-module is a sheaf $\mathcal{F}$ such that $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for each open set $U \subset X$, and if $V \supset U$, then the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear in the sense that if $a, b \in \mathcal{O}_X(U)$, $s, t \in \mathcal{F}(U)$, then

$$(a \cdot s)|_V = a|_V \cdot s|_V$$

A *map of $\mathcal{O}_X$-modules* or an *$\mathcal{O}_X$-linear map* is a map of sheaves $\alpha\colon \mathcal{F} \to \mathcal{G}$ between two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ such that for each open $U$ the map $\alpha_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear. The $\mathcal{O}_X$-modules on a scheme $X$ form a category, which we denote by $\mathsf{Mod}_{\mathcal{O}_X}$ .

We write $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the subgroup of $\mathrm{Hom}_{\mathsf{Ab}(X)}(\mathcal{F}, \mathcal{G})$ consisting of sheaf maps $\mathcal{F} \to \mathcal{G}$ which are $\mathcal{O}_X$-linear.

**Example 12.1** (Pushforwards)**.** For a morphism $f\colon X \to Y$ and an $\mathcal{O}_X$-module $\mathcal{F}$, the pushforward $f_*\mathcal{F}$ is naturally an $\mathcal{O}_Y$-module via the natural map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. That is, for a section $s \in f_*\mathcal{F}(V)$ and $a \in \mathcal{O}_Y(V)$, we define $a \cdot s \in f_*\mathcal{F}(V)$ to be section $f^\sharp(a) \cdot s \in \mathcal{F}(f^{-1}V)$. △

**Example 12.2** (Ideal sheaves)**.** Ideal sheaves are important examples of $\mathcal{O}_X$-modules. A sheaf $\mathcal{I}$ is an ideal sheaf if $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is an ideal for each open set $U \subset X$. For an ideal sheaf $\mathcal{I}$, the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ associated to an ideal sheaf $\mathcal{I}$ is an $\mathcal{O}_X$-module.

The primary example is the following. Let $\iota : Y \to X$ be a closed embedding, then the kernel $\mathcal{I}$ of the map $\iota^\sharp : \mathcal{O}_X \to \iota_*\mathcal{O}_Y$ is an ideal sheaf of $\mathcal{O}_X$, and there is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Y \longrightarrow 0$$

We also see that $\iota_*\mathcal{O}_Y \simeq \mathcal{O}_X/\mathcal{I}$ as $\mathcal{O}_X$-modules. See also Example 11.14. △

**Example 12.3.** If $\mathcal{F}$ is a sheaf obtained by gluing together sheaves $\mathcal{F}_i$ defined on a cover $\mathcal{U} = \{U_i\}_{i\in I}$, and each $\mathcal{F}_i$ is an $\mathcal{O}_{U_i}$-module, then $\mathcal{F}$ is an $\mathcal{O}_X$-module. Indeed, for $V \subseteq X$ open, $s \in \mathcal{F}(V)$, and $a \in \mathcal{O}_X(V)$, the product $a \cdot s$ is defined by gluing the local products $a|_{U_i\cap V} \cdot s|_{U_i\cap V}$ in $\mathcal{F}_i(U_i \cap V)$. △

**Example 12.4.** Write $\mathbb{P}^1_k$ for the projective line over a field $k$, and consider the sheaves



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$\mathcal{O}_{\mathbb{P}^1_k}(n)$ from Chapter 5; that is, $\mathcal{O}_{\mathbb{P}^1_k}(n)$ is the sheaf obtained by gluing $\mathcal{O}_{U_0}$ to $\mathcal{O}_{U_1}$ using the isomorphism $\mathcal{O}_{U_1}|_{U_0\cap U_1} \to \mathcal{O}_{U_0}|_{U_0\cap U_1}$ on $U_0 \cap U_1 = \operatorname{Spec} k[u,u^{-1}]$ given by multiplication by $u^n$. Then $\mathcal{O}_{\mathbb{P}^1_k}(n)$ is an $\mathcal{O}_{\mathbb{P}^1_k}$-module by Example 12.3. The map $\phi\colon \mathcal{O}_{\mathbb{P}^1_k}(-1) \to \mathcal{O}_{\mathbb{P}^1}$ is a map of $\mathcal{O}_{\mathbb{P}^1_k}$-modules, and the image of $\phi$ is an ideal sheaf of $\mathcal{O}_{\mathbb{P}^1}$. △

## 12.2 The tilde of a module

The most important examples of $\mathcal{O}_X$-modules are the sheaves of the form $\widetilde{M}$, which we introduced in Section 3.5. Let us briefly recall the construction. If $A$ is a ring, and $M$ is an $A$-module, the sheaf $\widetilde{M}$ on $X = \operatorname{Spec} A$ is the sheaf extending the following $\mathscr{B}$-sheaf

$$\widetilde{M}(D(f)) = M_f.$$

The restriction maps are the canonical localization maps, which are described as follows: when $D(g) \subset D(f)$, we may write $g^r = af$ for some $a \in A$ and some $r \in \mathbb{N}$, and the localization map $M_f \to M_g$ sends $mf^{-n}$ to $a^n m g^{-nr}$.

It is almost immediate that $\widetilde{M}$ is an $\mathcal{O}_X$-module. Over a distinguished open set $U = D(f)$, the group $\widetilde{M}(D(f)) = M_f$ is a module over $A_f$, and if $U \subset X$ is any open subset, we may cover it by distinguished open sets $D(f)$ and define an $\mathcal{O}_X(U)$-module structure on $\widetilde{M}(U)$ using the sheaf sequence. In the same way, one verifies that the restriction maps are maps of modules.

Sending an $A$-module $M$ to the $\mathcal{O}_X$-module $\widetilde{M}$ defines the *tilde functor* from $\mathsf{Mod}_A$ to $\mathsf{Mod}_X$. This functor has very good properties, as we are going to see. We start by explaining the universal property of $\widetilde{M}$ among $\mathcal{O}_X$-modules.

**Proposition 12.5.** Let $X = \operatorname{Spec} A$ be an affine scheme. For an $A$-module $M$ and an $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \mathcal{F}) \xrightarrow{\ \simeq\ } \operatorname{Hom}_A(M, \mathcal{F}(X))$$

that sends $\phi\colon \widetilde{M} \to \mathcal{F}$ to $\phi_X\colon M \to \mathcal{F}(X)$. It is functorial in both $M$ and $\mathcal{F}$.

*Proof* Let $f \in A$, and consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi_X} & \mathcal{F}(X) \\ \downarrow & & \downarrow \\ M_f & \xrightarrow{\phi_{D(f)}} & \mathcal{F}(D(f)) \end{array}$$

where the vertical maps are restriction maps. This gives the following relation:

$$\phi_{D(f)}(m/1) = \phi_X(m)|_{D(f)}.$$

Note that $\mathcal{F}(D(f))$ is an $A_f$-module, because $\mathcal{F}$ is an $\mathcal{O}_X$-module. Therefore, in the localizations at $f$, we have the following relation

$$\phi_{D(f)}(mf^{-n}) = \phi_X(m)|_{D(f)} \cdot f^{-n}, \tag{12.1}$$

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where $mf^{-n} \in M_f$. This means that the maps $\phi_{D(f)}$ are completely determined by $\phi_X : M \to \mathcal{F}(X)$. By Proposition 3.14, the map of sheaves $\phi$ is completely determined once it is specified over the $D(f)$'s. Hence $\phi$ is determined by $\phi_X$, and the map in the proposition is injective.

For the surjectivity, suppose we are given a map of $A$-modules $\alpha\colon M \to \mathcal{F}(X)$. As usual, to define a map $\widetilde{M} \to \mathcal{F}$ it suffices to tell what it does to sections over the distinguished open sets $D(f)$. Inspired by (12.1), we define $\alpha_{D(f)}$ by

$$\alpha_{D(f)}(mf^{-n}) = \alpha(m)|_{D(f)} \cdot f^{-n}.$$

(Note that $\alpha(m)$ is a section of $\mathcal{F}(D(g))$, so the multiplication makes sense because $\mathcal{F}$ is an $\mathcal{O}_X$-module). Hence $\alpha_{D(f)}$ is simply the composition of the two maps of $A_f$-modules

$$M_f \xrightarrow{\alpha_f} \mathcal{F}(X)_f \longrightarrow \mathcal{F}(D(f)),$$

where the right-hand map is induced from the restriction map $\mathcal{F}(X) \to \mathcal{F}(D(f))$ by localization (note that $\mathcal{F}(D(f))$ is an $A_f$-module). This is compatible with the restriction maps, so we get a well-defined map of sheaves $\phi : \widetilde{M} \to \mathcal{F}$. Taking $f = 1$, we see that we recover $\alpha$ from $\phi$ on global sections.

The statement about the functoriality also follows from formula (12.1). The details are left to the reader. □

If we apply Proposition 12.5 to $M = \mathcal{F}(X)$ and consider the preimage of the identity map $\mathcal{F}(X) \to \mathcal{F}(X)$, we obtain the following corollary:

**Corollary 12.6.** For each $\mathcal{O}_X$-module $\mathcal{F}$ on an affine scheme $X$, there is a *unique* $\mathcal{O}_X$-module homomorphism

$$\beta_{\mathcal{F}}\colon \widetilde{\mathcal{F}(X)} \longrightarrow \mathcal{F} \tag{12.2}$$

that induces the identity on the spaces of global sections. The map $\beta_{\mathcal{F}}$ is functorial in $\mathcal{F}$.

In concrete terms, the map $\beta_{\mathcal{F}}$ is defined over a distinguished open subset $D(f)$ as follows. A section of the sheaf $\widetilde{\mathcal{F}(X)}$ over $D(f)$ is an element of the form $s/f^n$ where $s \in \mathcal{F}(X)$. Regarding $f^{-n}$ as a section of $\mathcal{O}_X(D(f)) = A_f$, we may send $s/f^n$ to the product $s|_{D(f)} \cdot f^{-n}$, which, because $\mathcal{F}$ is an $\mathcal{O}_X$-module, defines a section of $\mathcal{F}(D(f))$.

**Proposition 12.7.** Let $A$ be a ring and let $X = \operatorname{Spec} A$. Then:

(i) The functor $M \mapsto \widetilde{M}$ is exact.

(ii) If $M$ and $N$ are $A$-modules, the map $\alpha \mapsto \widetilde{\alpha}$ gives an isomorphism

$$\operatorname{Hom}_A(M, N) \simeq \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}),$$

with inverse $\phi \mapsto \phi_X$.

*Proof* Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of $A$-modules. Applying tilde gives the sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \widetilde{M'} \longrightarrow \widetilde{M} \longrightarrow \widetilde{M''} \longrightarrow 0. \tag{12.3}$$

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To check that (12.3) is exact, it suffices to check that it is exact on stalks for every point $x \in X$. But if $x \in X$ corresponds to the prime ideal $\mathfrak{p} \subset A$, the stalks of (12.3) is simply the localization of (**??**) at $\mathfrak{p}$ (which is exact, because localization is an exact functor).

The last statement follows from Proposition 12.5 with $\mathcal{F} = \widetilde{N}$ and the fact that by definition $(\widetilde{\alpha})_X = \alpha$. □

Item (ii) above says that the tilde functor is fully faithful. Hence it establishes an equivalence between the category $\mathsf{Mod}_A$ of $A$-modules and a subcategory of $\mathsf{Mod}_X$. This subcategory is usually a strict subcategory; most $\mathcal{O}_X$-modules are not of tilde-type.

## 12.3 Quasi-coherent sheaves

The following is the most important definition in this chapter.

**Definition 12.8.** Let $X$ be a scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *quasi-coherent* if for every affine $U = \operatorname{Spec} A \subset X$, the restriction $\mathcal{F}|_U$ is of tilde type, that is, there exists an $A$-module $M$ such that $\mathcal{F}|_U \simeq \widetilde{M}$ as $\mathcal{O}_U$-modules.

We let $\mathsf{QCoh}_X$ denote the category of quasi-coherent sheaves on $X$, that is, the subcategory of $\mathsf{Mod}_X$ where the objecs are quasi-coherent sheaves and the morphisms are morphisms of $\mathcal{O}_X$-modules.

The next theorem is an important result in the theory of quasi-coherent sheaves. It says that to verify that a sheaf $\mathcal{F}$ is quasi-coherent, it suffices to check that $\mathcal{F}$ is locally of tilde-type for the opens in a single open affine cover; this is a much easier condition to check in practice.

**Theorem 12.9.** Let $X$ be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Suppose that there exists an open affine covering $\{U_i\}_{i\in I}$ of $X$, where $U_i = \operatorname{Spec} A_i$, and $A_i$-modules $M_i$ such that $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ as $\mathcal{O}_{U_i}$-modules for each $i$. Then $\mathcal{F}$ is quasi-coherent.

*Proof* Let $U = \operatorname{Spec} A$ be an affine open set of $X$. We need to show that the canonical map

$$\beta\colon \widetilde{\mathcal{F}(U)} \longrightarrow \mathcal{F} \tag{12.4}$$

is an isomorphism. It suffices to check this over every distinguished open set $D(f) \subset U$. This reduces to showing that the map

$$\mathcal{F}(U)_f \longrightarrow \mathcal{F}(D(f)) \tag{12.5}$$

which sends $s/f^n$ to $s|_{D(f)} \cdot f^{-n}$ is an isomorphism.

By the assumption on the covering $U_i$, the restrictions $\mathcal{F}|_{U_i}$ is of 'tilde type'. The covering $U \cap U_i$ forms an open cover of $U$. Using Lemma 6.22, we may cover each $U \cap U_i$ by affine open sets which are distinguished in both $U$ and $U_i$. Doing so, we obtain an affine cover of $U$ of open sets $D(g_1), \ldots, D(g_r)$ such that $\mathcal{F}|_{D(g_i)}$ is of tilde type over each $D(g_i)$.

As $\mathcal{F}|_{D(g_i)}$ is of tilde type on $D(g_i)$ for each $i$, the canonical maps

$$\beta_{\mathcal{F}|_{D(g_i)}}\colon \widetilde{\mathcal{F}(D(g_i))} \longrightarrow \mathcal{F}|_{D(g_i)} \tag{12.6}$$

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are isomorphisms. In particular, over each open set $D(f) \cap D(g_i) = D(fg_i)$, the map

$$\mathcal{F}(D(g_i))_f \longrightarrow \mathcal{F}(D(fg_i)) \tag{12.7}$$

which sends $s/f^n$ with $s \in \mathcal{F}(D(g_i))$ to $s|_{D(fg_i)} \cdot f^{-n}$, is an isomorphism. For each $i$ and $j$, the restriction of $\mathcal{F}$ to $D(g_ig_j)$ is also of tilde-type, so we also have the isomorphisms

$$\mathcal{F}(D(g_ig_j))_f \longrightarrow \mathcal{F}(D(fg_ig_j)) \tag{12.8}$$

Consider the sheaf exact sequence for the cover $D(g_1), \ldots, D(g_r)$:

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \textstyle\bigoplus_{i=1}^r \mathcal{F}(D(g_i)) \longrightarrow \bigoplus_{i,j=1}^r \mathcal{F}(D(g_ig_j)). \tag{12.9}$$

There is a similar sequence for the cover of $D(f)$ given by $D(fg_1), \ldots, D(fg_r)$. These sequences give the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{F}(U)_f & \longrightarrow & \bigoplus_{i=1}^r \mathcal{F}(D(g_i))_f & \longrightarrow & \bigoplus_{i,j=1}^r \mathcal{F}(D(g_ig_j))_f \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F}(D(f)) & \longrightarrow & \bigoplus_{i=1}^r \mathcal{F}(D(fg_i)) & \longrightarrow & \bigoplus_{i,j=1}^r \mathcal{F}(D(fg_ig_j))
\end{array}$$

The top row in the diagram is the localization of (12.26) with respect to $f$, so both rows are exact. The three vertical maps are direct sums of the appropriate $\beta$-maps. By (12.7) and (12.8), the two vertical maps to the right are isomorphisms. By the 5-lemma, the left-most vertical map, which equals (12.5) is also an isomorphism, as we wanted to show. □

For an affine scheme $X = \operatorname{Spec} A$, the assignment $M \mapsto \widetilde{M}$ defines a functor $\widetilde{(-)}$ from $A$-modules to the category of quasi-coherent sheaves $\mathsf{QCoh}_X$. By definition, every quasi-coherent sheaf is of the form $\widetilde{M}$, and we can recover the module $M$ by $M = \Gamma(X, \mathcal{F})$. In fact, this induces an equivalence of categories:

**Theorem 12.10.** Let $X = \operatorname{Spec} A$ be an affine scheme. Then the tilde functor

$$\widetilde{(-)} \colon \mathsf{Mod}_A \longrightarrow \mathsf{QCoh}_X$$

is an equivalence of categories. The global sections functor $\Gamma \colon \mathsf{QCoh}_X \to \mathsf{Mod}_A$ that sends $\mathcal{F}$ to $\mathcal{F}(X)$ defines a quasi-inverse.

Note that we always have an equality of $A$-modules $\Gamma(X, \widetilde{M}) = M$. On the other hand, the functorial map $\beta_{\mathcal{F}} \colon \widetilde{\mathcal{F}(X)} \to \mathcal{F}$ from Corollary 12.6 on page 223 is merely an isomorphism, hence the term 'quasi-inverse' in the theorem.

This result has several imporant consequences for quasi-coherent sheaves on affine schemes. The first is that the global section functor preserves exact sequences.

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**Corollary 12.11.** Let $X = \operatorname{Spec} A$ and let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a sequence of quasi-coherent $\mathcal{O}_X$-modules. Then the sequence on global sections

$$0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow 0 \tag{12.10}$$

is also exact.

*Proof* Since the global section functor is left exact, we need only show that (12.10) is right exact, i.e., that the cokernel $C = \operatorname{Coker}(\mathcal{F}(X) \to \mathcal{F}''(X))$ is zero. In any case, there is an exact sequence

$$\mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow C \longrightarrow 0.$$

Applying the tilde functor, which is exact (Proposition 12.7), we get an exact sequence

$$\mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow \widetilde{C} \longrightarrow 0.$$

By assumption, $\mathcal{F} \to \mathcal{F}''$ is surjective, so $\widetilde{C} = 0$ and hence $C = \Gamma(X, \widetilde{C}) = 0$. □

**Example 12.12** (Quasi-coherent sheaves on $\mathbb{P}^1$)**.** Consider the projective line $\mathbb{P}^1_k$ over $k$, with the usual covering by $U_0 = \operatorname{Spec} k[u]$ and $U_1 = \operatorname{Spec} k[u^{-1}]$, glued together along their common open set $\operatorname{Spec} k[u, u^{-1}]$.

The sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$, as defined in Chapter **??**, are all quasi-coherent. This follows because $\mathcal{O}_{\mathbb{P}^1_k}(n)|_{U_i} \simeq \mathcal{O}_{U_i}$ for each $i = 1, 2$ and $\mathcal{O}_{U_i}$ is of course quasi-coherent.

More generally, we can classify all quasi-coherent sheaves on $\mathbb{P}^1_k$ as follows. A quasi-coherent sheaf on $\mathbb{P}^1_k$ is given by a triple $(M_0, M_1, \tau)$, where $M_0$ is a module over $\mathcal{O}_X(U_0) = k[u]$, where $M_1$ is a module over $\mathcal{O}_X(U_1) = k[u^{-1}]$ and where

$$\tau\colon M_1 \otimes_{k[u^{-1}]} k[u, u^{-1}] \longrightarrow M_0 \otimes_{k[u]} k[u, u^{-1}].$$

is an isomorphism of modules over $k[u, u^{-1}]$.

In terms of this description, the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ are given by the triples $M_0 = k[u]$, $M_1 = k[u^{-1}]$ and the map $\tau\colon k[u, u^{-1}] \to k[u, u^{-1}]$ is multiplication by $u^n$.

This description of the sheaf is not unique: given any isomorphisms $\psi_0 : M_0 \to M_0$ and $\psi_0 : M_1 \to M_1$, we can define $\tau' = (\psi_1 \otimes \mathrm{id}) \circ \tau \circ (\psi_0 \otimes \mathrm{id})$ which defines an isomorphic sheaf.

△

### *Quasi-coherent sheaves and localization*

There is a big difference between quasi-coherent sheaves and $\mathcal{O}_X$-modules in general. One way to explain the term 'coherence' is that if $U$ is an affine subset, then the groups of sections of $\mathcal{F}(D(g))$ over smaller affines $D(g) \subset U$ are completely determined its sections over $U$ itself, through the natural map

$$\mathcal{F}(U)_g \longrightarrow \mathcal{F}(D(g)). \tag{12.11}$$

given by $s/g^n \mapsto s|_V \cdot g^{-n}$.

In contrast, Exercise (12.12.13) shows that the sections of a general $\mathcal{O}_X$-module can vary rather wildly.

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In fact, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if the localization maps (12.11) are isomorphisms for every $U$ and $D(g)$. If $\mathcal{F}$ is quasi-coherent, then $\mathcal{F}|_U \simeq \widetilde{M}$ for some $A$-module $M$ and (12.11) follows by the description of the sections of $\widetilde{M}$ over distinguished open sets. Conversely, if (12.11) holds then the natural map $\beta_{\mathcal{F}|_U} \colon \widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$ is an isomorphism, and so $\mathcal{F}$ is quasi-coherent.

**Example 12.13** (Modules on spectra of DVR's)**.** Let $A$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and fraction field $K$. The scheme $X = \operatorname{Spec} A$ consists of two points: the closed point $s$ (corresponding to $\mathfrak{m}$) and the generic point $\xi$ (corresponding to $(0)$). The nonempty open sets are $\{\xi\}$ and $X$ itself.

For this simple scheme we can describe all $\mathcal{O}_X$-modules as follows. Every $\mathcal{O}_X$-module $\mathcal{F}$ is uniquely determined by the following data:

(i) An $A$-module $M = \mathcal{F}(X)$,
(ii) A $K$-vector space $N = \mathcal{F}(\{\xi\})$,
(iii) An $A$-linear map $\rho \colon M \to N$ given by the restriction map $\mathcal{F}(X) \to \mathcal{F}(\{\xi\})$.

Conversely, any such triple $(M, N, \rho)$ determines an $\mathcal{O}_X$-module $\mathcal{F}$ by setting $\mathcal{F}(X) = M$, $\mathcal{F}(\{\xi\}) = N$, $\mathcal{F}(\varnothing) = 0$, with the restriction map $\mathcal{F}(X) \to \mathcal{F}(\{\xi\})$ given by $\rho$. Since $M$ and $N$ are modules over $\mathcal{O}_X(X) = A$ and $\mathcal{O}_X(\{\xi\}) = K$ respectively, this makes $\mathcal{F}$ into an $\mathcal{O}_X$-module.

Note that $\rho \colon M \to N$ can be arbtrary – it need not be injective, surjective or even nonzero. It can even be the zero map, and in that case $M$ and $N$ can be completely arbitrary modules.

In this setting, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if the induced map $\rho \otimes \mathrm{id}_K \colon M \otimes_A K \to N$ is an isomorphism (of $K$-vector spaces). To see this, note that if $\mathcal{F}$ is quasi-coherent, then it must be equal to $\widetilde{M}$. This forces $N = \mathcal{F}(\{\xi\}) = M_{(0)} = M \otimes_A K$ and $\rho$ to induce an isomorphism after tensoring with $K$. Conversely, if $\rho \otimes \mathrm{id}_K \colon M \otimes_A K \to N$ is an isomorphism, then $\mathcal{F}$ is isomorphic to $\widetilde{M}$, with $\mathcal{F}(\{\xi\}) = M_{(0)} = M \otimes_A K$. $\triangle$

## 12.4 Kernels, Images and Cokernels

Most of the constructions for modules over a ring have analogues for $\mathcal{O}_X$-modules and quasi-coherent sheaves. For instance, given a map of $\mathcal{O}_X$-modules $\phi \colon \mathcal{F} \to \mathcal{G}$, the kernel, image and cokernel of $\alpha$, have natural $\mathcal{O}_X$-modules structures. Here it is clear that the image and cokernel *presheaves* have $\mathcal{O}_X$-module structures, and then Exercise 12.12.4 shows that also the associated sheaves are $\mathcal{O}_X$-modules.

If $\phi \colon \mathcal{F} \to \mathcal{G}$ is a map between quasi-coherent sheaves, then $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$ and $\operatorname{Coker}\phi$ are also quasi-coherent. To see this, we may restrict to an open affine $U = \operatorname{Spec} A$. On $U$, the restriction of $\phi$ is of the form $\widetilde{\alpha}$ for some map of $A$-modules $\alpha \colon M \to N$. Since $\widetilde{(-)}$ is an equivalence of categories, we have:

$$\operatorname{Ker}\phi = \widetilde{\operatorname{Ker}\alpha}, \qquad \operatorname{Im}\phi = \widetilde{\operatorname{Im}\alpha}, \qquad \operatorname{Coker}\phi = \widetilde{\operatorname{Coker}\alpha}.$$

In particular, the global sections over an affine open set $U$ are computed as follows:

(i) $\Gamma(U, \operatorname{Ker}\phi) = \operatorname{Ker}\alpha_U$,
(ii) $\Gamma(U, \operatorname{Im}\phi) = \operatorname{Im}\alpha_U$,
(iii) $\Gamma(U, \operatorname{Coker}\phi) = \operatorname{Coker}\alpha_U$.

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The same applies to quotients: if $\mathcal{G} \subset \mathcal{F}$ is a subsheaf of an $\mathcal{O}_X$-module $\mathcal{F}$, then the quotient sheaf $\mathcal{F}/\mathcal{G}$ is naturally an $\mathcal{O}_X$-module. If $\mathcal{F}$ and $\mathcal{G}$ are both quasi-coherent, then so is $\mathcal{F}/\mathcal{Q}$ and for every affine open subset $U \subset X$, we have

$$\Gamma(U, \mathcal{F}/\mathcal{G}) = \mathcal{F}(U)/\mathcal{G}(U).$$

Note that this formula does not hold for arbitrary open subsets (see Example 11.14).

## 12.5 Direct sums and products

For a finite collection of sheaves $\mathcal{F}_1, \ldots, \mathcal{F}_n$, their direct sum $\bigoplus_{i=1}^n \mathcal{F}_i$ is defined by:

$$\Gamma(U, \bigoplus_{i=1}^n \mathcal{F}_i) = \bigoplus_{i=1}^n \mathcal{F}_i(U).$$

This is a sheaf with restriction maps defined componentwise. Locality holds because if $s = (s_1, \ldots, s_n) \in \bigoplus_{i=1}^n \mathcal{F}_i(U)$ restricts to 0 on a covering, then all $s_1 = \cdots = s_n = 0$ by Locality for the $\mathcal{F}_i$'s. Likewise, given local sections matching on the overlaps, one can glue componentwise.

It is also possible to define direct sums of an arbitrary collection of sheaves $\{\mathcal{F}_i\}_{i\in I}$, see Exercise 12.12.10.

If the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, the direct sum $\bigoplus_{i=1}^n \mathcal{F}_i$ is also an $\mathcal{O}_X$-module in a natural way, with multiplication being defined componentwise. Moreover, if the $\mathcal{F}_i$ are quasi-coherent, the direct sum is also quasi-coherent:

**Proposition 12.14.** If $\mathcal{F}_1, \ldots, \mathcal{F}_n$ is a collection of quasi-coherent sheaves, then the direct sum $\bigoplus_{i=1}^n \mathcal{F}_i$ is quasi-coherent. If $X = \operatorname{Spec} A$, then for all $A$-modules $M_i$, we have

$$\widetilde{\bigoplus_{i=1}^n M_i} \simeq \bigoplus_{i=1}^n \widetilde{M}_i. \tag{12.12}$$

*Proof* We may reduce to the affine case and write $X = \operatorname{Spec} A$. It suffices to prove the formula (12.12); quasi-coherence follows from this. As direct sums commute with localization, we have for every $f \in A$, a natural isomorphism of $A_f$-modules

$$(\bigoplus_{i=1}^n M_i)_f \longrightarrow \bigoplus_{i=1}^n (M_i)_f \tag{12.13}$$

These maps are compatible with the restriction maps, so there is an induced map of $\mathcal{O}_X$-modules $\phi\colon \widetilde{\bigoplus_{i=1}^n M_i} \to \bigoplus_{i=1}^n \widetilde{M}_i$. The map $\phi$ is an isomorphism, as it induces an isomorphism on stalks. □

For a collection of sheaves $\{\mathcal{F}_i\}_{i\in I}$, we define the *direct product* $\prod_{i\in I} \mathcal{F}_i$ by:

$$\Gamma\left(U, \prod_{i\in I} \mathcal{F}_i\right) = \prod_{i\in I} \mathcal{F}_i(U). \tag{12.14}$$

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This is a sheaf with componentwise restriction maps. Moreover, if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, $\prod_{i\in I}\mathcal{F}_i$ is naturally an $\mathcal{O}_X$-module.

If $\mathcal{F}_1,\dots,\mathcal{F}_n$ are quasi-coherent sheaves, then the product $\prod_{i=1}^n \mathcal{F}_i$ is also quasi-coherent. In fact, it is isomorphic to the direct sum $\bigoplus_{i=1}^n \mathcal{F}_i$ Moreover, if $X = \operatorname{Spec} A$, then

$$\left(\prod_{i=1}^n M_i\right)^{\sim} = \prod_{i=1}^n \widetilde{M}_i$$

For an infinite collection of sheaves, however, it can happen that the direct product is not quasi-coherent. The reason is that direct products do not commute with localization in general (see Exercise 12.12.10).

## 12.6 Tensor products

Given two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we define their *tensor product* $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ as the sheaf associated to the presheaf

$$T(U) = \mathcal{F}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{G}(U) \tag{12.15}$$

When the context is clear, we may simply write $\mathcal{F}\otimes\mathcal{G}$ for brevity.

In the definition, it necessary to sheafify, as the presheaf $T$ is not always a sheaf (see Example 12.16 below). Fortunately, the following proposition allows us to understand the sections of tensor products of *quasi-coherent sheaves*, at least over affine open subsets.

**Proposition 12.15.** Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves. Then:

(i) $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G}$ is quasi-coherent.

(ii) For any open affine subset $U\subset X$, there is a canonical isomorphism

$$(\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{G})(U) = \mathcal{F}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{G}(U) \tag{12.16}$$

(iii) For $X=\operatorname{Spec}A$ and $A$-modules $M$ and $N$,

$$\widetilde{M\otimes_A N} \simeq \widetilde{M}\otimes_{\mathcal{O}_X}\widetilde{N}. \tag{12.17}$$

*Proof* As in Proposition 12.14, we reduce to the affine case assume $X=\operatorname{Spec}A$ and to proving (12.17). The main point is that tensor products commute with localization. More precisely, for each $f\in A$, there is a natural isomorphism of $A_f$-modules

$$M_f\otimes_{A_f}N_f \longrightarrow \Gamma(U,\widetilde{M\otimes_A N}) = (M\otimes_A N)_f \tag{12.18}$$

given by the assignment $m/f^a\otimes n/f^b\mapsto(m\otimes n)/f^{a+b}$. These isomorphisms are compatible with the restriction maps, so there is an induced map of $\mathcal{O}_X$-modules $\phi\colon \widetilde{M\otimes_A N}\to \widetilde{M\otimes_A N}$. As the maps (12.18) are isomorphisms, $\phi$ induces an isomorphism on stalks, and hence it is an isomorphsm. □

**Example 12.16.** Let $\mathcal{F}=\mathcal{O}_{\mathbb{P}^1_k}(1)$ and $\mathcal{G}=\mathcal{O}_{\mathbb{P}^1_k}(-1)$. Then the global sections of the tensor product presheaf (12.15) is given by

$$\mathcal{F}(\mathbb{P}^1_k)\otimes_k\mathcal{G}(\mathbb{P}^1_k) = k^2\otimes_k 0 = 0.$$

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However, there is an isomorphism of sheaves

$$\mathcal{F} \otimes \mathcal{G} = \mathcal{O}_{\mathbb{P}^1_k}(1) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1_1}(-1) \simeq \mathcal{O}_{\mathbb{P}^1_k},$$

which has global sections equal to $k$. This means that the presheaf (12.15) is not a sheaf. △

**Example 12.17.** Let $A = k[x, y, z]$, and $X = \mathbb{A}^3_k = \operatorname{Spec} A$. Consider the $A$-modules

$$M = k[x, y, z]/(x) \quad \text{and} \quad N = k[x, y, z]/(y).$$

Then $M \otimes_A N = k[x, y, z]/(x, y) \simeq k[z]$.

Let $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$. If we let $U = \mathbb{A}^3_k - \{(0, 0, 0)\}$, then arguing as in Example 4.22, we find that $\mathcal{F}(U) = M$ and $\mathcal{G}(U) = N$, so

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \simeq k[z].$$

On the other hand,

$$(\mathcal{F} \otimes_{\mathcal{O}_X(U)} \mathcal{G})(U) = \Gamma(U, \widetilde{k[z]}) \simeq k[z, z^{-1}].$$

△

## 12.7 Finiteness conditions for $\mathcal{O}_X$-modules

Just as with modules over a ring, we can impose finite finiteness conditions on $\mathcal{O}_X$-modules on a scheme $X$.

**Definition 12.18.** A quasi-coherent sheaf $\mathcal{F}$ is *finitely generated*, or of *finite type*, if for each open affine $U \subset X$, the group $\mathcal{F}(U)$ is a finitely generated $\mathcal{O}_X(U)$-module.

As in the case of quasi-coherence, the conditions hold for any open affine cover provided that it holds for one:

**Proposition 12.19.** Let $X$ be a scheme and let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. If there is an affine cover $\{U_i\}_{i \in I}$ with $U_i = \operatorname{Spec} A_i$ such that for each $i$, $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ for some finitely generated $A_i$-module, then $\mathcal{F}$ is of finite type.

*Proof* See Exercise 12.12.51. □

**Example 12.20.** If $X = \operatorname{Spec} A$, then an $\mathcal{O}_X$-module $\mathcal{F}$ is of finite type if and only if $\mathcal{F} \simeq \widetilde{M}$ for some finitely generated $A$-module. △

**Example 12.21.** Let $X = \operatorname{Spec} \mathbb{Z}$. If $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, then $\mathcal{F} = \widetilde{M}$ for some finitely generated $\mathbb{Z}$-module $M$, and by the structure theorem for finitely generated abelian groups, we may write $M = \mathbb{Z}^r \oplus T$, where $T$ is a finite direct product of groups of the form $\mathbb{Z}/n\mathbb{Z}$. Hence we may write

$$\mathcal{F} = \mathcal{O}_X^r \oplus \mathcal{T} \tag{12.19}$$

where $\mathcal{T}$ is a sheaf having stalks $\mathcal{T}_p = 0$ for all but finitely many $p$ and $T_{(0)} = 0$. ($T$ is a *torsion sheaf*, see Exercise 12.12.41.) △

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**Example 12.22.** The argument of the previous example in fact applies over any PID $A$: for $X = \operatorname{Spec} A$, any $\mathcal{O}_X$-module of finite type sheaf must have the form $\widetilde{M}$ for $M = A^r \oplus T$ where $T$ is a finitely generated torsion module. In particular, any finite type sheaf on the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ decomposes as

$$\mathcal{F} = \mathcal{O}_X^r \oplus \mathcal{T} \tag{12.20}$$

where $\mathcal{T}$ is a torsion sheaf. △

**Example 12.23.** Let $X = \operatorname{Spec} \mathbb{C}[x]$, $Y = \operatorname{Spec} \mathbb{C}$, and let $f\colon X \to Y$ be the structure morphism $\mathbb{A}^1_{\mathbb{C}} \to \operatorname{Spec} \mathbb{C}$. Then $f_*\mathcal{O}_X$ is not of finite type, as it equals the tilde of $\mathbb{C}[x]$ which is not finitely generated as a $\mathbb{C}$-module.

In fact, it follows by definition that for an affine morphism $f\colon X \to Y$, $f_*\mathcal{O}_X$ is of finite type if and only if $f$ is of finite type. △

**Definition 12.24** (Coherent sheaves)**.** Let $X$ be a Noetherian scheme. We say that a quasi-coherent sheaf is *coherent* if it is of finite type.

*Remark* 12.25. There is a more general definition of 'coherence' for $\mathcal{O}_X$-modules on arbitrary schemes. Specifically, an $\mathcal{O}_X$-module $\mathcal{F}$ is defined to be coherent if it is of finite type, and for every open $U \subset X$ and every morphism of $\mathcal{O}_U$-modules $\phi\colon \mathcal{O}_U^n \to \mathcal{F}|_U$, the kernel of $\phi$ is of finite type. The main reason for introducing this stronger notion of coherence, is that the category of coherent sheaves forms an abelian category even in the non-Noetherian setting. The downside is that the coherence condition is very difficult to check in general, and in fact, for some schemes, even affine ones, the structure sheaf $\mathcal{O}_X$ itself is not coherent.

In this book, we will be interested in coherent modules exclusively in the Noetherian setting, and here the notion is equivalent to being of finite type.

## 12.8 The Hom-sheaf

Recall that we defined the Hom-sheaf of two sheaves $\mathcal{F}$ and $\mathcal{G}$ as the sheaf

$$\mathcal{H}om(\mathcal{F},\mathcal{G})(U) = \operatorname{Hom}_{\mathsf{Ab}(U)}(\mathcal{F}|_U, \mathcal{G}|_U).$$

For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we define the Hom-sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$, by

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})(U) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U). \tag{12.21}$$

where the right-hand side means all the $\mathcal{O}_U$-linear maps.

The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is an $\mathcal{O}_X$-module is a natural way, but it is not always quasi-coherent, even if both $\mathcal{F}$ and $\mathcal{G}$ are. This is because of the fact that Hom does not commute with localization in general. However, if $X$ is Noetherian and $\mathcal{F}$ is of finite type, then it is quasi-coherent:

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**Proposition 12.26.** Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves on a Noetherian scheme $X$, and assume that $\mathcal{F}$ is coherent.

(i) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is quasi-coherent.
(ii) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is coherent if $\mathcal{F}$ and $\mathcal{G}$ are.
(iii) For every open affine $U = \operatorname{Spec} A$,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})|_U = \operatorname{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U),\mathcal{G}(U))\,\tilde{}\,.$$

(iv) The stalk at a point $x \in X$ is given by

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})_x = \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x). \tag{12.22}$$

*Proof* The key point is that if $A$ is Noetherian, and $M$ is finitely generated, then $\operatorname{Hom}(M,-)$ commutes with localization (see Proposition A.8). More precisely, for every $f \in A$, there is a canonical isomorphism

$$\operatorname{Hom}_A(M,N)_f = \operatorname{Hom}_{A_f}(M_f,N_f) \tag{12.23}$$

for each $A$-module $N$. These isomorphisms are compatible with restriction maps, so we get a map of sheaves $\widetilde{\operatorname{Hom}_A(M,N)} \to \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$. By (12.23), the induced map is an isomorphism on stalks, and so it is an isomorphism of sheaves, giving us (iii). The remaining statements follow from this formula. □

**Example 12.27.** Let $X = \operatorname{Spec}\mathbb{Z}$ and consider the $\mathbb{Z}$-modules $M = \mathbb{Q}$ and $N = \mathbb{Z}$. Then the global sections of $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$ is given by:

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})(X) = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0.$$

On the other hand, the stalk at the generic point $\eta$ is equal to

$$\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})_\eta = \operatorname{Hom}_{S^{-1}\mathbb{Z}}(S^{-1}\mathbb{Q},S^{-1}\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q},\mathbb{Q}) = \mathbb{Q}.$$

Here $S = \mathbb{Z} - 0$. From this it follows that $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M},\widetilde{N})$ is not a quasi-coherent sheaf. △

## 12.9 Pushforwards

For a morphism of schemes $f\colon X \to Y$, and an $\mathcal{O}_X$-module $\mathcal{F}$, the pushforward sheaf $f_*\mathcal{F}$ defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$, is an $\mathcal{O}_Y$-module (see Example 12.1). In this section, we will explore when the pushforward $f_*\mathcal{F}$ of a quasi-coherent sheaf is again quasi-coherent.

We begin by considering the affine case. Consider a morphism $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ induced by a ring map $\phi\colon A \to B$. Via $\phi$, any $B$-module $M$ can also be considered as a module over $A$. We write $M_A$ for $M$ as an $A$-module. This works well with respect to localization: if $g \in A$, then there is an isomorphism of $A_g$-modules $M_{\phi(g)} = (M_A)_g$ (where $M_{\phi(g)}$ is considered as an $A_g$-module).

**Proposition 12.28.** Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism of affine schemes, and let $M$ be a $B$-module. Then

$$f_*\widetilde{M} = \widetilde{M_A}. \tag{12.24}$$

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*Proof* By Proposition 2.29, we have $f^{-1}D(g) = D(\phi(g))$. This means that we have equalities

$$(f_*\widetilde{M})(D(g)) = \widetilde{M}(f^{-1}D(g)) = \widetilde{M}(D(\phi(g))) = M_{\phi(g)} = (M_A)_g,$$

where the last equality holds by the above paragraph. These equalities are compatible with the restriction maps of the sheaves involved, so we conclude that $f_*\widetilde{M} = \widetilde{M_A}$. □

**Example 12.29.** If $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, and $f\colon X \to Y$ is a morphism induced by $\phi\colon A \to B$, then canonical map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is the map $\widetilde{A} \to \widetilde{B_A}$, where we consider $B$ as an $A$-module. Over the open set $U = D(g) \subset Y$, $f^\sharp_U$ is given by the localized map $A_g \to B_{\phi(g)}$. △

**Corollary 12.30.** Let $f\colon X \to Y$ be an affine morphism and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $f_*\mathcal{F}$ is quasi-coherent.

*Proof* As 'quasi-coherence' is a property which can be checked locally on $Y$, we may assume $Y = \operatorname{Spec} A$, and the result follows from the above proposition. □

The pushforward of a quasi-coherent sheaf is not always quasi-coherent. However, the counterexamples to this tend to be somewhat pathological (see Example 12.34). For most morphisms encountered in practice, quasi-coherence is preserved. The following theorem gives sharp sufficient conditions for this property to hold.

**Theorem 12.31 (Quasi-coherence of pushforwards).**
Let $f\colon X \to Y$ be a morphism, and suppose that for every affine open $V \subset Y$, the inverse image $f^{-1}(V)$ admits a finite affine cover $\{U_1, \dots, U_r\}$ such that each $U_i \cap U_j$ also admits a finite affine cover. Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the pushforward $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

*Proof* To show that $f_*\mathcal{F}$ is quasi-coherent, it suffices to show that $f_*\mathcal{F}|_V$ is quasi-coherent for any affine open subset $V \subset Y$. Therefore, we may assume that $Y = \operatorname{Spec} A$ is affine.

Let $M = \Gamma(Y, f_*\mathcal{F}) = \mathcal{F}(X)$. Note that $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module, so we may view $M$ as an $A$-module. We claim that the natural map

$$\widetilde{M} \longrightarrow f_*\mathcal{F} \tag{12.25}$$

which sends $s/g^n$ with $s \in \mathcal{F}(X)$ to $s|_{f^{-1}D(g)} \cdot g^{-n}$, is an isomorphism. It suffices to check this over the distinguished open subsets $D(g) \subset Y$ for $g \in A$.

By our assumptions, we may cover $X$ by finitely many affine open subsets $U_i = \operatorname{Spec} B_i$, and then cover the intersections $U_i \cap U_j$ by finitely many affines $U_{ijk} = \operatorname{Spec} B_{ijk}$.

Let $\phi_i\colon A \to B_i$ and $\phi_{ijk}\colon A \to B_{ijk}$ be the ring maps that induce the morphisms $U_i \to Y$ and $U_{ijk} \to Y$. We also consider the distinguished open sets $U_{i,g} = D(\phi_i(g)) \subset U_i$ and $U_{ijk,g} = D(\phi_{ijk}(g)) \subset U_{ijk}$.

With these definitions, a section of $(f_*\mathcal{F})(Y) = \mathcal{F}(X)$ is given by a collection of elements $s_i \in \mathcal{F}(U_i)$ that agree over the various intersections $U_{ijk}$. Equivalently, the following

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sequence is exact:

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j,k} \mathcal{F}(U_{ijk}). \tag{12.26}$$

There is a similar sequence for the cover of $f^{-1}D(g)$ given by $U_{i,g}$, for $i = 1, \ldots, r$. These sequences fit into the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F}(X)_g & \longrightarrow & \prod_i \mathcal{F}(U_i)_g & \longrightarrow & \prod_{i,j} \mathcal{F}(U_{ijk})_g \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F}(f^{-1}D(g)) & \longrightarrow & \prod_i \mathcal{F}(U_{i,g}) & \longrightarrow & \prod_{i,j,k} \mathcal{F}(U_{ijk,g}) \end{array}$$

The top row in the diagram is the localization of (12.26) with respect to $g$, so it is exact. The three vertical maps are induced by the restriction maps. As $U_i$ and $U_{ijk}$ are all affine and $\mathcal{F}$ is quasi-coherent, the second and third vertical maps are isomorphisms. By the 5-lemma, the left-most vertical map, $\mathcal{F}(X)_g \longrightarrow \mathcal{F}(D(g))$, which is exactly the map $\Gamma(D(g), \widetilde{M}) \to \Gamma(D(g), f_*\mathcal{F})$ is an isomorphism as well. □

**Corollary 12.32.** Let $f\colon X \to Y$ be a morphism with $X$ Noetherian and let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Then $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.

*Proof* Let $V \subset Y$ be any affine open subset. Since $X$ is Noetherian, the open set $f^{-1}(V)$ is quasi-compact and therefore admits a finite affine open cover $\{U_1, \ldots, U_r\}$. Similarly, each intersection $U_i \cap U_j$ is quasi-compact, so admits a finite affine cover. Hence the conditions of Theorem 12.31 apply and $f_*\mathcal{F}$ is quasi-coherent. □

**Example 12.33.** Consider the projective line $\mathbb{P}^1_k$ and the '$n$-th power map' map $f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$, given by $(x_0 : x_1) \mapsto (x_0^n : x_1^n)$. We claim that

$$f_*\mathcal{O}_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{n-1}.$$

Over the open set $U_0 = \operatorname{Spec} k[u]$, where $u = x_1/x_0$, the morphism $f$ is induced by the ring map $k[v] \to k[u]$ which sends $v \mapsto u^n$. Restricted to $V_0 = \operatorname{Spec} k[v]$, the pushforward $f_*\mathcal{O}_{\mathbb{P}^1}$ is the tilde of $k[u]$ as a $k[v]$-module, or in other words, the tilde of

$$k[v][u]/(u^n - v) = k[v] \oplus k[v]u \oplus \cdots \oplus k[v]u^{n-1}. \tag{12.27}$$

Likewise, in the other affine chart, the morphism is induced by the ring map $k[v^{-1}] \to k[u^{-1}]$ sending $v^{-1} \mapsto u^{-n}$, and when restricted to $V_1 = \operatorname{Spec} k[v^{-1}]$, $f_*\mathcal{O}_{\mathbb{P}^1}|_{V_1}$ is the tilde of

$$k[v^{-1}][u^{-1}]/(u^{-n} - v^{-1}) = k[v^{-1}] \oplus k[v^{-1}]u^{-1} \oplus \cdots \oplus k[v^{-1}]u^{-n+1}. \tag{12.28}$$

and on $V_0 \cap V_1 = \operatorname{Spec} k[v^{\pm 1}]$, it is the tilde of

$$k[v^{\pm 1}][u^{-1}]/(u^n - v) = k[v^{\pm 1}] \oplus k[v^{\pm 1}]u^{-1} \oplus \cdots \oplus k[v^{\pm 1}]u^{-n+1}.$$

In this ring, we have $u^{i-n} = v^{-1}u^i$ for $i = 1, \ldots, n-1$ and we can use this to pass between the two bases $\{1, u, \ldots, u^{n-1}\}$ and $\{1, u^{-1}, \ldots, u^{-n+1}\}$ in (12.27) and (12.28). From this,

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we find that $f_*\mathcal{O}_{\mathbb{P}^1}$ is obtained by gluing together $\mathcal{O}_{V_0}^n$ and $\mathcal{O}_{V_1}^n$ using the isomorphism $\mathcal{O}_{V_{01}}^n \xrightarrow{M} \mathcal{O}_{V_{01}}^n$ defined by the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & v^{-1} \\ \vdots & \vdots & \iddots & \iddots & 0 \\ 0 & 0 & \iddots & \iddots & \vdots \\ 0 & v^{-1} & 0 & \cdots & 0 \end{pmatrix}$$

This means that $f_*\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{n-1}$.

△

The following example shows that even $f_*\mathcal{O}_X$ can fail to be quasi-coherent in general.

**Example 12.34.** Let $X = \coprod_{i\in\mathbb{N}} \mathbb{A}^1_k$ be the disjoint union of countably infinitely many copies of $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ and let $f\colon X \to \mathbb{A}^1_k$ be the morphism that is the identity on each component. Then $f_*\mathcal{O}_X$ is not quasi-coherent.

To see this, note that the global sections of $f_*\mathcal{O}_X$ satisfy

$$\Gamma(\mathbb{A}^1_k, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \prod_{i\in\mathbb{N}} k[x]. \tag{12.29}$$

On the other hand, over the open set $D(x) \subset \mathbb{A}^1_k$, we have

$$\Gamma(D(x), f_*\mathcal{O}_X) = \Gamma(f^{-1}D(x), \mathcal{O}_X) = \prod_{i\in\mathbb{N}} k[x, x^{-1}]. \tag{12.30}$$

If $f_*\mathcal{O}_X$ were quasi-coherent, then the group (12.30) would be isomorphic to the localization $(\prod_{i\in\mathbb{N}} k[x])_x$. However, this is not the case, as elements in the localization have bounded denominators (of the form $p/x^n$ where $p \in \prod_{i\in\mathbb{N}} k[x]$). In contrast, the elements of $\prod_{i\in\mathbb{N}} k[x, x^{-1}]$ may have unbounded negative exponents (of the form $(p_i x^{-n_i})_{i\in\mathbb{N}}$, where $n_i \in \mathbb{Z}$ can vary). Therefore $f_*\mathcal{O}_X$ is not quasi-coherent. △

## 12.10 Closed subschemes and closed embeddings

As a first application of the material in this chapter, we will give a more systematic treatment of closed subschemes. As we discussed in Section **??**, the prototype example of a closed subscheme is the affine subscheme $\operatorname{Spec} A/I \to \operatorname{Spec} A$ defined by an ideal $I \subset A$. The general definition will involve *ideal sheaves* rather than ideals. That is, closed subschemes will correspond to ideal sheaves $\mathcal{I}$, so that $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is an ideal for each $U$. In order to obtain a scheme, it is important that $\mathcal{I}$ is *quasi-coherent*. Before explaining this, we need the notion of the support of a sheaf.

For a sheaf $\mathcal{F}$ on a space $X$ we define the *support* of $\mathcal{F}$, denoted by $\operatorname{Supp}(\mathcal{F})$, by

$$\operatorname{Supp}(\mathcal{F}) = \left\{ P \in X \;\middle|\; \mathcal{F}_P \neq 0 \right\}.$$

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The support may not be closed in general, but it will be when $\mathcal{F}$ is a quasi-coherent sheaf of finite type, by the following example.

**Example 12.35.** If $X = \operatorname{Spec} A$ and $M$ is some finitely generated $A$-module, then

$$\operatorname{Supp}(\widetilde{M}) = \operatorname{Supp}(M) = \{\, \mathfrak{p} \mid M_{\mathfrak{p}} \neq 0 \,\} = V(\operatorname{Ann}(M))$$

where $\operatorname{Ann}(M)$ is the annihilator ideal

$$\operatorname{Ann}(M) = \{\, a \in A \mid a \cdot m = 0 \text{ for all } m \in M \,\}.$$

For instance, if $\mathfrak{a} \subset A$ is an ideal, then

$$\operatorname{Supp}(\widetilde{A/\mathfrak{a}}) = V(\mathfrak{a})$$

△

Let $X$ be a scheme and let $\iota\colon Y \to X$ be a closed embedding. Then $\iota$ is an affine morphism, so the pushfoward $\iota_*\mathcal{O}_Y$ is a quasi-coherent sheaf on $X$ (Corollary 12.32). If we define $\mathcal{I}$ to be the kernel of $\iota^\sharp$, we have the *ideal sheaf sequence*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Y \longrightarrow 0. \tag{12.31}$$

Note that $\mathcal{I}$ is quasi-coherent being the kernel of a map of quasi-coherent sheaves. So every closed subscheme determines a quasi-coherent ideal sheaf $\mathcal{I}$.

Conversely, if $\mathcal{I} \subset \mathcal{O}_X$ is a quasi-coherent ideal sheaf, we can construct a closed subscheme $V(\mathcal{I})$ of $X$ as follows. The underlying set of $V(\mathcal{I})$ will be the support

$$Z = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I}) \subset X.$$

If $U = \operatorname{Spec} A \subset X$ is any open affine, then $\mathcal{I}|_U = \widetilde{\mathfrak{a}}$ where $\mathfrak{a} = \mathcal{I}(U)$ is an ideal in $A = \mathcal{O}_X(U)$. As the tilde functor is exact, we have

$$\mathcal{O}_X/\mathcal{I}|_U = \widetilde{A/\mathfrak{a}} \tag{12.32}$$

Therefore, $Z \cap U = \operatorname{Supp}(A/\mathfrak{a}) = V(\mathfrak{a}) \subset U$. This shows that $Z$ is a closed subset of $X$. We equip $Z$ with the subspace topology.

Next, we define the structure sheaf $\mathcal{O}_Z$. The $\mathcal{O}_X$-module $\mathcal{O}_X/\mathcal{I}$ is a sheaf on $X$, but it can in fact be considered as a sheaf on $Z$: over each affine open set $U$, we may identify $Z \cap U$ with $\operatorname{Spec}(A/\mathfrak{a})$ and then $\mathcal{O}_X/\mathcal{I}|_U$ is identified with the structure sheaf on $\operatorname{Spec}(A/\mathfrak{a})$. We define $\mathcal{O}_Z$ be $\mathcal{O}_X/\mathcal{I}$, but viewed as a sheaf on $Z$. Note that the stalks of $\mathcal{O}_Z$ at a point $x \in Z$, are given by $\mathcal{O}_{X,x}/\mathcal{I}_x$, which is a quotient of a local ring, hence a local ring. Hence $(Z, \mathcal{O}_Z)$ is a locally ringed space. It is by construction a scheme, as if $U = \operatorname{Spec} A \subset X$ is affine, then $Z \cap U$ is isomorphic to the affine scheme $\operatorname{Spec}(A/\mathfrak{a})$. We shall denote this scheme by $V(\mathcal{I})$.

It is not hard to check that the assignments $\iota \mapsto \mathcal{I}$ and $\mathcal{I} \mapsto V(\mathcal{I})$ are inverses, so we get the following important theorem:

**Theorem 12.36.** For a scheme $X$, any closed subscheme is of the form $V(\mathcal{I}) = (\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}), \mathcal{O}_X/\mathcal{I})$ for some unique quasi-coherent ideal sheaf $\mathcal{I}$.

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In general, if $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf, the quotient $\mathcal{O}_X/\mathcal{I}$ is always the structure sheaf of a locally ringed space. The quasi-coherence of $\mathcal{I}$ is what guarantees that this in fact a scheme. (See Exercise 12.12.53 for an ideal sheaf which is not quasi-coherent.)

In the affine case, we get a new proof of Proposition 4.26:

**Corollary 12.37 (Closed subschemes of affine schemes).** The assignment $I \mapsto \operatorname{Spec}(A/I)$ gives a one-to-one correspondence between the set of ideals of a ring $A$ and the set of closed subschemes of $\operatorname{Spec} A$.

**Example 12.38.** If $Y$ and $Z$ are two closed subschemes defined by ideal sheaves $\mathcal{I}$ and $\mathcal{J}$ respectively, then the scheme-theoretic intersection is defined by the ideal sheaf $\mathcal{I} + \mathcal{J} \subset \mathcal{O}_X$. △

**Proposition 12.39.** Each closed subset $Z \subset X$ of a scheme is the support of a unique reduced closed subscheme.

*Proof* Define a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ by the formula

$$\mathcal{I}(U) = \{\, s \in \mathcal{O}_X(U) \mid s(P) = 0 \text{ for all } P \in Z \cap U \,\},$$

where as usual $s(P)$ denotes the image of the germ $s_P$ in the residue field $\kappa(P) = \mathcal{O}_{X,P}/\mathfrak{m}_P$. It is straightforward to check that $\mathcal{I}(U)$ is an ideal, and that this assignment defines an ideal sheaf. We claim that $\mathcal{I}$ is quasi-coherent.

If $U = \operatorname{Spec} A$ is affine, then $Z \cap U$ is a closed subset, and so it is of the form $V(I)$ for a unique radical ideal $I \subset A$. The ideal $I$ is equal to the intersection $I = \bigcap_{I \subset \mathfrak{p}} \mathfrak{p}$ of all the prime ideals containing $I$. This intersection is precisely the set of elements $a \in A$ that vanish at all points in $V(I)$, and hence $\mathcal{I}(U) = I$. Being a radical ideal is a property that localizes, so $\mathcal{I}(D(g)) = \widetilde{I}|_{D(g)}$ for all distinguished open subsets $D(g) \subset U$, and consequently $\mathcal{I}|_U = \widetilde{I}$. This shows that $\mathcal{I}$ is quasi-coherent.

The corresponding subscheme $Z$ has structure sheaf given by $\mathcal{O}_X/\mathcal{I}$. This scheme is reduced, because for each affine $U \subset X$, $\mathcal{I}(U)$ is radical, and so $\mathcal{O}_X(U)/\mathcal{I}(U)$ is a reduced ring.

The uniqueness statement is clear in the affine case, as the ideal defining $Z$ is uniquely determined by $Z$. In general, if $\mathcal{I}$ and $\mathcal{I}'$ are two quasi-coherent ideals as in the proposition, their restrictions to each open affine subset must be equal. This implies that the inclusion $\mathcal{I} \subset \mathcal{I} + \mathcal{I}'$ is an equality (using Lemma 11.9), and hence $\mathcal{I} = \mathcal{I}'$. □

## 12.11 Pullbacks

Let $f\colon X \to Y$ be a morphism of schemes. Recall that we defined the pushforward functor which produces an $\mathcal{O}_Y$-module $f_*\mathcal{F}$ from an $\mathcal{O}_X$-module $\mathcal{F}$. There is an opposite operation, called the *pullback*, which produces an $\mathcal{O}_X$-module $f^*\mathcal{G}$ on $X$ from an $\mathcal{O}_Y$-module $\mathcal{G}$ on $Y$.

The sheaf $f^*\mathcal{G}$ satisfies a universal property similar to that of $f^{-1}\mathcal{G}$. Namely, maps of $\mathcal{O}_X$-modules $f^*\mathcal{G} \to \mathcal{F}$ are in one-to-one correspondence with maps of $\mathcal{O}_Y$-modules $\mathcal{G} \to f_*\mathcal{F}$. The precise statement is the following theorem:

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**Theorem 12.40.** Let $f\colon X \to Y$ be a map of schemes and let $\mathcal{G}$ be an $\mathcal{O}_Y$-module. Then there exists an $\mathcal{O}_X$-module $f^*\mathcal{G}$ on $X$ along with functorial bijections

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) \tag{12.33}$$

for each $\mathcal{O}_X$-module $\mathcal{F}$. The sheaf $f^*\mathcal{G}$ is unique up to isomorphism.

By Yoneda's Lemma (Lemma 11.32), the pullback $f^*\mathcal{G}$ is determined up to isomorphism by the universal property (12.33). We will therefore refer to a sheaf satisfying this condition as the *pullback of $\mathcal{G}$ by $f$*.

### *Pullbacks of quasi-coherent sheaves on affines*

We begin by studying the pullback in the most important special case, namely for quasi-coherent sheaves via morphisms of affine schemes.

Suppose $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and $f\colon X \to Y$ is induced by a ring map $\phi\colon A \to B$. Consider a quasi-coherent sheaf of the form $\mathcal{G} = \widetilde{N}$ on $Y$, where $N$ is an $A$-module. As $B$ is an $A$-algebra, the tensor product $N \otimes_A B$ is naturally a $B$-module, and we define the pullback of $\mathcal{G}$ by the formula

$$f^*\widetilde{N} = \widetilde{N \otimes_A B}. \tag{12.34}$$

This defines a functor $f^*\colon \mathsf{QCoh}_Y \to \mathsf{QCoh}_X$. Indeed, any a map of quasi-coherent sheaves $\widetilde{N} \to \widetilde{N'}$ is induced by a map of $A$-modules $N \to N'$. This in turn induces a map of $B$-modules $N \otimes_A B \to N' \otimes_A B$ and consequently a map of $\mathcal{O}_X$-modules $f^*\widetilde{N} \to f^*\widetilde{N'}$.

In this case, the identity (12.33) is a consequence of the formal properties of Hom and the tensor product. More precisely, we recall the following natural bijection, which holds for all $A$-modules $N$ and $B$-modules $M$:

$$\operatorname{Hom}_B(N \otimes_A B, M) = \operatorname{Hom}_A(N, M_A) \tag{12.35}$$

This bijection sends a $B$-linear map $\phi$ on the left-hand side to the $A$-linear map $N \to M_A$ given by $n \mapsto \phi(n \otimes 1)$. This map is functorial in $M$ and $N$. (See Exercise 12.12.34.)

Using this, and Proposition 12.5, we have for any $\mathcal{O}_X$-module $\mathcal{F}$,

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) &= \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{\mathcal{G}(Y) \otimes_A B}, \mathcal{F}) \\
&= \operatorname{Hom}_A(\mathcal{G}(Y) \otimes_A B, \mathcal{F}(X)) \\
&= \operatorname{Hom}_A(\mathcal{G}(Y), \mathcal{F}(X)_A) \\
&= \operatorname{Hom}_{\mathcal{O}_Y}(\widetilde{\mathcal{G}(Y)}, f_*\mathcal{F}) \\
&= \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).
\end{aligned}$$

The fact that these isomorphisms are functorial in $\mathcal{F}$ and $\mathcal{G}$ follows from the functoriality of the isomorphism in (12.35) and the isomorphism $\beta_{\mathcal{F}}$ in Proposition 12.5.

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### *General pullbacks*

We will now define the pullback for an arbitrary $\mathcal{O}_Y$-module $\mathcal{G}$ for a general morphism of schemes $f\colon X \to Y$. To do this, we recall the image sheaf in Section 11.6. Given a sheaf $\mathcal{G}$ on $Y$, the inverse image sheaf $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$f_p^{-1}\mathcal{G}(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V).$$

If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, each $\mathcal{G}(V)$ is an $\mathcal{O}_Y(V)$-module for each $V \supset f(U)$. This means that $f_p^{-1}\mathcal{G}(U)$ inherits the structure of a module over the ring $f_p^{-1}\mathcal{O}_Y(U)$. Explicitly, over an open set $U \subset X$, we can represent elements $f_p^{-1}\mathcal{O}_Y(U)$ and $f_p^{-1}\mathcal{G}(U)$ by $a \in \mathcal{O}_Y(V)$ and $s \in \mathcal{G}(V)$ for some open set $V \subset Y$ containing $f(U)$. Then the product $a \cdot s \in \mathcal{G}(V)$ is well-defined, and its class defines an element in $f_p^{-1}\mathcal{G}(U)$.

Also the ring $\mathcal{O}_X(U)$ becomes a module over the ring $f_p^{-1}\mathcal{O}_Y(U)$ via the natural map $f_p^{-1}\mathcal{O}_Y(U) \to \mathcal{O}_X(U)$ which sends a pair $(a, V)$ to $f^\sharp(a)|_U$. We then define the presheaf $f_p^{-1}\mathcal{G}$ by

$$f_p^*\mathcal{G}(U) = f_p^{-1}\mathcal{G}(U) \otimes_{f_p^{-1}\mathcal{O}_Y(U)} \mathcal{O}_X(U)$$

Note that $f_p^*\mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module. We define $f^*\mathcal{G}$ to be the associated sheaf. It is an $\mathcal{O}_X$-module by Exercise 12.12.4.

While this definition of $f^*\mathcal{G}$ is general and natural, it is not very practical for concrete computations. The next step will be to show that $f^*\mathcal{G}$ satisfies the universal property (12.12.32). This will imply that when $\mathcal{G}$ is quasi-coherent, the sheaf $f^*\mathcal{G}$ restricts to the pullback sheaf defined earlier for quasi-coherent sheaves. It is this description that is used in most computations.

*Proof of Proposition 12.40* By the universal property of the inverse image presheaf (see Theorem 11.29), we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{PAb}(X)}(f_p^{-1}\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_{\mathsf{Ab}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

Under this bijection, $\mathcal{O}_Y$-linear maps $\mathcal{G} \to f_*\mathcal{F}$ on the right correspond to $f_p^{-1}\mathcal{O}_X$-linear maps $f_p^{-1}\mathcal{G} \to \mathcal{F}$ on the left. Furthermore, by the usual change-of-rings formula (12.35), we have a natural bijection

$$\mathrm{Hom}_{f_p^{-1}\mathcal{O}_Y}(f_p^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(f_p^{-1}\mathcal{G} \otimes_{f_p^{-1}} \mathcal{O}_X, \mathcal{F}).$$

Finally, by sheafification, $f_p^{-1}\mathcal{O}_Y$-linear maps $f_p^{-1}\mathcal{G} \to \mathcal{F}$ are in one-to-one correspondence with $f^{-1}\mathcal{O}_Y$-linear maps $f^{-1}\mathcal{G} \to \mathcal{F}$. □

**Example 12.41** (Restrictions)**.** Let $i\colon U \to X$ be the inclusion of an open subset and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then the pullback is given by the restriction $i^*\mathcal{F} = \mathcal{F}|_U$, which is naturally an $\mathcal{O}_U$-module. △

Here are a few useful properties of the pullback:

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**Proposition 12.42.** Let $f\colon X \to Y$ be a morphism of schemes. Then:

(i) $f^*\mathcal{O}_Y = \mathcal{O}_X$.

(ii) $f^*$ is a right-exact functor.

(iii) If $g\colon W \to X$ is a morphism, then

$$(f \circ g)^* = g^* \circ f^*.$$

(iv) For any two $\mathcal{O}_X$-modules $\mathcal{G}$ and $\mathcal{H}$, we have

$$f^*(\mathcal{G} \oplus \mathcal{H}) = f^*\mathcal{G} \oplus f^*\mathcal{H} \quad \text{and} \quad f^*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{H}) = f^*\mathcal{G} \otimes_{\mathcal{O}_X} f^*\mathcal{H}.$$

(v) For $x \in X$, we have

$$(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.$$

*Proof* (i): To show that $f^*\mathcal{O}_Y = \mathcal{O}_X$, it suffices to show that $\mathcal{O}_X$ satisfies the required universal property (12.33). This follows from the identities

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{O}_Y, \mathcal{F}) &= \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, f_*\mathcal{F}) \\ &= f_*\mathcal{F}(Y) = \mathcal{F}(X) \\ &= \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})\end{aligned}$$

Also the item (iii) follows from the universal property. The sheaf $f^*(g^*\mathcal{G})$ is an $\mathcal{O}_X$-module, and for any $\mathcal{O}_X$-module $\mathcal{F}$, we have

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_X}(f^*(g^*\mathcal{G}), \mathcal{F}) &= \operatorname{Hom}_{\mathcal{O}_Y}(g^*\mathcal{G}, f_*\mathcal{F}) \\ &= \operatorname{Hom}_{\mathcal{O}_Z}(\mathcal{G}, g_*f_*\mathcal{F}) \\ &= \operatorname{Hom}_{\mathcal{O}_Z}(\mathcal{G}, (g \circ f)_*\mathcal{F}).\end{aligned}$$

Here all equalities are canonical bijections. Therefore $f^*(g^*\mathcal{G})$ and $(g \circ f)^*\mathcal{G}$ are isomorphic. Similar arguments show the identities in (iv).

(iv): the description of the stalks of $f^*\mathcal{G}$ follows because $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$, and $(f^{-1}\mathcal{O}_Y)_x = \mathcal{O}_{X,x}$, so that

$$(f^*\mathcal{G})_x = (\mathcal{G}_{f(x)}) \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.$$

Here we have also used the fact that sheafification preserves stalks.

(ii): If $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ is an exact sequence of $\mathcal{O}_Y$-modules, then the induced sequence obtained by applying $f^*$ is obtained by applying $f_p^{-1}(-)$ (which is right-exact) and then sheafifying (which is exact). □

**Proposition 12.43.** Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{G}$ be a quasi-coherent sheaf on $Y$. Then $f^*\mathcal{G}$ is quasi-coherent on $X$ and for each pair of affine open subsets $U \subset X$, $V \subset Y$ such that $f(U) \subset V$, we have

$$f^*\mathcal{G}|_U = \widetilde{\mathcal{G}(V) \otimes_{\mathcal{O}_Y(V)} \mathcal{O}_X(U)} \tag{12.36}$$

*Proof* This follows formally from the universal property. Let $i\colon U \to X$, $j\colon V \to Y$

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denote the inclusions and write $g = f|_U$. Then we have bijections

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_U}(g^*j^*\mathcal{G}, \mathcal{F}) &= \operatorname{Hom}_{\mathcal{O}_V}(j^*\mathcal{G}, g_*\mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, j_*g_*\mathcal{F}) \\ &= \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*i_*\mathcal{F}) = \operatorname{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, i_*\mathcal{F}) \\ &= \operatorname{Hom}_{\mathcal{O}_U}(i^*f^*\mathcal{G}, \mathcal{F}).\end{aligned}$$

This shows that $i^*f^*\mathcal{G} \simeq g^*j^*\mathcal{G}$, which by Example 12.41 means that $f^*\mathcal{G}|_U = (f|_U)^*(\mathcal{G}|_V)$. But by the affine case, the pullback $(f|_U)^*(\mathcal{G}|_V)$ is isomorphic to the sheaf to the right in (12.36). □

### *Pulling back sections*

As for the inverse image sheaf, the universal property (12.33) induces canonical maps

$$\eta\colon \mathcal{G} \longrightarrow f_*f^*\mathcal{G} \qquad \text{and} \qquad \varepsilon\colon f^*f_*\mathcal{F} \longrightarrow \mathcal{F}. \tag{12.37}$$

Here $\eta$ is map of $\mathcal{O}_Y$-modules and $\varepsilon$ is a map of $\mathcal{O}_X$-modules. These are obtained by applying (12.33) to the two identity maps $f^*\mathcal{G} \to f^*\mathcal{G}$ and $f_*\mathcal{F} \to f_*\mathcal{F}$.

For a section $s \in \mathcal{G}(V)$, we define the *pullback* to be the section

$$f^*(s) = \eta(s) \in \Gamma(f^{-1}V, f^*\mathcal{G}).$$

In the affine setting, the adjunction maps take a concrete form:

**Example 12.44.** Let $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ be induced by a ring map $A \to B$, and $\mathcal{F} = \widetilde{M}$ and $\mathcal{G} = \widetilde{N}$. Then $\eta\colon \widetilde{N} \to f_*f^*\widetilde{N}$ corresponds to the map of $A$-modules

$$N \to (N \otimes_A B)_A;\ n \mapsto (n \otimes 1).$$

Likewise, $\varepsilon\colon f_*f^*\widetilde{M} \to \widetilde{M}$ corresponds to the map of $B$-modules

$$M_A \otimes_A B \longrightarrow M;\ m \otimes b \longmapsto bm$$

For $s \in \mathcal{G}(Y) = N$, the pullback $f^*(s) \in (f^*\mathcal{G})(X) = N \otimes_A B$ is given by $s \otimes 1$. △

For a general morphism, we can compute $f^*(s)$ by covering $V$ and $f^{-1}V$ by affine schemes, and then reducing to the affine case.

**Example 12.45.** In the special case when $\mathcal{G} = \mathcal{O}_Y$, we have $f^*\mathcal{O}_Y = \mathcal{O}_X$ and the pullback of $s \in \mathcal{O}_Y(V)$ is simply given by

$$f^*(s) = f^\sharp(s) \in \mathcal{O}_X(f^{-1}V).$$

△

**Example 12.46.** Let $X = \operatorname{Spec} k[x, y]$, $Y = \operatorname{Spec} k[u, v]$, and the morphism

$$f\colon \operatorname{Spec} k[x, y] \longrightarrow \operatorname{Spec} k[u, v]; \qquad (u, v) \mapsto (x^2, y^2)$$

The map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ acts on sections by composition: it sends a local section $a(u, v)/b(u, v)$ to $a(x^2, y^2)/b(x^2, y^2)$.

In this example, there is no natural $\mathcal{O}_X$-structure on $f^{-1}\mathcal{O}_Y$: the element $x^2$ defines a section of $f^{-1}\mathcal{O}_Y$ over $D(x)$, but multiplying this with $y/x \in \mathcal{O}_X(D(x))$ gives $xy$, which is

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not of the form $a(x^2, y^2)/b(x^2, y^2)$. This is the reason why one considers the tensor product with $\mathcal{O}_X$ in the definition of $f^*\mathcal{O}_X$. △

**Example 12.47.** Let $X$ be a scheme over a field $k$ and let $f\colon X \to \operatorname{Spec} k$ be the structure morphism. Then for an $\mathcal{O}_X$-module $\mathcal{F}$, we have $f_*\mathcal{F} = \mathcal{F}(X)$ (the constant sheaf on $\operatorname{Spec} k$), and $f^{-1}f_*\mathcal{F} = \mathcal{F}(X)$ is the constant sheaf on $X$. The canonical map $f^*f_*\mathcal{F} \to \mathcal{F}$ is given by the 'multiplication map'

$$\mathcal{F}(X) \otimes_k \mathcal{O}_X(U) \longrightarrow \mathcal{F}(U)$$

induced by $s \otimes f \mapsto f \cdot s$. △

**Example 12.48.** While the inverse image functor $f^{-1}$ is exact, the pullback $f^*$ is only right-exact. The reason is that tensoring by a module is only a right-exact operation in general. Consider for instance the ideal sheaf sequence of the origin $P \in \mathbb{A}^1_k = \operatorname{Spec} k[t]$:

$$0 \longrightarrow \mathcal{I}_P \longrightarrow \mathcal{O}_{\mathbb{A}^1_k} \longrightarrow \iota_*\mathcal{O}_P \longrightarrow 0. \tag{12.38}$$

where $\iota\colon \operatorname{Spec} \kappa(P) \to \mathbb{A}^1_k$ is the closed embedding. This sequence is obtained by applying tilde to the sequence

$$0 \longrightarrow k[t] \xrightarrow{\ t\ } k[t] \longrightarrow k[t]/(t) \longrightarrow 0.$$

Applying $\iota^*$ corresponds to tensoring by $\kappa(P) = k[t]/(t)$. Applying this to (12.38) the sequence is the tilde of the following sequence of $\kappa(p)$-modules

$$0 \longrightarrow k[t] \otimes_{k[t]} \kappa(P) \xrightarrow{t\otimes \mathrm{id}} k[t] \otimes_{k[t]} \kappa(P) \longrightarrow k[t]/(t) \otimes_{k[t]} \kappa(P) \longrightarrow 0.$$

However the latter sequence is not exact because the map $t \otimes \mathrm{id}$ equals the zero map. △

## 12.12 Exercises

**Exercise 12.12.1.** Let $A = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ and $X = \operatorname{Spec} A$. Describe all $\mathcal{O}_X$-modules, all quasi-coherent and all coherent sheaves on $X$.

**Exercise 12.12.2.** Let $X = \mathbb{A}^1_{\mathbb{C}}$ and let $\mathcal{F}$ be the constant sheaf on $\mathbb{Z}$. Is $\mathcal{F}$ an $\mathcal{O}_X$-module?

**Exercise 12.12.3.** For each of the schemes below, describe the $\mathcal{O}_X$-modules on $X$.

a) $X$ is the scheme obtained by gluing $\operatorname{Spec} \mathbb{Z}_{(2)}$ and $\operatorname{Spec} \mathbb{Z}_{(3)}$ along their common open subscheme $\operatorname{Spec} \mathbb{Q}$.
b) $X$ is the scheme obtained by gluing two copies of $\operatorname{Spec} \mathbb{Z}_{(2)}$ along $\operatorname{Spec} \mathbb{Q}$.
c) Let $X$ be the scheme obtained by gluing the schemes $X_i = \operatorname{Spec} \mathbb{Z}_{(p_i)}$ together along their common open subschemes $\operatorname{Spec} \mathbb{Q}$. Describe the $\mathcal{O}_X$-modules on $X$.

**Exercise 12.12.4.** Suppose that $\mathcal{F}$ is a *presheaf* of $\mathcal{O}_X$-modules (i.e. a presheaf satisfying the usual $\mathcal{O}_X$-module axioms). Show that the associated sheaf $\mathcal{F}^+$ is an $\mathcal{O}_X$-module in a natural way. HINT: One can use the universal properties of sheafification, or the explicit description of $\mathcal{F}^+$.

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**Exercise 12.12.5.** Let $X$ be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on $X$. Show that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(X)$.

**Exercise 12.12.6.** Show in detail that the kernel, cokernel and image of a map of $\mathcal{O}_X$-modules indeed are $\mathcal{O}_X$-modules.

**Exercise 12.12.7.** Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of sheaves.

a) Show that $\mathrm{Ker}\,\phi$ satisfies the following universal property: Any map of sheaves $\nu : \mathcal{H} \to \mathcal{F}$ such that $\nu \circ \phi = 0$ factors via a unique map $\overline{\nu} : \mathcal{H} \to \mathrm{Ker}\,\phi$.
b) Show that $\mathrm{Im}\,\phi$ satisfies the following universal property: Given a map of sheaves $\alpha : \mathcal{F} \to \mathcal{H}$ and $\beta : \mathcal{H} \to \mathcal{G}$ such that $\beta \circ \alpha = \phi$, there is a unique morphism $t : \mathcal{H} \to \mathrm{Im}\,\phi$ factoring $\beta$.
c) Show that $\mathrm{Coker}\,\phi$ satisfies the following universal property: Given a map $\psi\colon \mathcal{G} \to \mathcal{H}$ with $\psi \circ \phi = 0$, there is a unique map $t : \mathrm{Coker}\,\phi \to \mathcal{H}$ factoring $\psi$.
d) Show that if $\phi$ is a map of $\mathcal{O}_X$-modules, the corresponding universal properties in a),b) and c) hold in the category of $\mathcal{O}_X$-modules as well.
e) Show that a sequence of $\mathcal{O}_X$-modules is exact if and only it is exact as a sequence of sheaves.

HINT: The arguments in a), b) and c) are rather different. For b), use the explicit description of $\mathrm{Im}\,\phi$. For c), the universal property of sheafification may be helpful.

**Exercise 12.12.8.** Show that the sequence (11.8) is exact. HINT: Show that it is exact on stalks.

**Exercise 12.12.9** (General direct sums)**.** Let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of sheaves. We define the direct sum $\bigoplus_{i\in I} \mathcal{F}_i$ as the sheaf associated to the presheaf

$$\left(\bigoplus_{i\in I} \mathcal{F}_i\right)'(U) = \bigoplus_{i\in I} \mathcal{F}_i(U). \tag{12.39}$$

a) Show that if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, then $\bigoplus_{i\in I} \mathcal{F}_i$ is an $\mathcal{O}_X$-module.
b) Show that the direct sum has canonical inclusions $\epsilon_i\colon \mathcal{F}_i \to \bigoplus_i \mathcal{F}_i$, which have the following universal property: for any family of maps $\eta_i\colon \mathcal{F}_i \to \mathcal{G}$ there is a unique map $\eta\colon \bigoplus_i \mathcal{F}_i \to \mathcal{G}$ such that $\eta_i = \eta \circ \epsilon_i$.
c) Show that if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, then the sheaves $\bigoplus_{i\in I} \mathcal{F}_i$, and the maps $\eta_i$ and $\epsilon_i$ are maps of $\mathcal{O}_X$-modules.
d) Show that if $X = \mathrm{Spec}\,A$, then for all $A$-modules $M_i$, we have $\widetilde{\bigoplus_{i\in I} M_i} \simeq \bigoplus_{i\in I} \widetilde{M}_i$..
e) Show that if the $\mathcal{F}_i$ are quasi-coherent, then $\bigoplus_{i\in I} \mathcal{F}_i$ is again quasi-coherent and for each open *affine* $U \subset X$,

$$\left(\bigoplus_{i\in I} \mathcal{F}_i\right)(U) = \bigoplus_{i\in I} \mathcal{F}_i(U).$$

f) Let $X = \coprod_{n=1}^{\infty} \mathrm{Spec}\,\mathbb{C}$ be the disjoint union of countably many copies of $\mathrm{Spec}\,\mathbb{C}$. For each $n \in \mathbb{N}$, let $\iota_n\colon p_n = \mathrm{Spec}\,\mathbb{C} \to X$ be the open embedding of

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the $n$-th copy of $\operatorname{Spec}\mathbb{C}$ and let $\mathcal{F}_n = \iota_{n*}\mathbb{C}$ the skyscraper sheaf at $p_n$. Consider the direct sum presheaf $\mathcal{S} = \bigoplus_{n=1}^{\infty} \mathcal{F}_n$. Show that $\mathcal{S}(X) \neq \bigoplus_{n=1}^{\infty} \mathcal{F}_n(X)$. Deduce that $\mathcal{S}$ is not a sheaf.

g) Let $\{\mathcal{F}_i\}$ be a family of sheaves on $X$ and $U \subset X$ an open set. If $U$ is quasi-compact, show that $(\bigoplus_i \mathcal{F}_i)(U) = \bigoplus_i \mathcal{F}_i(U)$.

h) Conclude that if $X$ is Noetherian, then the presheaf defined in (12.39) is a sheaf.

**Exercise 12.12.10** (Direct products)**.** Let $\{\mathcal{F}_i\}_{i\in I}$ be a collection of sheaves.

a) Show that the direct product presheaf $\prod_{i\in I} \mathcal{F}_i$ defined in the text is a sheaf.

b) Show that $\prod_{i\in I} \mathcal{F}_i$ is an $\mathcal{O}_X$-module if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules.

c) Show that the direct product has canonical projections $\pi_i \colon \prod_i \mathcal{F}_i \to \mathcal{F}_i$ having the universal property: for any collection of maps $\epsilon_i \colon \mathcal{F}_i \to \mathcal{G}$ there is a map $\eta \colon \mathcal{F}_i \to \prod_i \mathcal{F}_i$ such that $\pi_i \circ \eta = \epsilon_i$.

d) Let $X = \operatorname{Spec} A$ and let $\{M_i\}_{i\in I}$ be a collection of $A$-modules. Show that $\widetilde{\prod_i M_i}$ satisfies the universal property of $\prod_{i\in I} \widetilde{M}_i$ in the subcategory $\mathsf{QCoh}_X$.

e) Let $X = \operatorname{Spec}\mathbb{Z}$ and $M_i = \mathbb{Z}$ for $i \in \mathbb{N}$. Show that $\prod_{i\in\mathbb{N}} \widetilde{M}_i$ and $\widetilde{\prod_{i\in\mathbb{N}} M_i}$ are not isomorphic. Deduce that a direct product of quasi-coherent sheaves need not be quasi-coherent in general. HINT: Infinite direct products do not commute with localization.

**Exercise 12.12.11.** Show that the direct sum sheaf can be defined as the image sheaf of the natural map $\bigoplus_{i\in I} \mathcal{F}_i \to \prod_{i\in I} \mathcal{F}_i$ where the left-hand side is regarded as a presheaf. HINT: Use Example 11.17 and the universal property of $\bigoplus$.

**Exercise 12.12.12.** Let $X$ be a scheme. A directed system of sheaves is a family of sheaves $\{\mathcal{F}_i\}_{i\in I}$ indexed by a directed set $I$, together with morphisms of sheaves $\phi_{ij} \colon \mathcal{F}_i \to \mathcal{F}_j$ for each $i \leqslant j$ such that (i) $\phi_{ii}$ is the identity map on $F_i$; and (ii) For $i \leqslant j \leqslant k$, we have $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$. We define the direct limit sheaf as the sheaf associated to the presheaf

$$(\varinjlim \mathcal{F}_i)(U) = \varinjlim \mathcal{F}_i(U). \tag{12.40}$$

a) Show that there are sheaf maps $\phi_i \colon \mathcal{F}_i \to \varinjlim \mathcal{F}_i$ satisfying the following compatability condition: For all $i \leqslant j$, we have $\phi_j \circ \phi_{ij} = \phi_i$.

b) Show that $\varinjlim \mathcal{F}_i$ satisfies the following universal property: if there exists another sheaf $\mathcal{G}$ with morphisms $\psi_i \colon \mathcal{F}_i \to \mathcal{G}$ satisfying the compatibility condition, then there exists a unique morphism of sheaves $u \colon \mathcal{F} \to \mathcal{G}$ such that $u \circ \phi_i = \psi_i$ for all $i \in I$.

c) Show that if the $\mathcal{F}_i$ are $\mathcal{O}_X$-modules, then $\varinjlim \mathcal{F}_i$ is naturally an $\mathcal{O}_X$-module.

d) Show that if each $\mathcal{F}_i$ is quasi-coherent, show that $\varinjlim \mathcal{F}_i$ is quasi-coherent, and:
   (i) For each affine open set $U \subset X$, $(\varinjlim \mathcal{F}_i)(U) = \varinjlim \mathcal{F}_i(U)$.
   (ii) For each $x \in X$, we have $(\varinjlim \mathcal{F}_i)_x = \varinjlim (\mathcal{F}_i)_x$.

**Exercise 12.12.13** (Godement sheaves)**.** Given any collection of abelian groups $\{A_x\}_{x\in X}$ indexed by the points $x$ of $X$, we can define a sheaf $\mathcal{A}$ by

$$\mathcal{A}(U) = \prod_{x\in U} A_x,$$

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and whose restriction maps to smaller open subsets are just the projections onto the corresponding smaller products.

a) Show that $\mathcal{A}$ is a sheaf.
b) If we assume that each $A_x$ be a module over the local ring $\mathcal{O}_{X,x}$, show that sheaf $\mathcal{A}$ becomes an $\mathcal{O}_X$-module.

**Exercise 12.12.14.** For a presheaf $\mathcal{F}$ consider the sheaf $\Pi(\mathcal{F})$ defined by $\Pi(\mathcal{F})(U) = \prod_{x\in U} \mathcal{F}_x$. This is sometimes called the 'sheaf of discontinuous sections'.

Show that there is a canonical map of presheaves $\sigma : \mathcal{F} \to \Pi(\mathcal{F})$ so that $\mathcal{F}^+ = \operatorname{Im}\sigma$.

**Exercise 12.12.15.** Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Show that the assignment $U \mapsto \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}(U), \mathcal{G}(U))$ does not define a presheaf in general.

**Exercise 12.12.16** (Stalks of the Hom-sheaf)**.** Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathcal{O}_X$-modules.

a) For $x \in X$, show that there is a natural map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \longrightarrow \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \tag{12.41}$$

b) Show that the map (12.41) may fail to be surjective. HINT: Let $\mathcal{F}$ be a skyscraper sheaf and $\mathcal{G}$ a constant sheaf; then $\mathcal{H}om(F, G)$ is the zero-sheaf.
c) Show that the map (12.41) may fail to be injective. HINT: Let $U = X - x$ and let $\mathcal{F} = \mathcal{G} = i_!\mathbb{Z}$, where $i : U \to X$ is the inclusion; then $\mathcal{F}_x = \mathcal{G}_x = 0$.

**Exercise 12.12.17.** If $s \in \mathcal{F}(U)$, we define the *support of* $s$ as the subset of points $x \in X$ such that $s_x \in \mathcal{F}_x$ is non-zero.

a) Show that the support of a section $s \in \mathcal{F}(U)$ is closed in $U$.
b) If $s, t \in \mathcal{F}(U)$, show that $\operatorname{Supp}(st) \subset \operatorname{Supp}(s) \cap \operatorname{Supp}(t)$ and that $\operatorname{Supp}(s+t) \subset \operatorname{Supp}(s) \cup \operatorname{Supp}(t)$.
c) Show that if $\phi\colon \mathcal{F} \to \mathcal{G}$ is a map of $\mathcal{O}_X$-modules, then $\operatorname{Supp}(\phi(s)) \subset \operatorname{Supp}(s)$.

**Exercise 12.12.18.** For a sheaf $\mathcal{F}$ on a space $X$, we define the *support* as the subset of points $x \in X$ such that the stalk $\mathcal{F}_x$ is nonzero.

a) Show that if $\mathcal{F}$ is a quasi-coherent sheaf of finite type, then $\operatorname{Supp}(\mathcal{F})$ is closed.
b) Let $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec}\mathbb{C}[t]$, and consider the $\mathbb{C}[t]$-module

$$M = \bigoplus_{i\in\mathbb{N}} \mathbb{C}[t]/(t-i).$$

Compute $\operatorname{Supp}(\widetilde{M})$ and show that it is not closed.
c) Show that if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-algbras, then $\operatorname{Supp}(\mathcal{F})$ is closed. HINT: A ring is the zero ring if and only if $1 = 0$.

**Exercise 12.12.19** (Subsheaf with support)**.** Let $Z \subset X$ be a closed subset and $\mathcal{F}$ a sheaf on $X$. For an open set $U \subset X$, define $\Gamma_{Z\cap U}(U, \mathcal{F})$ to be the subgroup of $\mathcal{F}(U)$ consisting of sections $s$ with $\operatorname{Supp}(s) \subset Z \cap U$. Show that this is a sheaf. This is usually denoted by $\mathcal{H}^0_Z(\mathcal{F})$. Show that the functor $\mathcal{F} \mapsto \Gamma_{Z\cap U}(U, \mathcal{F})$ is left exact.

**Exercise 12.12.20.** Let $X$ be the nodal cubic curve and consider the normalization morphism $f\colon \mathbb{A}^1_k \to X$ of Example 10.37, which identifies the two points 0 and 1 in $\mathbb{A}^1_k$.

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a) Show that the presheaf $f_p^{-1}\mathcal{O}_X$ consists of the sections of $\mathcal{O}_{\mathbb{A}^1_k}$ which take the same value on 0 and 1. Show that this is not a sheaf.
b) Compute $f^{-1}\mathcal{O}_X$ and $f^*\mathcal{O}_X$.

**Exercise 12.12.21.** Show that a sheaf $\mathcal{F}$ on a scheme $X$ is quasi-coherent if and only if there is an open cover $\{U_i\}_{i\in I}$ such that each of the restrictions $\mathcal{F}|_{U_i}$ may be presented as the cokernel of a map between free $\mathcal{O}_X$-modules; that is, they appear in exact sequences

$$\mathcal{O}_{U_i}^J \longrightarrow \mathcal{O}_{U_i}^I \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0,$$

where $\mathcal{G}^I$ stands for the direct sum $\bigoplus_{i\in I}\mathcal{G}$ of copies of a sheaf $\mathcal{G}$ (and where $I$ and $J$ may be infinite and dependent on $i$). Conclude that being quasi-coherent is a local property for an $\mathcal{O}_X$-module.

**Exercise 12.12.22.** Let $X = \operatorname{Spec} A$ be an affine scheme and let

$$\cdots \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow \mathcal{F}_{i+1} \longrightarrow \cdots$$

be an exact sequence of quasi-coherent sheaves. Show that

$$\cdots \longrightarrow \mathcal{F}_{i-1}(X) \longrightarrow \mathcal{F}_i(X) \longrightarrow \mathcal{F}_{i+1}(X) \longrightarrow \cdots$$

is also exact.

**Exercise 12.12.23.** Let $X = \operatorname{Spec} A$ and consider an distinguished open subscheme $D(g) \subset X$. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Show that:

(i) If $s \in \mathcal{F}(X)$ and $s|_{D(g)} = 0$, then $g^n s = 0$ for some $n \in \mathbb{N}$.
(ii) If $s$ is a section of $\widetilde{M}$ over $D(g)$, then for some $n \in \mathbb{N}$ the section $g^n s$ extends to $X$; that is, there is a $t \in \mathcal{F}(X)$ so that $t|_{D(g)} = g^n s$.

HINT: Use the formula $\mathcal{F}(D(g)) = \mathcal{F}(X)_g$.

**Exercise 12.12.24.** Show the following:

(i) The skyscraper sheaf of $k$ on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ at the origin 0 is quasi-coherent.
(ii) The skyscraper sheaf of $k(t)$ on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ at the origin 0 is *not* quasi-coherent. HINT: Consider sections over $U = D(t)$.

**Exercise 12.12.25.** Let $X \subset \mathbb{A}^3_k = \operatorname{Spec} k[x,y,z]$ be the twisted cubic curve, defined the ideal $I = (y - x^2, z - x^3)$. Consider the morphism $\pi\colon X \to \mathbb{A}^1_k = \operatorname{Spec} k[z]$ induced by the ring map $k[z] \to k[x,y,z]/I$.

(i) Show that $\pi$ is a finite morphism.
(ii) Compute the sheaves $\pi_*\mathcal{O}_C$, $\pi^*\mathcal{O}_{\mathbb{A}^1_k}$ and $\pi^*\mathcal{J}$ where $\mathcal{J}$ is the ideal sheaf of the closed point $0 \in \mathbb{A}^1_k$.

**Exercise 12.12.26** (Flat morphisms)**.** Let $f\colon X \to Y$ be a morphism of schemes and let $x \in X$ be a point. We say that $f$ is flat if the map $f_x^\sharp\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ into a flat $\mathcal{O}_{Y,f(x)}$-module for every $x \in X$.

(i) Show that open embeddings are flat. What about closed embeddings?
(ii) Show that a morphism of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat if and only if the map of rings $A \to B$ is flat. More generally, a quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Spec} B$ is flat over $\operatorname{Spec} A$ if and only if $M$ is flat as an $A$-module.

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(iii) Which of the morphisms in Exercise 2.7.28 are flat?
(iv) Prove that the blow-up morphism $\pi\colon Bl_0\mathbb{A}^2 \to \mathbb{A}^2$ is not flat.

**Exercise 12.12.27** (Flat base change)**.** Let $A$ be a ring and let $B$ be a flat $A$-algebra. Let $X$ be an $A$-scheme and let $X_B$ denote the base change to $B$. Show that for any quasi-coherent sheaf $\mathcal{F}$, there is a natural isomorphism

$$\Gamma(X_B, \mathcal{F}_B) = \Gamma(X, \mathcal{F}) \otimes_A B$$

where $\mathcal{F}_B = q^*\mathcal{F}$ and $q\colon X_B \to X$ is morphism obtained by base change. HINT: Tensor the sheaf exact sequence.

**Exercise 12.12.28** (Morphisms to a closed subscheme)**.** Let $Z$ be a closed subscheme of $X$ given by sheaf of ideals $\mathcal{I}$. Suppose $f\colon Y \to X$ is a morphism of schemes. Show that $f$ factors through a map $g\colon Y \to Z$ if and only if
(i) $f(Y) \subset Z$.
(ii) $\mathcal{I} \subset \operatorname{Ker}(f^\sharp\colon \mathcal{O}_X \to f_*(\mathcal{O}_Y))$.
For a morphism of schemes $f\colon Y \to X$, we can define the *scheme-theoretic image* of $f$ as a subscheme $Z \subset X$ satisfying the universal property that if $f$ factors through a subscheme $Z' \subset Z$, then $Z \subset Z'$. To define $Z$ it is tempting to use the ideal sheaf $\mathcal{I} = \operatorname{Ker}(\mathcal{O}_X \to f_*(\mathcal{O}_Y))$, but this may fail to be quasi-coherent for a general morphism $f$. Show however that if $X$ and $Y$ are Noetherian, then it is quasi-coherent.

**Exercise 12.12.29.** Let $X$ be a scheme.
a) Show that the direct sum of two $\mathcal{O}_X$-modules of finite type sheaves is again of finite type.
b) Show that the '2-out-of-3'-property holds for $\mathcal{O}_X$-module of finite type. That is, if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, and if two of $\mathcal{F}, \mathcal{F}', \mathcal{F}''$ are of finite type, then so is the third.
c) Let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of $\mathcal{O}_X$-module of finite type. Show that $\operatorname{Ker}\phi$, $\operatorname{Im}\phi$ and $\operatorname{Coker}\phi$ are all of finite type. HINT: Apply Exercise 12.12.29.

**Exercise 12.12.30.** Let $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules and let $x \in X$ be a point. Show that

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.$$

**Exercise 12.12.31.** Use Yoneda's lemma for sheaves to prove the identity $\widetilde{M \otimes_A N} = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$. HINT: Proposition 12.5 may also be useful.

**Exercise 12.12.32.** Let $f\colon X \to Y$ be a morphism of schemes, $\mathcal{F}$ an $\mathcal{O}_X$-module, and $\mathcal{G}$ an $\mathcal{O}_Y$-module.
a) Given a map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{F}$, show that $f^\sharp$ induces a map of $\mathcal{O}_Y$-modules $\mathcal{O}_Y \to f_*\mathcal{F}$.
b) Given a map of $\mathcal{O}_Y$-modules $\mathcal{O}_Y \to f_*\mathcal{F}$, show that there is an induced map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{F}$.
c) Show that the constructions in a) and b) are inverse to each other and conclude that $f^*\mathcal{O}_Y = \mathcal{O}_X$.

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**Exercise 12.12.33.** Let $f\colon X \to Y$ be a morphism and let $\mathcal{G}$ a be a quasi-coherent $\mathcal{O}_Y$-module. Show that the stalk of $f^*\mathcal{G}$ at $x \in X$ is given by $\mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$.

**Exercise 12.12.34.** Show that the map in (12.35) is a bijection, and that the isomorphism is functorial in $M$ and $N$.

**Exercise 12.12.35.** Show directly that the pullback $f^*\mathcal{G}$ can be constructed by gluing together sheaves of the form $\widetilde{M \otimes_A B}$.

**Exercise 12.12.36.** Let $f\colon X \to Y$ be a morphism of schemes and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type.

a) Show that $f_*\mathcal{F}$ need not be of finite type in general.
b) Show that if $f$ is finite, then $f_*\mathcal{F}$ is an finite type $\mathcal{O}_Y$-module. In particular, this holds for closed embeddings.

**Exercise 12.12.37.** Let $f\colon X \to Y$ be a morphism and let $\mathcal{J}$ be an ideal sheaf of $\mathcal{O}_Y$.

a) Show that there is a map $f^*\mathcal{J} \to \mathcal{O}_X$
b) Give an example to show that $f^*\mathcal{J}$ needs not be an ideal of $\mathcal{O}_X$ in general.
c) Let $\mathcal{I} \subset \mathcal{O}_X$ denote the image of $f^*\mathcal{J}$ in $\mathcal{O}_X$. Show that $\mathcal{I}$ is an ideal sheaf and that
$$\operatorname{Supp}(\mathcal{O}_X/\mathcal{I}) = f^{-1}\operatorname{Supp}(\mathcal{O}_Y/\mathcal{J})$$
HINT: Reduce to the affine case.

**Exercise 12.12.38.** Consider the affine plane $\mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ over a field $k$ and let $P \in \mathbb{A}^2_k$ be the closed point corresponding to the origin. Define the sheaf $\mathcal{I}$ by
$$\mathcal{I}(U) = \begin{cases} 0 & P \in U \\ \mathcal{O}_X(U) & P \notin U \end{cases}$$

a) Show that $\mathcal{I}$ is an ideal sheaf and that $\operatorname{Supp}(\mathcal{O}_X/\mathcal{I})$ is a closed subset of $X$.
b) Show that $\mathcal{I}$ is not quasi-coherent. In fact, show that $\mathcal{I}$ is not even locally generated by sections. HINT: The idea is that any section of $\mathcal{I}$ is 0 in any neighbourhood of $P$, but $\mathcal{F}$ is not 0 when restrcted to any neighourhood of $P$.
c) Show directly that $\mathcal{I}$ is not quasi-coherent by showing that $\mathcal{I}(X) = 0$, but $\mathcal{I} \neq 0$.
d) Show that there is an exact sequence
$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow i_*\mathcal{O}_{\mathbb{A}^2_k,P} \longrightarrow 0$$
where $i_*\mathcal{O}_{\mathbb{A}^2_k,P}$ is the skyscraper sheaf on $P$.
e) Show that $i_*\mathcal{O}_{\mathbb{A}^2_k,P}$ is an $\mathcal{O}_X$-module of finite type, but not quasi-coherent.

**Exercise 12.12.39.** Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms and let $\mathcal{G}$ be a sheaf on $Z$. Show that $f^{-1}(g^{-1}\mathcal{G}) = (g \circ f)^{-1}\mathcal{G}$.

**Exercise 12.12.40.** Let $X$ be a scheme and let $\mathcal{F}$ be a quasi-coherent sheaf of finite type. Let $s_1, \dots, s_n$ be sections of $\mathcal{F}$ and consider the map of $\mathcal{O}_X$-modules $\rho\colon \mathcal{O}_X^n \to \mathcal{F}$ sending $e_i$ to $s_i$. Show that $\rho$ is surjective in a neighbourhood of $x$ if and only if the images $s_1(x), \dots, s_n(x)$ generate $\mathcal{F}(x)$. HINT: Apply the previous exercise to the cokernel.

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**Exercise 12.12.41** (Torsion sheaves)**.** Let $X$ be an integral scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Define for each open set $U \subset X$, a subgroup $T(U) \subset \mathcal{F}(U)$ consisting of all the elements $m \in \mathcal{F}(U)$ such that the germ $m_x$ is torsion in $\mathcal{F}_x$ for all $x \in X$, i.e., $a_x \cdot m_x = 0$ for some non-zero $a_x \in \mathcal{O}_{X,x}$.

a) Show that $T$ is a subsheaf of $\mathcal{F}$. Also, show that $T$ is quasi-coherent. $T$ is called the *torsion subsheaf of* $\mathcal{F}$. Another notation for it is $\mathcal{F}_{\text{tors}}$.

b) Let $\mathscr{K}$ denote the constant sheaf on $K = K(X)$. Define a map of sheaves

$$\nu : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_X} \mathscr{K}\,.$$

Show that $T = \operatorname{Ker} \nu$.

c) A sheaf is called *torsion free if* $\mathcal{F}_{\text{tors}} = 0$. Show that the quotient $\mathcal{F}/T$ is always torsion free, i.e., $(\mathcal{F}/T)_{\text{tors}} = 0$.

d) Show that any locally free sheaf is torsion free.

**Exercise 12.12.42.** Let $X = \operatorname{Spec} \mathbb{Z}$ and consider the $\mathbb{Z}$-modules $M = \bigoplus_{\mathbb{N}} \mathbb{Z}$ and $N = \mathbb{Z}$. Show that $\mathcal{H}om_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ is not quasi-coherent. HINT: Use the identity $\operatorname{Hom}_A(\bigoplus_{\mathbb{N}} A, A) = \prod_{\mathbb{N}} A$.

**Exercise 12.12.43.** Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$ and let $U \subset X$ be an open set. Show that the restriction $\mathcal{F}|_U$ is also quasi-coherent.

**Exercise 12.12.44.** Let $X$ be a Noetherian scheme. Show that the direct sum presheaf $\bigoplus_{i \in I} \mathcal{F}_i$ of (**??**) is a sheaf.

**Exercise 12.12.45.** Let $X$ be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on $X$. Show that $\mathcal{F}$ is quasi-coherent if and only if for any pair $V \subset U$ open affine subsets the natural map

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \to \mathcal{F}(V) \tag{12.42}$$

that sends $s \otimes g$ to $gs|_V$, is an isomorphism.

**Exercise 12.12.46.** Show that a morphism $f\colon X \to Y$ of schemes is a closed embedding if and only if it is affine and map $\iota^\sharp\colon \mathcal{O}_Y \to \iota_* \mathcal{O}_X$ is surjective.

**Exercise 12.12.47.** Find a scheme $X$ such that $f_* \mathcal{O}_X$ is not quasi-coherent sheaf on $\operatorname{Spec} \mathbb{Z}$, where $f\colon X \to \operatorname{Spec} \mathbb{Z}$ is the canonical map.

**Exercise 12.12.48.** Let $\phi\colon \mathcal{F} \to \mathcal{G}$ be a map of quasi-coherent sheaves. Show that $\phi$ is injective (resp. surjective) if and only if $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective (resp. surjective) for each affine open $U \subset X$.

**Exercise 12.12.49.** A morphism $f\colon X \to Y$ is called *quasi-compact* if for every affine open subset $U \subset Y$, the inverse image $f^{-1}(U)$ is quasi-compact. $f$ is called *quasi-separated* if for any pair of affine open subsets $U, V \subset Y$, the intersection $f^{-1}(U) \cap f^{-1}(V)$ can be covered by finitely many affine open subsets.

a) Show that the properties of being quasi-compact and quasi-separated are stable under base change.

b) Let $f\colon X \to Y$ be a morphism which is both quasi-compact and quasi-separated. Show that $f_* \mathcal{F}$ is quasi-coherent if $\mathcal{F}$ is.

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**Exercise 12.12.50.** Let $X$ be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module.

a) Let $\mathcal{P}$ be the following property of an open affine set $U \subset X$: the canonical map $\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$. Show that $\mathcal{P}$ is a distinguished property.
b) Give another proof of Theorem 12.9.

**Exercise 12.12.51.** Show that a quasi-coherent sheaf $\mathcal{F}$ on a scheme $X$ is of finite type if and only if there exists an affine open cover $\{U_i\}_{i\in I}$ with $U_i = \operatorname{Spec} A_i$ such that for each $i$, $\mathcal{F}|_{U_i} \simeq \widetilde{M}_i$ for some finitely generated $A_i$-module. HINT: See the proof of Proposition 6.49.

**Exercise 12.12.52** (The 2-out-of-3 property)**.** Let $X$ be a scheme and consider a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{12.43}$$

Show that if two of the sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are quasi-coherent, then the third is quasi-coherent as well.

HINT: Two of the three implications are straightforward. For the third, the most direct way is to use a future result in Chapter 15: that in Corollary 12.11, exactness holds even if only the leftmost sheaf is quasi-coherent.

**Exercise 12.12.53** (A non-quasi-coherent ideal sheaf)**.** Let $X = \operatorname{Spec} A$ where $A = k[t]_{(t)}$. Let $\xi$ be the generic point and let $s$ be the generic point. Define a presheaf $\mathcal{I}$ by $\mathcal{I}(\{\xi\}) = \mathcal{I}(\varnothing) = 0$ and $\mathcal{I}(X) = A$.

a) Show that $\mathcal{I}$ is an ideal sheaf.
b) Show that the quotient presheaf $\mathcal{O}_X/\mathcal{I}$ is a sheaf and compute the sections of $\mathcal{O}_X/\mathcal{I}$ over $\varnothing$, $\{\xi\}$ and $X$.
c) Show that the support of $\mathcal{O}_X/\mathcal{I}$ is equal to $\{\xi\}$.
d) Does $\mathcal{I}$ correspond to a closed subscheme?
e) Show directly that $\mathcal{I}$ is not quasi-coherent.

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# 13

# Sheaves on projective space

Projective schemes are often preferred over affine ones due to their more favorable properties. However, affine schemes exhibit better behavior in certain aspects. For instance, while morphisms of affine schemes are completely determined by ring homomorphisms, projective morphisms lack such a straightforward characterization. Also, for affine schemes, closed subschemes correspond bijectively to ideals in the coordinate ring. For projective schemes, however, multiple homogeneous ideals (even in different gradings) can define the same closed subscheme.

## 13.1 Quasi-coherent sheaves on projective schemes

Let $R$ be a graded ring. If $M$ is a graded $R$-module, we define a sheaf $\widetilde{M}$ by setting, for each $D_+(f) \subset X$ where $f$ is homogeneous of positive degree,

$$\Gamma(D_+(f), \widetilde{M}) = (M_f)_0$$

For an inclusion $D_+(fg) \subset D_+(f)$, there is a canonical localization map of graded modules $(M_f)_0 \to (M_{fg})_0$. Taking degree 0 parts, we have a canonical map

$$(M_f)_0 \longrightarrow (M_g)_0. \tag{13.1}$$

this defines a restriction map $\Gamma(D_+(f), \widetilde{M}) \to \Gamma(D_+(fg), \widetilde{M})$. The two $\mathscr{B}$-sheaf axioms are easily verified, for instance by taking the degree zero part of the sheaf sequence (**??**) for the sheaf $\widetilde{M}$ on $\operatorname{Spec} R$. We continue to denote this sheaf by $\widetilde{M}$.

**Proposition 13.1.** The sheaf $\widetilde{M}$ is a quasi-coherent sheaf on $\operatorname{Proj} R$.

*Proof* The argument is very similar to the proof that $\mathcal{O}_{\operatorname{Proj} R}|_{D_+(f)} \simeq \mathcal{O}_{\operatorname{Spec}(R_f)_0}$ on page **??**. The main thing to note is that Lemma **??** holds by replacing $R$ by an $R$-module $M$, which implies that the restriction of $\widetilde{M}$ to $D_+(f)$ and $\widetilde{(M_f)_0}$ on takes the same values on distinguished opens $D_+(fg)$ contained in $D_+(f)$. □

Note that a map of graded modules $M \to N$ induces maps between the degree 0 part of the localizations $(M_f)_0 \to (N_f)_0$. These are compatible with further localizations, so we get induced maps of $\mathcal{O}_X$-modules $\widetilde{M} \to \widetilde{N}$. It follows that the graded tilde operation $M \mapsto \widetilde{M}$ defines a functor from the category of graded $R$-modules to quasi-coherent sheaves on $X$.

Each element $m \in M_0$ determines a section of $\Gamma(D_+(f), \widetilde{M}) = (M_f)_0$, by taking the



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image of $m$ in $(M_f)_0$. If $g$ is another homogenous element, then the two sections clearly agree when restricted to the overlaps $D_+(fg)$, (they both map to the image of $m$ in $(M_{fg})_0$). Therefore, they glue together to a global section of $\widetilde{M}$.

The following proposition summarizes the basic properties of the tilde-functor.

**Proposition 13.2.** Let $R$ be a graded ring. The graded tilde functor $M \mapsto \widetilde{M}$ has the following properties:

(i) It is additive and exact.
(ii) For each $\mathfrak{p} \in \operatorname{Proj} R$, we have $(\widetilde{M})_{\mathfrak{p}} = (M_{\mathfrak{p}})_0$.
(iii) If $M$ is finitely generated, then $\widetilde{M}$ is of finite type.

*Proof* The item (i) holds because localization and taking degree zero parts are exact operations that commute with forming direct limits and direct sums. The second item is a consequence of $\widetilde{M}|_{D_+(f)} = \widetilde{(M_f)_0}$; that a prime $\mathfrak{p} \in D_+(f) \subset \operatorname{Proj} R$ corresponds to $\mathfrak{q} = (\mathfrak{p}R_f)_0$; and that we have the equality

$$\widetilde{M}_{\mathfrak{p}} = ((M_f)_0)_{\mathfrak{q}} = (M_{\mathfrak{q}})_0 = (M_{\mathfrak{p}})_0.$$

If $M$ is finitely generated, then choosing a graded surjection $R^n \to M \to 0$, we get $\mathcal{O}_X^n \to \widetilde{M} \to 0$, showing $\widetilde{M}$ is of finite type. □

From now on, we will let $X \subset \mathbb{P}_A^n$ be a projective scheme over $A$. That is $X = \operatorname{Proj} R$, where $R$ is a graded ring of the form

$$R = A[x_0, \ldots, x_n]/\mathfrak{a} \qquad (\bigstar)$$

where $x_i$ have degree 1 and $\mathfrak{a} \subset A[x_0, \ldots, x_n]$ is a homogeneous ideal. (This assumption is not too restrictive, see Exericse 13.1.1).

**Exercise 13.1.1.** Let $R = \bigoplus_{d \geqslant 0} R_d$ be a graded ring which is finitely generated as an algebra over $A = R_0$, and let $X = \operatorname{Proj} R$. By considering a suitable Veronese subring $R^{(e)}$, show that $X$ is isomorphic to $\operatorname{Proj} S$ where $S$ is of the form $A[x_0, \ldots, x_n]/\mathfrak{a}$ where the $x_i$ have degree 1.

In many aspects the projective tilde-functor behaves as the one of $\operatorname{Spec}$, but there are also many important differences. The most striking is that different non-isomorphic modules can give rise to the same sheaf. This can be explained from the fact that primes contained in $V(R_+)$ are disregarded in the Proj-construction – this has the effect that graded modules supported in $V(R_+)$ must give the zero sheaf after applying the tilde-functor.

For any integer $d$, we let $M_{>d}$ be the $R$-module $M_{>d} = \bigoplus_{i>d} M_i$.

**Lemma 13.3.** Let $M$ and $N$ be two graded $R$-modules,

(i) $\widetilde{M} = 0$ if and only if $\operatorname{Supp} M \subset V(x_0, \ldots, x_n)$.
(ii) Assume that $M_{>d} \simeq N_{>d}$ for some $d$. Then $\widetilde{M} \simeq \widetilde{N}$. The converse also holds if $M$ and $N$ are finitely generated.

*Proof* To prove (i), suppose that $\operatorname{Supp} M \subset V(R_+)$. Then item (ii) of Proposition 13.2

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implies that the stalks of $\widetilde{M}$ are zero for every $\mathfrak{p} \in \operatorname{Proj}(R)$, and hence $\widetilde{M} = 0$. Conversely, if the support of $M$ is not contained in $V(R_+)$, there is a homogeneous prime ideal $\mathfrak{p} \in \operatorname{Proj} R$ such that $M_{\mathfrak{p}} \neq 0$. When $R$ is generated in degree 1, this in turn implies that $(M_{\mathfrak{p}})_0 \neq 0$. Indeed, $\operatorname{Proj} R$ is covered by distinguished open sets $D_+(f)$ where $f$ has degree 1, and so there is an $f$ of degree 1 not lying in $\mathfrak{p}$. Then for an non-zero homogeneous element $x \in M_{\mathfrak{p}}$, the element $x/f^{\deg x}$ yields a non-zero element of degree zero in $M_{\mathfrak{p}}$.

To prove (ii), note that the quotient $N = M/M_{>d}$ is a graded module which is killed by the power $(R_+)^d$ and consequently has support in $V(R_+)$. By (i), we have $\widetilde{N} = 0$, and hence $\widetilde{M_{>d}} = \widetilde{M}$. As this holds for $N$ as well, we get $\widetilde{M} \simeq \widetilde{M_{>d}} \simeq \widetilde{N_{>d}} \simeq \widetilde{N}$. □

**Example 13.4.** On $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$, the module $M = k[x_0, x_1]/(x_0^2, x_1^2)$ satisfies $\widetilde{M} = 0$, but it is non-zero. △

### *Tensor products*

Let $M$ and $N$ be two graded $R$-modules. There is a natural way of defining a grading on the tensor product $M \otimes_R N$, by defining the degree $d$ piece $(M \otimes_R N)_d$ to be the additive subgroup of $M \otimes_R N$ generated by elements $x \otimes y$ where $x \in M_p$ and $y \in N_q$ where $p + q = d$. One checks that $M \otimes_R N$ is the direct sum of these graded parts (as an $R_0$-module).

**Example 13.5.** Let $R = k[x]$ with the standard grading, and let $M = N = R$. Then $(M \otimes_R N)_d = k[x]_d$. Note that the graded pieces of $(M \otimes_R N)_d$ are not equal to $\bigoplus_{p+q=d}(M_p \otimes_{R_0} N_q)$; that would give a vector space isomorphic to $k[x, y]_d$. △

**Proposition 13.6.** For graded $R$-modules $M$ and $N$, there is a natural isomorphism

$$\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Proj} R}} \widetilde{N} \longrightarrow \widetilde{M \otimes_R N}$$

*Proof* For each homogeneous element $f \in R$ of positive degree, there is a canonical map

$$M_f \otimes_{(R_f)_0} N_f \to M_f \otimes_{R_f} N_f \simeq (M \otimes_R N)_f$$

induced by sending $x/f^n \otimes y/f^m$ to $(x \otimes y)/f^{n+m}$. When restricted to elements of degree zero, we get a map

$$(M_f)_0 \otimes_{(R_f)_0} (N_f)_0 \to ((M \otimes_R N)_f)_0, \tag{13.2}$$

which one checks is compatible with the restriction maps induced from inclusions $D_+(g) \subset D_+(f)$, and so it is a map of $\mathscr{B}$-sheaves with $\mathscr{B}$ being the basis of distinguished open subsets. Therefore, we get a natural map of sheaves

$$\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Proj} R}} \widetilde{N} \longrightarrow \widetilde{M \otimes_R N}. \tag{13.3}$$

This is an isomorphism, because it is an isomorphism over each $f$ of degree 1. □

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## 13.2 The sheaves $\mathcal{O}(m)$

The most important sheaves on $\operatorname{Proj} R$ are the so-called *twisting sheaves*, denoted by $\mathcal{O}_{\operatorname{Proj} R}(d)$. These generalize the sheaves $\mathcal{O}_{\mathbb{P}^n_A}(d)$ on $\mathbb{P}^n_A$ which were introduced in Chapter 17.33. Just as homogeneous polynomials in $A[x_0, \ldots, x_n]$ define global sections of $\mathcal{O}_{\mathbb{P}^n_A}(d)$, the degree $d$ elements of $R$ will define sections of the sheaf $\mathcal{O}_{\operatorname{Proj} R}(d)$.

If $M$ is a graded $R$-module, and $n$ is an integer, we define an $R$-module $M(n)$ as follows: the underlying $R$-module of $M(n)$ is just $M$, but the grading is shifted by $d$ steps:

$$M(n)_d = M_{d+n}. \tag{13.4}$$

That is, $N = M(n)$ is a graded $R$-module such that $N_0 = M_n$, $N_1 = M_{n+1}$ and so on, and elements from $M_d$ considered as elements in $N$ will be of degree $d - n$.

In the special case when $M = R$, this gives a graded and free $R$-module $R(n)$, which is generated by the element $1 \in R_{-n}$. Note the canonical isomorphism $M(n) = M \otimes_R R(n)$: both have $M$ as underlying module, and the gradings coincide by the way we have defined the grading on the tensor product. Moreover, for each $m, n \in \mathbb{Z}$, we have

$$R(n) \otimes_R R(m) = R(n + m). \tag{13.5}$$

Applying the tilde-functor to $R(n)$ gives us a quasi-coherent $\mathcal{O}_{\operatorname{Proj} R}$-module on $\operatorname{Proj} R$:

**Definition 13.7.** For each integer $n \in \mathbb{Z}$, and for $X = \operatorname{Proj} R$, we define

$$\mathcal{O}_X(n) = \widetilde{R(n)}.$$

For an $\mathcal{O}_X$-module $\mathcal{F}$, we define the *twist of $\mathcal{F}$ by $n$* to be the $\mathcal{O}_X$-module

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

If $h \in R$ is homogeneous of degree $d$, then it defines a section of $\mathcal{O}_X(d)$ via the map in Lemma **??**. Abusing notation slightly, we will continue to write $h$ for this section.

For an $\mathcal{O}_X$-module $\mathcal{F}$, the tensor product gives us natural multiplication maps

$$\mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{F}(e) \longrightarrow \mathcal{F}(d + e)$$

In the case $\mathcal{F} = \widetilde{M}$, this is simply the map of sheaves induced by the multiplication maps

$$(R_f)_d \otimes_{(R_f)_0} (M_f)_e \longrightarrow (M_f)_{d+e}; \quad h/f^a \otimes m/f^b \mapsto (h \cdot m)/f^{a+b}$$

over each distinguished open.

The sheaf $\mathcal{O}_X(m)$ is an invertible sheaf. As $x_i$ is invertible in $R_{x_i}$, we have $R_{x_i} \cdot x_i^m \simeq R_{x_i}$. In degree 0, we find that

$$(R(m)_{x_i})_0 = (R_{x_i})_m = x_i^m \cdot (R_{x_i})_0.$$

Therefore, on the distinguished affine open set $D_+(x_i)$ we see that

$$\mathcal{O}_X(m)|_{D_+(x_i)} = x_i^m \cdot \mathcal{O}_X|_{D_+(x_i)}.$$

Moreover, for each $m, n \in \mathbb{Z}$, there are canonical isomorphisms

$$\mathcal{O}_X(m + n) \simeq \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

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For graded rings with non-standard gradings, one can still define $\widetilde{M}$ and $\mathcal{O}_X(m)$ but some of the results are more complicated. For instance, it can happen that $\widetilde{M} \otimes \widetilde{N} \not\simeq \widetilde{M \otimes_R N}$ and $\mathcal{O}(m)$ is not be locally free. Here is an example:

**Example 13.8.** Consider the polynomial ring $R = k[x_0, x_1, x_2]$ with a grading defined by $\deg(x_0) = 1$, $\deg(x_1) = 2$, and $\deg(x_2) = 3$. After localizing at $x_2$, the degree zero part of $R_{x_2}$ is given by

$$(R_{x_2})_0 = k\left[\frac{x_0^3}{x_2}, \frac{x_0x_1}{x_2}, \frac{x_1^3}{x_2^2}\right].$$

We claim that $\widetilde{M} \otimes \widetilde{N} \neq \widetilde{M \otimes N}$ for the two graded $R$-modules $M = R(1)$ and $N = R(1)$. Of course, we have $R(1) \otimes R(1) = R(2)$. However, we have the following equalities of $(R_{x_2})_0$-modules:

$$(R(1)_{x_2})_0 = (R_{x_2})_0 \cdot x_0 + (R_{x_2})_0 \cdot x_1^2/x_2$$
$$(R(2)_{x_2})_0 = (R_{x_2})_0 \cdot x_1 + (R_{x_2})_0 \cdot x_0^2$$

Next, if we compute the tensor product of $(R(1)_{x_2})_0$ with itself over $(R_{x_2})_0$ we get an $(R_{x_2})_0$-module which does not contain the monomial $x_1$. However, $x_1$ clearly belongs to $(R(2)_{x_2})_0$. This shows that $\widetilde{R(1)} \otimes \widetilde{R(1)} \not\simeq \widetilde{R(2)}$, as the two sheaves are different over the open set $U = D_+(x_2)$. △

## 13.3 Sheaves on projective schemes

Here is the main theorem of this section:

**Theorem 13.9.** Every quasi-coherent sheaf $\mathcal{F}$ on $X$ is of the form $\widetilde{M}$ for some graded $R$-module $M$. If $\mathcal{F}$ is of finite type, then $\mathcal{F} \simeq \widetilde{M}$ for some finitely generated graded $R$-module.

In the course of the proof, we will see that we may take $M$ to be the module

$$\Gamma_*(\mathcal{F}) = \bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{F}(d)).$$

This is called the *associated graded module* to $\mathcal{F}$. The $R$-module structure is defined via the multiplication maps

$$\mathcal{O}_X(d) \otimes \mathcal{F}(e) \to \mathcal{F}(d+e).$$

More precisely, if $a \in R_d$, and $s \in \Gamma_*(\mathcal{F})_e$, we regard $a$ as a section of $\mathcal{O}_X(d)$ and define the product $h \cdot s$ to be the image of $h \otimes s$ in $\Gamma(X, \mathcal{F}(d+e)) = \Gamma_*(\mathcal{F})_{d+e}$.

*Proof* Let $U = \operatorname{Spec} R - V(x_0, \ldots, x_n)$, and consider the two morphisms $\iota\colon U \to \operatorname{Spec} R$ and $\pi\colon U \to X$. These fit into the diagram:

$$\begin{array}{ccc} U & \xrightarrow{\ \iota\ } & \operatorname{Spec} R \\ \downarrow{\scriptstyle \pi} & & \\ X & & \end{array}$$

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Since $\mathcal{F}$ is quasi-coherent on $X$, the pullback $\pi^*\mathcal{F}$ is quasi-coherent on $U$. As $\iota$ is quasi-compact, the pushforward $\iota_*\pi^*\mathcal{F}$ is quasi-coherent on $\operatorname{Spec} R$. Therefore,

$$\iota_*\pi^*\mathcal{F} = \widetilde{M}$$

for some $R$-module $M$. We can identify this module as

$$M = \Gamma(\operatorname{Spec} R, \iota_*\pi^*\mathcal{F}) = \Gamma(U, \pi^*\mathcal{F}).$$

The next lemma identifies this group of global sections.

**Lemma 13.10.** There is a natural isomorphism of $\mathcal{O}_X$-modules $\pi_*\pi^*\mathcal{F} = \bigoplus_{d\in\mathbb{Z}} \mathcal{F}(d)$.

*Proof* We use the standard affine covering of $X$ by the distinguished open sets $D_+(x_i)$. For each $i$, the preimage $\pi^{-1}(D_+(x_i))$ is the distinguished open set $D(x_i)$ in $\operatorname{Spec} R$, and the collection of these form an open cover of $U$. The projection $\pi|_{D(x_i)}\colon D(x_i) \to D_+(x_i)$ is induced by the inclusion $(R_{x_i})_0 \subset R_{x_i}$. The key point is that $R_{x_i}$ decomposes as an $(R_{x_i})_0$-module as

$$R_{x_i} = \bigoplus_{d\in\mathbb{Z}} (R_i)_0 \cdot x_i^d.$$

The restriction of $\pi^*\mathcal{F}$ to $D(x_i)$ is the sheaf associated to the $(R_{x_i})$-module

$$\begin{aligned} \Gamma(D(x_i), \pi^*\mathcal{F}) &= \Gamma(D_+(x_i), \mathcal{F}) \otimes_{(R_{x_i})_0} R_{x_i} \\ &= \bigoplus_{d\in\mathbb{Z}} \Gamma(D_+(x_i), \mathcal{F}) \cdot x_i^d. \end{aligned} \tag{13.6}$$

Now, for each twist $\mathcal{F}(d)$, we have

$$\begin{aligned} \Gamma(D_+(x_i), \mathcal{F}(d)) &= \Gamma(D_+(x_i), \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(d)) \\ &= \Gamma(D_+(x_i), \mathcal{F}) \otimes_{(R_{x_i})_0} \Gamma(D_+(x_i), \mathcal{O}_X(d)) \\ &= \Gamma(D_+(x_i), \mathcal{F}) \otimes_{(R_{x_i})_0} (R_{x_i})_0 \cdot x_i^d \\ &= \Gamma(D_+(x_i), \mathcal{F}) \cdot x_i^d \end{aligned} \tag{13.7}$$

Comparing (13.6) and (13.7), we obtain a natural isomorphism

$$\Gamma(D_+(x_i), \pi_*\pi^*\mathcal{F}) = \bigoplus_{d\in\mathbb{Z}} \Gamma(D_+(x_i), \mathcal{F}(d)). \tag{13.8}$$

The isomorphisms (13.8) induce isomorphisms

$$\pi_*\pi^*\mathcal{F}|_{D_+(x_i)} \longrightarrow \bigoplus_{d\in\mathbb{Z}} \mathcal{F}(d)|_{D_+(x_i)}$$

The isomorphisms over $D(x_i)$ and $D(x_j)$ restrict to the same isomorphism

$$\Gamma(D_+(x_ix_j), \pi_*\pi^*\mathcal{F}) = \bigoplus_{d\in\mathbb{Z}} \Gamma(D_+(x_ix_j), \mathcal{F}(d)).$$

over the overlap $D(x_ix_j)$. Therefore, we obtain an isomorphism $\pi_*\pi^*\mathcal{F} \to \bigoplus_{d\in\mathbb{Z}} \mathcal{F}(d)$. □

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As a byproduct of the above proof, we see that $M = \Gamma(U, j_*\pi^*\mathcal{F})$ has the structure of a *graded* $R$-module, with degree $d$-component equal to $\Gamma(X, \mathcal{F}(d))$. Hence we may consider the sheaf $\widetilde{M}$ on $X$.

We claim that $\mathcal{F} \simeq \widetilde{M}$. To show this, it suffices to show that the two sheaves agree over each $D_+(f)$ where $f \in R$ is homogeneous of positive degree. For such an $f$, we have

$$\begin{aligned}
\Gamma(D_+(f), \widetilde{M}) &= \Gamma(D(f), \widetilde{M})_0 \\
&= \Gamma(D(f), \iota_*\pi^*\mathcal{F})_0 \\
&= \Gamma(D(f), \pi^*\mathcal{F})_0 \\
&= \Gamma(D_+(f), \pi_*\pi^*\mathcal{F})_0 \\
&= \Gamma(D_+(f), \bigoplus_{d\in\mathbb{Z}} \mathcal{F}(d))_0 \\
&= \Gamma(D_+(f), \mathcal{F}).
\end{aligned}$$

These isomorphisms are compatible with restrictions to $D_+(fg)$ so they define an isomorphism $\widetilde{M} \simeq \mathcal{F}$.

□

**Corollary 13.11.** Let $X \subset \mathbb{P}^n_A$ be a projective scheme. If $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, then there exists an integer $d > 0$ and a surjective map of $\mathcal{O}_X$-modules

$$\mathcal{O}_X(-d)^{\oplus N} \longrightarrow \mathcal{F}.$$

To summarize, we have defined two functors, $\widetilde{(-)}\colon \mathsf{GrMod}_R \to \mathsf{QCoh}_X$ and $\Gamma_*\colon \mathsf{QCoh}_X \to \mathsf{GrMod}_R$. Since $\widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ is an isomorphism, it follows that the tilde functor is essentially surjective; that is, every quasi-coherent sheaf on $X$ is the tilde of a graded module. However, unlike the affine case, the functors do not give mutual inverses. The functor $\sim$ is not faithful as the tilde of any module $M$ with support in $V(R_+)$ is the zero sheaf. By Lemma 13.3 however, this is the only source of ambiguity.

The module $\Gamma_*(\mathcal{F})$ plays the role of the global sections functor on $\operatorname{Spec} A$. For projective schemes, it is not enough to consider the global sections alone; one has to consider the global sections of all twists of $\mathcal{F}$ at once, to be able to recover $\mathcal{F}$.

**Exercise 13.3.1.** Let $X = \operatorname{Proj} R$, where $R = A[x_0, \dots, x_n]/\mathfrak{a}$ where the $x_i$ have degree 1.

a) Show that if each $x_i$ is a nonzerodivisor, then

$$\bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)) = \bigcap_{i=0}^{n} R_{x_i} \subset R_0[x_0, x_0^{-1}, \dots, x_n, x_n^{-1}]. \tag{13.9}$$

b) Show that if each $x_i$ is a prime element, then $\bigoplus_{d\in\mathbb{Z}} \Gamma(X, \mathcal{O}_X(d)) = R$.

When $R$ is not a polynomial ring, it can easily happen that $\Gamma_*(\mathcal{O}_X)$ is different than $R$.

**Exercise 13.3.2.** Consider $X = \mathbb{P}^0_A = \operatorname{Proj} A[t]$. Then $\mathcal{O}(1) \simeq \mathcal{O}$, so $\Gamma(X, \mathcal{O}_X(m)) = A$ for all $m$, even $m < 0$. So $\Gamma_*(\mathcal{O}_X) \neq A[t]$.

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**Example 13.12.** Let $R$ be $\mathbb{C}[x_0, x_1]$, but with with the degree 0 piece replaced by $R_0 = \mathbb{Z}$. Then $X = \operatorname{Proj} R$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, because $(R_{x_0})_0 = \mathbb{C}[x_1/x_0]$ and $(R_{x_1})_0 = \mathbb{C}[x_0/x_1]$. However, $\Gamma_*(X) = \mathbb{C}[x_0, x_1]$, which is not isomorphic to $R$. Note that $R$ is not finitely generated by over $R_0$ in this example, so the proposition above does not apply. △

**Example 13.13** (A quartic rational space curve)**.** A systematic way of producing examples of rings so that $\Gamma_*(\mathcal{O}_X) \neq R$, is to start with a closed subscheme $X \subset \mathbb{P}^n_k$ and project it into $\mathbb{P}^{n-1}_k$. In some cases this will again be a closed embedding of $X$, but in this new embedding $X$ will be equal to $\operatorname{Proj} S$ with a different graded ring $S$.

The simplest example of this set-up is the rational normal quartic curve in $\mathbb{P}^4_k$. This scheme is defined as $X = \operatorname{Proj} R$ where $R$ is the ring

$$R = k[u^4, u^3v, u^2v^2, uv^3, v^4] \subset k[u, v],$$

where all of the indicated monomials are given degree 1. Note that this equals the Veronese ring $k[u, v]^{(4)}$ from Section **??**.

Projecting into a lower projective space corresponds to discarding some of the generators, and in our example we discard the monomial $u^2v^2$ and work with $X = \operatorname{Proj} S$ where

$$S = k[u^4, u^3v, uv^3, v^4].$$

Evidently, $S_1$ is of dimension 4, and we shall se that the monomial $u^2v^2$ reappears in $\Gamma(X, \mathcal{O}_X(1))$, and so $\Gamma(X, \mathcal{O}_X(1))$ will be of dimension 5 as a $k$-vector space.

Let us compute $\Gamma(X, \mathcal{O}_X(1))$ using the sheaf sequence, using the open affine cover consisting of $U_0 = D_+(u^4)$ and $U_1 = D_+(v^4)$ (these two cover $X$ because $S^4_+ \subset (u^4, v^4)$). Moreover, we have equalities $\mathcal{O}_X(U_0) = (S_{u^4})_0 = k[v/u]$ and $\mathcal{O}_X(U_1) = k[u/v]$. We have isomorphisms $\mathcal{O}_X(1)|_{U_0} \simeq \mathcal{O}_{U_0} \cdot u^4$ and $\mathcal{O}_X(1)|_{U_1} \simeq \mathcal{O}_{U_1} \cdot v^4$. The sheaf sequence then takes the form

$$0 \to \Gamma(X, \mathcal{O}_X(1)) \to k[v/u]u^4 \oplus k[u/v]v^4 \to k[u/v, v/u]u^4.$$

Note that $u^2v^2 = (v/u)^2u^4 = (u/v)^2v^4$, so the monomial $u^2v^2$ belongs to both the rings $k[u/v]v^4$ and $k[v/u]u^4$ and hence defines a global section in $\Gamma(X, \mathcal{O}_X(1))$. In fact,

$$\Gamma(X, \mathcal{O}_X(1)) = ku^4 \oplus ku^3v \oplus ku^2v^2 \oplus kuv^3 \oplus v^4.$$

Therefore $\Gamma(X, \mathcal{O}_X(1))$ contains all 5 monomials, while $u^2v^2$ is missing from $S_1$. In this example, the graded ring $\Gamma_*(\mathcal{O}_X)$ is isomorphic to $R$, and $R$ is the integral closure of $S$. Exercise 13.7.5 shows that this is not a coincidence. △

**Exercise 13.3.3.** Let $X = \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. Show that the functor $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$ is not right exact, by considering the exact sequence

$$0 \longrightarrow R(-n) \longrightarrow R \longrightarrow R/(x_0^n) \longrightarrow 0.$$

## 13.4 Closed subschemes of projective space

We saw in Chapter **??**, that every homogeneous ideal $\mathfrak{a} \subset A[x_0, \ldots, x_n]$ determines a closed subscheme $V_+(\mathfrak{a}) \subset \mathbb{P}^n_A$. The main result of this section is that every closed subscheme is of this form. However, as in the discussion on quasi-coherent sheaves, it can happen that different ideals correspond to the same subscheme.

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**Example 13.14.** In $k[x_0, x_1]$, the two ideals $\mathfrak{a} = (x_0)$ and $\mathfrak{b} = (x_0^2, x_0x_1)$ both define the closed subscheme given by the point $(0:1)$ in $\mathbb{P}^1_k$. To see this, note that $\mathfrak{b}$ dehomogenizes to $(x_0^2/x_1^2, x_0x_1/x_1^2) = (x_0/x_1)$ in $k[x_1/x_0]$ and to the unit ideal in $k[x_0/x_1]$. Note that $\mathfrak{b} = (x_0^2, x_1) \cap (x_0)$. So while both $\mathfrak{a}$ and $\mathfrak{b}$ define the same subscheme of $\mathbb{P}^1_k$, the ideal $\mathfrak{b}$ is in some sense inferior to $\mathfrak{a}$, as it has a component in the irrelevant ideal $(x_0, x_1)$. △

To deal with this ambiguity, we introduce the *saturation* of an ideal $\mathfrak{a} \subset A[x_0, \ldots, x_n]$ with respect to the irrelevant ideal $(x_0, \ldots, x_n)$, defined as

$$\mathfrak{a}^{\mathrm{sat}} = \{f \in A[x_0, \ldots, x_n] \;\mid\; \exists N \in \mathbb{N} \text{ such that } f \cdot x_i^N \in \mathfrak{a} \text{ for every } i\}$$

We always have an inclusion $\mathfrak{a} \subset \mathfrak{a}^{\mathrm{sat}}$, and we say that $\mathfrak{a}$ is *saturated* if $\mathfrak{a} = \mathfrak{a}^{\mathrm{sat}}$.

In the example above, the saturation of $(x_0^2, x_0x_1)$ is the ideal $(x_0)$. This example is typical: taking the saturation throws away components of $\mathfrak{a}$ supported in the irrelevant ideal. In particular, the saturation of the irrelevant ideal $(x_0, \ldots, x_n)$ is the unit ideal.

Consider a homogeneous ideal $\mathfrak{a} \subset A[x_0, \ldots, x_n]$ and its associated closed subscheme $Y = V_+(\mathfrak{a})$, with ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n_A}$. Note that $\widetilde{\mathfrak{a}} = \mathcal{I}_Y$: this follows from comparing the ideal sheaf sequence with the exact sequence of $A$-modules

$$0 \longrightarrow \mathfrak{a} \longrightarrow A[x_0, \ldots, x_n] \longrightarrow A[x_0, \ldots, x_n]/\mathfrak{a} \longrightarrow 0$$

We also consider the associated graded module

$$\Gamma_*(\mathcal{I}_Y) = \bigoplus_{d \geqslant 0} \Gamma(\mathbb{P}^n_A, \mathcal{I}_Y(d)) \subset A[x_0, \ldots, x_n].$$

Since $\widetilde{\mathfrak{a}} = \mathcal{I}_Y$, we can compute this in terms of $\mathfrak{a}$ as follows:

$$\begin{aligned}
\Gamma_*(\widetilde{\mathfrak{a}}) &= \bigoplus_{d \geqslant 0} \{\, f \in A[x_0, \ldots, x_n]_d \mid f/x_i^d \in (\mathfrak{a}_{x_i})_0 \,\} \\
&= \bigoplus_{d \geqslant 0} \{\, f \in A[x_0, \ldots, x_n]_d \mid \exists N \in \mathbb{N} \text{ such that } f \cdot x_i^N \in \mathfrak{a} \text{ for every } i \,\} \\
&= \mathfrak{a}^{\mathrm{sat}}. \qquad (13.10)
\end{aligned}$$

That is, composing $\widetilde{(-)}$ with $\Gamma_*(-)$ corresponds to taking the saturation.

**Theorem 13.15.** For any ring $A$, there is a bijection

$$\left\{ \begin{array}{c} \text{closed subschemes} \\ Y \subset \mathbb{P}^n_A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{saturated homogeneous} \\ \text{ideals } \mathfrak{a} \subset A[x_0, \ldots, x_n] \end{array} \right\}$$

given by $Y \mapsto \Gamma_*(\mathcal{I}_Y)$ and $\mathfrak{a} \mapsto V_+(\mathfrak{a})$.

*Proof* Let $Y \subset \mathbb{P}^n_A$ be a closed subscheme with ideal sheaf $\mathcal{I}_Y$. We claim that $Y = V_+(\mathfrak{a})$ where $\mathfrak{a} = \Gamma_*(\mathcal{I}_Y)$ and that $\mathfrak{a}$ is saturated.

The first part follows because the subschemes $V_+(\mathfrak{a})$ and $Y$ have the same ideal sheaf: the ideal sheaf of $V_+(\Gamma_*(\mathcal{I}_Y))$ is given by $\widetilde{\Gamma_*(\mathcal{I}_Y)}$, which is equal to $\mathcal{I}_Y$ by Proposition **??**.

To see that $\mathfrak{a}$ is saturated, take $f \in \mathfrak{a}^{\mathrm{sat}}$ homogeneous of degree $d$. By definition, there exists $N \in \mathbb{N}$ such that for each $i$, $x_i^N f \in \mathfrak{a}$, meaning, $x_i^N f \in \Gamma(\mathbb{P}^n_A, \mathcal{I}_Y(d+N))$. On $D_+(x_i)$, this implies $f|_{D_+(x_i)} \in \mathcal{I}_Y(d)|_{D_+(x_i)}$ since $x_i$ is invertible there. This means that

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$f$ is a section of $\mathcal{O}_{\mathbb{P}^n_A}(d)$ which lies in $\mathcal{I}_Y(d)$ when restricted to each $D_+(x_i)$ and hence $f \in \Gamma(\mathbb{P}^n_A, \mathcal{I}_Y(d))$. Therefore, $\mathfrak{a}$ is saturated.

Conversely, if $\mathfrak{a}$ is a saturated homogeneous ideal, the equality (13.10) shows that $\Gamma_*(\mathcal{I}_{V_+(\mathfrak{a})}) = \mathfrak{a}$. Therefore, the two maps are mutually inverse. □

**Exercise 13.4.1.** Show that if $\mathfrak{a} \subset R = k[x_0, \ldots, x_n]$ is a *radical* homogeneous ideal, and $\mathfrak{a}$ is not equal to $(x_0, \ldots, x_n)$, then $\mathfrak{a}$ is saturated.

## 13.5 Sheaves on projective space

We recall the following fundamental theorem from commutative algebra, which gives the structure of graded modules over $R$ in terms of free modules.

**Theorem 13.16 (Hilbert's syzygy theorem).** Let $k$ be a field and let $R = k[x_0, \ldots, x_n]$. Then if $M$ is a finitely generated graded $R$-module, then there is a finite free resolution (that is, an exact sequence)

$$0 \longrightarrow F_n \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0, \tag{13.11}$$

where $F_j = \bigoplus_{i=1}^{b_k} R(-d_{ij})$ is a free graded $R$-module. $F_i$ is called the $i$-th *syzygy module* of the resolution.

By the theorem, $M$ is the cokernel of a map $F_1 \to F_0$ of *free* $R$-modules of finite rank. Any such map is defined by a matrix with entries being homogenous elements of $R$, so this is a convenient way to present $M$: the generators come from $F_0$ and the relations come from $F_1$. Furthermore, the relations among the relations, and so on, are defined by matrices between free modules. This sort of resolution is immensely useful in the study of $M$, as it allows us to apply linear algebra techniques which can simplify computations and define invariants of $M$.

If we apply the $\sim$-functor to the sequence (13.11), we obtain an exact sequence of sheaves on $\mathbb{P}^n_k$

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \widetilde{M} \longrightarrow 0$$

where $\mathcal{E}_j = \bigoplus_{i=1}^{b_k} \mathcal{O}_{\mathbb{P}^n_k}(-d_{ij})$ is a direct sum of sheaves of the form $\mathcal{O}_{\mathbb{P}^n_k}(d)$. Hence any finite type $\mathcal{O}_X$-module admits a resolution where the terms are direct sums of invertible sheaves. This shows very clearly why the invertible sheaves $\mathcal{O}(d)$ are so important: they are the building blocks of all finite type sheaves on $\mathbb{P}^n_k$.

Here are a few important special cases:

**Example 13.17** (Hypersurfaces)**.** Let $F \in R$ denote an homogeneous polynomial of degree $d > 0$. Then $F$ determines a projective hypersurface $X = \operatorname{Proj}(R/F) \subset \mathbb{P}^n_k$. Write $\iota\colon X \to \mathbb{P}^n_A$ for the closed embedding.

Consider the map $R(-d) \to R$ given by multiplication by $F$. Note the shift in degrees here, in order to make this into a map of graded modules (the constant '1' gets sent to $F$, which should have degree $d$ on both sides). We obtain an exact sequence

$$0 \longrightarrow R(-d) \longrightarrow R \longrightarrow R/(F) \longrightarrow 0.$$

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Applying the tilde to this sequence, we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow \widetilde{R/F} \longrightarrow 0. \tag{13.12}$$

Note that $\widetilde{R/F}$ is isomorphic to the sheaf $\iota_*\mathcal{O}_X$; both sheaves take the value $(R_f/(F))_0$ over each distinguished open set $D_+(f)$. Therefore, comparing (13.12) to the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \iota_*\mathcal{O}_X \longrightarrow 0,$$

we see that there is an isomorphism of $\mathcal{O}_{\mathbb{P}^n_A}$-modules $\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^n_A}(-d)$. △

**Example 13.18** (The twisted cubic curve)**.** Let $k$ be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will consider the *twisted cubic curve* $C = V(I)$ where $I \subset R$ is the ideal generated by the $2 \times 2$-minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

i.e., $I = (q_0, q_1, q_2) = (x_1^2 - x_0x_2, x_0x_3 - x_1x_2, -x_2^2 + x_1x_3)$.

Consider the map of $R$-modules $R^3 \to I$ sending $e_i \mapsto q_i$. This is clearly surjective, since the $q_i$ generate $I$. Let us consider the kernel of this map, that is, the module of relations of the form $a_0q_0 + a_1q_1 + a_2q_2 = 0$ for $a_i \in R$. There are two obvious relations of this form, i.e., the ones we get from expanding the determinants of the two matrices

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \qquad \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

(So the first matrix gives $x_0q_2 - x_1q_1 + x_2q_2 = 0$ for instance). These give a map $R^2 \xrightarrow{A\cdot} R^3$, where $A$ is the matrix above. This map is injective, and it turns out that there is an exact sequence of $R$-modules

$$0 \longrightarrow R^2 \xrightarrow{A} R^3 \longrightarrow I \longrightarrow 0.$$

Again, we should consider these as *graded* modules, so we must shift the degrees according to the degrees of the maps above

$$0 \longrightarrow R(-3)^2 \xrightarrow{A} R(-2)^3 \longrightarrow I \longrightarrow 0.$$

This gives the resolution of the ideal $I$ of $C$. Then applying $\sim$, and using the fact that $\mathcal{I} = \widetilde{I}$, we get a resolution of the ideal sheaf of $C$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-3)^2 \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3_k}(-2)^3 \longrightarrow \mathcal{I} \longrightarrow 0.$$

We will see later in Chapter 15 how to use sequences like this to extract geometric information about $C$. △

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## 13.6 Projective morphisms

A morphism $f\colon X \to \operatorname{Spec} A$ is said to be projective if it factors as

$$X \longrightarrow \mathbb{P}^n_A \longrightarrow \operatorname{Spec} A,$$

where the first map is a closed embedding of $A$-schemes and the second is the structure morphism.

Like separatedness and properness, projectivity is a relative notion: it is the morphism $X \to \operatorname{Spec} A$ which is projective, not $X$ itself. Intuitively, it is the fibers of $X \to \operatorname{Spec} A$ which are projective. For instance, $\mathbb{P}^1_{k[t]}$ is projective over $\operatorname{Spec} k[t]$, but it is not over $\operatorname{Spec} k$. In the example, the (scheme-theoretic) fiber over $s \in S = \operatorname{Spec} k[t]$ equals the projective line $\mathbb{P}^1_{\kappa(s)}$ over $\kappa(s)$. Still, if we are working in the category of schemes over, say, a field $k$ or $\mathbb{Z}$, we still refer to a scheme $X$ being 'projective' when it is projective over the base scheme.[1]

**Example 13.19.** For $A = \mathbb{C}[t]$, the scheme $X = V_+(zy^2 - x^3 - txz^2)$ in $\mathbb{P}^2_A$ is projective over $\mathbb{A}^1_{\mathbb{C}} = \operatorname{Spec} A$. The fiber of $X \to \operatorname{Spec} \mathbb{C}[t]$ over any closed point $a \in \operatorname{Spec} \mathbb{C}[t]$ is an integral projective subscheme of dimension 1, namely $V_+(zy^2 - x^3 - axz^2) \subset \mathbb{P}^2_{\mathbb{C}}$. $X$ is however not projective over $\mathbb{C}$ (e.g., as it admits many non-constant maps to $\mathbb{A}^1_{\mathbb{C}}$). △

We are ready to prove the following important theorem:

**Theorem 13.20.** Let $S$ be a scheme and let $X \subset \mathbb{P}^n_S$ be a closed subscheme. Then the structure morphism $X \to S$ is proper.

*Proof* Projective morphisms are separated and of finite type, so to prove that they are proper it remains to show that they are universally closed.

By the base change property (item (i) of Proposition 9.23) it suffices to show that $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is universally closed. So let $T$ be any scheme, and consider the base change $\pi\colon \mathbb{P}^n_T \to T$. Since 'being closed' is a local property on $T$, it suffices to assume that $T = \operatorname{Spec} A$ is affine.

Let $Z \subset \mathbb{P}^n_A$ be a closed subscheme. We need to show that $\pi(Z)$ is closed in $\operatorname{Spec} A$. By Proposition **??** in Chapter **??**, $Z$ is given by a homogeneous ideal $I \subset A[x_0, \ldots, x_n]$. Let $\tau : Z \to \operatorname{Spec} A$ denote the composition $Z \to \mathbb{P}^n_A \to \operatorname{Spec} A$.

Let $\mathfrak{p} \in \operatorname{Spec} A$ be a point which is *not* in the image of $Z$. We want to show that there is some open neighbourhood $U$ of $\mathfrak{p}$ so that $\pi^{-1}(y) \cap Z$ is empty for all $y \in U$. Equivalently, we want to show that $\tau^{-1}(y) = \varnothing$ for all $y \in U$.

Note that the fiber of $\mathfrak{p}$ in $\mathbb{P}^n_A$ is given by the Proj of the ring

$$A[x_0, \ldots, x_n]/I \otimes_A \kappa(\mathfrak{p}) \tag{13.13}$$

To say that this Proj is empty means that the graded ring (13.13) is zero in high degrees. In other words, we have

$$I_N \otimes_A \kappa(\mathfrak{p}) = A[x_0, \ldots, x_n]_N \otimes_A \kappa(\mathfrak{p}) \tag{13.14}$$

[1] The notion of projectivity can be extended to more general morphisms $f\colon X \to S$. In fact, there exist two slightly different definitions in the literature, see (Stacks Project, 2018, Tag 01W7). These notions agree in most cases, for instance when $S$ is affine or when $S$ satisfies some reasonable finiteness conditions.

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for some large $N > 0$. Note that $I_N$ and $A[x_0, \dots, x_n]_N$ are both finitely generated modules over $A$. Tensoring these by $A_{\mathfrak{p}}$ and using Nakayama's lemma, (13.14) implies that

$$(I_N)_{\mathfrak{p}} = (A[x_0, \dots, x_n]_N)_{\mathfrak{p}}$$

In particular, there exists an $f \notin \mathfrak{p}$ so that

$$(I_N)_f = (A[x_0, \dots, x_n]_N)_f = A_f[x_0, \dots, x_n]_N$$

But then

$$\tau^{-1}(D(f)) = \operatorname{Proj}(A_f[x_0, \dots, x_m]/I) = \varnothing$$

because $(I_N)_f$ contains all monomials of degree $N$. This completes the proof. □

A consequence of this is that the 'image of a projective variety is projective':

**Corollary 13.21.** Let $k$ be a field, and let $f\colon X \to Y$ be a morphism of projective schemes over $k$. Then the image $f(X) \subset Y$ is closed in $Y$.

This statement is a priori not at all obvious, especially given how badly it fails with 'projective' replaced by 'affine' (see Example 9.20). It is yet another reason why one prefers projective varieties to non-projective ones.

*Proof of Corollary 13.21* By assumption $X$ and $Y$ are closed subvarieties of projective spaces over $k$, and so they are proper over $k$. But then the morphism $f\colon X \to Y$ is also proper, and hence closed. □

**Example 13.22.** Let $A = \mathbb{C}[a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n]$ be the polynomial ring, and consider the closed subscheme of $\mathbb{P}^1_A = \operatorname{Proj} A[x, y]$ given by

$$Z = V\left(a_0x^m + a_1x^{m-1}y + \cdots + a_my^m, b_0x^n + b_1xy^{n-1} + \cdots + a_ny^n\right).$$

If $\pi\colon \mathbb{P}^1_A \to \operatorname{Spec} A$ denotes the structure morphism, the image $\pi(Z) \subset \operatorname{Spec} A$ contains the set of points $(a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n) \in \mathbb{A}^{m+n+2}(\mathbb{C})$ such that the two polynomials of degrees $m$ and $n$ have a common root. This locus is indeed closed; it is given by $V(\det R)$, where $\det R$ is the *resultant* of the two polynomials, i.e., the determinant of the Sylvester matrix. This matrix is shown below for $m = 3$ and $n = 2$:

$$R = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 \end{pmatrix}.$$

△

**Example 13.23.** Let $X$ be a proper scheme over an algebraically closed field $k$. Let us show that $\mathcal{O}_X(X) = k$. We showed in Example **??** that elements of $\mathcal{O}_X(X)$ can be understood as the set of morphisms $f\colon X \to \mathbb{A}^1_k$, so it suffices to show that all such morphisms are constant. Let $f\colon X \to \mathbb{A}^1_k$ be a morphism and consider the morphism $\bar{f}\colon X \to \mathbb{P}^1_k$ given by composing $f$ with the embedding $\mathbb{A}^1_k \to \mathbb{P}^1_k$. Then since $\mathbb{P}^1_k$ is proper, the image $\bar{f}(X)$ is

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closed. But the closed subsets of $\mathbb{P}^1_k$ are either closed points or all of $\mathbb{P}^1_k$. The latter case is impossible because $\bar{f}$ factors via $\mathbb{A}^1_k \to \mathbb{P}^1_k$, and so $f$ maps to a point. △

**Proposition 13.24.** Let $f : X \to Y$ be a morphism of projective schemes over an algebraically closed field $k$. Suppose that $f^{-1}(y)$ is finite for every $k$-point $y \in Y$. Then $f$ is a finite morphism.

*Proof* Since the property of 'being finite' is local on the target, and the $k$-points are dense in $Y$, we may assume that $Y = \operatorname{Spec} A$ is affine. As $f$ factors as $X \xrightarrow{\iota} \mathbb{P}^n_A \xrightarrow{\pi} Y = \operatorname{Spec} A$, we may regard $X$ as a closed subscheme of $\mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$ and $f$ is the restriction of $\pi\colon \mathbb{P}^n_A \to \operatorname{Spec} A$ to $X$.

We first show that $f$ is affine. Let $y \in Y$ be a $k$-point. By assumption, the topolocal fiber $f^{-1}(y)$ is a finite set in $\pi^{-1}(y) \simeq \mathbb{P}^n_k$. Pick an element $h \in A[x_0, \ldots, x_n]$, so that $V(h)$ is disjoint from $f^{-1}(y)$. By Corollary 13.21, the image $f(V(h) \cap X)$ is a closed set in $Y$, and this set does not contain $y$. Therefore, there exists a distinguished open set $D(g) \subset Y$ containing $y$, so that $D(g) \subset Y - f(V(h))$. This means that $f^{-1}(D(g)) \subset X - V(h)$. Note that $U = X - V(h) = X \cap D_+(h)$ is an affine scheme, being a closed subset of $D_+(h)$. If $\phi\colon A \to U$ is the ring map which induces $f|_U\colon U \to Y$, we have $f^{-1}(D(g)) = D(\phi(g))$ in $U$, and so $f$ is affine.

Now, write $X = \operatorname{Spec} B$ for some $A$-algebra $B$. To conclude, we need to show that $B$ is a finite $A$-module. This follows from Serre's theorem 15.28, because $X$ is projective over $A$ and $H^0(X, \mathcal{O}_X) = B$. □

**Example 13.25.** The projectivity assumption here is essential. For instance, if $f\colon U \to X$ is the inclusion of an open subset, then certainly $f^{-1}(x)$ is finite for every $x \in X$, but $f$ is typically not finite (see Example 6.35). △

**Example 13.26.** The morphism $f\colon \operatorname{Spec}(\overline{\mathbb{Q}}) \to \operatorname{Spec} \mathbb{Q}$ is not finite, but of course the fibers are finite. △

## 13.7 Exercises

**Exercise 13.7.1.** Let $R = \mathbb{Q}[x, y, z]$ with $\deg x = 1, \deg y = 2, \deg z = 3$. Show that the map (13.3) is not an isomorphism for $M = R(1)$ and $N = R(2)$.

**Exercise 13.7.2.** Suppose $R$ is a graded ring finitely generated in degree 1 and let $M$ and $N$ be two finitely generated $R$-modules. Show that $\widetilde{M} \simeq \widetilde{N}$ if and only if $M_{>d} \simeq N_{>d}$ for some $d$.

**Exercise 13.7.3.** Let $\mathbb{P}^1_k = \operatorname{Proj} R$ where $R = k[x_0, x_1]$. Show that the functor $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$ is not right exact, by considering the exact sequence

$$0 \longrightarrow R(-n) \longrightarrow R \longrightarrow R/(x_0^n) \longrightarrow 0.$$

**Exercise 13.7.4.** Let $k$ be a field and let $R = k[x_0, \ldots, x_n]$. Let $\pi\colon \mathbb{A}^{n+1} - 0 \to \mathbb{P}^n_k = \operatorname{Proj} R$ denote the 'quotient morphism' (see page 97). Show that for a graded $R$-module $M$,

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we have

$$\pi_*(\widetilde{M}|_{\mathbb{A}_k^{n+1}-0}) = \bigoplus_{n\in\mathbb{Z}} \widetilde{M}(d)$$

**Exercise 13.7.5.** Let $R$ be a graded Noetherian integral domain generated in degree 1 by $x_0, \ldots, x_n$. The aim of this exercise is to show that $R' = \Gamma_*(\mathcal{O}_X)$ is an integral extension of $R$.

a) Show that $\alpha : R \to R'$ is injective.
b) $s \in R'$ be a homogeneous element of non-negative degree. Show that we can find an $n > 0$, so that $\alpha(x_i^n)s \in \alpha(R)$ for every $i$. Deduce that $\alpha(R_m)s \subset \alpha(R)$ for $m$ large.
c) Let $J \subset R$ be the generated by elements of degree $\geqslant kn$. Show that $J$ is finitely generated, and that $\alpha(J)s \subset \alpha(J)$.
d) Use the Cayley–Hamilton theorem to show that $s$ satisfies an integral equation over $R$.

**Exercise 13.7.6.** Let $k$ be an algebraically closed field and let $X \subset \mathbb{P}_k^n$ be a projective $k$-scheme. Show that $\Gamma(X, \mathcal{O}_X) = k$.

**Exercise 13.7.7.** Let $X = \operatorname{Proj}(\mathbb{R}[x, y]/(x^2 + y^2))$. Show that $X$ is affine, and that

$$\Gamma(X, \mathcal{O}_X) \simeq \mathbb{C}.$$

**Exercise 13.7.8.** Let $\mathbb{P}_k^1 = \operatorname{Proj} k[u_0, u_1]$ and $t = u_0/u_1$. Consider the closed subscheme $Z \subset \mathbb{P}_k^1$ which is supported at $(0, 1)$ and which is locally given as $\operatorname{Spec} k[t]/t^n \subset D_+(u_0) = \operatorname{Spec} k[t]$. Describe the $R$ module $\Gamma_*\mathcal{O}_Z$ and the canonical map $R = \Gamma_*\mathcal{O}_{P_k^1} \to \Gamma_*\mathcal{O}_Z$.

**Exercise 13.7.9.** Check that the saturation of a homogeneous ideal is homogengeous.

**Exercise 13.7.10.** Assume $R$ be generated over $R_0$ by finitely many elements of degree 1. Let $M$ and $N$ be two finitely generated graded $R$-modules. In this exercise you will show that $\widetilde{M} \simeq \widetilde{N}$ if and only if $M_{>d} \simeq N_{>d}$ for some $d$. HINT: Consider the images of $M$ and $N$ in $\Gamma_*(\widetilde{M}) = \Gamma(\widetilde{N})$.

**Exercise 13.7.11.** Assume $R$ be generated over $R_0$ by finitely many elements of degree 1. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\operatorname{Proj} R$. Show that $\mathcal{F}$ is of finite type if an only if it is the tilde of a finitely generated $R$-module. HINT: Write $\mathcal{F} = \widetilde{M}$ and pick surjections $\mathcal{O}_{U_i}^n \to \widetilde{M}|_{U_i}$. Lift these to a map $\phi : \bigoplus_{i,j} R(-d_{ij}) \to M$ and show that the cokernel has zero tilde.

**Exercise 13.7.12.** Let $\phi : R \to S$ be a map of graded rings and let $f : U \to \operatorname{Proj} S$ be the morphism given by $\phi$, where $U = \operatorname{Proj}(S) - V(\phi(R_+))$.

a) Show that $f^*\widetilde{M} = \widetilde{M \otimes_R S}|_U$ for each graded $R$-module $M$.
b) Show that $f_*(\widetilde{N}|_U) = \widetilde{N_R}$ for each graded $S$-module $N$ (and where $N_R$ is as usual, $N$ regarded as a graded $R$-module).

**Exercise 13.7.13** (The Veronese surface)**.** Let $X = \mathbb{P}_k^2 = \operatorname{Proj} k[x_0, x_1, x_2]$. Show that

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$L = \mathcal{O}_{\mathbb{P}^2}(2)$ is globally generated by the sections

$$x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2$$

Show that the corresponding morphism $\phi\colon \mathbb{P}^2 \to \mathbb{P}^5$ is a closed embedding.

**Exercise 13.7.14** (The quadric surface)**.** Let $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$ and $L = p^*\mathcal{O}_{\mathbb{P}^1_k}(1) \otimes q^*\mathcal{O}_{\mathbb{P}^1_k}(1)$ where $p, q\colon X \to \mathbb{P}^1$ are the two projections. If $x_0, x_1$ is a basis for $\Gamma(X, p^*\mathcal{O}_{\mathbb{P}^1_k}(1))$, and $y_0, y_1$ is a basis for $\Gamma(X, q^*\mathcal{O}_{\mathbb{P}^1_k}(1))$, show that $p^*\mathcal{O}_{\mathbb{P}^1_k}(1) \otimes q^*\mathcal{O}_{\mathbb{P}^1_k}(1)$ is globally generated by the four sections $s_0 = x_0y_0$, $s_1 = x_0y_1$, $s_2 = x_1y_0$, $s_3 = x_1y_1$. Describe the corresponding morphism $X \to \mathbb{P}^3_k$.

**Exercise 13.7.15** (Serre's theorem)**.** Let $X \subset \mathbb{P}^n_A$ be a projective scheme over a Noetherian ring $A$ and let $L$ be an ample invertible sheaf. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then there is an integer $m_0$ such that $\mathcal{F} \otimes L^m$ is globally generated (by a finite set of global sections) for all $m \geqslant m_0$. HINT: Reduce to the case $X = \mathbb{P}^n_A$ and $L = \mathcal{O}(1)$. Then apply Corollary 13.11.

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# 14

# Locally free sheaves

The most important examples of quasi-coherent sheaves are the locally free sheaves. As the name suggests, these are sheaves which are locally isomorphic to a direct sum of copies of the structure sheaf of the scheme. Because of this 'freeness' property, these sheaves are in many respects the nicest examples of sheaves on a scheme and the easiest to work with. They are also the algebraic counterpart to the vector bundles in topology.

## 14.1 Locally free sheaves

Let $X$ be a scheme. We say that an $\mathcal{O}_X$-module $\mathcal{E}$ is *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$ and it is *locally free* if there is an open cover $\{U_i\}_{\in I}$ of $X$ such that the restriction $\mathcal{E}|_{U_i}$ is free for every $i$. The cover $\{U_i\}$ is called a 'trivializing cover', and a locally free sheaf which is globally free is sometimes said to be *trivial*.

If $P \in X$ is a point, say contained in $U_i$, the *rank* of $\mathcal{E}$ at $P$ is the number of copies of $\mathcal{O}_{U_i}$ needed to express $\mathcal{E}|_{U_i}$ as a direct sum of $\mathcal{O}_{U_i}$'s. The rank may be finite or infinite, but we will almost exclusively concern ourself with the case of finite rank. A locally free sheaf $\mathcal{E}$ is also allowed to have different ranks at different points of $X$, but the rank will be constant on each connected component of $X$. If $\mathcal{E}$ has rank equal to $r$ at every point of $X$, we say that $\mathcal{E}$ is locally free of rank $r$ and write $\operatorname{rank}(\mathcal{E}) = r$.

The rank is perhaps better viewed as a function $X \to \{0, 1, 2, \dots\} \cup \{\infty\}$. To discuss this function more precisely, we make the following defintion, which applies to any coherent sheaf.

**Definition 14.1.** Let $\mathcal{E}$ be a coherent sheaf and let $P \in X$ be a point. The *fiber of $\mathcal{E}$ at $P$* is defined as the $\kappa(P)$-vector space

$$\mathcal{E}(P) = \mathcal{E}_P/\mathfrak{m}_P\mathcal{E}_P.$$

If $U \subset X$ is an open subset containing $P$ and $s \in \Gamma(U, \mathcal{E})$ is a section of $\mathcal{E}$ over $U$, we denote by $s(P)$ the image of the germ $s_P \in \mathcal{E}_P$ in the fiber $\mathcal{E}(P)$.

Geometrically, $\mathcal{E}(P) \simeq \mathcal{E}_P \otimes_{\mathcal{O}_{X,P}} \kappa(P)$ is the $\kappa(P)$-vector space which corresponds to the pullback of $\mathcal{E}$ via the map $\iota\colon \operatorname{Spec} \kappa(P) \to X$.

Clearly, if $\mathcal{E}$ is locally free of rank $r$, then $\mathcal{E}_P$ is a free $\mathcal{O}_{X,P}$-module of rank $r$ and $\mathcal{E}(P)$ is a free $\kappa(P)$-vector space of dimension $r$ at every point $P$. In the appendix of this chapter, we will prove the following proposition, which says that the converse holds under some mild hypotheses.



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**Proposition 14.2.** Let $X$ be an integral Noetherian scheme and let $\mathcal{E}$ be a coherent sheaf. The following are equivalent:

(i) $\mathcal{E}$ is locally free.
(ii) $\mathcal{E}_P$ is a free $\mathcal{O}_{X,P}$-module for every $P \in X$.
(iii) The function $P \mapsto \dim_{\kappa(P)} \mathcal{E}(P)$ is constant.

The next proposition summarises some of the basic properties of locally free sheaves.

**Proposition 14.3.** Let $X$ be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be two locally free $\mathcal{O}_X$-modules of rank $e$ and $f$ respectively. Then

(i) The direct sum $\mathcal{E} \oplus \mathcal{F}$ is locally free of rank $e + f$.
(ii) The tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ is locally free of rank $e \cdot f$.
(iii) The Hom sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$ is locally free of rank $e \cdot f$.
(iv) The dual sheaf $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O})$ is locally free of rank $e$. The double dual $(E^\vee)^\vee$ is canonically isomorphic to $\mathcal{E}$. Moreover, there is a canonical isomorphism

$$\mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}). \tag{14.1}$$

*Proof* Let $P \in X$ be a point and let $U = \operatorname{Spec} A$ be an open affine neighbourhood of $P$ such that $\mathcal{E}|_U$ and $\mathcal{F}_U$ are trivial $\mathcal{O}_U$-modules. Then the restrictions of the sheaves in question are given by

$$\begin{aligned} \mathcal{E} \oplus \mathcal{F}|_U &= (E \oplus F)^\sim \\ \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}|_U &= (E \otimes_A F)^\sim \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})|_U &= \operatorname{Hom}_A(E, F)^\sim. \end{aligned}$$

where $E$ and $F$ are free $A$-modules. The modules on the right-hand side are all free the ranks indicated in the proposition, so we get (i), (ii) and (iii).

(iv): If $M$ is free, then the dual module $M^\vee = \operatorname{Hom}_A(M, A)$ is also free of the same rank. The evaluation map $M \to (M^\vee)^\vee$ defined by $m \mapsto (\phi \mapsto \phi(m))$ is an isomorphism when $M$ is free of finite rank. Moreover, for any $A$-module $N$, there is a canonical map

$$M^\vee \otimes_A N \longrightarrow \operatorname{Hom}_A(M, N), \tag{14.2}$$

defined by $\phi \otimes n \mapsto (m \mapsto \phi(m)n)$. This map is an isomorphism if $M$ and $N$ are both free of finite rank. (This is clearly the case when $N = A$, and as both sides are additive, it holds in general.) Finally, since the isomorphisms (14.2) are compatible with localization, they glue together to a global isomorphism (14.1). □

**Example 14.4.** On the projective line $X = \mathbb{P}^1_A$, one has the sheaves $\mathcal{O}_{\mathbb{P}^1_A}(m)$ constructed on page 101. These were constructed by gluing together two copies of $\mathcal{O}_{\mathbb{A}^1_A}$, so they are invertible sheaves. These sheaves are non-trivial for all $m \neq 0$, as we computed the global sections of $\mathcal{O}_{\mathbb{P}^1_A}(m)$ is not isomorphic to $\Gamma(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}) = A$.

The isomorphism $\mathcal{O}_{\mathbb{P}^1_A}(m) \otimes \mathcal{O}_{\mathbb{P}^1_A}(-m) \simeq \mathcal{O}_{\mathbb{P}^1_A}$ shows that $\mathcal{O}_{\mathbb{P}^1_A}(-m)$ is identified with the dual of $\mathcal{O}_{\mathbb{P}^1_A}(m)$. △

**Example 14.5.** Let $A = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $X = \operatorname{Spec} A$. Consider the ideal $I = \mathbb{Z}/2 \times (0)$

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and the sheaf $\mathcal{E} = \widetilde{I}$. Note that $\operatorname{Spec} A$ is the disjoint union of two copies of $\operatorname{Spec} \mathbb{Z}/2$, and $\mathcal{E}$ restricts to the structure sheaf on one of these and to the zero sheaf on the other. $\mathcal{E}$ is therefore locally free. However, $\mathcal{E}$ is not free, because $\mathcal{E}(X) = \mathbb{Z}/2$, whereas any nonzero free $A$-module must have at least four elements. △

**Example 14.6** (The integers)**.** Let $X = \operatorname{Spec} \mathbb{Z}$. If $\mathcal{E}$ is a $\mathcal{O}_X$-module of finite type, then $\mathcal{E} = \widetilde{M}$ for some finitely generated $\mathbb{Z}$-module $M$, and by the structure theorem for finitely generated abelian groups, we may write $M = \mathbb{Z}^r \oplus T$, where $T$ is a finite direct product of groups of the form $\mathbb{Z}/n\mathbb{Z}$ . If $\mathcal{E}$ in addition is required to be locally free, it must hold that $T = 0$ (otherwise, if $p$ is a prime factor of an $n$ appearing in one of the summands of $T$, the stalk at $(p)$ will not be free). Hence $\mathcal{E} = \widetilde{\mathbb{Z}^r} = \mathcal{O}_X^r$, and we conclude that every locally free sheaf on $\operatorname{Spec} \mathbb{Z}$ is trivial. △

**Example 14.7** (The affine line)**.** The argument of the previous example in fact applies over any PID $A$. Every $\mathcal{O}_X$-module of finite type on $X = \operatorname{Spec} A$ must have the form $\widetilde{M}$ for $M = A^r \oplus T$ where $T$ is a finitely generated torsion module, and if we require $\widetilde{M}$ to be locally free, the torsion part must vanish; i.e. we must have $T = 0$. In particular, this applies to locally free sheaves on the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ over any field $k$. △

**Example 14.8** (The tangent bundle of the 2-sphere)**.** Let $A = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ and consider the $A$-linear map

$$\phi\colon A^3 \longrightarrow A, \quad (p, q, r) \mapsto xp + yq + zr.$$

The kernel $M = \operatorname{Ker}(\phi)$ defines a quasi-coherent sheaf $\widetilde{M}$ which is locally free of rank 2.

To see this, note that $M$ consists of triples $(p, q, r) \in A^3$ satisfying $xp + yq + zr = 0$. Localizing at $x$, any such triple is uniquely determined by $q$ and $r$ since $p = -x^{-1}(yq + zr)$. This gives an isomorphism $M_x \simeq A_x^2$, and hence $\widetilde{M}|_{D(x)} \simeq \mathcal{O}^2_{D(x)}$. The same argument applies to $y$ and $z$, so we find that $\widetilde{M}$ is locally free.

While $\widetilde{M}$ is locally trival, it is in fact globally non-trivial – $M$ is not a free $A$-module. This is not so easy to prove: it reflects the famous topological fact that the tangent bundle of $S^2$ admits no nowhere-vanishing section (the *Hairy Ball Theorem*). In contrast, if we replace $\mathbb{R}$ by $\mathbb{C}$ in this example, the corresponding sheaf *is* trivial. △

## 14.2 Zero sets of sections

Let $\mathcal{E}$ be an locally free sheaf of rank $r$ on the scheme $X$, and let $s \in \Gamma(X, \mathcal{E})$ be a global section. We define the *zero set of* $s$ by

$$V(s) = \{P \in X \mid s(P) = 0 \text{ in } \mathcal{E}(P)\}.$$

Equivalently, $V(s)$ consists of the points $P \in X$ such that $s_P \in \mathfrak{m}_P \mathcal{E}_P$.

The set $V(s)$ is a closed subset of $X$. In fact, it carries a canonical structure as a closed subscheme of $X$, which we will also denote by $V(s)$. To see this, consider an affine open set $U = \operatorname{Spec} A$ such that $\mathcal{E}|_U \simeq \mathcal{O}_X^r|_U$. Under this isomorphism, the restriction of $s$ corresponds to an $r$-tuple of elements $(f_1, \ldots, f_r) \in A^r$. This $r$-tuple generates an ideal $I = (f_1, \ldots, f_r)$, which defines a closed subscheme $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$. One can check

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that that as $U$ runs over all open affines where $\mathcal{E}$ is trivial, these closed subschemes glue together to form a closed subscheme of $X$.

The ideal sheaf describing $V(s)$ has the following description. If we identify $\Gamma(X, \mathcal{E}) = \operatorname{Hom}(\mathcal{O}_X, \mathcal{E})$, we may regard $s$ as a map of $\mathcal{O}_X$-modules $s\colon \mathcal{O}_X \to \mathcal{E}$. Applying $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, -)$, we obtain a map of $\mathcal{O}_X$-modules

$$s^\vee\colon \mathcal{E}^\vee \longrightarrow \mathcal{O}_X. \tag{14.3}$$

The image of $s^\vee$ is a quasi-coherent ideal sheaf $\mathcal{I}$ of $\mathcal{O}_X$, and this is the ideal sheaf that defines $V(s)$. Concretely, over an affine open $U = \operatorname{Spec} A$, after identifying $s$ with $(f_1, \ldots, f_r) \in A^r$, the map $s^\vee$ is simply the tilde of the map $A^r \to A$ that sends the $i$-th basis vector $e_i$ to $f_i$. In other words, $\mathcal{I}|_U$ corresponds to the ideal generated by the $f_i$'s.

**Proposition 14.9.** Let $X$ be an integral scheme and let $s \in \Gamma(X, \mathcal{E})$ be a global section of locally free sheaf $\mathcal{E}$ of rank $r$. The zero scheme $V(s)$ has codimension at most $r$ in $X$.

**Example 14.10.** The monomial $s = x_0^2 x_1$ defines a global section of $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k}(3)$. The zero scheme $V(s)$ consists of the two points $(0:1)$ and $(1:0)$. △

In the special case when $\mathcal{E}$ has rank 1, then the map $s^\vee\colon \mathcal{E}^\vee \to \mathcal{O}_X$ is given by multiplication by $s$. In particular, if $X$ is integral, this map is injective and identifies $\mathcal{E}^\vee$ with the ideal sheaf $\mathcal{I}$ of $V(s)$. The ideal sheaf sequence takes the form

$$0 \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_{V(s)} \longrightarrow 0 \tag{14.4}$$

where $\iota\colon V(s) \to X$ is the inclusion.

## 14.3 Transition functions

Let $\mathcal{E}$ be a locally free sheaf of rank $r$, and $\{U_i\}_{i\in I}$ be a trivializing open cover of $\mathcal{E}$. Then by definition, there are isomorphisms of $\mathcal{O}_{U_i}$-modules

$$\phi_i\colon \mathcal{O}_{U_i}^r \xrightarrow{\simeq} \mathcal{E}|_{U_i}. \tag{14.5}$$

Over the intersections $U_{ij} = U_i \cap U_j$ the maps $\tau_{ji} = \phi_j^{-1} \circ \phi_i$ are well-defined, and they give isomorphisms

$$\tau_{ji}\colon \mathcal{O}_{U_{ij}}^r \xrightarrow{\simeq} \mathcal{O}_{U_{ij}}^r,$$

which restricted to triple intersections $U_{ijk} = U_i \cap U_j \cap U_k$ satisfy the cocycle condition

$$\tau_{ki} = \tau_{kj} \circ \tau_{ji}. \tag{14.6}$$

Indeed, we have $\phi_k^{-1} \circ \phi_i = (\phi_k^{-1} \circ \phi_j) \circ (\phi_j^{-1} \circ \phi_i)$.

Conversely, we know from the Gluing Lemma for sheaves that given isomorphisms $\tau_{ji}$ as above, satisfying (14.6) on the triple overlaps, the sheaves $\mathcal{O}_{U_{ij}}^r$ may be glued together to a sheaf $\mathcal{E}$, which by definition is locally free of rank $r$.

Note that any isomorphism of $\mathcal{O}_{U_{ij}}$-modules $\mathcal{O}_{U_{ij}}^r \to \mathcal{O}_{U_{ij}}^r$ is given by some $r \times r$-matrix

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with entries in $\mathcal{O}_X(U_{ij})$. Thus it is possible to define $\mathcal{E}$ by specifying a collection of matrices $\tau_{ji}$ satisfying the cocycle condition (14.6). These are sometimes referred to as the *transition functions* of $\mathcal{E}$.

The isomorphisms $\tau_{ji}$ defining $\mathcal{E}$ are not unique. For instance, if we choose any isomorphisms $\psi_i : \mathcal{O}_{U_i}^r \to \mathcal{O}_{U_i}^r$, we can form new isomorphisms $\tau'_{ji} = \psi_j^{-1} \circ \tau_{ji} \circ \psi_i$, satisfying the cocycle conditions and which define the same sheaf $\mathcal{E}$. Furthermore, $\mathcal{E}$ could admit isomorphisms defined on a different open cover. This is therefore not an ideal way of *classifying* locally free sheaves, and in practice it is very difficult to decide when two constructions yield isomorphic sheaves.

**Example 14.11.** Suppose $\mathcal{E}$ and $\mathcal{F}$ are locally free of ranks $r$ and $s$ respectively. After refining, we may assume that they admit the same trivializing cover. Suppose that the gluing functions are given by $\tau_{ji}$ and $\nu_{ji}$ respectively. Then $\mathcal{E} \oplus \mathcal{F}$ is obtained by gluing together the different $\mathcal{O}_{U_i}^r \oplus \mathcal{O}_{U_i}^s$ with help of the matrices

$$\Phi_{ji} = \begin{pmatrix} \tau_{ji} & 0 \\ 0 & \nu_{ji} \end{pmatrix}$$

For instance, the sheaf $\mathcal{O}_{\mathbb{P}^1_A} \oplus \mathcal{O}_{\mathbb{P}^1_A}(-1)$ on the projective line is obtained using the gluing matrix

$$\tau_{01} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$$

over $U_0 \cap U_1 = \operatorname{Spec} A[u, u^{-1}]$ with $U_0 = \operatorname{Spec} A[u]$ and $U_1 = \operatorname{Spec} A[u^{-1}]$. △

## 14.4 Locally free sheaves on $\mathbb{P}^1$

In this section, we prove that every locally free sheaf on the projective line over a field decomposes into a direct sum of invertible sheaves. This result was first established by Dedekind and Weber in 1882 and has since been rediscovered multiple times, most notably by Birkhoff in 1913 and Grothendieck in 1955.

**Theorem 14.12.** Let $k$ be a field. If $\mathcal{E}$ is a locally free sheaf of rank $r$ on $\mathbb{P}^1_k$, then there are integers $a_1, \ldots, a_r$ such that

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_r). \tag{14.7}$$

This decomposition is unique up to ordering of the factors.

*Proof* The result follows from the classification of quasi-coherent sheaves on $\mathbb{P}^1_k$ of Example 12.12. In the notation there, we have $M_0 = k[x]^r$, $M_1 = k[x^{-1}]^r$ and

$$\tau \colon k[x^{\pm 1}]^r \longrightarrow k[x^{\pm 1}]^r.$$

As $k[x, x^{-1}]$ is a PID, using the *Smith Normal Form* (Theorem A.60), one can find invertible matrices $P$ and $Q$ with entries in $k[x, x^{-1}]$, so that $D = PAQ$ is a diagonal matrix. Moreover, as $\tau$ is an isomorphism, the diagonal entries must be units in $k[x, x^{-1}]$, hence powers of $x$. Therefore, $D = \operatorname{diag}(x^{e_1}, \ldots, x^{e_r})$ for some integers $e_1, \ldots, e_r$. By multiplying $P$ by

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a suitable power of $x$, and $Q$ by the same power of $x^{-1}$, we can even assume that their entries lie in $k[x]$ and $k[x^{-1}]$ respectively. This means that the data defining $\mathcal{E}$ are equivalent to the data defining the locally free sheaf $\mathcal{O}_{\mathbb{P}^1_k}(-e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(-e_r)$ and hence $\mathcal{E}$ is isomorphic as a direct sum as above. For the uniqueness part, see Exercise 14.9.16. □

## 14.5 Invertible sheaves and the Picard group

Among the locally free sheaves, the ones of rank 1, are of particular importance. These are called *invertible sheaves*. By definition, an $\mathcal{O}_X$-module $L$ is invertible whenever there exists an open cover $\{U_i\}_{i\in I}$ and isomorphisms

$$\phi_i \colon \mathcal{O}_{U_i} \longrightarrow L|_{U_i}.$$

These send the element $1 \in \mathcal{O}_X(U_i)$ to a section $s_i = \phi_i(1) \in L(U_i)$, which we call a *local generator* for $L$ over $U_i$. For $i, j \in I$, both $s_i$ and $s_j$ generate $L$ over $U_i \cap U_j$, so there is a relation

$$s_i = g_{ij} s_j$$

for some invertible element $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$.

By construction, we have $g_{ii} = \mathrm{id}$, $g_{ij} = g_{ji}^{-1}$. Moreover, over the triple overlaps $U_i \cap U_j \cap U_k$,

$$s_i = g_{ik} s_k, \quad s_i = g_{ij} s_k, \quad s_j = g_{jk} s_k,$$

which implies that the $g_{ij}$ satisfy the *cocycle condition*

$$g_{ik} = g_{ij} \cdot g_{jk}$$

for all $i, j, k \in I$.

The invertible sheaf $L$ is determined up to isomorphism by the collection of units $g_{ij} \in \mathcal{O}_X(U_{ij})^\times$. If $L$ is trivial, i.e., there is an isomorphism $\mathcal{O}_X \to L$, then the image of $1 \in \mathcal{O}_X(X)$ gives a section $s \in L(X)$ that generates $L$ everywhere. This means that there exist $g_i \in \mathcal{O}_X(U_i)^\times$ such that $s|_{U_i} = g_i s_i$ for every $i$. In terms of the $g_{ij}$, this means that $g_{ij} = g_j g_i^{-1}$.

**Example 14.13.** Consider $\mathbb{P}^1_k$ with the standard covering $U_0 = \operatorname{Spec} k[x_1/x_0]$ and $U_1 = \operatorname{Spec} k[x_0/x_1]$. Then $L = \mathcal{O}_{\mathbb{P}^1_k}(1)$ is invertible. We have $L|_{U_0} = \mathcal{O}_{U_0} x_0$ and $L|_{U_1} = \mathcal{O}_{U_1} x_1$. These are related by $x_0 = (x_1/x_0)x_1$, so the transition function is given by $g_{01} = x_1/x_0$.

More generally, the sheaf $\mathcal{O}(m)$ on $\mathbb{P}^n_A$ is free over the distinguished open sets $U_i = \operatorname{Spec}(A[x_0/x_i, \ldots, x_n/x_i])$, and the transition functions are given by

$$g_{ij} = \left(\frac{x_j}{x_i}\right)^m$$

Note that the ordering of $i$ and $j$ matters: $g_{ij} = (x_i/x_j)^m$ defines the sheaf $\mathcal{O}(-m)$. △

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### The Picard group

As a special case of Proposition 14.3, we have:

**Proposition 14.14.** Let $X$ be a scheme and $L$ and $M$ two invertible sheaves on $X$. Then:
(i) $L \otimes_{\mathcal{O}_X} M$ is also an invertible sheaf.
(ii) $\mathcal{H}om_{\mathcal{O}_X}(L, M)$ is also invertible. In particular, $\mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$ is invertible, and

$$\mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X) \otimes_{\mathcal{O}_X} M \simeq \mathcal{H}om_{\mathcal{O}_X}(L, M). \tag{14.8}$$

This implies that the tensor product acts as a sort of binary operation on the set of invertible sheaves; $L \otimes M$ is invertible if $L$ and $M$ are, and the tensor product is associative. Tensoring an invertible sheaf by $\mathcal{O}_X$ gives an isomorphism $L \otimes_{\mathcal{O}_X} \mathcal{O}_X \simeq L$, so $\mathcal{O}_X$ serves as the identity. For an invertible sheaf $L$, we define $L^{-1} = \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$. By the proposition, we see that $L^{-1}$ is again invertible, and serves as a multiplicative inverse of $L$ under $\otimes$. In particular, this explains the term 'invertible'.

**Definition 14.15.** For a scheme $X$, the *Picard group* $\operatorname{Pic}(X)$ of $X$ is the group of isomorphism classes of invertible sheaves on $X$ under the tensor product.

Note that it is the set of isomorphism classes of invertible sheaves that form a group, not the invertible sheaves themselves. That is, $L \otimes_{\mathcal{O}_X} L^{-1}$ is isomorphic, but strictly speaking, not *equal* to $\mathcal{O}_X$. Note also that $\operatorname{Pic}(X)$ is an abelian group, because $L \otimes_{\mathcal{O}_X} M$ is canonically isomorphic to $M \otimes_{\mathcal{O}_X} L$.

**Example 14.16** ($\operatorname{Spec}\mathbb{Z}$)**.** By Example 14.6, any invertible sheaf on $X = \operatorname{Spec}\mathbb{Z}$ is trivial, so

$$\operatorname{Pic}(\operatorname{Spec}\mathbb{Z}) = 0.$$

On the other hand, $\operatorname{Pic}(\mathbb{Z}[\sqrt{-5}]) \neq 0$, as we will show in Proposition 16.48. △

**Example 14.17** (The affine line)**.** By Example 12.22, any invertible sheaf on $\mathbb{A}^1_k$ is trivial. In particular,

$$\operatorname{Pic}(\mathbb{A}^1_k) = 0.$$

We will prove more generally that $\operatorname{Pic}(\mathbb{A}^n_k) = 0$ for any $n \geqslant 0$ in Chapter 16. △

**Example 14.18** (The projective line)**.** By Theorem 14.12, every invertible sheaf on $\mathbb{P}^1_k$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(m)$ for some $m \in \mathbb{Z}$. Hence we have an isomorphism

$$\operatorname{Pic}\mathbb{P}^1_k \simeq \mathbb{Z}.$$

△

**Example 14.19.** Let $X \subset \mathbb{A}^2_k$ denote the plane curve defined by $y^2 = x^3 + x$ and let $P = (0, 0)$. The ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ of sections vanishing at $P$ is invertible. To see this, we show that for every $Q \in X$, the ideal $\mathcal{I}_Q$ is a principal ideal in $\mathcal{O}_{X,Q}$.

At the point $P$, the stalk $\mathcal{I}_P = (x, y)\mathcal{O}_{X,P}$ is the maximal ideal in $\mathcal{O}_{X,P}$. This ideal is

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generated by the element $y$. Indeed, since $y^2 = x(x-1)(x+1)$, and $(x-1)$ and $(x+1)$ are invertible in $\mathcal{O}_{X,P}$, we can express $x$ in terms of $y$.

For any other point $Q$, at least $x$ or $y$ is invertible in $\mathcal{O}_{X,Q}$ and so $\mathcal{I}_Q = (1)$ in $\mathcal{O}_{X,Q}$.

In either case, the ideal $\mathcal{I}_Q$ is principal, hence $\mathcal{I}$ is an invertible sheaf. It is however not trivial, as any ideal sheaf of $\mathcal{O}_X$ is trivial if and only if it is principal and $(x, y)$ requires two generators.

On the other hand, $\mathcal{I} \otimes \mathcal{I}$ is trivial. Indeed, it corresponds to the tilde of the ideal $(x, y)^2$ in $k[x, y]/(y^2 - x^3 - x)$, and

$$(x, y)^2 = (x^2, xy, y^2) = (x^2, xy, x^3 + x) = (x).$$

In terms of the Picard group, what we have shown is that the class $[\mathcal{I}]$ in $\mathrm{Pic}(X)$ is nonzero, by $2[\mathcal{I}] = 0$ in $\mathrm{Pic}(X)$. In fact, when $k$ is algebraically closed, $\mathrm{Pic}(X)$ is an infinitely generated abelian group, as we will see in Section 20.5. △

## 14.6 Pullbacks of invertible sheaves

Let $f\colon X \to Y$ be a morphism of schemes and $L$ be an invertible sheaf on $Y$. Then $f^*L$ is invertible on $X$. Indeed, if $L$ is trivial over an open cover $\{V_i\}_{i\in I}$, then $f^*L$ is trivial over the open sets $f^{-1}(V_i)$. Moreover, if $L$ is described the local data $g_{ij} \in \mathcal{O}_X(V_i \cap V_j)^\times$, then $f^*L$ is described by the pullbacks $f^\sharp(g_{ij}) \in \mathcal{O}_X(f^{-1}(V_i \cap V_j))$.

If $L$ and $M$ are two invertible sheaves on $Y$, then

$$f^*(L \otimes_{\mathcal{O}_Y} M) = f^*(L) \otimes_{\mathcal{O}_X} f^*(M)$$

Therefore, the assignment $L \mapsto f^*L$ induces a morphism of groups

$$f^*\colon \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X).$$

The pullback operation extends naturally to sections. For any open subset $W \subset Y$ and section $s \in L(W)$, we obtain a section $f^*s \in f^*L(f^{-1}(W))$ as follows: locally on $W \cap V_i$ where $L$ is trivial, we may write $s$ as $a_i \cdot s_i$ for some $a_i \in \mathcal{O}_Y(W \cap V_i)$ and local generator $s_i$ of $L|_{V_i}$. The pullback section is then given by $f^*s = f^\sharp(a_i) \cdot f^*e_i$ on $f^{-1}(W \cap V_i)$, where $f^*e_i$ is the corresponding local generator of $f^*L|_{f^{-1}(V_i)}$. This construction is independent of the choice of trivialization and glues to give a well-defined global section.

**Example 14.20.** Consider the morphism

$$f\colon \mathbb{P}^1_k \longrightarrow \mathbb{P}^2_k; \quad (x_0 : x_1) \mapsto (x_0^2 : x_0x_1 : x_1^2).$$

In this example, $f^*\mathcal{O}_{\mathbb{P}^2_k}(1) = \mathcal{O}_{\mathbb{P}^2_k}(2)$. To see this, let $U_0, U_1$ and $V_0, V_1, V_2$ be the standard coverings of $\mathbb{P}^1_k$ and $\mathbb{P}^2_k$ respectively. Then we the image of $f$ is covered by the two open sets $V_0$ and $V_2$, and $f^{-1}V_0 = U_0$ and $f^{-1}V_2 = U_1$. We have $\mathcal{O}_{\mathbb{P}^2_k}(1)|_{V_0} = \mathcal{O}_{V_0}y_0$ and $\mathcal{O}_{\mathbb{P}^2_k}(1)|_{V_2} = \mathcal{O}_{V_2}y_2$. The transition function between the two is given by $g_{02} = y_2/y_0$. Here we have $f^\sharp(g_{02}) = (x_1/x_0)^2$, so $f^*\mathcal{O}_{\mathbb{P}^2_k}(1)$ is defined by exactly the same transition functions as $\mathcal{O}_{\mathbb{P}^1_k}(2)$.

Note that the global sections of $\mathcal{O}_{\mathbb{P}^2_k}(1)$ are of the form $s = ay_0 + by_1 + cy_2$ for $a, b, c \in k$. This pulls back to

$$f^*(s) = ay_0^2 + by_0y_1 + cy_2^2.$$

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△

**Example 14.21.** Consider the closed subscheme $X = V(x_0) \subset \mathbb{P}^2_k$ and let $\iota\colon X \to \mathbb{P}^2_k$ be the inclusion. Then $\iota^*\mathcal{O}_{\mathbb{P}^2_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(1)$. Hence

$$\Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(1)) \simeq kx_0 \oplus kx_1 \oplus kx_2 \text{ and } \Gamma(X, \iota^*\mathcal{O}_{\mathbb{P}^2_k}(1)) \simeq kx_1 \oplus kx_2$$

In particular, contrary to $f_*$, the pullback functor does not always give an isomorphism on global sections. △

## 14.7 Morphisms to projective space

One reason to care about invertible sheaves is that they help us understanding morphisms to projective space. More precisely, given a scheme $X$, it is natural to ask what sort of data one needs to specify a morphism

$$f\colon X \longrightarrow \mathbb{P}^n. \tag{14.9}$$

The corresponding question for $\mathbb{A}^n$ has already been answered: m morphisms $X \to \mathbb{A}^n$ are in one-to-one correspondence with elements of $\Gamma(X, \mathcal{O}_X)^n$, i.e., an $n$-tuple of regular functions on $X$.

For a morphism into projective space it will not be the global sections $\Gamma(X, \mathcal{O}_X)$, but rather sections of an *invertible sheaf* that will be the analogous data we need in order to specify a morphism.

To explain this properly, let $L$ be an invertible sheaf and let $s \in \Gamma(X, L)$ be a global section. Consider the open set

$$D(s) = \{\, P \in X \mid s(P) \neq 0 \text{ in } L(P) \,\}.$$

That is, $D(s)$ is the set of points $P \in X$ where $s_P \notin \mathfrak{m}_P L_P$. Being the complement of $V(s)$, this set is indeed an open set of $X$.

We say that a set of sections $\{s_i\}_{i\in I}$ *generate* $L$ if for any $P \in X$, there is an $s_i$ such that $s_i(P) \neq 0$ in $L(P)$. Equivalently, the open sets $D(s_i)$ form an open cover of $X$. If this holds, we say that $L$ is *globally generated*.

The next lemma says that any nonzero section generates $L$ over the open set $D(s)$.

**Lemma 14.22.** Let $X$ be a scheme and let $L$ be an invertible sheaf on $X$. If $s \in \Gamma(X, L)$ is a nonzero global section, then there is an isomorphism of $\mathcal{O}_X$-modules

$$\phi\colon \mathcal{O}_X|_{D(s)} \longrightarrow L|_{D(s)}$$

which sends 1 to $s$.

*Proof* We define $\phi$ over an open set $U \subset D(s)$, by sending $1 \in \mathcal{O}_X(U)$ to $s|_U \in L(U)$ and extend $\mathcal{O}_X$-linearly. This is an isomorphism if and only if it is an isomorphism locally, so we may reduce to the case where $X = \operatorname{Spec} A$ and $L = \mathcal{O}_X$. In that case, $D(s) = \operatorname{Spec} A_s$, $s$ is invertible in $A_s$ and so multiplication by $s$ is an isomorphism $A_s \to A_s$. □

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**Lemma 14.23.** Let $f\colon X \to Y$ be a morphism of schemes and let $L$ be an invertible sheaf on $Y$. Then if $L$ is an invertible sheaf which is generated by global sections $s_0, \ldots, s_n$, then $f^*L$ is generated by the sections $t_0 = f^*s_0, \ldots, t_n = f^*s_n$, and $X$ is covered by the open sets $D(t_0), \ldots, D(t_n)$.

*Proof* As $L$ is invertible, we may reduce to the case $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, $L = \mathcal{O}_Y$ and $f$ is induced by a ring map $\phi\colon A \to B$. Then we can view the sections $s_0, \ldots, s_n$ as elements of $A$, and the pullback $f^*(s)$ of a section $s$ is simply given by $\phi(s)$.

Let $P \in X$ be a point. Then there is an index $j$ so that $s_j(P) \neq 0 \in \kappa(Q)$ is nonzero at $Q = f(P)$. Then $f^*(s_j) = \phi(s_j)$ is then also nonzero at $P$, as its class in $\mathcal{O}_{X,P}$ coincides with the image of $s_j$ via the map of local rings $f_P\colon \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$.

As the elements $s_0, \ldots, s_n$ generate $L$, they generate the unit ideal in $A$, and so the pullbacks $\phi(s_0), \ldots, \phi(s_n)$ generate the unit ideal in $B$. Hence $X = \operatorname{Spec} B$ is covered by the open sets $D(f^*(s_0)), \ldots, D(f^*(s_n))$. □

**Example 14.24.** If $X = \operatorname{Spec} A$ is an affine scheme, then any invertible sheaf $L$ is globally generated. Indeed, $L = \widetilde{M}$ for some $A$-module $M$. Choosing a (possibly infinite) set of generators $\{m_i\}_{i\in I}$ for $M$, these define global sections of $L$ which generate $L$ in every stalk. △

**Example 14.25.** On $\mathbb{P}^n_k$, the sections $x_0, \ldots, x_n$ generate $\mathcal{O}_{\mathbb{P}^n_k}(1)$. Indeed, every point $P \in \mathbb{P}^n_k$ lies in some open set $U_i \subset \mathbb{P}^n_k$, and for this $i$, we have $x_i(P) \neq 0$. △

Combining the previous example with Lemma 14.23, we see that given a morphism $f\colon X \to \mathbb{P}^n$, we obtain an invertible sheaf $L = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ on $X$, as well as $n+1$ distinguished global sections $s_i = f^*x_i$ by pulling back the sections $x_0, \ldots, x_n$ of $\mathcal{O}(1)$. As $\mathcal{O}(1)$ is globally generated by $x_0, \ldots, x_n$, $L$ will be globally generated by the pullbacks $s_0, \ldots, s_n$.

The main result in this section is that there is a way to reverse this process. In other words, from a given invertible sheaf $L$ and $n+1$ global sections $s_i \in \Gamma(X, L)$ that generate $L$ at every point, we can uniquely construct a morphism $f\colon X \to \mathbb{P}^n$ so that $f^*\mathcal{O}_{\mathbb{P}^n}(1) = L$ and $f^*x_i = s_i$. Hence $(L, s_0, \ldots, s_n)$ is the data we are after.

**Theorem 14.26.** Let $X$ be a scheme over a ring $A$, and let $L$ be an invertible sheaf on $X$ with global sections $s_0, \ldots, s_n \in \Gamma(X, L)$ that generate $L$. Then there is a *unique* morphism of $A$-schemes

$$f\colon X \longrightarrow \mathbb{P}^n_A = \operatorname{Proj} A[x_0, \ldots, x_n]$$

and an isomorphism $f^*\mathcal{O}(1) \simeq L$ which maps $f^*x_i$ to $s_i$ for $i = 0, \ldots, n$.

*Proof of Theorem 14.26* Write $\mathbb{P}^n_A = \operatorname{Proj} R$, where $R = A[x_0, \ldots, x_n]$ and denote $\mathcal{O}_{\mathbb{P}^1_A}(1)$ by $\mathcal{O}(1)$.

By assumption, the $n+1$ sections $s_0, \ldots, s_n$ generate invertible sheaf $L$. By Lemma 14.22, the open sets $D(s_i)$ provide a local trivializing cover of $L$, with isomorphisms $\phi_i\colon \mathcal{O}_X|_{D(s_i)} \to L|_{D(s_i)}$ which sends 1 to the section $s_i$. Since $s_i$ is a local genera-

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tor for $L$ over $D(s_i)$ we may for each $j = 0, \dots, n$, express $s_j|_{D(s_i)}$ as $r_{ij}s_i$ for some $r_{ij} \in \mathcal{O}_X(D(s_i))$. For simplicity, we write $s_j/s_i$ for $r_{ij}$.

These sections define a map of $A$-algebras

$$(R_{x_i})_0 \longrightarrow \Gamma(D(s_i), \mathcal{O}_X); \qquad x_j/x_i \longmapsto s_j/s_i \tag{14.10}$$

By the correspondence between ring homomorphisms and maps into affine schemes, we obtain a morphism of $A$-schemes $f_i \colon D(s_i) \to D_+(x_i)$.

To check that the morphisms $f_i$ glue to a morphism $f \colon X \to \mathbb{P}^n_A$, we check the overlaps $D(s_i) \cap D(s_j)$ and $D_+(x_i) \cap D_+(x_j) = D_+(x_ix_j)$. Note that there is a commutative diagram

$$\begin{array}{ccc} (R_{x_i})_0 & \longrightarrow & \Gamma(D_+(s_i), \mathcal{O}_X) \\ \downarrow & & \downarrow \\ (R_{x_ix_j})_0 & \longrightarrow & \Gamma(D_+(s_is_j), \mathcal{O}_X) \end{array} \tag{14.11}$$

As the bottom the map sends $\frac{x_k}{x_j} = \frac{x_k/x_i}{x_j/x_i}$ to $\frac{s_k}{s_j} = \frac{s_k/s_i}{s_j/s_i}$, we see that both $f_i$ and $f_j$ restrict to the same map $D(s_is_j) \to \mathbb{P}^n_A$, namely the one induced by the bottom map in (14.11). Hence they glue to a morphism $f \colon X \to \mathbb{P}^n$.

Next, we show that there is an isomorphism $f^*\mathcal{O}(1) \simeq L$. On $\mathbb{P}^n_A$, we have $\mathcal{O}(1)|_{D_+(x_i)} = \mathcal{O}_{D_+(x_i)}x_i$, and $x_i$ serves as the local generator. Over the open sets $D(s_i) = f^{-1}(D(x_i))$, we therefore have $f^*\mathcal{O}(1)|_{D(s_i)} \simeq \mathcal{O}_{D(s_i)}$ with local generator $f^*x_i$. Likewise, $L|_{D(s_i)} \simeq \mathcal{O}_{D(s_i)}$ with local generator $s_i$. As both $L$ and $f^*\mathcal{O}(1)$ are trivial over $D(s_i)$, we may define an isomorphism $f^*\mathcal{O}(1)|_{D(s_i)} \to L|_{D(s_i)}$ by sending $f^*x_i$ to $s_i$. To check that these isomorphisms glue, we need to check that they are compatible with the gluing maps. The sheaf $\mathcal{O}(1)$ is obtained by gluing together the $\mathcal{O}_{D(x_i)}$ using the transition functions $g_{ij} = x_j/x_i$. Therefore $f^*\mathcal{O}(1)$ is obtained by gluing together $\mathcal{O}_{D(s_i)}$ using multiplication by $f^*g_{ij} = f^\sharp(x_j/x_i)$ as transition functions. However, by construction, $f^\sharp(x_j/x_i) = s_j/s_i$. Hence $f^*\mathcal{O}(1)$ and $L$ are defined by the same gluing data, and the local isomorphisms define an isomorphism $f^*\mathcal{O}(1) \simeq L$. It is clear that the generator $f^*x_i$ maps to $s_i$ via this identification, because this happens over each open set $D(s_i)$.

The uniqueness part is clear, because the sections $s_0, \dots, s_n$ determine the ratios $s_j/s_i$, which in turn determine $f$. □

We will refer to a morphism $\phi \colon X \to \mathbb{P}^n_A$ as given by the data $(L, s_0, \dots, s_n)$ and informally write

$$\begin{aligned} X &\longrightarrow \mathbb{P}^n_A \\ x &\longmapsto (s_0(x) : \cdots : s_n(x)) \end{aligned}$$

One should still keep in mind that the sections $s_i$ are sections of $L$, not regular functions. In light of the above proof, we see that it is the ratios $s_j/s_i$ which can be interpretated as regular functions, locally on $D(s_i) = \{x \in X \mid s_i(x) \neq 0\}$.

We also see that two sets of data $(L, s_0, \dots, s_n)$, $(L, t_0, \dots, t_n)$ give rise to the same morphism $f \colon X \to \mathbb{P}^n_A$ if and only there is a unit $\lambda \in \mathcal{O}_X^\times(X)$ so that $t_i = \lambda s_i$ for each $i$. Thus morphisms $f \colon X \to \mathbb{P}^n_A$ are in bijective correspondence with the data $(L, s_0, \dots, s_n)$

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modulo this equivalence relation. One can compare this with the description of the $A$-valued points of $\mathbb{P}^n_A$ from Chapter **??**:

**Example 14.27.** Let $A = \mathbb{Z}$ and consider $X = \operatorname{Spec}\mathbb{Z}$. Since any invertible sheaf on $\operatorname{Spec}\mathbb{Z}$ is isomorphic to the structure sheaf, we see that $\mathbb{Z}$-morphisms $f\colon X \to \mathbb{P}^n_{\mathbb{Z}}$ are in one-to-one correspondance with $(n+1)$-tuples of elements in $\Gamma(X, \mathcal{O}_X) = \mathbb{Z}$. In other words, any $\mathbb{Z}$-point $\operatorname{Spec}\mathbb{Z} \to \mathbb{P}^n_{\mathbb{Z}}$ is in 'homogeneous coordinate form', as defined in Chapter **??**. The same applies when $A$ is a local ring, or any ring with $\operatorname{Pic}(\operatorname{Spec} A) = 0$. On the other hand, $\operatorname{Pic}(\operatorname{Spec}\mathbb{Z}[\sqrt{-5}]) \neq 0$, and we saw that there were 'non-obivious' maps $\operatorname{Spec}\mathbb{Z}[\sqrt{-5}] \to \mathbb{P}^1_{\mathbb{Z}}$. △

**Example 14.28.** Let $X = \mathbb{P}^1_k = \operatorname{Proj} k[s,t]$ and $L = \mathcal{O}_{\mathbb{P}^1_k}(2)$. Then $L$ is globally generated by $s^2, st, t^2$ and the corresponding morphism

$$\begin{aligned} \phi\colon \mathbb{P}^1_k &\longrightarrow \mathbb{P}^2_k \\ (s:t) &\longmapsto (s^2 : st : t^2) \end{aligned}$$

has image $V(x_0x_2 - x_1^2)$ which is an irreducible degree 2 curve. △

**Example 14.29** (Cuspidal cubic)**.** Let $X = \mathbb{A}^1_k$ and $L = \mathcal{O}_X$. Then, $\Gamma(X, L) = k[t]$. Choosing the three sections $1, t^2, t^3$, we get a map of schemes

$$\begin{aligned} \phi\colon X &\longrightarrow \mathbb{P}^2_k \\ t &\longmapsto (1 : t^2 : t^3) \end{aligned}$$

whose image in $\mathbb{P}^2$ is the cuspidal cubic $V(x_0x_2^2 - x_1^3)$ minus the point at infinity. △

Given a scheme $X$ with $s_0, \ldots, s_n$ of an invertible sheaf $L$, there is a maximal open subset $U$ such that the sections generate $L$ for all points in $U$, namely $U = \bigcup_{i=0}^n D(s_i)$. Even if we do not assume that the $s_i$ globally generate $L$, we still get a morphism $\phi\colon U \to \mathbb{P}^n_A$. In other words, the sections define a *rational map* $\phi\colon X \dashrightarrow \mathbb{P}^n_A$, which is a morphism when restricted to $U$.

**Example 14.30.** Let $X = \mathbb{A}^{n+1}_k = \operatorname{Spec} k[x_0, \ldots, x_n]$ and $L = \mathcal{O}_X$. Then $\Gamma(X, L) = k[x_0, \ldots, x_n]$. If we take the sections $x_0, \ldots, x_n$, then they generate $L$ outside $V(x_0, \ldots, x_n)$. Hence we get a morphism of schemes

$$\begin{aligned} \mathbb{A}^{n+1}_k - V(x_0, \ldots, x_n) &\longrightarrow \mathbb{P}^n_k \\ (x_0, \ldots, x_n) &\mapsto (x_0 : \cdots : x_n) \end{aligned}$$

which is exactly the 'quotient space' description of $\mathbb{P}^n$ from (5.5). △

## 14.8 Application: Automorphisms of $\mathbb{P}^n_k$

If $k$ is a field, then any invertible $(n+1) \times (n+1)$ matrix $A$ with entires in $k$ acts on $k[x_0, \ldots, x_n]$ and gives rise to a linear automorphism $\mathbb{P}^n_k \to \mathbb{P}^n_k$. Moreover, two matrices $A$ and $A'$ determine the same automorphism if and only if $m = \lambda m'$ for some non-zero scalar $\lambda \in k^\times$. The *projective linear group* is defined by as the quotient group

$$\mathrm{PGL}_n(k) = \mathrm{GL}_n(k)/k^\times$$

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We will now prove that all automorphisms of $\mathbb{P}^n_k$ are given by linear transformations.

**Theorem 14.31.** $\mathrm{Aut}_K(\mathbb{P}^n_k) = \mathrm{PGL}_n(k)$.

*Proof* The above paragraph shows that there is an injective map from the righthand side to the left. Conversely, let $\phi\colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ be any automorphism. Then $\phi$ induces an isomorphism

$$\mathrm{Pic}(\mathbb{P}^n) \longrightarrow \mathrm{Pic}(\mathbb{P}^n_k)$$

Since $\mathrm{Pic}(\mathbb{P}^n_k)$ is generated by $\mathcal{O}_{\mathbb{P}^n_k}(1)$, we must have either $\phi^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^n_k}(1)$ or $\phi^*\mathcal{O}_{\mathbb{P}^n_k}(1) = \mathcal{O}_{\mathbb{P}^n_k}(-1)$. The latter case is however impossible, because $\phi^*\mathcal{O}_{\mathbb{P}^n_k}(1)$ has many global sections, whereas $\mathcal{O}_{\mathbb{P}^n_k}(-1)$ has none. Hence $\phi^*\mathcal{O}_{\mathbb{P}^1_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(1)$. Evaluating $\phi^*$ over global sections gives a $k$-linear isomorphism

$$\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)) \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n_k}(1)),$$

Now, we may choose $\{x_0, \ldots, x_n\}$ as a basis for $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(1))$, and so in this basis $\phi^*$ gives rise to an invertible $(n+1)\times(n+1)$-matrix $A$. By construction $A$ induces the same linear transformation $\mathbb{P}^n_k \to \mathbb{P}^n_k$ as $\phi$, and so $\phi$ comes from an element of $\mathrm{PGL}_n(k)$. □

## Appendix: Locally free sheaves and stalks

If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank $r$, then the stalk $\mathcal{E}_P$ is clearly isomorphic to $\mathcal{O}^r_{X,P}$ for every $P \in X$. In fact, the converse holds if $\mathcal{E}$ is coherent:

**Lemma 14.32.** Let $X$ be an integral Noetherian scheme and let $\mathcal{E}$ be a coherent sheaf on $X$. Let $P \in X$ be a point. If $\mathcal{E}_P$ is a free $\mathcal{O}_{X,P}$-module, then there is an open set $U \subset X$ such that $\mathcal{E}|_U$ is free.

*Proof* We may assume that $X$ is affine, say $X = \mathrm{Spec}\, A$, and $\mathcal{E} = \widetilde{M}$ for some finitely generated $A$-module $M$. Suppose that $P$ corresponds to the prime ideal $\mathfrak{p} \subset A$. By assumption, $M_\mathfrak{p} \simeq A^r_\mathfrak{p}$. Pick elements $m_1, \ldots, m_r \in M$ so that their classes generate $M_\mathfrak{p}$ and consider the map of $A$-modules $\phi\colon A^r \to M$ sending the $i$-th basis vector to $m_i$. Letting $K = \mathrm{Ker}(\phi)$ and $C = \mathrm{Coker}\,\phi$, there is an exact sequence

$$0 \longrightarrow K \longrightarrow A^r \xrightarrow{\phi} M \longrightarrow C \longrightarrow 0 \tag{14.12}$$

As $A$ is Noetherian and $M$ is finitely generated, both $K$ and $C$ are also finitely generated. This implies that the supports $\mathrm{Supp}(K)$ and $\mathrm{Supp}(C)$ are closed in $\mathrm{Spec}(A)$ (they are equal to $V(\mathrm{Ann}(K))$ and $V(\mathrm{Ann}(C))$ respectively). These subsets are are moreover proper closed subsets, because $\phi_\mathfrak{p}$ is an isomorphism. Therefore, we may pick an $f \in A$ so that $D(f) \subset \mathrm{Spec}\, A - V(\mathrm{Ann}(C)) \cup V(\mathrm{Ann}(K))$ is an open set containing $\mathfrak{p}$. This means that $C_f = K_f = 0$ and hence that $\phi_f\colon A^r_f \to M_f$ is an isomorphism. Consequently, $\mathcal{E}$ is free over the open set $D(f) \subset \mathrm{Spec}\, A$. □

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**Corollary 14.33.** Let $X$ be a Noetherian integral scheme and let $\mathcal{E}$ be a coherent sheaf. Then the following are equivalent:
(i) $\mathcal{E}$ is locally free of rank $r$
(ii) $\mathcal{E}_P$ is a free $\mathcal{O}_{X,P}$-module of rank $r$ for every $P \in X$.

**Example 14.34.** Let $A$ be DVR with fraction field $K$, and let $s$ and $\xi$ be respectively the closed and the open point of $X = \operatorname{Spec} A$. Let $\mathcal{E}$ be the $\mathcal{O}_X$-module with $\Gamma(X, \mathcal{E}) = A$ and $\Gamma(\{\xi\}, \mathcal{E}) = K$, and with the restriction map $A \to K$ equal to the zero map. Then $\mathcal{E}$ is an $\mathcal{O}_X$-module with exactly the same stalks as the structure sheaf $\mathcal{O}_X$, but it is not locally free (in fact, it is not even quasi-coherent). △

**Corollary 14.35.** If $X$ is a Noetherian integral scheme and $\mathcal{E}$ is a coherent sheaf, then there is a dense open set $U \subset X$ such that $\mathcal{E}|_U$ is locally free.

*Proof* This follows by applying Lemma 14.32 to the generic point. In that case the local ring is a field, so any finitely generated module is free. □

Let $\mathcal{E}$ be a coherent sheaf and let $P \in X$ be a point. The *fiber of $\mathcal{E}$ at $P$* is defined as the $\kappa(P)$-vector space

$$\mathcal{E}(P) = \mathcal{E}_P / \mathfrak{m}_P \mathcal{E}_P.$$

Geometrically, $\mathcal{E}(P) \simeq \mathcal{E}_P \otimes_{\mathcal{O}_{X,P}} \kappa(P)$ is the $\kappa(P)$-vector space which corresponds to the pullback of $\mathcal{E}$ via the map $\iota\colon \operatorname{Spec} \kappa(P) \to X$.

If $\mathcal{E}$ is locally free of rank $r$, then $\mathcal{E}(P)$ is a $\kappa(P)$-vector space of dimension $r$ at every point $P$. The next result states that the converse holds, that is, a coherent sheaf with the property have the fibers have the same dimension at every point is locally free.

**Proposition 14.36.** Let $X$ be an integral Noetherian scheme with generic point $\xi$ and let $\mathcal{E}$ be a coherent sheaf. Then for all $P \in X$,

$$\dim_{\kappa(P)} \mathcal{E}(P) \geqslant \dim_{\kappa(\xi)} \mathcal{E}(\xi)$$

If equality holds for every $P \in X$, then $\mathcal{E}$ is locally free of rank $r = \dim_{\kappa(\xi)} \mathcal{E}(\xi)$.

To prove this, we immediately reduce to the affine case $X = \operatorname{Spec} A$, where $\mathcal{E} = \widetilde{M}$ for a finitely generated $A$-module $M$. Using Theorem **??**, we reduce to proving the following:

**Lemma 14.37.** Let $A$ be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and fraction field $K$. Let $M$ be a finitely generated $A$-module. Then

$$\dim_{A/\mathfrak{m}}(M \otimes_A A/\mathfrak{m}) \geqslant \dim_K(M \otimes_A K) \tag{14.13}$$

with equality if and only if $M$ is free.

*Proof* Let $x_1, \ldots, x_s$ be elements of $M$ so that $\overline{x_1}, \ldots, \overline{x_s}$ form a basis for $M/\mathfrak{m}M = M \otimes_A A/\mathfrak{m}$ as a $A/\mathfrak{m}$-vector space. By Nakayama's Lemma, the $x_1, \ldots, x_s$ generate $M$ as

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an $A$-module. Then the elements $x_1 \otimes 1, \ldots, x_r \otimes 1$ also generate $M \otimes_A K$ as a $K$-vector space, and hence (14.13) holds.

We next need to show that if $\dim_\kappa M \otimes_A A/\mathfrak{m} = \dim_K M \otimes_A K = r$, then $M$ is free of rank $r$. Choose generators $x_1, \ldots, x_r \in M$ so that their classes define a basis for $M \otimes_A A/\mathfrak{m}$ as before. Consider the map $\phi\colon A^r \to M$ defined by $(a_1, \ldots, a_r) \mapsto \sum_{i=1}^r a_i x_i$. There is an exact sequence

$$0 \longrightarrow \operatorname{Ker}(\phi) \longrightarrow A^r \xrightarrow{\phi} M \longrightarrow 0$$

Tensoring the sequence by $K$ (which corresponds to localizing at $(0)$, and hence preserves exactness), gives a sequence of finite-dimensional $K$-vector spaces

$$0 \longrightarrow \operatorname{Ker}(\phi) \otimes_A K \longrightarrow K^r \longrightarrow M \otimes_A K \longrightarrow 0$$

As $K^r$ and $\dim_K(M \otimes_A K) = r$ have the same dimension, and the sequence is exact, we must have $\operatorname{Ker}(\phi) \otimes_A K = 0$. Hence $\operatorname{Ker}(\phi)$ is torsion module. But being a submodule of $A^r$, where $A$ is an integral domain, it must be torsion free, and hence $\operatorname{Ker}(\phi) = 0$. Therefore, $\phi$ is an isomorphism and hence $M$ is free.

Finally, if $M \simeq A^r$ then $M \otimes_A A/\mathfrak{m} \simeq (A/\mathfrak{m})^r$ and $M \otimes_A K \simeq K^r$, so equality holds in (14.13). □

In the affine case, where $X = \operatorname{Spec} A$ and $\mathcal{E} = \widetilde{M}$, we may understand this open set as follows. Choose a presentation

$$A^m \xrightarrow{\phi} A^n \longrightarrow M \longrightarrow 0. \tag{14.14}$$

For $r \in \mathbb{N}$, define the ideal $I_r(M)$ generated by the $(n-r) \times (n-r)$-minors of the matrix representing $\phi$. By (**?**, Section 20.3), the ideal $I_r(M)$ is independent of the chosen presentation of $M$, and

$$V(I_r(M)) = \{\mathfrak{p} \in \operatorname{Spec} A \mid \operatorname{rank}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geqslant r+1\}.$$

In particular, setting $r = \dim_K(M \otimes_A K)$, then $\widetilde{M}$ will be locally free over the open subset

$$U = \operatorname{Spec} A - V(I_r).$$

In particular, $\widetilde{M}$ is locally free of rank $r$ if and only if $I_{r-1}(M) = 0$ and $I_r(M) = A$.

This criterion is often useful for proving local freeness. Here's an example:

**Example 14.38.** Let $k$ be a field and let $R = k[x_0, \ldots, x_5]$. Consider the matrix

$$\phi = \begin{pmatrix} 0 & x_0 & x_1 & x_2 \\ -x_0 & 0 & x_3 & x_4 \\ -x_1 & -x_3 & 0 & x_5 \\ -x_2 & -x_4 & -x_5 & 0 \end{pmatrix}.$$

The determinant of $\phi$ is the square of the quadratic polynomial

$$q = x_0x_5 - x_1x_4 + x_2x_3$$

This means that the locus of points where $\phi$ has rank at most 3 is given by the hypersurface

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$X = V(q) \subset \mathbb{A}^6_k$. The locus of points where $\phi$ has rank at most 2 is defined by the ideal generated by the $2 \times 2$-minors, which by direct calculation has radical equal to the ideal $(x_0, \ldots, x_5)$.

Let $A = R/(q)$ and $X = \operatorname{Spec} A$. The matrix $\phi$ defines an exact sequence of $A$-modules

$$A^4 \xrightarrow{\phi} A^4 \longrightarrow \operatorname{Coker}(\phi) \longrightarrow 0.$$

Applying $\sim$, we get a sequence

$$\mathcal{O}_X^4 \longrightarrow \mathcal{O}_X^4 \longrightarrow \mathcal{E} \longrightarrow 0$$

Proposition 14.36 then shows that $\mathcal{E}$ is locally free of rank 2 on $X - \{(0, \ldots, 0)\}$. △

## 14.9 Exercises

**Exercise 14.9.1.** Let $\phi : L \to M$ be a map of invertible sheaves.

a) Show that if $\phi$ is surjective, then it is an isomorphism.
b) Give an example of an injective map $\phi$ which is not an isomorphism.

**Exercise 14.9.2.** Show that tensoring by a locally free sheaf of finite rank preserves exact sequences.

**Exercise 14.9.3.** Let $A$ be a ring, let $M$ be a finitely generated $A$-module and let $S \subset A$ is a multiplicative set. Show that there are isomorphisms of $S^{-1}A$-modules

a) $S^{-1}(\operatorname{Sym}^n M) = \operatorname{Sym}^n(S^{-1}M)$.
b) $S^{-1}(\bigwedge^n M) = \bigwedge^n(S^{-1}M)$.

**Exercise 14.9.4.** Let $\mathcal{E}$ be a quasi-coherent sheaf. Define the *$n$-th symmetric power* $\operatorname{Sym}^n(\mathcal{E})$ and the *$n$-th exterior power*, $\bigwedge^n \mathcal{E}$, to be the sheaves associated to the presheaves

$$U \mapsto \operatorname{Sym}^n(\mathcal{E}(U)) \quad \text{and} \quad U \mapsto \bigwedge^n \mathcal{E}(U).$$

respectively.

a) Show that $\operatorname{Sym}^n(\mathcal{E})$ and $\bigwedge^n(\mathcal{E})$ are quasi-coherent and that

$$\Gamma(U, \operatorname{Sym}^n(\mathcal{E})) = \operatorname{Sym}^n(\mathcal{E}(U)) \quad \text{and} \quad \Gamma(U, \bigwedge^n \mathcal{E}) = \bigwedge^n(\mathcal{E}(U)).$$

b) Show that if $\mathcal{E}$ is locally free of rank $r$, then $\operatorname{Sym}^n(\mathcal{E})$ and $\bigwedge^n \mathcal{E}$ are locally free of ranks $\binom{n+r-1}{r-1}$ and $\binom{r}{n}$ respectively.
c) Compute the following sheaves:
   (i) $\operatorname{Sym}^n(\mathcal{O}_{\mathbb{P}^1_k}(a) \oplus \mathcal{O}_{\mathbb{P}^1_k}(b))$ for $n \geqslant 1$.
   (ii) $\bigwedge^2(\mathcal{O}_{\mathbb{P}^1_k}(a) \oplus \mathcal{O}_{\mathbb{P}^1_k}(b))$
   (iii) $\bigwedge^2(\mathcal{O}_{\mathbb{P}^1_k}(a) \oplus \mathcal{O}_{\mathbb{P}^1_k}(b) \oplus \mathcal{O}_{\mathbb{P}^1_k}(c))$.
d) Suppose $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of locally free sheaves. Show that for each $n \geqslant 0$, there is a filtration of $\bigwedge^n \mathcal{E}$ whose associated graded pieces are:

$$\bigoplus_{k=0}^{n} \left( \bigwedge^k \mathcal{E}' \otimes \bigwedge^{n-k} \mathcal{E}'' \right).$$

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e) Let $\mathcal{E}$ be a locally free of rank $r$. The sheaf $\det(\mathcal{E}) = \wedge^r \mathcal{E}$ is called the *determinant* of $\mathcal{E}$. Show that $\det \mathcal{E}$ is an invertible sheaf.

f) Show that if $f\colon X \to Y$ is a morphism, then for each locally free sheaf $\mathcal{E}$ on $Y$,
$$f^* \det(\mathcal{E}) = \det(f^*\mathcal{E})$$

g) Show that if $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1_k}(a_r)$ for $a_i \in \mathbb{Z}$, then
$$\det(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^1_k}(a_1 + \cdots + a_r).$$

h) Show that the determinant is additive on exact sequences: if
$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$
is exact, then
$$\det(\mathcal{E}) = \det(\mathcal{E}') \otimes_{\mathcal{O}_X} \det(\mathcal{E}''). \tag{14.15}$$

**Exercise 14.9.5.** Let $A$ be a ring. Let $0 \to P \to N \to M \to 0$ be an exact sequence of $A$-modules. Show that there are exact sequences
$$P \otimes_A \operatorname{Sym}^{n-1}(N) \to \operatorname{Sym}^n(N) \to \operatorname{Sym}^n(M) \to 0$$
and
$$P \otimes_A \bigwedge^{n-1} N \to \bigwedge^n N \to \bigwedge^n M \to 0.$$
Generalize these to sequences of locally free sheaves.

**Exercise 14.9.6.** Let $f\colon X \to Y$ and let $\mathcal{E}$ be a locally free sheaf of finite rank on $Y$. Find examples where the pullback map $f^*\colon \Gamma(Y, \mathcal{E}) \to \Gamma(X, f^*\mathcal{E})$ may fail to be injective/surjective.

**Exercise 14.9.7.** Let $X = \operatorname{Spec} A$, where $A = \prod_{i=0}^{\infty} \mathbb{Z}$. Show that $M = \mathbb{Z}$ admits the structure of an $A$-module which is projective, but not free.

**Exercise 14.9.8.** If $Y \subset X$ is a closed subscheme and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type with support contained in $Y$, show that $\mathcal{F}$ admits a finite filtration of $\mathcal{O}_X$-modules of finite type
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$
such that each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the pushforward of an $\mathcal{O}_Y$-module of finite type.

**Exercise 14.9.9.** Let $X$ be a Noetherian scheme. We define the *Grothendieck group* $K_0(X)$ as the quotient of the free abelian group on symbols $[\mathcal{F}]$ for $\mathcal{F}$ an $\mathcal{O}_X$-module of finite type, by the subgroup generated by the expressions $[\mathcal{F}] - [\mathcal{F}'] - [\mathcal{F}'']$ whenever there is an exact sequence
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$
of finite type $\mathcal{O}_X$-modules.

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a) If $X$ is integral and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, we define the rank of $\mathcal{F}$ as the number
$$rank(\mathcal{F}) = \dim_{\mathcal{O}_{X,\xi}}(\mathcal{F}_\xi)$$
where $\xi$ is the generic point of $X$. Show that the rank defines a surjective map of groups $K_0(X) \to \mathbb{Z}$.

b) If $Y \subset X$ is a closed subscheme, show that there is an exact sequence of groups
$$K_0(Y) \xrightarrow{\alpha} K_0(X) \xrightarrow{\beta} K_0(X-Y) \to 0$$
where $\alpha$ is induced by $i_*$ and $\beta$ is induced by restriction. HINT: Compute that $\beta \circ \alpha = 0$, so that $\beta$ induces a map $\bar{\beta} : K_0(X)/\alpha(K_0(Y)) \to K_0(X-Y)$. Use Exercise 14.9.8 and Exercise **??**.

c) Let $k$ be a field. Compute $K_0(\operatorname{Spec} k)$ and $K_0(\mathbb{A}^1_k)$.

d) Compute $K_0(\mathbb{P}^1_k)$.

**Exercise 14.9.10** (The projection formula)**.** Let $f\colon X \to Y$ be a morphism of schemes, and let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{O}_X$-modules. Use the adjunction between $f_*$ and $f^*$ to define a map
$$f_*(\mathcal{F} \otimes f^*\mathcal{E}) \longrightarrow f_*(\mathcal{F}) \otimes \mathcal{E}. \tag{14.16}$$
Show that if $\mathcal{E}$ is locally free of finite rank then (14.16) is an isomorphism.

**Exercise 14.9.11.** Show that the subscheme $Z(s)$ satisfies the following universal property: A morphism $f : T \to X$ satisfies $f^*s = 0$ if and only if it factors through $Z(s)$. (Hint: Understand the subscheme on each open affine $\operatorname{Spec} A \subset X$ first. Reduce to the case $\mathcal{E} = \mathcal{O}_X$.)

**Exercise 14.9.12** (Projective modules)**.** Let $A$ be a Noetherian integral domain. An $A$-module $M$ is called *projective* if the functor $N \mapsto \operatorname{Hom}_A(M,N)$ is exact. Show that a finitely generated $A$-module $M$ is projective if and only if it is a direct summand in a free module; that is, if there is another module $N$ so that $M \oplus N \simeq A^I$.

**Exercise 14.9.13.** Let $A$ be a Noetherian ring. Show that a finitely generated module is projective if and only if $M_\mathfrak{p}$ is projective for all $\mathfrak{p} \in \operatorname{Spec} A$. HINT: Show that $\operatorname{Hom}_A(M,N)_\mathfrak{p} = \operatorname{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$.

**Exercise 14.9.14.** Let $R = \mathbb{Z}[\sqrt{-5}]$ and let $\mathfrak{a} = (2, 1+\sqrt{-5})$. Define two maps $p\colon R^2 \to \mathfrak{a}$ and $s\colon \mathfrak{a} \to R^2$ by $p(a,b) = 2a + (1+\sqrt{-5})b$, and $s(x) = (-x, x\frac{1-\sqrt{-5}}{2})$.

a) Show that $p \circ s = \mathrm{id}$ and deduce that
$$R^2 \simeq \mathfrak{a} \oplus \operatorname{Ker} p.$$

b) Show that $\mathfrak{a}$ is projective, but not free.

**Exercise 14.9.15.** Let $R = \prod_{i=1}^\infty \mathbb{F}_2$ be a direct product of countably many copies of the field $\mathbb{F}_2$ with two elements, and let $I = \bigoplus_{i=1}^\infty \mathbb{F}_2$.

a) Show that $I$ is an ideal which is not finitely generated.

b) Show that $R/I$ is not projective. HINT: $I$ is generated by the elements $e_i \in R$ with a 1 in the $i$-th entry and 0 in all other entries. No non-zero element in $R$ is killed by all the $e_i$'s.

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c) For a prime $\mathfrak{p} \in \operatorname{Spec} R$, show that $(R/I)_{\mathfrak{p}}$ is free. HINT: If $I \subset \mathfrak{p}$ and $\alpha \in I$, let $n \in \mathbb{N}$ be such that $\alpha_i = 0$ for every $i > n$ and define $\beta$ by $\beta_i = 0$ for $i < n$ and $\beta_i = 1$ for $i > n$. Then $\beta \cdot \alpha = 0$ and $1 = \sum_{i=1}^{n} e_i + \beta$. Show that $\beta \notin \mathfrak{p}$ and deduce that $\alpha$ maps to 0 in $I_{\mathfrak{p}}$ and hence that $I_{\mathfrak{p}} = 0$.

**Exercise 14.9.16.** Show that the integers appearing in Theorem 14.12 are unique. HINT: Consider the global sections of a suitable twists $\mathcal{E}(m)$.

**Exercise 14.9.17.** Find an example showing that if $\mathcal{E}$ is locally free, then $\mathcal{E}|_U$ needs not be free for *every* affine $U \subset X$.

**Exercise 14.9.18** (Vector bundles)**.** A *vector bundle of rank* $r$ over a scheme $X$ is a scheme $V$ together with a morphism $\pi\colon V \to X$ such that there is an open cover $\{U_i\}_{i\in I}$ such that for each $i \in I$, there is an isomorphism $\phi_i\colon \pi^{-1}(U_i) \simeq U_i \times \mathbb{A}^r$ so that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times \mathbb{A}^r \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U_i & \xrightarrow{=} & U_i \end{array}$$

and for any pair $i, j \in I$, and affine $V = \operatorname{Spec} A \subset \mathcal{U}_i \cap U_j$ the automorphism $\phi_{ji} = \phi_j \circ \phi_i^{-1}$ induces an isomorphism $V \times \mathbb{A}^r \to V \times \mathbb{A}^r$ which is $A$-linear, i.e., given by $\theta\colon A[x_1, \ldots, x_r] \to A[x_1, \ldots, x_r]$ where $\theta(a) = a$ for all $a \in A$ and $\theta(x_i) = \sum a_{ij}x_j$ for every $i = 1, \ldots, r$.

a) Given a vector bundle $\pi\colon V \to X$, show that $\pi_*\mathcal{O}_V$ is a locally free sheaf of rank $r$.

b) Conversely, given a locally free sheaf $\mathcal{E}$ of rank $r$ on $X$, construct a vector bundle $\pi\colon V \to X$ such that $\pi_*\mathcal{O}_V = \mathcal{E}$. HINT: Define $V$ by gluing, using the transition functions of $\mathcal{E}$.

**Exercise 14.9.19.** Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves of ranks $e$ and $f$ respectively, and let $\{U_i\}$ be an affine cover trivializing both $\mathcal{E}$ and $\mathcal{F}$. Suppose the gluing isomorphisms for $\mathcal{E}$ be given by $\tau_{ji} : \mathcal{O}_{U_i}^r \to \mathcal{O}_{U_j}^r$ and the gluing isomorphisms for $\mathcal{F}$ are given by $\nu_{ji} : \mathcal{O}_{U_i}^s \to \mathcal{O}_{U_j}^s$. Show that the gluing isomorphisms for the tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ are given by $\tau_{ji} \otimes \nu_{ji}$. Show that the matrix representing $\mathcal{E} \otimes \mathcal{F}$ is given by the Kronecker product of the matrices representing $\tau_{ji}$ and $\nu_{ji}$.

**Exercise 14.9.20.** Let $\iota\colon Y \to X$ be a closed subscheme. Show that $\iota_*\mathcal{O}_Y$ is globally generated.

**Exercise 14.9.21.** Consider the scheme $U = \mathbb{P}^2_k - \{(0:0:1)\}$, covered by the two affine open sets

$$U_0 = \operatorname{Spec} k[x, y] \text{ and } U_1 = \operatorname{Spec} k[x^{-1}, x^{-1}y].$$

with $U_0 \cap U_1 = D(x) = D(x^{-1})$. Consider the matrix $\tau_{01} = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$. Show that $\tau_{01}$ defines a locally free sheaf $\mathcal{E}$ on $U$ which fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_U \longrightarrow 0.$$

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Compute $\Gamma(X, \mathcal{E})$. Is $\mathcal{E}$ globally generated?

**Exercise 14.9.22.** Compute $\mathrm{Aut}(\mathbb{A}^1_k)$ by showing that every automorphism $\phi\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ extends uniquely to an automorphism $\bar{\phi}\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ which fixes the point at infinity $(0:1)$.

**Exercise 14.9.23.** Show that the map $(x, y) \mapsto (x, y + p(x))$ defines an automorphism of $\mathbb{A}^2_k$. Conclude that $\mathrm{Aut}(\mathbb{A}^n_k)$ is enormous for $n \geqslant 2$.

**Exercise 14.9.24** (Globally generated sheaves)**.** An $\mathcal{O}_X$-module $\mathcal{F}$ is *globally generated*, if there exist sections $s_i \in \mathcal{F}(X)$, $i \in I$, and the map of $\mathcal{O}_X$-modules

$$\bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \mathcal{F}$$

sending $e_i \mapsto s_i$ is surjective.

a) Show that $\mathcal{F}$ is globally generated if and only if there is a set of sections $s_i \in \mathcal{F}(X)$, $i \in I$, such that for each $x \in X$, the germs of $s_i$ generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module.
b) Show that on an affine scheme, any quasi-coherent sheaf is globally generated.
c) Show that $\mathcal{O}_{\mathbb{P}^n_k}(d)$ is globally generated if and only if $d \geqslant 0$.
d) Let $f\colon X \to Y$ and let $\mathcal{G}$ be a globally generated $\mathcal{O}_Y$-module. Show that $f^*\mathcal{G}$ is globally generated on $X$.
e) Let $f\colon X \to Y$ be a closed embedding. Show that $f_*\mathcal{O}_X$ is globally generated.
f) Find an example of a morphism $f\colon X \to Y$ such that $f_*\mathcal{O}_X$ is not globally generated.

**Exercise 14.9.25.** Suppose $\mathcal{E}$ is locally free of rank $r$. Let $U_i$ be a trivializing cover, and let $\tau_{ji}$ denote the transition functions for $\mathcal{E}$, interpretated as $r \times r$ matrices with entries in $\mathcal{O}_X(U_i \cap U_j)$. Show that $\mathcal{E}^\vee$ is obtained by the transition matrices $\nu_{ji} = (\tau^t_{ji})^{-1}$.

**Exercise 14.9.26** (Koszul complex)**.** Let $A$ be an integral domain and let $x, y \in A$. Assume that $x, y$ form a *regular sequence*, that is, $x$ and $y$ are nonzerodivisors, and $x$ is a nonzerodivisor modulo $y$. Consider the ideal $I = (x, y)$. Show that the sequences

$$0 \longrightarrow A \xrightarrow{\rho} A \oplus A \xrightarrow{\sigma} I \longrightarrow 0 \tag{14.17}$$

and

$$0 \longrightarrow A \xrightarrow{\rho} A \oplus A \xrightarrow{\sigma} A \longrightarrow A/I \longrightarrow 0. \tag{14.18}$$

are exact.

**Exercise 14.9.27** (Koszul complex II)**.** Let $X$ be an integral scheme and let $L$ be an invertible sheaf on $X$. For two sections $s, t \in \Gamma(X, L)$, let $Y = V(s, t)$. Assume that the germs of $s$ and $t$ form a regular sequence in $L_p$ for all $p \in X$.

a) Show that there are exact sequences

$$0 \longrightarrow L^{-2} \xrightarrow{\rho} L^{-1} \oplus L^{-1} \xrightarrow{\sigma} I \longrightarrow 0 \tag{14.19}$$

and

$$0 \longrightarrow L^{-2} \xrightarrow{\rho} L^{-1} \oplus L^{-1} \xrightarrow{\sigma} \mathcal{O}_X \longrightarrow \iota_*\mathcal{O}_Y \longrightarrow 0. \tag{14.20}$$

where $\mathcal{I}$ is the ideal sheaf of the zero scheme $Y = V(s, t)$ of $s$ and $t$.

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b) Study this sequence when $X = \mathbb{P}^1_k$, when $L = \mathcal{O}_{\mathbb{P}^1_k}(1)$ and $s = x_0$, $t = x_1$. What happens on global sections?

**Exercise 14.9.28.** Find a locally free sheaf $\mathcal{E}$ on a scheme and two points $p, q \in X$ such that the ranks of $\mathcal{E}$ at $p$ and $q$ are different.

**Exercise 14.9.29.** Consider the ring $A = \mathbb{Z}[\sqrt{-5}]$ and the ideal $\mathfrak{a} = (2, 1 + \sqrt{-5})$. Show that $\mathfrak{a}$ is projective, but not free. Compare this with Exercise 14.9.14 and Example 5.7.

**Exercise 14.9.30.** Let $k$ be a field, $R = k[x_0, \ldots, x_n]$ and $\mathbb{A}^{n+1}_k = \operatorname{Spec} R$. Consider the exact sequence of $R$-modules

$$0 \longrightarrow R \xrightarrow{\phi} R^{n+1} \longrightarrow M \longrightarrow 0 \tag{14.21}$$

where the map $\phi$ sends a polynomial $p$ to $\sum_i p x_i e_i$ where $e_i$ is the $i$-th standard basis vector. Show that the restriction $\mathcal{E} = \widetilde{M}|_U$ is a locally free sheaf of rank $n$ over the open set $U = \mathbb{A}^{n+1}_k - \{0\}$

**Exercise 14.9.31.** Let $L$ be an invertible sheaf and $Y = V(s)$ the zero scheme of $s$. Show that there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow L \longrightarrow L|_Y \longrightarrow 0 \tag{14.22}$$

where we write $L|_Y = \iota_* \iota^* L$.

**Exercise 14.9.32.** In light of the construction of projective space, it is tempting to consider the functor $F\colon \mathsf{Rings} \to \mathsf{Sets}$ given by

$$F(R) = \{\text{free direct summands } L \subset R^{n+1} \text{ of rank } 1\}$$

However, show that $F$ is *not* representable by a scheme. HINT: Consider $R = \mathbb{Z}[\sqrt{-5}]$ and the open sets $D(2), D(3)$ of $\operatorname{Spec} R$.

**Exercise 14.9.33.** The aim of this exercise is to investigate the functor of points of projective space $\mathbb{P}^n$. We will associate to a scheme $T$, its set of data $(L, s_0, \ldots, s_n)$ where $L$ is an invertible sheaf $L$, with an $(n+1)$-tuple of sections $s_0, \ldots, s_n$ that locally generate $L$ everywhere. We declare $(L, s_0, \ldots, s_n) \sim (M, t_0, \ldots, t_n)$ if there is an isomorphism $f : L \to M$ so that $f^*(s_i) = \lambda \cdot t_i$ for some $\lambda \in \mathcal{O}_T^\times(T)$.

a) Show that $\sim$ is an equivalence relation.

b) Consider the assignment

$$F(T) = \big\{(L, s_0, \ldots, s_n) \big| s_0, \ldots, s_n \in \Gamma(T, L) \text{ generate } L \text{ everywhere}\big\} / \sim$$

Show that $F$ is a functor.

c) Show that there is a natural transformation

$$\Phi(X) : \operatorname{Hom}(T, \mathbb{P}^n) \to F(T)$$

sending a morphism $f : T \to \mathbb{P}^n$ to the equivalence class of the data

$$(L, s_0, \ldots, s_n) = (f^*\mathcal{O}(1), f^*x_0, \ldots, f^*x_n)$$

d) Construct an inverse to $\Phi$ and deduce that $F$ is represented by $\mathbb{P}^n$.

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e) Show that elements of $F(\operatorname{Spec} k)$ are in correspondence with $(n+1)$-tuples $(a_0, \ldots, a_n) \in k^{n+1}$, so that not all $a_i$ are zero. Thus we recover the usual description of the $k$-points of projective space as '1-dimensional subspaces of $k^{n+1}$'.
f) Show that the previous exercise also holds for a local ring.
g) Show that for a ring $R$, the set $F(\operatorname{Spec} R)$ is in bijection with the set of projective rank 1 summands of $R^{n+1}$, i.e., modules of rank 1 such that $M \oplus E \simeq R^{n+1}$ for some module $E$. This is the right generalization of a "line in $k^n$" for general rings.

**Exercise 14.9.34.** This is a continuation of Exercise 14.9.33. We will consider the product $X = \mathbb{P}^m \times \mathbb{P}^n$ and give a new interpretation of the Segre embedding $X \to \mathbb{P}^{(m+1)(n+1)-1}$ in terms of the functor of points.

a) Let $T$ be a scheme and let $(L, s_0, \ldots, s_m)$ and $(M, t_0, \ldots, t_n)$ be elements of $h_{\mathbb{P}^m}(T)$ and $h_{\mathbb{P}^n}(T)$ respectively. Show that the $(m+1)(n+1)$ tensor products $u_{ij} = pr_1^* s_i \otimes pr_2^* t_j$ generate $pr_1^* L \otimes pr_2^* M$ on $T \times T$.
b) Show that
$$(pr_1^* L \otimes pr_2^* M, u_{00}, \ldots, u_{mn}) \tag{14.23}$$
defines an element of $h_{\mathbb{P}^{(m+1)(n+1)-1}}(T)$, and that this defines a functor from $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$.
c) Deduce that there is a morphism $\phi\colon X \to \mathbb{P}^{(m+1)(n+1)-1}$.
d) Show that $\phi$ is an embedding. HINT: Show that the morphism $\phi$ has the property that $\phi^{-1}(D_+(u_{ij})) = D_+(x_0) \times D_+(y_0)$, and show that $\phi$ restricts to an embedding on distinguished subsets.

**Exercise 14.9.35.** This is a continuation of Exercise 14.9.33. We will consider the projective space $\mathbb{P}^n$ and give a new interpretation of the Veronese embedding $X \to \mathbb{P}^N$ in terms of the functor of points.

a) Let $T$ be a scheme and let $(L, s_0, \ldots, s_n)$ be an element of $h_{\mathbb{P}^m}(T)$. Show that for each $d \geqslant 1$, the $N = \binom{n+d}{d}$ monomials
$$s_0^{\otimes e_0} \otimes s_1^{\otimes e_1} \otimes \cdots \otimes s_n^{\otimes e_n} \tag{14.24}$$
for $e_0 + \cdots + e_n = d$, generate $L^{\otimes d}$.
b) Show that $L^{\otimes}$ together with the $N$ sections in (14.24) defines an element in $h_{\mathbb{P}^{N-1}}(T)$ and that this defines a functor from $\mathsf{Sch}^{\mathrm{op}} \to \mathsf{Sets}$.
c) Deduce that there is a morphism $\phi\colon X \to \mathbb{P}^{N-1}$.
d) Show that $\phi$ is an embedding. HINT: Consider distinguished open sets.

**Exercise 14.9.36.** Consider the projective line $\mathbb{P}^1_k$ and the squaring map $f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$, which restricts to the squaring map $U_0 \to U_0$ on each $U_i \simeq \mathbb{A}^1_k$. Show that $f^* \mathcal{O}_{\mathbb{P}^1_k}(1) = \mathcal{O}_{\mathbb{P}^1_k}(2)$.

**Exercise 14.9.37** (A weighted projective plane)**.** Define the weighted projective plane $X = \mathbb{P}(1,1,2)$ over a field $k$.

Show that $X$ is isomorphic to $V(xz - y^2) \subset \mathbb{P}^3_k$.

Use the standard exact sequences on $\mathbb{P}^3$ to compute $H^1(X, \mathcal{O}_X)$ and $H^2(X, \mathcal{O}_X)$.

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Let $L = V(x, y)$ be the line on $X \subset \mathbb{P}^3_k$. Is the ideal sheaf $\mathcal{I}_L$ invertible? HINT: Consider the local ring at the origin in one of the affine opens in $X$.

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# 15

# First steps in sheaf cohomology

One of the main challenges when working with sheaves is that surjective maps of sheaves do not always induce surjections on global sections. Given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

one has a sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \tag{15.1}$$

which is exact at each stage except on the right, but the right-most map may fail to be surjective. In many situations in algebraic geometry, knowing that $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'')$ is surjective is of fundamental importance. For instance, if $U \subset X$ is an open subscheme, it is useful to know when a regular function defined on $U$ extends to a regular function on all of $X$.

Cohomology groups can be seen as a partial response to this behavior of $\Gamma$, and in good situations, they allow us to say something about the missing cokernel. More precisely, the sequence (15.1), induces a *long exact sequence of cohomology groups*

$$\begin{aligned} 0 \longrightarrow \Gamma(X, \mathcal{F}') &\longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \\ \hookrightarrow H^1(X, \mathcal{F}') &\longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^1(X, \mathcal{F}'') \\ \hookrightarrow H^2(X, \mathcal{F}') &\longrightarrow H^2(X, \mathcal{F}) \longrightarrow H^2(X, \mathcal{F}'') \longrightarrow \cdots \end{aligned}$$

This means that the failure of surjectivity of the above is controlled by the group $H^1(X, \mathcal{F}')$ and the other groups in the sequence.

In addition to problems such as lifting sections, cohomology groups allow us to define geometric invariants of $\mathcal{F}$ and $X$. These in turn allow us to distinguish schemes, that is, if two schemes have different cohomology groups they can not be isomorphic.

Cohomology groups can be defined in a completely general setting, for any topological space and a (pre)sheaf on it. There are several ways to define them. The modern approach uses the theory of derived functors. This is in most respects the 'right way' to define the groups in general, but going through the whole machinery of derived functors would take



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us too far astray. We therefore take a more down-to-earth approach using *Cech cohomology* which is better suited for computations.

## 15.1 Some homological algebra

A *complex of abelian groups* $A^\bullet$ is a collection of groups $A^i$, indexed by the integers, together with maps between them

$$\cdots \xrightarrow{d^{p-2}} A^{p-1} \xrightarrow{d^{p-1}} A^p \xrightarrow{d^p} A^{p+1} \xrightarrow{d^{i+1}} \cdots$$

called differentials, such that $d^p \circ d^{p-1} = 0$ for each $i$. A *map of complexes* $f\colon A^\bullet \to B^\bullet$ is a collection of group maps $f_p\colon A^p \to B^p$ that commute with the differentials, meaning that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{p-1} & \xrightarrow{d_A^{p-1}} & A^p & \xrightarrow{d_A^p} & A^{p+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{p-1}} & & \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f_{p+1}} & & \\
\cdots & \longrightarrow & B^{p-1} & \xrightarrow{d_B^{p-1}} & B^p & \xrightarrow{d_B^p} & B^{p+1} & \longrightarrow & \cdots
\end{array}$$

In this way, we may talk about kernels, images, cokernels, exact sequences of complexes, etc.

An element $\sigma \in A^p$ is called a *cocycle* if $d^p\sigma = 0$ and it is a *coboundary* if it lies in the image of $d^{p-1}$, that is, $\sigma = d^{p-1}\tau$ for some $\tau \in A^{p-1}$. The condition $d^p \circ d^{p-1} = 0$ implies that

$$\operatorname{Im} d^{p-1} \subset \operatorname{Ker} d^p,$$

so every coboundary is a cocycle. The *cohomology groups* of the complex $A^\bullet$ are designed to measure the difference between these two notions. The *$p$-th cohomology group* is defined as the quotient

$$H^p(A^\bullet) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

One may think of $H^p(A^\bullet)$ as a group that measures the failure of exactness of the complex $A^\bullet$ at stage $p$: the complex $A^\bullet$ is exact at $p$ if and only if $H^p(A^\bullet) = 0$.

A map of complexes $f\colon A^\bullet \to B^\bullet$ takes cocycles to cocycles and coboundaries to coboundaries. Explicitly, if $a \in A^p$ is a cocycle (so that $d_A^p a = 0$), then $d_B^p(f_p a) = f_{p+1}(d_A^p a) = 0$, so $f_p a$ is also a cocycle. Similarly, if $a = d_A^{p-1} b$ is a coboundary, then $f_p a = d_B^{p-1}(f_{p-1} b)$ is also a coboundary. Consequently, $f$ induces a map on cohomology groups

$$H^p(A^\bullet) \longrightarrow H^p(B^\bullet); \qquad [a] \mapsto [f_p a].$$

The following result is fundamental in the theory of cohomology:

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**Proposition 15.1.** Let $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ be an exact sequence of complexes. Then there is a *long exact sequence* of cohomology groups

$$\cdots \longrightarrow H^p(A^\bullet) \longrightarrow H^p(B^\bullet) \longrightarrow H^p(C^\bullet) \longrightarrow H^{p+1}(A^\bullet) \longrightarrow H^{p+1}(B^\bullet) \longrightarrow H^{p+1}(C^\bullet) \longrightarrow \cdots$$

*Proof* For each $p \in \mathbb{Z}$, consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^p & \xrightarrow{f_p} & B^p & \xrightarrow{g_p} & C^p & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_A^p} & & \downarrow{\scriptstyle d_B^p} & & \downarrow{\scriptstyle d_C^p} & & \\ 0 & \longrightarrow & A^{p+1} & \xrightarrow{f_{p+1}} & B^{p+1} & \xrightarrow{g_{p+1}} & C^{p+1} & \longrightarrow & 0 \end{array}$$

Applying the Snake Lemma, we obtain an exact sequence

$$0 \longrightarrow \operatorname{Ker} d_A^p \xrightarrow{f_p} \operatorname{Ker} d_B^p \xrightarrow{g_p} \operatorname{Ker} d_C^p \xrightarrow{\delta} A^{p+1}/\operatorname{Im} d_A^p \xrightarrow{f_{p+1}} B^{p+1}/\operatorname{Im} d_B^p \xrightarrow{g_{p+1}} C^p/\operatorname{Im} d_C^{p+1} \longrightarrow 0.$$

Now consider the diagram:

$$\begin{array}{ccccccccc} & & A^p/\operatorname{Im} d_A^{p-1} & \xrightarrow{f_p} & B^p/\operatorname{Im} d_B^{p-1} & \xrightarrow{g_p} & C^p/\operatorname{Im} d_C^{p-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{d}_A^p} & & \downarrow{\scriptstyle \overline{d}_B^p} & & \downarrow{\scriptstyle \overline{d}_C^p} & & \\ 0 & \longrightarrow & \operatorname{Ker} d_A^{p+1} & \xrightarrow{f_{p+1}} & \operatorname{Ker} d_B^{p+1} & \xrightarrow{g_{p+1}} & \operatorname{Ker} d_C^{p+1} & & \end{array}$$

where the rows are exact by the above. Observe that

$$H^p(A^\bullet) = \operatorname{Ker} d_A^p/\operatorname{Im} d_A^{p-1} = \operatorname{Ker} \overline{d}_A^p, \quad H^{p+1}(A^\bullet) = \operatorname{Coker} \overline{d}_A^p,$$

and similarly for $B^\bullet$ and $C^\bullet$. Applying the Snake Lemma again, we get the desired exact sequence. □

Two maps $f, g\colon A^\bullet \to B^\bullet$ are said to be *chain homotopic* if there are maps $h_p\colon A^p \to B^{p-1}$, such that

$$f - g = d_B \circ h + h \circ d_A.$$

For any cocycle $a \in \operatorname{Ker} d_A^p$, this implies $f_p(a) - g_p(a) = d_B^{p-1}(h_p(a))$, an so $[f_p a] = [g_p a]$ in cohomology. Consequently, chain homotopic maps induce the *same* maps on cohomology.

**Example 15.2.** As common strategy for proving that $H^p(A^\bullet) = 0$ is to show that the identity map is chain homotopic to the zero map. Explicitly, this means that one constructs maps $h\colon A^\bullet \to A^{\bullet-1}$ such that

$$d^{p-1}(h_p(a)) + h_{p+1}(d^p(a)) = a \quad \text{for all } a \in A^p \tag{15.2}$$

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If $a \in A^p$ is a cocycle (so that $d^p a = 0$), then this equation implies that $a$ is a coboundary, because $a = d^{p-1}(h_p a)$, and so $H^p(A^\bullet) = 0$. $\triangle$

## 15.2 Cech cohomology

Let $X$ be a topological space and let $\mathcal{U}$ be an open cover of $X$ consisting of finitely many open sets $U_1, \ldots, U_r$. For a sheaf $\mathcal{F}$ on $X$, we have the sheaf exact sequence

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i<j} \mathcal{F}(U_i \cap U_j). \tag{15.3}$$

The *Cech complex* is essentially the continuation of this sequence; it is a complex obtained by adjoining all the groups $\mathcal{F}(U_{i_1} \cap \cdots \cap U_{i_r})$ over all possible intersections $U_{i_0} \cap \cdots \cap U_{i_r}$.

**Definition 15.3.** For a sheaf $\mathcal{F}$ on $X$, the *Cech complex* $C^\bullet(\mathcal{U}, \mathcal{F})$ of $\mathcal{F}$ (with respect to the open cover $\mathcal{U}$) is the complex

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{d^2} \ldots$$

where the $p$-th term is given by

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0<i_1<\cdots<i_p} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}),$$

and the differentials $d^p \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ are defined by

$$(d^p\sigma)_{i_0,\ldots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}\big|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}$$

where $i_0, \ldots \widehat{i_j}, \ldots, i_{p+1}$ means $i_0, \ldots, i_{p+1}$ with the index $i_j$ omitted.

Note that since we assume that the open cover is finite, we have $C^p(\mathcal{U}, \mathcal{F}) = 0$ whenever $p \notin \{0, \ldots, r\}$, where $r$ is the number of open sets.

**Example 15.4.** The two first terms in the Cech complex are given by

$$C^0(\mathcal{U}, \mathcal{F}) = \prod_{i_0} \mathcal{F}(U_{i_0}) \quad \text{and} \quad C^1(\mathcal{U}, \mathcal{F}) = \prod_{i_0<i_1} \mathcal{F}(U_{i_0} \cap U_{i_1}).$$

The group $C^0(\mathcal{U}, \mathcal{F})$ consists of $r$-tuples of sections $\sigma = (\sigma_1, \ldots, \sigma_r)$, with $\sigma_i \in \mathcal{F}(U_i)$ for each $i$. Likewise, an element $\sigma = (\sigma_{ij}) \in C^1(\mathcal{U}, \mathcal{F})$ is a collection of sections $\sigma_{ij} \in \mathcal{F}(U_i \cap U_j)$, one for each pair $i < j$.

The map $d^0 \colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$ sends an element $\sigma = (\sigma_1, \ldots, \sigma_r)$, to the element $d^0\sigma \in C^1(\mathcal{U}, F)$ whose $ij$-th component is equal to

$$(d^0\sigma)_{ij} = \sigma_j - \sigma_i\,\big|_{U_{ij}}. \tag{15.4}$$

The map $d^1 \colon C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$ sends $\sigma = (\sigma_{ij})$ to the element with $ijk$-th component equal to

$$(d^1\sigma)_{ijk} = \sigma_{jk} - \sigma_{ik} + \sigma_{ij}\,\big|_{U_{ijk}} \tag{15.5}$$

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Substituting (15.4) into (15.5), we see that all the terms cancel, so that $d^1 \circ d^0 = 0$. The same happens also in higher degrees:

**Lemma 15.5.** For every $p$, we have

$$d^p \circ d^{p-1} = 0.$$

*Proof* For $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ and $i_0 < \cdots < i_{p+1}$, we compute:

$$\begin{aligned} d^p(d^{p-1}(\sigma))_{i_0,\ldots,i_{p+1}} &= \sum_{j=0}^{p+1} (-1)^j (d^{p-1}\sigma)_{i_0,\ldots,\widehat{i_j},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}} \\ &= \sum_{j=0}^{p+1} (-1)^j \left( \sum_{l=0}^{j-1} (-1)^l \sigma_{i_0,\ldots,\widehat{i_l},\ldots,\widehat{i_j},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}} \right) \\ &+ \sum_{j=0}^{p+1} (-1)^j \left( \sum_{l=j}^{p} (-1)^l \sigma_{i_0,\ldots,\widehat{i_j},\ldots,\widehat{i_{l+1}},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}} \right). \end{aligned}$$

For integers $m < n$, the term $\sigma_{i_0,\ldots,\widehat{i_m},\ldots,\widehat{i_n},\ldots,i_{p+1}}$ appears exactly twice in the sum: for $(l, j) = (m, n)$ in the first sum and $(l, j) = (n-1, m)$ in the second. As the signs are different, the two terms cancel, and so $d^p \circ d^{p-1} = 0$.. □

Therefore, the Cech complex is indeed a complex of abelian groups.

**Definition 15.6.** The $p$-th *Cech cohomology* of $\mathcal{F}$ with respect to $\mathcal{U}$ is defined as the cohomology of the Cech complex:

$$H^p(\mathcal{U}, \mathcal{F}) = \operatorname{Ker} d^p / \operatorname{Im} d^{p-1}.$$

The Čech cohomology groups depend on the choice of open cover $\mathcal{U}$, but are independent of the ordering of the open sets $U_i$ in the cover. Given two orderings, there is an isomorphism between the two associated Cech complexes which commutes with the differentials, and the cohomology groups are canonically isomorphic (see Exercise ).

**Example 15.7.** The group $H^0(\mathcal{U}, \mathcal{F})$ is the kernel of the map $d^0 \colon C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$, which is simply the usual map

$$\prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i<j} \mathcal{F}(U_i \cap U_j).$$

This kernel is equal to $\mathcal{F}(X)$ by the sheaf axioms, so $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$. △

**Example 15.8** ($H^1$ and lifting of sections)**.** The most interesting cohomology group is arguably $H^1(\mathcal{U}, \mathcal{F})$. It is the group of elements $(\sigma_{ij})$ such that $\sigma_{ik} = \sigma_{ij} + \sigma_{jk}$ modulo the elements of the form $\sigma_{ij} = \tau_j - \tau_i$ (restricted to $U_i \cap U_j$). As mentioned in the introduction, this group is closely related to the lifting of sections, as we now explain.

Suppose that we have a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

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Suppose we want to try to lift a section $c \in \mathcal{C}(X)$ to a section of $\mathcal{B}(X)$. Since the sequence is exact, we can at least find lifts locally, i.e. there is an open cover $\mathcal{U} = \{\mathcal{U}_i\}$ and sections $b_i \in \mathcal{B}(U_i)$ that map to $c|_{U_i}$ over each $U_i$. Now we ask if we can assemble the $b_i$ to a section $b \in \mathcal{B}(X)$. For this to be the case, we must have $b_j|_{U_{ij}} - b_i|_{U_{ij}} = 0$. In any case,

$$\sigma = \big(b_j|_{U_{ij}} - b_i|_{U_{ij}}\big)$$

defines an element of $C^1(\mathcal{U}, \mathcal{A})$ (because $b_i$ and $b_j$ map to the same element in $\mathcal{C}(U_{ij})$). Furthermore, $d\sigma = 0$, because

$$(d\sigma)_{ijk} = (b_k - b_j) - (b_k - b_i) + (b_j - b_i) = 0$$

(all terms restricted to $U_{ijk}$). When is $\sigma$ zero in $H^1(\mathcal{U}, \mathcal{A})$? This occurs if and only if there is an element $a = (a_i) \in C^0(\mathcal{U}, \mathcal{A})$ such that

$$b_j|_{U_{ij}} - b_i|_{U_{ij}} = a_j|_{U_{ij}} - a_i|_{U_{ij}},$$

which is equivalent to saying that the elements $b_i - a_i \in \mathcal{B}(U_i)$ agree over the overlaps $U_{ij}$, or in other words, that they glue together to a section $b \in \mathcal{B}(X)$. Note that since $a_i \in \mathcal{A}(U_i)$, the image of $b_i - a_i$ is the same as that of $b_i$, i.e. $b$ maps to $c$.

In summary, the section $c \in \mathcal{C}(X)$ can be lifted if and only if the associated element in $H^1(\mathcal{U}, \mathcal{A})$ equals 0. If the latter group is zero, any section of $\mathcal{C}(X)$ lifts.

In Example **??** we will see a concrete example of a section which does not lift. △

A morphism of sheaves $\phi\colon \mathcal{F} \to \mathcal{G}$ induces a morphism of Čech cohomology groups. For each open set $U \subset X$, we have a map $\phi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$, and for each $p \geqslant 0$, we may define the map $\phi^p\colon C^p(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{G})$ by applying $\phi$ componentwise. To see that this is a map of complexes, we check that $\phi^{p+1} \circ d^p_{\mathcal{F}} = d^p_{\mathcal{G}} \circ \phi^p$. For $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ and any $i_0 < \cdots < i_{p+1}$, we have

$$(\phi^{p+1}(d^p_{\mathcal{F}}\sigma))_{i_0,\ldots,i_{p+1}} = \phi_{U_{i_0,\ldots,i_{p+1}}}\left(\sum_{k=0}^{p+1}(-1)^k \sigma_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}\right).$$

Since $\phi$ is a map of sheaves, it commutes with restrictions, so this equals

$$\sum_{k=0}^{p+1}(-1)^k \phi_{U_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}}(\sigma_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}})|_{U_{i_0,\ldots,i_{p+1}}}.$$

On the other hand,

$$\begin{aligned}(d^p_{\mathcal{G}}(\phi^p\sigma))_{i_0,\ldots,i_{p+1}} &= \sum_{k=0}^{p+1}(-1)^k(\phi^p\sigma)_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}} \\ &= \sum_{k=0}^{p+1}(-1)^k \phi_{U_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}}(\sigma_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}})|_{U_{i_0,\ldots,i_{p+1}}}.\end{aligned}$$

Since these are equal, we see that the maps $\phi^p$ form a map of complexes. In particular, they induce a map between the corresponding cohomology groups

$$H^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(\mathcal{U}, \mathcal{G})$$

which shows that $H^p(\mathcal{U}, -)$ define functors from sheaves to abelian groups.

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### 15.3 Examples

**Example 15.9.** For a cover $\mathcal{U} = \{U_0, U_1\}$ consisting of just two open sets, the Cech complex takes the following simple form

$$0 \longrightarrow \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{d} \mathcal{F}(U_0 \cap U_1) \longrightarrow 0 \tag{15.6}$$

where $d(s_0, s_1) = s_1|_{U_0 \cap U_1} - s_0|_{U_0 \cap U_1}$. Then

$$H^0(\mathcal{U}, \mathcal{F}) = \operatorname{Ker} d \quad \text{and} \quad H^1(\mathcal{U}, \mathcal{F}) = \operatorname{Coker} d,$$

and there is an exact sequence

$$0 \to H^0(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{d} \mathcal{F}(U_0 \cap U_1) \to H^1(\mathcal{U}, \mathcal{F}) \to 0. \tag{15.7}$$

Concretely, $H^1(\mathcal{U}, \mathcal{F})$ is the group of sections $\mathcal{F}(U_0 \cap U_1)$ modulo the sections of the form $s_1 - s_0$ where $s_0$ and $s_1$ are restrictions of sections in $\mathcal{F}(U_0)$ and $\mathcal{F}(U_1)$ respectively.

If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, we can understand the connecting map $\delta$ in the long exact sequence by applying the Snake Lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^0(\mathcal{U}, \mathcal{F}') & \longrightarrow & C^0(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^0(\mathcal{U}, \mathcal{F}'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C^1(\mathcal{U}, \mathcal{F}') & \longrightarrow & C^1(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^1(\mathcal{U}, \mathcal{F}'') & \longrightarrow & 0 \end{array} \tag{15.8}$$

Concretely, for $s'' \in H^0(\mathcal{U}, \mathcal{F}'') \subset C^0(\mathcal{U}, \mathcal{F}'')$, lift to $(s_0, s_1) \in C^0(\mathcal{U}, \mathcal{F})$, then $\delta(s'')$ is represented by the class of $s_1|_{U_0 \cap U_1} - s_0|_{U_0 \cap U_1}$ in $H^1(\mathcal{U}, \mathcal{F}')$. △

**Example 15.10** (The projective line)**.** Consider the projective line $\mathbb{P}^1_A$ over a ring $A$, with the standard affine cover given by $U_0 = \operatorname{Spec} A[x_1/x_0]$ and $U_1 = \operatorname{Spec} A[x_0/x_1]$ with intersection $U_0 \cap U_1 = \operatorname{Spec} A[x_0/x_1, x_1/x_0]$. Let us compute the cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}^1_A}(m)$ on $\mathbb{P}^1_A$. The Cech complex takes the form:

$$0 \longrightarrow A\left[\frac{x_1}{x_0}\right] x_0^m \times A\left[\frac{x_0}{x_1}\right] x_1^m \xrightarrow{d} A\left[\frac{x_1}{x_0}, \frac{x_0}{x_1}\right] x_0^m = A\left[\frac{x_1}{x_0}, \frac{x_0}{x_1}\right] x_1^m \longrightarrow 0$$

The kernel of $d$ consists of pairs $(fx_0^m, gx_1^m)$ which are equal on the overlaps, which corresponds precisely to the space of homogeneous polynomials of degree $m$ in $A[x_0, x_1]$.

To analyze the cokernel of $d$, we examine the monomials in each term. The module $A[x_1/x_0]x_0^m$ is generated by $x_0^a x_1^b$ for $a + b = m$ and $b \geqslant 0$, while $A[x_0/x_1]x_1^m$ is generated by $x_0^a x_1^b$ for $a + b = m$ and $a \geqslant 0$. The target module $A[x_1/x_0, x_0/x_1]x_0^m$ is generated by all monomials $x_0^a x_1^b$ with $a + b = m$.

When $m \geqslant 0$, every monomial in $A[x_1/x_0, x_0/x_1]x_0^m$ is obtained from either $A[x_1/x_0]x_0^m$ or $A[x_0/x_1]x_1^m$, so

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_A}(m)) = 0 \qquad \text{for all } m \geqslant 0.$$

In particular, $H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_A}) = 0$.

When $m < 0$, the cokernel is generated as an $A$-module by the monomials $x_0^a x_1^b$ with $a, b < 0$, namely, the monomials $x_0^{-1}x_1^{m+1}, x_0^{-2}x_1^{m+2}, \ldots, x_0^{m+1}x_1^{-1}$. Hence

$$H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_A}(m)) \simeq A^{-m+1} \qquad \text{for all } m < 0.$$

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**Example 15.11** (The cuspidal cubic)**.** Consider the cuspidal cubic curve $X = V_+(x_2^3 - x_0x_1^2)$ in $\mathbb{P}^2_k$. This admits an open cover $\mathcal{U}$ with two open sets, $U_0 = D_+(x_0)$ and $U_1 = D_+(x_1)$. Writing $u = x_1/x_0$, $v = x_2/x_0$ and $w = x_2/x_1$, we have

$$\begin{aligned}\mathcal{O}_X(U_0) &= k[x_1/x_0, x_2/x_0]/((x_2/x_0)^3 - (x_1/x_0)^2) = k[u,v]/(v^3 - u^2)\\ \mathcal{O}_X(U_1) &= k[x_0/x_1, x_2/x_1]/((x_2/x_1)^3 - (x_0/x_1)) = k[w]\\ \mathcal{O}_X(U_0 \cap U_1) &= k[w, w^{-1}].\end{aligned}$$

where we used the relation $x_0/x_1 = (x_2/x_1)^3 = w^3$ to eliminate $x_0/x_1$. Note that the images of $u$ and $v$ in $k[w, w^{-1}]$ are given by $x_1/x_0 = (x_2/x_1)^3 = w^3$ and $(x_2/x_1)(x_1/x_0) = w \cdot w^{-3} = x^{-2}$ respectively. Therefore, the differential is given by

$$d(p(u,v), q(w)) = q(w) - p(w^{-3}, w^{-2}).$$

The cokernel of $d$ is generated by the monomial $w^{-1}$. Hence,

$$H^1(\mathcal{U}, \mathcal{O}_X) \simeq k.$$

△

**Example 15.12.** Let $\mathcal{U} = \{U_0, \dots, U_r\}$ be a finite open cover such that one of the members is the whole space $X$. Then

$$H^p(\mathcal{U}, \mathcal{F}) = 0 \qquad \text{for all } p \geqslant 1$$

To see this, we use the trick described in Example 15.2. Suppose for simplicity that $U_0 = X$ (otherwise, reorder the indexes) and define the map $h\colon C^{p+1}(\mathcal{U}, \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F})$ by

$$h(\sigma)_{j_0,\dots,j_p} = \begin{cases} \sigma_{0,j_0,\dots,j_p} & \text{if } j_0 \neq 0 \\ 0 & \text{if } j_0 = 0. \end{cases}$$

This is well-defined, because $\sigma_{0,j_0,\dots,j_p} \in \mathcal{F}(U_{j_0} \cap \cdots \cap U_{j_p})$.

We proceed to check that $dh + hd = \mathrm{id}$. For $i_0 \neq 0$, we have

$$\begin{aligned}(dh + hd)(\sigma)_{i_0,\dots,i_p} &= \textstyle\sum_{j=0}^p (-1)^j h(\sigma)_{i_0,\dots,\hat{i}_j,\dots,i_p} + d(\sigma)_{0,i_0,\dots,i_p}\\ &= \textstyle\sum_{j=0}^p (-1)^j \sigma_{0,i_0,\dots,\hat{i}_j,\dots,i_p} + \sigma_{i_0,\dots,i_p} + \sum_{j=0}^p (-1)^{j+1} \sigma_{0,i_0,\dots,\hat{i}_j,\dots,i_p}\\ &= \sigma_{i_0,\dots,i_p}.\end{aligned}$$

Likewise, if $i_0 = 0$, we have

$$\begin{aligned}(dh + hd)(\sigma)_{0,i_1,\dots,i_p} &= \textstyle\sum_{j=0}^p (-1)^j h(\sigma)_{0,i_1,\dots,\hat{i}_j,\dots,i_p} + 0\\ &= \sigma_{0,i_1,\dots,i_p}.\end{aligned}$$

Hence $h$ is a chain homotopy between the identity map on $C^{p+1}(\mathcal{U}, \mathcal{F})$ and the zero map, and the cohomology group $H^{p+1}(\mathcal{U}, \mathcal{F})$ is zero by Example 15.2. △

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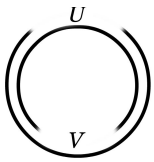
**Example 15.13** (The unit circle)**.** Here is an example from topology. Consider the unit circle $X = S^1$ with the Euclidean topology, equipped with the open cover $\mathcal{U} = \{U, V\}$ consisting of two overlapping intervals as shown in the figure blow. Consider the constant sheaf $\mathbb{Z}_X$ on $\mathbb{Z}$.




The Cech complex takes the form

$$C^0(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}_X(U) \times \mathbb{Z}_X(V) \simeq \mathbb{Z} \times \mathbb{Z} \qquad C^1(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}_X(U \cap V) \simeq \mathbb{Z} \times \mathbb{Z}.$$

and the differential $d^0 \colon C^0(\mathcal{U}, \mathbb{Z}_X) \to C^1(\mathcal{U}, \mathbb{Z}_X)$ is the map $\mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$d^0(a, b) = (b - a, b - a).$$

Hence

$$H^0(\mathcal{U}, \mathbb{Z}_X) = \operatorname{Ker} d^0 = \mathbb{Z}(1, 1) \simeq \mathbb{Z},$$

and

$$H^1(\mathcal{U}, \mathbb{Z}_X) = \operatorname{Coker} d^0 = \mathbb{Z}^2/\mathbb{Z}(1, 1) \simeq \mathbb{Z}.$$

Readers familiar with algebraic topology may recognize that this gives the same answer as singular cohomology. More generally, the cohomology groups $H^p(\mathcal{U}, \mathbb{Z}_X)$ agree with the usual singular cohomology groups $H^p_{\text{sing}}(X, \mathbb{Z})$ for any topological space homotopy equivalent to a CW complex, provided that the open sets in the cover $\mathcal{U}$ are contractible (see e.g., (Griffiths and Harris, 1979, p. 42)). △

**Example 15.14** (Constant sheaves on irreducible spaces)**.** Let $X$ be an *irreducible* topological space and let $A_X$ denote the constant sheaf associated to an abelian group $A$. We claim that for any finite open cover $\mathcal{U} = \{U\}_{i \in I}$ of $X$,

$$H^p(\mathcal{U}, A_X) = 0 \qquad \text{for all } p \geqslant 1.$$

As $X$ is irreducible, all the intersections $U_{i_0} \cap \cdots U_{i_p}$ are nonempty and connected. Therefore, the Cech complex takes the following form:

$$\prod_i A \longrightarrow \prod_{i<j} A \longrightarrow \prod_{i<j<k} A \longrightarrow \cdots \tag{15.9}$$

Note that this complex does not depend on $X$ nor on the open cover $\mathcal{U}$; only the index set $I$ plays a role. We can therefore replace $\mathcal{U}$ by the trivial cover where all $U_i = X$. In this case, the higher cohomology groups vanish by Example 15.12.

For this reason, constant sheaves are not particularly useful for studying geometric properties of schemes with the Zariski topology. For schemes, it is the quasi-coherent sheaves which give us richer and more interesting geometric invariants. △

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## 15.4 Cech cohomology on schemes

As the previous examples illustrate, the cohomology groups $H^p(\mathcal{U}, \mathcal{F})$ can be computed if we have a good understaning of the sections of $\mathcal{F}$ over the open sets in the finite cover $\mathcal{U}$. In such cases, the maps in the Cech complex are entirely explicit, and computing their kernels and images can often be done quite algorithmically.

On the other hand, the current definition of the cohomology groups is unsatisfactory for a few reasons. First of all, we have been restricted to working with finite open covers. Second, the groups $H^p(\mathcal{U}, \mathcal{F})$ depend on the open cover $\mathcal{U}$, whereas we want intrisic invariants that only depend on $\mathcal{F}$. This dependency is problematic: for instance, $\mathcal{U}$ could consist of the single open set $X$, and so $H^i(\mathcal{U}, \mathcal{F}) = 0$ for all $i \geqslant 1$. Moreover, Example 15.12 shows that adding $X$ to any cover $\mathcal{U}$ has the effect of making all the higher cohomology groups zero.

The solution to these issues is typically to use more refined covers, where the open sets are 'simple' in some sense. In the context of schemes, this means considering open covers $\mathcal{U}$ consisting of affine open sets. We will show that in good situations, that is, $X$ is Noetherian and separated and the sheaf $\mathcal{F}$ is quasi-coherent, the group $H^i(\mathcal{U}, \mathcal{F})$ will, in fact, turn out to be independent of the cover $\mathcal{U}$.

**Theorem 15.15 (Main properties of Cech cohomology).** Let $X$ be a separated scheme and let $\mathcal{U} = \{U_i\}$ be a finite affine cover of $X$. Then

(i) The Cech cohomology groups define functors $H^i(\mathcal{U}, -)\colon \mathsf{Ab}(X) \to \mathsf{Ab}$.

(ii) $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

(iii) Short exact sequences of quasi-coherent sheaves induce long exact sequences

$$\cdots \to H^p(\mathcal{U}, \mathcal{F}') \to H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{U}, \mathcal{F}'') \to H^{p+1}(\mathcal{U}, \mathcal{F}') \to \cdots$$

(iv) For any other finite affine cover $\mathcal{V} = \{V_j\}$ of $X$ and quasi-coherent sheaf $\mathcal{F}$, there are canonical isomorphisms

$$H^p(\mathcal{U}, \mathcal{F}) = H^p(\mathcal{V}, \mathcal{F}) \qquad \text{for all} p \geqslant 0$$

(v) If $X$ has dimension $n$, then $H^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > n$ and all quasi-coherent $\mathcal{F}$.

We have already proved (i) and (ii). These statements hold in complete generality – we do not need to assume that $X$ is separated, nor that the cover is finite. For the remaining properties, the proofs rely on working with a finite affine cover $\{U_i\}$ such that all the intersections $U_{i_0} \cap \cdots \cap C_{i_p}$ are affine. If $X$ is separated and Noetherian, then $X$ always admits such an affine cover. In light of item (iv), we make the following definition.

**Definition 15.16.** For a Noetherian separated scheme $X$, and a quasi-coherent sheaf $\mathcal{F}$, we write $H^p(X, \mathcal{F})$ for the group $H^p(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ is any finite affine cover of $X$.

Let us prove item (iii), namely that short exact sequences of sheaves induce long exact sequences in cohomology. Consider a short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

By Corollary 12.11, for any affine open subset $U = \operatorname{Spec} A$, the sequence of sections is

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exact:

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0 \tag{15.10}$$

For our cover, $\mathcal{U} = \{U_i\}_{i\in I}$, where all intersections are affine. Taking products of the exact sequences (15.10) over all relevant intersections, we obtain exact sequences

$$0 \longrightarrow C^p(\mathcal{U}, \mathcal{F}') \longrightarrow C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^p(\mathcal{U}, \mathcal{F}'') \longrightarrow 0,$$

and consequently an exact sequence of Cech complexes

$$0 \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}') \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{F}'') \longrightarrow 0.$$

Finally, applying Lemma 15.1, we obtain the desired long exact sequence

$$\cdots \longrightarrow H^i(\mathcal{U}, \mathcal{F}') \longrightarrow H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{U}, \mathcal{F}'') \longrightarrow \cdots.$$

## 15.5 Cohomology of sheaves on affine schemes

The following result is fundamental in the study of sheaf cohomology groups on schemes.

**Theorem 15.17 (Cohomology on affine schemes).** Let $X = \operatorname{Spec} A$ and let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Then for any finite affine cover $\mathcal{U}$ of $X$,

$$H^p(\mathcal{U}, \mathcal{F}) = 0 \text{ for all } p > 0.$$

This is the first example of a *vanishing theorem* for cohomology. The fact that higher cohomology groups vanish is desirable thing – after all, these groups were introduced to measure the 'failure' of certain desirable statements.

*Proof* Let $\mathcal{U} = \{U_1, \ldots, U_r\}$ be the affine cover. As $X$ is affine, it is separated, so all intersections $U_{i_0} \cap \cdots \cap U_{i_p}$ are also affine. We want to show that the complex of $A$-modules

$$0 \longrightarrow \mathcal{F}(X) \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots \tag{15.11}$$

is exact. We know that the theorem holds in the 'trivial case' when one of the $U_i$, say, $U_0$ is equal to $X$ (see Example 15.12). For the general case, we reduce to the trivial case as follows.

As (15.11) is a complex of $A$-modules, exactness can be checked by localizing at each prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Without loss of generality, we may assume $\mathfrak{p} \in U_0$.

Consider the restriction $\mathcal{F}|_{U_0}$ and the induced cover $\mathcal{U} \cap U_0$ of $U_0$. The corresponding Čech complex

$$0 \longrightarrow \mathcal{F}(U_0) \longrightarrow C^0(\mathcal{U} \cap U_0, \mathcal{F}|_{U_0}) \longrightarrow C^1(\mathcal{U} \cap U_0, \mathcal{F}|_{U_0}) \longrightarrow \cdots \tag{15.12}$$

is exact by the 'trivial case', by Example 15.12. Localization commutes with finite products and preserves exactness, so thefollowing sequence is also exact:

$$0 \longrightarrow \mathcal{F}(U_0)_\mathfrak{p} \longrightarrow C^0(\mathcal{U} \cap U_0, \mathcal{F}|_{U_0})_\mathfrak{p} \longrightarrow C^1(\mathcal{U} \cap U_0, \mathcal{F}|_{U_0})_\mathfrak{p} \longrightarrow \cdots \tag{15.13}$$

As $\mathcal{F}$ is quasi-coherent, the localization of (15.11) at $\mathfrak{p}$ coincides with the complex (15.13). Since the latter is exact, so is (15.11), completing the proof. □

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The fact that the higher cohomology groups vanish for all quasi-coherent sheaves is a distinctive property of affine schemes. In fact, this property characterizes affine schemes among all schemes (see (Stacks Project, 2018, Tag 01XE)). Example **??** showed that even the most basic non-affine scheme, $\mathbb{P}^1_k$, admits sheaves with non-vanishing higher cohomology. Here is another example:

**Example 15.18** (The affine line with two origins)**.** Consider the affine line with two origins $X$, constructed by gluing two copies of $\operatorname{Spec} k[x]$ along the open subset $D(x) = \operatorname{Spec} k[x, x^{-1}]$ via the identity map. The Čech complex for this cover takes the form

$$0 \longrightarrow k[x] \times k[x] \xrightarrow{d} k[x, x^{-1}] \longrightarrow 0,$$

As we verified previously, $H^0(\mathcal{U}, \mathcal{O}_X) = \operatorname{Ker} d = k[x]$. For the $H^1$, note that

$$H^1(\mathcal{U}, \mathcal{O}_X) = \operatorname{Coker} d = k[x, x^{-1}]/k[x] = \bigoplus_{i>0} k \cdot x^{-i}.$$

In particular, $H^1(\mathcal{U}, \mathcal{O}_X)$ is infinite-dimensional as a $k$-vector space. △

## 15.6 Independence of the cover

Our first application of the vanishing of higher cohomology groups on affine schemes will be the proof of item (iv) in Theorem 15.15, which states that the Cech cohomology groups are independent of the affine cover.

Let $\mathcal{U} = \{U_1, \ldots, U_r\}$ and $\mathcal{V} = \{V_1, \ldots, V_s\}$ be two finite affine covers, and consider the groups of sections over all the mixed intersections:

$$C^{m,n} = \prod_{|I|=m, |J|=n} \Gamma(U_I \cap V_J, \mathcal{F})$$

where $U_I = \bigcap_{i\in I} U_i$ and $V_J = \bigcap_{j\in J} V_j$.

For fixed $m$, we have

$$C^{m,\bullet} \simeq \prod_{|I|=m} C^\bullet(U_I \cap V_J, \mathcal{F}|_{U_I})$$

where $C^\bullet(U_I \cap V_J, \mathcal{F}|_{U_I})$ is the Cech complex of $\mathcal{F}|_{U_I}$ with respect to the cover $\{V_j \cap U_I\}_{j=1}^s$. Similarly,

$$C^{\bullet,n} \simeq \prod_{|J|=n} C^\bullet(V_J \cap U_i, \mathcal{F}|_{U_I})$$

where $C^\bullet(V_J \cap U_i, \mathcal{F}|_{U_I})$ is the Cech complex of $\mathcal{F}|_{V_J}$ with respect to the cover $\{U_i \cap V_J\}_{i=1}^r$.

The collection of groups $C^{m,n}$ form a *double complex*. This has two differentials, one written $d$ in the 'rightwards' direction (induced by the Cech differentials for $\mathcal{V}$), and one in the 'upwards' direction, $\delta$ (induced by the Cech differentials for $\mathcal{U}$). The figure in (15.14) illustrates this.

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When $m = 0$, we recover the Cech complex of $\mathcal{F}$ with respect to $\mathcal{V}$

$$C^{0,n} = \prod_{|J|=n} \Gamma(V_J, \mathcal{F}) \text{ and } C^{m,0} = \prod_{|I|=m} \Gamma(U_I, \mathcal{F})$$

whose cohomology in the $d$-direction is exactly $H^0(\mathcal{U}, \mathcal{F})$. Similarly, the cohomology in the $\delta$-direction is $H^0(\mathcal{V}, \mathcal{F})$. To show the independence of the cover, we want to show that the cohomology of these two complexes are equal. The key point is that the intersections $U_I \cap V_J$ are affine, as $X$ is assumed to be separated. By Theorem 15.17 this implies that all the cohomology groups in positive degrees in both directions are zero, i.e., the complexes $C^{m,\bullet}$ and $C^{\bullet,n}$ are exact when $m, n \geqslant 1$.

To conclude that $H^p(\mathcal{U}, \mathcal{F})$ and $H^p(\mathcal{V}, \mathcal{F})$ are isomorphic, we apply the following lemma from homological algebra:

**Lemma 15.19.** Let $C^{n,m}$ be a double complex with $H^i(C^{m,\bullet}) = H^i(C^{\bullet,n}) = 0$ for all $m, n \geqslant 1$. Then $A^m = H^0(C^{m,\bullet})$ and $B^n = H^0(C^{\bullet,n})$ are complexes and there is a canonical isomorphism between their cohomology:

$$H^p(A^\bullet) = H^p(B^\bullet).$$

*Proof* We augment the double complex by adding $A^p = \operatorname{Ker}(C^{p,0} \to C^{p,1})$ and $B^p = \operatorname{Ker}(C^{0,p} \to C^{1,p})$ to get the diagram below.

$$\begin{array}{ccccccccc}
B^2 & \longrightarrow & C^{0,2} & \xrightarrow{d} & C^{1,2} & \xrightarrow{d} & C^{2,2} & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
B^1 & \longrightarrow & C^{0,1} & \xrightarrow{d} & C^{1,1} & \xrightarrow{d} & C^{2,1} & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \\
B^0 & \longrightarrow & C^{0,0} & \xrightarrow{d} & C^{1,0} & \xrightarrow{d} & C^{2,0} & \xrightarrow{d} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & A^0 & \xrightarrow{d} & A^1 & \xrightarrow{d} & A^2 & \xrightarrow{d} & \cdots
\end{array} \tag{15.14}$$

Now all rows and columns are exact except along $A^\bullet$ and $B^\bullet$. One now applies a diagram chase to construct a map

$$H^p(A^\bullet) \longrightarrow H^p(B^\bullet).$$

To see how this works, consider the case $p = 1$. We perform a diagram chase of the form

$$\begin{array}{ccccc}
\exists c_3 \in B^1 & \longrightarrow & c_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow \\
 & & c_1 & \longrightarrow & c & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & c \in A^1 & \longrightarrow & 0
\end{array}$$

That is, starting with an element $c \in A^1$ so that $d(c) = 0$, send it to $c \in C^{1,0}$, which by

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commutativity of the diagram must map to 0 in $C^{2,0}$. By exactness, we may therefore lift $c$ to an element $c^1 \in C^{0,0}$, which in turn maps to an element $c_2 \in C^{0,1}$. By commutativity of the diagram, $c_2$ maps to zero in $C^{1,1}$. Hence by exactness, there is an $c_3 \in B^1$ such that $d(c_3) = c_2$. The element $c_3$ satisfies $\delta(c_3) = 0$ because $\delta(c_2) = 0$, and hence it defines an element in $H^1(B^\bullet)$. One checks that this assignment is independent of the choices of lifts, and that the map $H^1(A^\bullet) \to H^1(B^\bullet)$ is a map of groups. By a symmetric argument, one constructs a map $H^1(B^\bullet) \to H^1(A^\bullet)$ which defines the inverse. A similar argument works for $p > 1$.

For more details of this argument, see Exercise 15.18.18. For a full proof, see (**?**, p. 64). □

## 15.7 Pushforwards and cohomology

The following lemma is very useful, and will be applied several times later on. Given an affine morphism $f\colon X \to Y$, it will allow us to transport computations of cohomology groups on $X$ to computations on $Y$, at the cost of replacing $\mathcal{F}$ with $f_*\mathcal{F}$.

**Lemma 15.20.** Let $f\colon X \to Y$ be an affine morphism of Noetherian, separated schemes. Then for each quasi-coherent sheaf $\mathcal{F}$ on $X$, and $p \geqslant 0$, we have a canonical isomorphism

$$H^p(X, \mathcal{F}) = H^p(Y, f_*\mathcal{F}). \tag{15.15}$$

*Proof* Let $\mathcal{V} = \{V_i\}$ be a finite affine covering of $Y$ such that $H^p(X, f_*\mathcal{F})$ is computed by the Čech complex $C^\bullet(V_i, f_*\mathcal{F})$. Note that the latter complex equals $C^\bullet(f^{-1}V_i, \mathcal{F})$. As $f$ is affine, the affine subsets $U_i = f^{-1}(V_i)$ form a finite affine covering $\mathcal{U}$ of $X$, and the lemma follows simply because the Cech complexes computing the two sides of (15.15) are identical. □

**Example 15.21.** For a closed embedding $\iota\colon Y \to X$ (which is affine), we obtain

$$H^i(Y, \mathcal{F}) = H^i(X, \iota_*\mathcal{F}).$$

In particular, for the structure sheaf,

$$H^i(Y, \mathcal{O}_Y) = H^i(X, \iota_*\mathcal{O}_X).$$

△

**Example 15.22.** Let $Z \subset \mathbb{P}^1_k$ be the subscheme given by two closed points $p, q$ in $\mathbb{P}^1_k$ and let $\iota\colon Z \to \mathbb{P}^1_k$ be the inclusion. The ideal sheaf sequence 13.17 takes the following form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow \iota_*\mathcal{O}_Z \longrightarrow 0.$$

The element $(0, 1) \in k \oplus k$ defines a section of the sheaf $i_*\mathcal{O}_Z(\mathbb{P}^1_k) = k \oplus k$ corresponding to a function which vanishes at $p$ and takes the value 1 at $q$. This section cannot lift to a global section of $\mathcal{O}_{\mathbb{P}^1_k}$, since any regular function on $\mathbb{P}^1_k$ must be constant and cannot take the value 0 at one point and 1 at another.

This failure of ability to lift can be explained by the cohomology group $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2))$ which is 1-dimensional. Since $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = 0$, one can think of the elements of this group

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as the group of elements of $k \oplus k$ modulo those that lift to $\mathcal{O}_{\mathbb{P}^1_k}$. Here it is clear that an element $(a, b) \in k \oplus k$ lifts if and only if $a = b$. In fact, in this example, the connecting map

$$\delta\colon H^0(\mathbb{P}^1_k, i_*\mathcal{O}_Z) \longrightarrow H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2))$$

can be identified with the map $k \oplus k \to k$ sending $(a, b)$ to $a - b$. △

## 15.8 Cohomology and dimension

The next result is another 'vanishing theorem' for cohomology groups. It states that the cohomology groups vanish in degrees above the dimension of the scheme.

**Theorem 15.23.** Let $X$ be a quasi-projective scheme of dimension $n$ over a Noetherian ring $A$. Then:

(i) $X$ admits an open cover $\mathcal{U}$ by at most $n + 1$ affine open subsets.

(ii) For any quasi-coherent sheaf $\mathcal{F}$ on $X$, we have

$$H^p(X, \mathcal{F}) = 0 \quad \text{for all } p > n.$$

*Proof* (i): Since $X$ is quasi-projective, we may write $X = \overline{X} - W$ where $\overline{X} \subset \mathbb{P}^r_A$ where $\overline{X}$ and $\overline{W}$ are closed subschemes of $\mathbb{P}^r_A$. Also, we may assume that no irreducible component of $\widetilde{X}$ is contained in $W$ (simply by disregarding such components).

We prove (i) by induction on $n = \dim X$, showing that $X$ can be covered by at most $n + 1$ distuinguished open sets from $\mathbb{P}^n_A$.

When $n = 0$, the scheme $X$ consists of finitely many closed points, which correspond to homogeneous maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ in $A[x_0, \ldots, x_n]$. By prime avoidance, we may choose a homogeneous element of positive degree $f \in A[x_0, \ldots, x_n]$ not contained in any $\mathfrak{p}_i$. This means that $V_+(f)$ is disjoint from $X$, and $X$ is contained in $D_+(f)$, which is affine. Therefore, the statement holds for $n = 0$.

For $n > 0$, let $I_X, I_W \subset A[x_0, \ldots, x_r]$ be the homogeneous ideals defining $\overline{X}$ and $W$ respectively. Again by prime avoidance, we may choose a homogeneous polynomial $f \in I_W$ which does not vanish on any component of $\overline{X}$. By construction, the hypersurface $H = V_+(f)$ contains $W$ and $\mathbb{P}^r_A - H = D_+(f)$ is affine. The intersection $\overline{X} \cap D_+(f)$ is therefore an affine open subscheme of $X$. Since $f$ does not vanish identically on any component of $\overline{X}$, the intersection $\overline{X} \cap H$ has dimension $< n$. By the induction hypothesis, $\overline{X} \cap H$ can be covered by at most $n$ distinguished open sets in $\mathbb{P}^r_A$. Combining $\overline{X} \cap D_+(f)$ with these affine opens gives the required cover of $X$ by at most $n + 1$ affine opens.

(ii): The vanishing follows directly from the existence of the affine cover in (i). For any quasi-coherent sheaf $\mathcal{F}$, the Čech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ associated to this cover has length $n + 1$, and consequently $H^p(X, \mathcal{F}) = 0$ for all $p > n$. □

The vanishing $H^p(X, \mathcal{F}) = 0$ for $p > n$ holds more generally for arbitrary sheaves on Noetherian topological spaces, though the proof requires different techniques. The above special case for quasi-coherent sheaves is sufficient for the geometric applications in this book.

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## 15.9 Cohomology of sheaves on projective space

In Examples **??** and **??** we computed the cohomology groups of the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(m)$. For $m \geqslant 0$, we found that $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m))$ could be identified with the space of homogeneous polynomials of degree $m$, and $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m)) = 0$. On the other hand, for $m \leqslant -2$, $H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m)) = 0$, while $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(m))$ was non-zero.

We will now carry out a more general computation of the cohomology groups of $\mathcal{O}_{\mathbb{P}^n_A}(m)$ for any projective space $\mathbb{P}^n_A$ over a ring $A$. The strategy is the same: we use the Cech complex associated to the standard cover $D_+(x_0), \ldots, D_+(x_n)$ to to compute the cohomology groups.

Write $R = A[x_0, \ldots, x_n]$. Then the nonzero groups in the Cech complex are

$$\begin{aligned} C^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= \prod_{0 \leqslant i \leqslant n} (R_{x_i})_m \\ C^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= \prod_{0 \leqslant i < j \leqslant n} \left(R_{x_i x_j}\right)_m \\ &\vdots \\ C^n(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= \left(R_{x_0 x_1 \cdots x_n}\right)_m \end{aligned}$$

and the Cech complex takes the form

$$\prod_{0 \leqslant i \leqslant n} (R_{x_i})_m \xrightarrow{d^0} \prod_{0 \leqslant i < j \leqslant n} \left(R_{x_i x_j}\right)_m \xrightarrow{d^1} \prod_{0 \leqslant i < j < k \leqslant n} \left(R_{x_i x_j x_k}\right)_m \xrightarrow{d^2} \cdots \tag{15.16}$$

where the maps are as usual composed of alternating sums of localization maps.

**Theorem 15.24 (Cohomology of sheaves on $\mathbb{P}^n$).** Let $A$ be a ring and let $\mathbb{P}^n_A$ be the projective $n$-space over $A$. Then:

(i) For each $m \in \mathbb{Z}$,

$$H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = A[x_0, \ldots, x_n]_m.$$

(ii) For each $m \in \mathbb{Z}$,

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = \left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \ldots, x_n^{-1}]\right)_m \tag{15.17}$$

In particular, there is a canonical isomorphism

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)) \simeq A.$$

(iii) For each $m \in \mathbb{Z}$ and $p \neq 0, n$,

$$H^p(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = 0.$$

*Proof* (i): In degree 0, we recover the following isomorphism:

$$\begin{aligned} H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= \operatorname{Ker} d^0 \\ &= A[x_0, \ldots, x_n]_m. \end{aligned}$$

(ii): Observe that

$$C^n(\mathcal{U}, \mathcal{O}_{\mathbb{P}^n_A}(m)) = \left(A[x_0, \ldots, x_n]_{x_0 \cdots x_n}\right)_m$$

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is a free graded $A$-module spanned by monomials of the form $x_0^{e_0} \cdots x_n^{e_n}$ of degree $\sum e_i = m$. The image of $d^{n-1}$ is generated by such monomials where at least one $e_i$ is non-negative. Therefore, $H^n(X, \mathcal{O}_{\mathbb{P}^n_A}(m))$, which is the cokernel of $d^{n-1}$ is given by

$$\begin{aligned} H^n(X, \mathcal{O}_{\mathbb{P}^n_A}(m)) &= A\left\{ x_0^{e_0} \cdots x_n^{e_n} \;\middle|\; e_i < 0 \text{ for every } i \text{ and } \sum e_i = m \right\} \\ &= \left( x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \ldots, x_n^{-1}] \right)_m \end{aligned}$$

In degree $m = -n-1$, there is only one such monomial, namely $x_0^{-1} \cdots x_n^{-1}$, so

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-n-1)) = A \cdot x_0^{-1} \cdots x_n^{-1}.$$

(ii): We use induction on $n$. For $n = 0$ and $n = 1$, there is nothing to prove. Let $n \geqslant 2$, and write $R = A[x_0, \ldots, x_n]$. Rewriting to single out the variable $x_n$, we express the Cech complex as the degree $m$ part of the following complex:

$$\mathcal{C}^\bullet : \qquad 0 \to \prod_{i=0}^{n-1} R_{x_i} \times R_{x_n} \to \cdots \to \prod_{\substack{|I|=p \\ I \not\ni n}} R_{x_I} \times \prod_{\substack{|I|=p-1 \\ I \not\ni n}} R_{x_I x_n} \to \cdots \tag{15.18}$$

To finish the proof, we need to prove that $H^p(\mathcal{C}^\bullet) = 0$ for $p = 1, \ldots, n-1$.

If we localize the complex with respect to $x_n$, we see that $\mathcal{C}^\bullet$ embeds as a subcomplex of the following complex

$$\mathcal{D}^\bullet : \qquad 0 \to \prod_{i=0}^{n-1} R_{x_i x_n} \times R_{x_n} \to \cdots \to \prod_{\substack{|I|=p \\ I \not\ni n}} R_{x_I x_n} \times \prod_{\substack{|I|=p-1 \\ I \not\ni n}} R_{x_I x_n} \to \cdots \tag{15.19}$$

We recognize this as the same complex as the one given by the Cech complex of $\mathcal{O}_{D(x_n)}$ on $D(x_n) = \operatorname{Spec}(R_{x_n})$ associated to the covering with $n+1$ open affines

$$D(x_0 x_n), \ldots, D(x_{n-1} x_n), D(x_n).$$

In particular, since $D(x_n)$ is affine, the complex (15.19) is exact in all positive degrees, i.e., $H^p(\mathcal{D}^\bullet) = 0$ for all $p > 0$.

Let $S = A[x_0, \ldots, x_{n-1}]$. Decomposing the elements in $\mathcal{C}^\bullet$ and $\mathcal{D}^\bullet$ according to their $x_n$-degree, we see that the quotient complex $\mathcal{E}^\bullet = \mathcal{D}^\bullet / \mathcal{C}^\bullet$ decomposes as $\mathcal{E}^\bullet = \bigoplus_{l>0} \mathcal{E}_l^\bullet \cdot x_n^{-l}$, and $\mathcal{E}_l^\bullet$ is the complex

$$0 \to \prod_{i=0}^{n-1} S_{x_i} \times 0 \to \cdots \to \prod_{\substack{|I|=p \\ I \not\ni n}} S_{x_I} \times \prod_{\substack{|I|=p-1 \\ I \not\ni n}} 0 \to \cdots, \tag{15.20}$$

Disregaring the zeroes in (15.20), we see that the degree-$m$ part of the complex $\mathcal{E}_l^\bullet$ is exactly the Cech complex of $\mathcal{O}_{\mathbb{P}^{n-1}_A}(m)$ on $\mathbb{P}^{n-1}_A$. In particular, by induction, we have

$$H^p(\mathcal{E}^\bullet) = 0 \quad \text{for } p = 1, \ldots, n-2$$

Now, taking the long exact sequence of cohomology associated to the sequence

$$0 \to \mathcal{C}^\bullet \to \mathcal{D}^\bullet \to \mathcal{E}^\bullet \to 0, \tag{15.21}$$

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we obtain the exact sequence

$$0 \to H^0(\mathcal{C}^\bullet) \to H^0(\mathcal{D}^\bullet) \to H^0(\mathcal{E}^\bullet) \to H^1(\mathcal{C}^\bullet) \to 0 \tag{15.22}$$

and isomorphisms

$$H^p(\mathcal{C}^\bullet) \simeq H^{p-1}(\mathcal{E}^\bullet) \qquad \text{for } p > 1. \tag{15.23}$$

Hence $H^p(\mathcal{C}^\bullet) = 0$ for $p = 2, \ldots, n-1$.

For the remaining group $H^1(\mathcal{C}^\bullet)$, note that (15.22) takes the form

$$0 \to R \to R_{x_n} \to \bigoplus_{l>0} S \cdot x_n^{-l} \to H^1(\mathcal{C}^\bullet) \to 0$$

Here the map $R_{x_n} \to \bigoplus_{l>0} S \cdot x_n^{-l}$ is clearly surjective, so we deduce that $H^1(\mathcal{C}^\bullet) = 0$. This completes the proof of (ii). □

The proof of Theorem 15.24 also provides an important duality between $H^0$ and $H^n$ on $\mathbb{P}^n_A$. To state this precisely, we recall the following definition:

**Definition 15.25.** Let $A$ be a ring and let $M$ and $N$ be $A$-modules. A bilinear map $M \times N \to A$ is a called a *perfect pairing* if the induced map $M \mapsto \mathrm{Hom}_A(N, A)$ is an isomorphism.

Consider the pairing of $A$-modules defined by

$$(\,,\,) : A[x_0, \ldots, x_n] \times \left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \ldots, x_n^{-1}]\right) \to A, \tag{15.24}$$

sending a pair of Laurent polynomials $(p, q)$ to the coefficient of $x_0^{-1} \ldots x_n^{-1}$ in the product $pq$. Note that this mapping is $A$-linear in each factor. In terms of the standard monomial basis, we have

$$(x_0^{d_0} \ldots x_n^{d_n}, x_0^{e_0} \ldots x_n^{e_n}) = \begin{cases} 1 & \text{if } d_i + e_i = -1 \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

This pairing is perfect and allows us to canonically identify

$$\left(x_0^{-1} \cdots x_n^{-1} A[x_0^{-1}, \ldots, x_n^{-1}]\right)_d = \mathrm{Hom}_A(A[x_0, \ldots, x_n]_{-n-1-m}, A).$$

This allows us to regard the $n$-th cohomology group $H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m))$ as the *dual* of a corresponding $H^0$:

**Corollary 15.26 (Serre duality for $\mathbb{P}^n$).** For each $m \in \mathbb{Z}$, there is a canonical isomorphism

$$H^n(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m)) = \mathrm{Hom}_A(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n_A}(-m-n-1)), A). \tag{15.25}$$

When $A = k$ is a field, the dimensions of the cohomology groups are easily computed:

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**Corollary 15.27.** Let $k$ be a field. Then for $m \geqslant 0$

$$\dim_k H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m)) = \binom{m+n}{n}$$
$$\dim_k H^n(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(-m)) = \binom{m-1}{n}.$$

All other cohomology groups are 0.

## 15.10 Cohomology of sheaves on projective schemes

By the results of the previous section, the cohomology groups of $\mathcal{O}(m)$ on $\mathbb{P}^n_k$ over a field $k$ are always finite-dimensional $k$-vector spaces. This is part of a more general result, saying that on projective schemes of finite type over a ring, the cohomology groups of quasi-coherent sheaves of finite type are always $k$-vector spaces of finite dimension. Note that this is definitely not the case for affine schemes: even the $H^0$ of the structure sheaf on $\mathbb{A}^1_k$ is infinite dimensional, as it equals $k[t]$.

**Theorem 15.28 (Serre).** Let $X \subset \mathbb{P}^n_A$ be a projective scheme of finite type over a Noetherian ring $A$ and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then:

(i) For each $i$, the cohomology group $H^i(X, \mathcal{F})$ is a finite $A$-module.

(ii) There exists an $m_0 > 0$ such that

$$H^i(X, \mathcal{F}(m)) = 0.$$

for all $m \geqslant m_0$ and $i > 0$.

*Proof* Let $\iota\colon X \to \mathbb{P}^n_A$ denote the closed embedding and consider the sheaf $\iota_*\mathcal{F}$. Since $\iota$ is finite, the sheaf $\iota_*\mathcal{F}$ is again of finite type (Exercise 12.12.36) and

$$\begin{aligned} H^i(X, \mathcal{F}(d)) &= H^i(X, \mathcal{F} \otimes_{\mathcal{O}_X} \iota^*\mathcal{O}_{\mathbb{P}^n}(d)) \\ &= H^i(\mathbb{P}^n, \iota_*(\mathcal{F} \otimes_{\mathcal{O}_X} \iota^*\mathcal{O}_{\mathbb{P}^n}(d))) \\ &= H^i(\mathbb{P}^n, \iota_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(d)) \\ &= H^i(\mathbb{P}^n, (\iota_*\mathcal{F})(d)). \end{aligned} \tag{15.26}$$

Therefore we reduce to the case $X = \mathbb{P}^n_A$.

Note that both parts of the theorem are trivially satisfied if $i > n$ because the cohomology groups are zero in this case ($\mathbb{P}^n_A$ can be covered by $n+1$ affines). The proof will take this as the base case and proceed by downwards induction on $i$.

By Theorem **??**, any finite type $\mathcal{O}_{\mathbb{P}^n_A}$-module is of the form $\widetilde{M}$ for some finitely graded module $M$ over $R = A[x_0, \ldots, x_n]$.

(i): As $M$ is finitely genenerated, we may pick a graded surjection $\bigoplus_j R(-a_j) \to M$ for $M$. Letting $K$ be the kernel, we have an exact sequence of finitely generated graded $R$-modules

$$0 \longrightarrow K \longrightarrow \bigoplus_i R(-a_j) \longrightarrow M \longrightarrow 0$$

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Applying tilde, we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigoplus_j \mathcal{O}_{\mathbb{P}^n_A}(-a_j) \longrightarrow \mathcal{F} \longrightarrow 0.$$

If we take the long exact sequence of cohomology, we get

$$\cdots \to H^i(\mathbb{P}^n_A, \mathcal{K}) \to \bigoplus_j H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-a_j)) \to H^i(\mathbb{P}^n_A, \mathcal{F}) \to H^{i+1}(\mathbb{P}^n_A, \mathcal{K}) \to \dots$$

By induction on $i$, the cohomology group $H^{i+1}(\mathbb{P}^n_A, \mathcal{K})$ is a finitely generated $A$-module, and the same holds for each $H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(-a_j))$ by Theorem 15.24. $H^i(\mathbb{P}^n_A, \mathcal{F})$ is therefore squeezed between two finitely generated $A$-modules, so by exactness, it must itself be finitely generated.

(ii): Twist the above sequence by $\mathcal{O}_{\mathbb{P}^n_A}(m)$ and take the long exact sequence in cohomology to get

$$H^i(\mathbb{P}^n_A, \bigoplus_j \mathcal{O}_{\mathbb{P}^n_A}(m - a_j)) \longrightarrow H^i(\mathbb{P}^n_A, \mathcal{F}(m)) \longrightarrow H^{i+1}(\mathbb{P}^n_A, \mathcal{K}(m))$$

Again, by downward induction on $i$, and the fact that $H^i(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(m - a_j)) = 0$ for all $i > 0$ and $m > a_j$, we find that $H^i(\mathbb{P}^n_A, \mathcal{F}(m)) = 0$. □

**Corollary 15.29.** Let $f\colon X \to \operatorname{Spec} A$ be a projective scheme over $A$ and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module of finite type. Then $f_*\mathcal{F}$ is a coherent sheaf on $\operatorname{Spec} A$.

*Proof* The sheaf $f_*\mathcal{F}$ is in any case quasi-coherent by Theorem 12.31. In fact, $f_*\mathcal{F} = \widetilde{\Gamma(X, \mathcal{F})}$, so it suffices to show that $\Gamma(X, \mathcal{F}) = H^0(X, \mathcal{F})$ is finitely generated as an $A$-module. This is clear if $f$ is a closed embedding $X \to \mathbb{P}^n_A$, because $f$ is finite and $f_*\mathcal{F}$ is of finite type on $\mathbb{P}^n_A$ (Exericise 12.12.36). It is also clear if $f$ is the projection $\mathbb{P}^n_A \to \operatorname{Spec} A$, by item (i) in the above theorem. Now the general case follows by factoring $f$ as a closed embedding followed by the projection. □

The assumption that $X$ is projective over $A$ is essential: for the structure morphism $f\colon \operatorname{Spec} k[x] \to \operatorname{Spec} k$, the pushforward $f_*\mathcal{O}_{\operatorname{Spec} k[x]} = \widetilde{k[x]}$, but $k[x]$ is not of finite type over $k$.

### *The Euler characteristic*

If $X$ is a projective scheme of finite type over a field $k$ and $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, Serre's theorem tells us that the cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional $k$-vector spaces. In particular, we can ask about their dimensions. It turns out that the alternating sum of these dimensions has very good formal properties, so we make the following definition:

**Definition 15.30.** Let $X$ be a projective scheme over a field $k$. We define the *Euler characteristic* of $\mathcal{F}$ as

$$\chi(\mathcal{F}) = \sum_{k \geqslant 0} (-1)^k \dim_k H^k(X, \mathcal{F}).$$

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Note that the sum is well-defined, as there are only finitely many non-zero cohomology groups appearing on the right-hand side.

**Proposition 15.31.** The Euler characteristic $\chi$ is additive on exact sequences, i.e., if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules of finite type, then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

*Proof* This follows from the fact that if $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ is an exact sequence of $k$-vector spaces, then $\sum_i (-1)^i \dim_k V_i = 0$. Applying this to the long exact sequence in cohomology gives the claim. □

**Example 15.32.** Let $X = \mathbb{P}^n_k$ and $\mathcal{F} = \mathcal{O}(d)$ for $d \geqslant 0$. Then $\dim_k H^0(\mathbb{P}^n_k, \mathcal{F}) = \binom{n+d}{n}$ and all of the higher cohomology groups are zero. In the case when $d < 0$, only $H^n(X, \mathcal{F})$ can be non-zero, and the rank is given by $\binom{n+d}{n}$, where we use the extended binomial coefficient

$$\binom{x}{d} = x(x-1)\cdots(x-d+1)/d!$$

for any $x \in \mathbb{R}$. In particular,

$$\chi(\mathcal{O}_{\mathbb{P}^n_k}(d)) = \binom{n+d}{d}$$

is a polynomial in $d$ of degree $n$, which agrees with $\dim H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d))$ for all $d \geqslant 0$. △

The example shows that for a direct sum $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n_k}(a_r)$, the Euler characteristic $\chi(\mathcal{E}(m))$ is a polynomial in $m$. Even more generally, we can take any $\mathcal{O}_{\mathbb{P}^n_k}$-module of finite type and a free resolution of it:

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

where the $\mathcal{E}_i$ are direct sums of invertible sheaves of the form $\mathcal{O}(d)$. If we tensor this sequence by $\mathcal{O}_{\mathbb{P}^n_A}(m)$, we get[1]

$$0 \to \mathcal{E}_n(m) \to \cdots \to \mathcal{E}_1(m) \to \mathcal{E}_0(m) \to \mathcal{F}(m) \to 0$$

Note that each of the terms $\chi(\mathcal{E}_i(m))$ is a polynomial in $m$. Then since the Euler characteristic is additive on exact sequences, also $\chi(\mathcal{F}(m))$ is a polynomial in $m$. Moreover, again by Serre's theorem, we have $H^i(X, \mathcal{F}(m)) = 0$ for $m \gg 0$ and $i > 0$, and so $\chi(\mathcal{F}(m)) = H^0(\mathcal{F}(m))$ for $m$ large.

If we start with an $\mathcal{O}_X$-module $\mathcal{F}$ of finite type on $X \subset \mathbb{P}^n_k$, and apply the previous discussion to $i_*\mathcal{F}$ on $\mathbb{P}^n_k$, we have proved the following:

[1] Recall that tensoring by a locally free sheaf preserves exactness.

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**Corollary 15.33.** Let $X \subset \mathbb{P}^n_k$ be a projective scheme of finite type over $k$ and let $\mathcal{O}(1)$ be the Serre twisting sheaf. Then the function

$$P_{\mathcal{F}}(m) = \chi(\mathcal{F}(m))$$

is a polynomial in $m$, and for large $m$, $P_{\mathcal{F}}(m) = H^0(X, \mathcal{F}(m))$.

This polynomial is called the *Hilbert polynomial* of $\mathcal{F}$. While $\chi(\mathcal{F})$ is an intrinsic invariant of $\mathcal{F}$, the Hilbert polynomial is not, as it depends on the choice of embedding $X \subset \mathbb{P}^n_k$.

When $\mathcal{F} = \widetilde{M}$ for a graded module $M$, $P_{\mathcal{F}}(m)$ coincides with the usual Hilbert polynomial of $M$ as defined in commutative algebra.

## 15.11 Example: Plane curves

Let $X = V_+(f) \subset \mathbb{P}^2_k$ be a plane curve, defined by an homogeneous polynomial $f(x_0, x_1, x_2)$ of degree $d$. Let us compute the groups of the structure sheaf $H^i(X, \mathcal{O}_X)$. We have the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$$

where the ideal sheaf $\mathcal{I}_X$ is the kernel of the restriction $\mathcal{O}_{\mathbb{P}^2} \to i_*\mathcal{O}_X$. By Example 13.17, $\mathcal{I}_X \simeq \mathcal{O}_{\mathbb{P}^2}(-d)$, and the sequence can be rewritten as

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0 \tag{15.27}$$

From the short exact sequence, we get the long exact sequence as follows:

$$\begin{aligned} 0 \to H^0(\mathbb{P}^2, \mathcal{O}(-d)) &\to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^0(X, \mathcal{O}_X) \\ &\to H^1(\mathbb{P}^2, \mathcal{O}(-d)) \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \to H^1(X, \mathcal{O}_X) \\ &\to H^2(\mathbb{P}^2, \mathcal{O}(-d)) \to H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow 0. \end{aligned}$$

Using the results on cohomology of line bundles on $\mathbb{P}^2$, we deduce the equality $H^0(X, \mathcal{O}_X) \simeq k$ and hence

$$H^1(X, \mathcal{O}_X) \simeq k^{(d-1)(d-2)/2}.$$

The dimension of the cohomology group on the left is the *genus* of the curve $X$ (it will be introduced properly in Chapter 18). So the above can be rephrased as saying *the genus of a plane curve of degree $d$ is* $(d-1)(d-2)/2$.

Tensoring the sequence (15.27) by $\mathcal{O}_{\mathbb{P}^2}(m)$, we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(m) \longrightarrow i_*\mathcal{O}_X(m) \longrightarrow 0$$

and the long exact sequence gives that the Hilbert polynomial of $\mathcal{O}_X$ equals

$$P(m) = \binom{m+2}{2} - \binom{m-d+2}{2} = dm - \frac{d^2 - 3d}{2}.$$

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**Example 15.34.** A plane curve of degree 1, i.e., a projective line, is isomorphic to $\mathbb{P}^1_k$, and the genus is 0 in accordance with the above result. △

**Example 15.35.** Suppose $k$ is a field of characacteristic $\neq 2$ and assume there is an $i \in k$ with $i^2 = -1$. Then the curve

$$X = V_+(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2_k$$

is isomorphic to $\mathbb{P}^1_k$. This follows because the variable change $u_1 = x_1$, $u_0 = x_0 + ix_2$, $u_2 = x_0 - ix_2$, brings the equation into the form $u_1^2 - u_0u_2 = 0$, and so the image is isomorphic to $\mathbb{P}^1_k$ embedded into $\mathbb{P}^2_k$ via the standard embedding $(s : t) \to (s^2 : st : t^2)$.

A plane curve of degree 3, i.e., an elliptic curve, has genus 1. It follows for instance that the curve

$$X = V_+(x_0^3 + x_1^3 + x_2^3) \subset \mathbb{P}^2_k$$

is not isomorphic to $\mathbb{P}^1_k$, over any field. △

## 15.12 Example: The twisted cubic

Let $k$ be a field and consider $\mathbb{P}^3 = \operatorname{Proj} R$ where $R = k[x_0, x_1, x_2, x_3]$. We will continue Example 13.18 and consider the twisted cubic curve $X = V_+(I)$ where $I \subset R$ is the ideal generated by the $2 \times 2$-minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Let us compute the group $H^1(X, \mathcal{O}_X)$. Of course we know what the answer should be, because $X \simeq \mathbb{P}^1$, and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$.

Now, to compute $H^1(X, \mathcal{O}_X)$ on $X$, it is convenient to relate it to a cohomology group on $\mathbb{P}^3$. We have $H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^3, i_*\mathcal{O}_X)$ where $i\colon X \to \mathbb{P}^3$ is the inclusion. The sheaf $i_*\mathcal{O}_X$ fits into the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow i_*\mathcal{O}_X \to 0$$

where $\mathcal{I}$ is the ideal sheaf of $X$ in $\mathbb{P}^3$. Applying the long exact sequence in cohomology, we get

$$\cdots \longrightarrow H^1(\mathbb{P}^3, \mathcal{I}) \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow H^1(\mathbb{P}^3, i_*\mathcal{O}_X) \longrightarrow H^2(\mathbb{P}^3, \mathcal{I}) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow \cdots$$

By our description of sheaf cohomology on $\mathbb{P}^3$, $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0$, which implies that $H^1(X, \mathcal{O}_X) = H^2(\mathbb{P}^3, \mathcal{I})$. We can compute the latter cohomology group using the exact sequence of Example 13.18:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3)^2 \to \mathcal{O}_{\mathbb{P}^3}(-2)^3 \to \mathcal{I} \to 0.$$

Now, taking the long exact sequence we get

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$$\cdots \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}(-3)^2) \longrightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \longrightarrow H^2(\mathbb{P}^3, \mathcal{I})$$
$$\hookrightarrow H^3(\mathbb{P}^3, \mathcal{O}(-3)^2) \longrightarrow H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-2)^3) \longrightarrow H^3(\mathbb{P}^3, \mathcal{I}).$$

Here $H^2(\mathbb{P}^3, \mathcal{O}(-2)) = 0$ and $H^3(\mathbb{P}^3, \mathcal{O}(-3)) = 0$ by our previsous computations. Hence by exactness, we find $H^2(\mathbb{P}^3, \mathcal{I}) = 0$. It follows that $H^1(X, \mathcal{O}_X) = 0$ also, as expected.

## 15.13 Example: Hyperelliptic curves

Let us recall the hyperelliptic curves defined in Chapter **??**. For simplicity, we work over an algebraically closed field $k$ and consider hyperelliptic curve associated to the polynomial $f(x)$ in $k[x]$. More precisely, if the degree is $n = 2g + 1$ or $n = 2g + 2$, we consider the scheme $X$ glued together by the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, where

$$A = \frac{k[x, y]}{(y^2 - f(x))} \text{ and } B = \frac{k[u, v]}{(v^2 - u^{2g+2} f(u))}.$$

and before, we glue $D(x) \subset U$ to $D(u) \subset V$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

Let us compute the Cech cohomology groups of $\mathcal{O}_X$ with respect to the affine covering $\mathcal{U} = \{U, V\}$. As $\mathcal{U}$ has only two elements, the Cech complex has only two terms, $\mathcal{O}_X(U) \times \mathcal{O}_X(V) = A \times B$ and $\mathcal{O}_X(U \cap V) = A_x$. To simplify the computation, we use the relation $y^2 = f(x)$ to decompose $A$ as a $k[x]$-module as

$$\frac{k[x, y]}{(y^2 - f(x))} \simeq k[x] \oplus k[x]y$$

and similarly $B \simeq k[u] \oplus k[u]v$ as a $k[u]$-module. With these identifications, the Cech complex takes the form

$$(k[x] \oplus k[x]y) \oplus \big(k[x^{-1}] \oplus k[x^{-1}]x^{-g-1}y\big) \xrightarrow{d} k[x^{\pm 1}] \oplus k[x^{\pm 1}]y,$$

where the differential $d$ is given by

$$d^0(p(x) + q(x)y, r(x^{-1}) + s(x^{-1})x^{-g-1}y) \\ = p(x) - r(x^{-1}) + (q(x) - s(x^{-1})x^{-g-1})y.$$

Comparing monomials on each side, we deduce that

$$H^0(X, \mathcal{O}_X) = \operatorname{Ker} d^0 = k$$

and

$$H^1(X, \mathcal{O}_X) = \operatorname{Coker} d^0 = k\, x^{-1}y \oplus \cdots \oplus k\, x^{-g}y \simeq k^g.$$

For $g = 2$, we get a particularly interesting curve: it is an irreducible projective curve which cannot be embedded in $\mathbb{P}^2$. Indeed, we showed that for any irreducible curve in $\mathbb{P}^2$ of degree $d$, the corresponding cohomology group $H^1(X, \mathcal{O}_X)$ has dimension equal to $\frac{1}{2}(d-1)(d-2)$, but this does not take the value 2 if $d$ is an integer.

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**Proposition 15.36.** There exist nonsingular projective curves which cannot be embedded in the projective plane $\mathbb{P}^2$.

We still haven't yet proved that $X$ is projective. This follows because $X$ can be embedded into the *weighted* projective space $\mathbb{P}(1,1,g+1) = \operatorname{Proj} k[x_0, x_1, w]$ given by the equation

$$w^2 = a_{2g+2}x_0^{2g+2} + a_{2g+1}x_0^{2g+1}x_1 + \cdots + a_0 x_1^{2g+2}. \tag{15.28}$$

Note that this makes sense because $w$ has degree $g+1$; the equation (15.28) does not define a subscheme of $\mathbb{P}^2_k$. Then the two open sets $U$ and $V$ are isomorphic to the distinguished opens $D_+(x_0)$ and $D_+(x_1)$ respectively. The weighted projective plane $\mathbb{P}(1,1,g+1)$ is a projective variety, and hence so is $X$.

## 15.14 Example: Complete intersections

Let $k$ be a field and let $F$ and $G$ be two homogeneous polynomials in $k[x_0, \ldots, x_n]$ of degrees $d$ and $e$ respectively. Let $X$ be the closed subscheme of $\mathbb{P}^n_k$ defined by $I = (F, G)$. Let us use the exact sequence (**??**) to compute the cohomology groups of $X$.

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \iota_*\mathcal{O}_X \longrightarrow 0.$$

Note that $H^0(\mathbb{P}^n_k, \mathcal{I}) = \Gamma(\mathbb{P}^n_k, \mathcal{I}) = 0$, as there are no nonconstant global sections which vanish along $X$. Taking the long exact sequence, we get that

- $\dim_k H^0(X, \mathcal{O}_X) = \dim_k H^1(\mathbb{P}^n_k, \mathcal{I}) + 1$,
- $\dim_k H^p(X, \mathcal{O}_X) = H^{p+1}(\mathbb{P}^n_k, \mathcal{I})$ for $p = 1, \ldots, n-1$
- $H^n(\mathbb{P}^n_k, \mathcal{I}) = H^{n-1}(\mathbb{P}^n_k, \iota_*\mathcal{O}_X) = H^n(X, \mathcal{O}_X) = 0$, because $X$ has dimension $n-2$.

We proceed to study the cohomology groups $H^p(\mathbb{P}^n_k, \mathcal{I}_Z)$. Recall the exact sequence from Section **??**,

$$0 \to \mathcal{O}_{\mathbb{P}^n_k}(-d-e) \to \mathcal{O}_{\mathbb{P}^n_k}(-d) \oplus \mathcal{O}_{\mathbb{P}^n_k}(-e) \to \mathcal{I}_Z \to 0.$$

Taking the long exact sequence of cohomology, we obtain that

$$H^p(\mathbb{P}^n_k, \mathcal{I}) = 0$$

for $p = 0, \ldots, n-2$, and

$$0 \to H^{n-1}(\mathcal{I}) \to H^n(\mathcal{O}_{\mathbb{P}^n_k}(-d-e)) \to H^n(\mathcal{O}_{\mathbb{P}^n_k}(-d)) \oplus H^n(\mathcal{O}_{\mathbb{P}^n_k}(-e)) \to 0.$$

Therefore, we get

$$\dim_k H^{n-1}(X, \mathcal{O}_X) = \binom{d+e-1}{n} - \binom{d-1}{n} - \binom{e-1}{n}. \tag{15.29}$$

**Example 15.37.** When $n = 3$, $X \subset \mathbb{P}^3_k$ is a curve, with $H^1(X, \mathcal{O}_X)$ of dimension

$$\binom{d+e-1}{3} - \binom{d-1}{3} - \binom{e-1}{3} = \frac{1}{2}de(d+e-4) + 1$$

For instance, when $d = 1, e = 2$, $X$ is a conic and we recover $h^1(X, \mathcal{O}_X) = 0$. △

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## 15.15 Example: Bezout's theorem

Let $k$ be an algebraically closed field. Let $C$ and $D$ be two curves in $\mathbb{P}^2_k$ defined by homogeneous polynomials $F$ and $G$ of degrees $d$ and $e$ respectively, and assume that $C$ and $D$ have no common component, so that closed subscheme $Z = V_+(F,G)$ is a 0-dimensional subscheme. By a linear coordinate change, we may assume $Z = \{x_1, \ldots, x_r\}$ is contained in $D(x_0) \simeq k[x,y]$. Then the classical form of *Bezout's theorem* says that

$$\sum_{i=1}^{r} \operatorname{mult}_{x_i}(C,D) = de, \tag{15.30}$$

where

$$\operatorname{mult}_{x_i}(C,D) = \dim_k \mathcal{O}_{Z,x_i} = \dim_k \left( \frac{k[x,y]}{(f,g)} \right)_{\mathfrak{m}_{x_i}}$$

is the *intersection multiplicity* of $C$ and $D$ at $x_i$, and $f$ and $g$ are the dehomogenizations of $F$ and $G$.

There is a quick proof of the formula (15.30) by computing the cohomology group $H^0(Z, \mathcal{O}_Z)$ in two different ways. First of all, this group is the space of global sections $\mathcal{O}_Z(Z)$, so it decomposes as

$$\mathcal{O}_Z(Z) = \bigoplus_{i=1}^{r} \mathcal{O}_{Z,x_i} = \bigoplus_{i=1}^{r} \left( \frac{k[x,y]}{(f,g)} \right)_{\mathfrak{m}_{x_i}} \tag{15.31}$$

where $f, g$ are the dehomogenized equations for $C$ and $D$. Therefore,

$$\dim_k H^0(Z, \mathcal{O}_Z) = \sum_{i=1}^{r} \dim_k \left( \frac{k[x,y]}{(f,g)} \right)_{\mathfrak{m}_{x_i}}.$$

On the other hand, from (15.29), we find $\dim_k H^0(Z, \mathcal{O}_Z) = \dim_k H^1(\mathbb{P}^2_k, \mathcal{I}_Z) + 1$ and hence

$$\dim_k H^0(Z, \mathcal{O}_Z) = \binom{d+e-1}{2} - \binom{d-1}{2} - \binom{e-1}{2} + 1 = de.$$

## 15.16 Example: Non-split locally free sheaves

A locally free sheaf is said to be *split* if it is isomorphic to a direct sum of invertible sheaves. We have seen several examples of locally free sheaves that are not free, even on affine schemes, but a priori it is not so clear whether these are direct sums of invertible sheaves. In this section we will study the sheaf $\mathcal{E}$ from Section **??** and show that it is indeed non-split.

The sheaf $\mathcal{E}$ is the locally free sheaf of rank $n$ on $\mathbb{P}^n_k$ sitting in the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1) \longrightarrow \mathcal{O}^{n+1}_{\mathbb{P}^n_k} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Suppose that $\mathcal{E}$ is split, i.e., $\mathcal{E}$ is isomorphic to a direct sum of invertible sheaves. Since $\operatorname{Pic}(\mathbb{P}^n_k) = \mathbb{Z}$ is generated by the class of $\mathcal{O}(1)$, this would mean that

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^n_k}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n_k}(a_n)$$

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for some integers $a_1, \dots a_n \in \mathbb{Z}$.

Recall that for $n \geqslant 2$, we have $H^{n-1}(\mathbb{P}^n_k, \mathcal{O}(m)) = 0$ for any $m \in \mathbb{Z}$. So if we could show that $H^{n-1}(\mathbb{P}^n_k, \mathcal{E}) \neq 0$, we would have a contradiction. Actually, it is the case that $H^{n-1}(\mathbb{P}^n_k, \mathcal{E}) = 0$, but we can instead consider $\mathcal{F} = \mathcal{E}(-n)$, which fits into the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-n)^{n+1} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Taking the long exact sequence in cohomology, we get

$$\cdots \to H^{n-1}(\mathcal{O}^{n+1}_{\mathbb{P}^n_k}(-n)) \to H^{n-1}(\mathcal{F}) \xrightarrow{\delta} H^n(\mathcal{O}_{\mathbb{P}^n_k}(-n-1)) \to H^n(\mathcal{O}^{n+1}_{\mathbb{P}^n_k}(-n)) \to \cdots$$

Here the two outer cohomology groups are zero, by Theorem 15.24. Hence, by exactness, we find that $H^{n-1}(\mathbb{P}^n_k, \mathcal{F}(-1)) \simeq H^0(\mathbb{P}^n_k \mathcal{O}_{\mathbb{P}^n_k}) = k$. Therefore, we find that $\mathcal{F} = \mathcal{E}(-n)$ is not split, and hence $\mathcal{E}$ is not split either.

This gives an example of a non-split locally free sheaf of rank $n$ on $\mathbb{P}^n_k$. However, coming up with examples of non-split sheaves of *low rank* on projective space is much more difficult. In fact, a famous conjecture of Hartshorne says that any rank 2 vector bundle on $\mathbb{P}^n_k$ for $n \geqslant 5$ is split.

## 15.17 Example: The rational quartic curve

Let us consider the rational quartic curve $X$ of Example 13.13. In other words, $X \subset \mathbb{P}^3_k = \operatorname{Proj} k[x_0, x_1, x_2, x_3]$ is the closed subscheme defined by the ideal

$$I = (x_1x_2 - x_0x_3, x_2^3 - x_1x_3^2, x_0x_2^2 - x_1^2x_3, x_1^3 - x_0^2x_2) \tag{15.32}$$

We showed in the previous example that the natural map

$$k[x_0, x_1, x_2, x_3] \longrightarrow \Gamma_*(\mathcal{O}_X) \tag{15.33}$$

is not surjective and that $\Gamma(X, \mathcal{O}_X(1))$ is 5-dimensional as a $k$-vector space. We may explain this fact by looking at the cohomology of the ideal sheaf sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^3_k} \longrightarrow \iota_*\mathcal{O}_X \longrightarrow 0 \tag{15.34}$$

More precisely, we claim that the map $\mathcal{O}_{\mathbb{P}^3_k}(1) \to (\iota_*\mathcal{O}_X)(1)$ is not surjective on global sections, and that $H^1(\mathbb{P}^3_k, \mathcal{I}(1))$ is 1-dimensional.

Using a computer, we find that the ideal has a graded resolution given by

$$0 \to R(-5) \xrightarrow{U\cdot} R(-4)^4 \xrightarrow{V\cdot} R(-2) \oplus R(-3)^3 \xrightarrow{W\cdot} I \to 0 \tag{15.35}$$

where the matrices are given by

$$U = \begin{pmatrix} x_3 \\ -x_2 \\ -x_1 \\ x_0 \end{pmatrix}, \; V = \begin{pmatrix} -x_1^2 & -x_0x_2 & -x_1x_3 & -x_2^2 \\ x_2 & x_3 & 0 & 0 \\ x_0 & x_1 & -x_2 & -x_3 \\ 0 & 0 & x_0 & x_1 \end{pmatrix}$$

and $W$ is the $4 \times 1$-matrix with entries equal to the generators of $I$ in (15.32). Note that $W \cdot V = 0$ and $VU = 0$.

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If we twist the sequence by 1 and apply tilde, we get the following exact sequence of sheaves on $\mathbb{P}^3_k$.

$$0 \to \mathcal{O}(-4) \to \mathcal{O}(-3)^4 \xrightarrow{v} \mathcal{O}(-1) \oplus \mathcal{O}(-2)^3 \to \mathcal{I}(1) \to 0 \tag{15.36}$$

To compute the cohomology of $\mathcal{I}(1)$, we split this sequence into

$$0 \to \mathcal{O}(-4) \to \mathcal{O}(-3)^4 \to \operatorname{Im} v \to 0$$
$$0 \to \operatorname{Im} v \to \mathcal{O}(-1) \oplus \mathcal{O}(-2)^3 \to \mathcal{I}(1) \to 0$$

Taking the long exact sequence of the first sequence, we deduce that $H^2(\mathbb{P}^3_k, \operatorname{Im} v) = H^3(\mathbb{P}^3_k, \mathcal{O}(-4)) = k$. Then doing the same for the second, we see that $H^1(\mathbb{P}^3_k, \mathcal{I}(1)) = k$.

The cohomology group $H^1(\mathbb{P}^3_k, \mathcal{I}(1))$ explains the failure of surjectivity of the map (15.33). The fact that it is 1-dimensional explains the missing monomial $x^2y^2$ in the definition of $X$.

## 15.18 Exercises

**Exercise 15.18.1.** Generalize Example 15.11 to show that the curve $V(x_2^d - x_0x_1^{d-1}) \subset \mathbb{P}^2_k$ has an $H^1(X, \mathcal{O}_X)$ of dimension $\frac{1}{2}(d-1)(d-2)$ as a $k$-vector space.

**Exercise 15.18.2.** Let $X = S^1$ and let $\mathcal{U}$ be the covering of $X$ with three pairwise intersecting open intervals with empty intersection. Show that the Cech complex is of the form

$$\mathbb{Z}^3 \xrightarrow{d^0} \mathbb{Z}^3 \to 0.$$

Compute the map $d^0$ and use it to verify again that $H^i(\mathcal{U}, \mathbb{Z}_X) = \mathbb{Z}$ for $i = 0, 1$ as above.

**Exercise 15.18.3.** Let $X = \mathbb{A}^n_k - \{0\}$ be the complement of the origin.

a) Compute $H^i(X, \mathcal{O}_X)$ for all $i$.
b) Give a new proof that $X$ is not an affine scheme for $n \geqslant 2$.

**Exercise 15.18.4.** With reference to the twisted cubic example in Section 15.12, show that

- $H^0(\mathbb{P}^3, \mathcal{I}(2)) = k^3$ (find a basis!)
- $H^1(\mathbb{P}^3, \mathcal{I}(m)) = 0$ for all $m \in \mathbb{Z}$.
- $H^2(\mathbb{P}^3, \mathcal{I}(-1)) = k$.

**Exercise 15.18.5** (Künneth formula)**.** Let $X$ and $Y$ be Noetherian schemes of finite type over a field. Let $\mathcal{F}$ two two finite type quasi-coherent sheaves on $X$ and $Y$ respectively. Show that there is a canonical isomorphism

$$H^n(X \times_k Y, p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}) \simeq \bigoplus_{p+q=n} H^p(X, \mathcal{F}) \otimes_k H^q(X, \mathcal{G}).$$

HINT: Use a Chech complex using products of affine opens.

**Exercise 15.18.6.** Let $X \subset \mathbb{P}^5$ denote a quadric hypersurface (i.e., $X = V(q)$ for a homogeneous degree 2 polynomial). Recall the exact sequence **??**

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-1)^4 \to \mathcal{O}^4_{\mathbb{P}^5} \to i_*\mathcal{E} \to 0$$

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where $\mathcal{E}$ is a locally free sheaf of rank 2.

Use this exact sequence to show that $\mathcal{E}$ is not split.

**Exercise 15.18.7.** Let $n > 0$ be an integer and consider the integral projective scheme $X = \operatorname{Proj}(R)$, where $R$ is the ring

$$R = k[x, y, z, w]/(x^2, xy, y^2, u^n x - v^n y).$$

a) Show that $X$ is irreducible, non-reduced, and of dimension 1.
b) Compute $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$.

**Exercise 15.18.8.** Let $Y_r \subset \mathbb{P}^n_k$ be a complete intersection of $r$ hypersurfaces $H_1, \ldots, H_r$ of degrees $d_1, \ldots, d_r$.

a) Show that there is an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_{Y_{r-1}}(-H_r) \longrightarrow \mathcal{O}_{Y_{r-1}} \longrightarrow \mathcal{O}_{Y_r} \longrightarrow 0$$

b) The natural maps

$$H^i(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(d)) \longrightarrow H^i(Y_r, \mathcal{O}_{Y_r}(d))$$

are isomorphisms for all $i < n - r$ and all $d \in \mathbb{Z}$.
c) Show that $Y_r$ is connected if $\dim Y_r \geqslant 1$.

**Exercise 15.18.9.** Let $\mathcal{E}$ be a locally free sheaf of rank $r$ on $\mathbb{P}^1_k$.

a) Show that the set of integers $n$ such that there is an injection $\mathcal{O}(n) \subset \mathcal{E}$ is non-empty and bounded from above.
b) Show that if $n$ is maximal as in *a*) then $\mathcal{E}/\mathcal{O}(n)$ is locally free of rank $r - 1$
c) Show that if $r = 2$, and the maximal $n$ is $-1$, then $\mathcal{E}/\mathcal{O}(-1) \simeq \mathcal{O}(m)$ for some $m < 0$.
d) Show that, in general, there exists a a filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_r = \mathcal{E}$$

such that $\mathcal{E}_i$ is locally free of rank $i$ and $\mathcal{E}_i/\mathcal{E}_{i-1} \simeq \mathcal{O}(a_i)$ for some $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r$.
e) Show that for every $\mathcal{E}$ with a filtration as above, we have $\mathcal{E} = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ and that the $a_i$ are uniquely determined by $\mathcal{E}$.
f) Conclude that any locally free sheaf on $\mathbb{P}^1_k$ splits as a direct sum of invertible sheaves.

**Exercise 15.18.10.** Let $X$ be a projective scheme and let $Y, Z$ be closed subschemes of $X$ with $Z \subset Y$. Write $P_Y$ and $P_Z$ for the Hilbert polynomials of $\mathcal{O}_Y$ and $\mathcal{O}_Z$ respectively. Show that $P_Z(m) \leqslant P_Y(m)$ for $m \gg 0$.

**Exercise 15.18.11** (Length 2 subschemes)**.** If $Z$ is a 0-dimensional subscheme, then we define the length of $Z$ as the length of $\mathcal{O}_Z(Z)$, which is an Artinian ring.

Use a cohomological argument to show that any 0-dimensional subscheme of length 2 in $\mathbb{P}^2$ is a complete intersection of two curves of degree 1 and 2.

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**Exercise 15.18.12.** Let $X$ be a scheme and let $\mathcal{O}_X^\times$ denote the sheaf of units (Exercise 4.11.44). Show that there is a natural isomorphism

$$\operatorname{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$$

HINT: Invertible sheaves can be specified using units in $\mathcal{O}_X(U_i \cap U_j)$.

**Exercise 15.18.13.** Continuing Exercise 16.10.3, show that the map from Cartier divisors to $\operatorname{Pic}(X)$ (induced by $D \mapsto \mathcal{O}_X(D)$) can be identified with the connection map $H^0(X, \mathbf{K}(X)^\times/\mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times)$ from the exact sequence

$$0 \longrightarrow \mathcal{O}_X^\times \longrightarrow \mathbf{K}(X)^\times \longrightarrow\longrightarrow \mathbf{K}(X)/\mathcal{O}_X^\times \longrightarrow 0.$$

**Exercise 15.18.14** (Double structures)**.** Let $X$ be a scheme such that $Y = X_{\mathrm{red}} \subset X$ is defined by an ideal sheaf $\mathcal{I}$. Let us assume that $\mathcal{I}^2 = 0$.

a) Show that there is an exact sequence $0 \to \mathcal{I} \to \mathcal{O}_X^\times \to \mathcal{O}_Y^\times \to 0$, where the first map is $f \mapsto 1 + f$.

b) Assume that $X$ is projective. Show that there is an exact sequence

$$0 \to H^1(X, \mathcal{I}) \to \operatorname{Pic}(X) \to \operatorname{Pic}(Y) \to H^2(X, \mathcal{I})$$

c) Compute the Picard group of the scheme

$$X = \operatorname{Proj}(k[x, y, z, w]/(x^2, xy, y^2)).$$

**Exercise 15.18.15** ($\mathbb{P}^2$ minus a point)**.** Consider $X = \mathbb{P}^2_k - \{(0 : 0 : 1)\}$, with the affine covering given by $U_0 = \operatorname{Spec} k[x, y]$ and $U_1 = \operatorname{Spec} k[x^{-1}, x^{-1}y]$. Show that $H^0(X, \mathcal{O}_X) = k$ and

$$H^1(X, \mathcal{O}_X) = \bigoplus_{b > -a,\, a < 0} k \cdot x^a y^b.$$

Deduce that $\mathbb{P}^2_k - \{(0 : 0 : 1)\}$ is not an affine scheme.

**Exercise 15.18.16.** Continuing the previous exercise, let $C$ be the curve defined by $y^3 + x^2 + x$ in $U_0$ and $(x^{-1}y)^3 + x^{-2} + x^{-1}$ in $U_1$. Show that $H^0(C, \mathcal{O}_C) = k$ and

$$H^1(X, \mathcal{O}_C) = k \cdot x^{-1}y^2.$$

More generally, if $C$ is the curve defined by $y^d + x^{d-1} + x$ in $U_0$, and $(x^{-1}y)^d + x^{-d+1} + x^{-1}$ in $U_1$, then

$$H^1(C, \mathcal{O}_C) \simeq k^{\binom{d-1}{2}}.$$

**Exercise 15.18.17** (Interpolation problems)**.** Let $Z = \{p_1, \ldots, p_r\} \subset \mathbb{P}^n_k$ be a finite set of points, with the reduced scheme structure.

a) Show that the space of degree $d$ polynomials passing through $p_1, \ldots, p_r$ has dimension equal to

$$\dim H^0(\mathbb{P}^n, \mathcal{I}_Z(d)) = \dim_k H^0(\mathcal{I}_Z(d)) = \binom{n+d}{d} - r + H^1(\mathbb{P}^n_Z(d)).$$

b) Show that for two points in $\mathbb{P}^2_2$ there is a unique line passing through them.

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c) Let $Z = \{p_1, p_2, p_3\}$ be three distinct points in $\mathbb{P}^2_k$, show that

$$H^1(\mathbb{P}^2_k, \mathcal{I}_Z(1)) \neq 0$$

if and only if $p_1, p_2, p_3$ lie on a line.

d) Let $p_1, \ldots, p_5$ be 5 points in $\mathbb{P}^2_k$. When is $H^1(\mathbb{P}^2_k, \mathcal{I}_Z(2)) = 0$?.

**Exercise 15.18.18.** Consider a double complex $C^{m,n}$ as in the text, with horizontal and vertical coboundary maps $d$ and $\delta$. Assume that all rows $C^{m,n}$ are exact except when $n = 0$, and that all columns are exact except when $m = 0$.

For each $p$, consider the group $Z^p \subset \bigoplus_{m+n=p} C^{m,n}$ of 'zig-zags' $(c^{p,0}, c^{p-1,1}, \ldots, c^{0,p})$ such that

$$\delta(c^{i,j}) = d(c^{i-1,j+1}) \ \forall i \geqslant 1, \quad \text{and} \quad \delta(c^{0,p}) = d(c^{p,0}) = 0.$$

Consider the quotient $\mathbb{H}^i(C) = Z^i(C)/B^i(C)$ where $B^i(C) \subset Z^i(C)$ is the subgroup generated by the 'trivial zig-zags'

$$(d(c^{p-1,0}),\ d(c^{p-2,1}) + \delta(c^{p-1,0}), \ldots,\ d(c^{0,p-1}) + \delta(c^{1,p-2}),\ \delta(c^{0,p-1})),$$

a) Define maps $\alpha_p \colon \mathbb{H}^p(C) \to H^p(A^\bullet)$ and $\beta_p \colon \mathbb{H}^p(C) \to H^p(B^\bullet)$.

b) Use a diagram chase similar to the one above to show that $\alpha_p$ is surjective, and then injective.

c) Prove Lemma 15.19.

**Exercise 15.18.19.** Let $\mathcal{F}$ be a sheaf on a topological space $X$ and let $\mathcal{U} = \{U_i\}_{i=1}^r$ be an open cover. Let $\sigma \in S_r$ be a permutation and consider the permuted cover $\mathcal{U}' = \{U_{\sigma(i)}\}_{i=1}^r$.

For each $p$, define a map $\phi^p : C^p(\mathcal{U}', \mathcal{F}) \to C^p(\mathcal{U}, \mathcal{F})$ by

$$\phi^p(\alpha)_{\sigma(i_0),\sigma(i_1),\ldots,\sigma(i_p)} = \operatorname{sgn}(\sigma|_{i_0,\ldots,i_p}) \cdot \alpha_{i_0,i_1,\ldots,i_p}$$

where $\operatorname{sgn}(\sigma|_{i_0,\ldots,i_p})$ is the sign of the permutation $\sigma$ restricted to the indices $\{i_0, i_1, \ldots, i_p\}$.

a) Show that each $\sigma^p$ is bijective.

b) Show that $\phi^p$ is a map of complexes, i.e., $\phi^{p+1} \circ d^p = d'^p \circ \phi^p$.

c) Conclude that that $\phi^\bullet$ induces a canonical isomorphism $H^p(\mathcal{U}, \mathcal{F}) \simeq H^p(\mathcal{U}', \mathcal{F})$

**Exercise 15.18.20.** Let $X$ be the scheme obtained by gluing two copies $U, V$ of $\operatorname{Spec}(\mathbb{Z}_{(2)})$ to itself over the generic point. Compute $H^1(\mathcal{U}, \mathcal{O}_X)$ for the covering $\{U, V\}$.

**Exercise 15.18.21.** Give a second proof of Corollary 12.11, using the fact that $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent sheaf $\mathcal{F}$ on $X = \operatorname{Spec} A$.

**Exercise 15.18.22** (Cohomology of $\mathbb{P}^3_k$ minus a line)**.** Consider the scheme $X = \mathbb{P}^3_k - V_+(x_0, x_1)$. Compute the cohomology groups of $\mathcal{O}_X$.

**Exercise 15.18.23.** Let $X$ be the gluing of two copies of $\operatorname{Spec} \mathbb{Z}$ along the open set $U = \operatorname{Spec} \mathbb{Z} - \{(2)\}$.

Show that $X$ is not separated and describe the diagonal map.

Compute the Cech cohomology groups $H^i(\mathcal{U}, \mathcal{O}_X)$ with respect to the cover given by two copies of $\operatorname{Spec} \mathbb{Z}$.

**Exercise 15.18.24** (Products of curves)**.** Let $C \subset \mathbb{P}^2_k$ be plane curve of degree 4.

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a) Compute the cohomology groups $H^i(C, \mathcal{O}_C)$ for $i = 0, 1, 2$.
b) Compute the cohomology groups $H^i(C \times_k \mathbb{A}^1_k, \mathcal{O}_C)$ for $i = 0, 1, 2$.
c) Compute the cohomology groups $H^i(C \times_k \mathbb{P}^1_k, \mathcal{O}_C)$ for $i = 0, 1, 2$.

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# 16

# Divisors

## 16.1 Weil divisors

Throughout this chapter we will assume that $X$ is a separated Noetherian integral normal scheme. A *prime divisor* is an integral subscheme $Z \subset X$ of codimension 1. A *Weil divisor*, or simply *divisor*, is a finite formal sum

$$D = m_1 Z_1 + \cdots + m_r Z_r$$

where $m_i \in \mathbb{Z}$ and the $Z_i$'s are prime divisors. The set of divisors form a group, which will be denoted by $\operatorname{Div}(X)$.

We say that a divisor $D$ is *effective* if each $m_i \geqslant 0$, and we write $D \geqslant 0$. The *support* of $D$ is defined to be the closed subset $\operatorname{Supp}(D) = Z_1 \cup \cdots \cup Z_r$.

**Example 16.1.** If $X = \operatorname{Spec} A$ is affine, then every prime divisor is of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of height 1. Hence $\operatorname{Div}(\operatorname{Spec} A) = \bigoplus_{\operatorname{ht}\mathfrak{p}=1} \mathbb{Z} \cdot V(\mathfrak{p})$ can be identified with the free abelian group on all height 1 prime ideals. △

**Example 16.2.** On $X = \mathbb{P}^1_{\mathbb{C}}$, the prime divisors are simply the closed points. Here are some examples of Weil divisors:

$$D_1 = 3 \cdot (1:0) - 5 \cdot (0:1), \qquad D_2 = (1:1) + 5 \cdot (0:1)$$
$$D_1 + D_2 = 3 \cdot (1:0) + (1:1).$$ △

The reader may wonder why we include the assumption that $X$ is normal. Certainly, the above definition can be made for any scheme, but the concept of a Weil divisor is not particularly useful without the normality assumption. The main reason is that there is a well-behaved notion of 'order of vanishing', or 'multiplicity' for rational functions.

More precisely, as the scheme $X$ is normal, all the local rings $\mathcal{O}_{X,x}$ are integrally closed in their fraction field, $K(X)$. This means that if $Z \subset X$ is a prime divisor, and $\zeta \in X$ is the generic point of $Z$, then the local ring $\mathcal{O}_{X,\zeta}$ is a 1-dimensional normal ring, and hence a discrete valuation ring (see Appendix **?**). We write

$$\operatorname{ord}_Z \colon K(X) \longrightarrow \mathbb{Z} \cup \{\infty\}$$

for the corresponding valuation. More explicitly, the function $\operatorname{ord}_Z$ can be defined as follows. Given a nonzero $f \in \mathcal{O}_{X,\zeta}$, we can write it as $f = u \cdot t^m$, where $t$ is the generator for the maximal ideal of $\mathcal{O}_{X,\zeta}$ and $u$ is a unit, and define the valuation of $f$ at $Z$ to be $\operatorname{ord}_Z(f) = m$. Finally, we extend this to elements of $K(X)$ by setting $\operatorname{ord}_Z(f/g) = \operatorname{ord}_Z(f) - \operatorname{ord}_Z(g)$ and $\operatorname{ord}_Z(0) = \infty$. The number $\operatorname{ord}_Z(f)$ is called the *order of vanishing* of $f$ at $Z$. If



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$\operatorname{ord}_Z(f)$ is non-negative we say that $f$ vanishes to that order along $Z$. If $\operatorname{ord}_Z(f)$ is negative, then we say that $f$ has a *pole* of that order along $Z$.

If $X = \operatorname{Spec} A$ is an affine scheme, then all prime divisors are of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of height 1. For such a prime $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ is a discrete valuation ring, consisting of exactly the fractions $f = a/b \in K(A)$ where $b \notin \mathfrak{p}$, or equivalently, $\operatorname{ord}_{V(\mathfrak{p})}(f) \geqslant 0$. In light of Theorem 16.3, which says that $A = \bigcap_{\operatorname{ht}\mathfrak{p}=1} A_\mathfrak{p}$, we conclude that $\operatorname{ord}_{V(\mathfrak{p})}(f) \geqslant 0$ for every prime divisor $V(\mathfrak{p})$ if and only if $f \in A$. Moreover, $f \in K(A)$ has zero order of vanishing along every prime divisor if and only if $f$ is invertible in $A^\times$.

In general, for an integral scheme $X$, a rational function $f \in K(X)$ is regular if and only if $f \in \mathcal{O}_{X,x}$ for every $x \in X$, which by the affine case above means that $\operatorname{ord}_Z(f) \geqslant 0$ for every prime divisor $Z$. Therefore, we have:

**Proposition 16.3.** For a rational function $f \in K(X)^\times$, we have
(i) $\operatorname{ord}_Z(f) \geqslant 0$ for all prime divisors $Z \subset X$ if and only if $f \in \mathcal{O}_X(X)$.
(ii) $\operatorname{ord}_Z(f) = 0$ for all prime divisors $Z$ if and only if $f \in \mathcal{O}_X(X)^\times$.

This is illustrates one of the close links between rational functions and codimension 1 subschemes on normal schemes. If a rational function is regular outside a closed subset of codimension at least 2, then it is regular everywhere.

**Definition 16.4** (Principal divisors)**.** For a rational function $f \in K(X)^\times$, we define its corresponding divisor by

$$\operatorname{div}(f) = \sum_Z \operatorname{ord}_Z(f) Z, \tag{16.1}$$

where the sum runs over all prime divisors. Divisors of the form $\operatorname{div}(f)$ are called *principal divisors*, and they form a subgroup of $\operatorname{Div}(X)$.

To ensure that this is well-defined, we need to verify that for a nonzero rational function $f \in K(X)$, there are only finitely many prime divisors $Z \subset X$ such that $\operatorname{ord}_Z(f) \neq 0$. To see this, let $U = \operatorname{Spec} A$ be any open affine subset such that $f|_U \in \mathcal{O}_X(U)$. For a prime divisor $Z \subset X$, there are two possible cases: (i) $Z \subset X - U$, or (ii) $Z \cap U$ is a prime divisor of $U$. As we assume $X$ is Noetherian and integral, there can be only finitely many codimension 1 components of the closed subset $X - U$, so there are only finitely many $Z$'s in case (i). For the ones in case (ii), note that $\operatorname{ord}_Z(f) \geqslant 0$ automatically, because $f$ is regular in $U$, and $\operatorname{ord}_Z(f) > 0$ if and only if $f$ vanishes along $Z$. Again because $X$ is Noetherian, the zero set $V(f)$ has only finitely many components, so we conclude.

Note that if $f$ is a regular function, i.e., $f \in \mathcal{O}_X(X)$, then $\operatorname{div}(f)$ is an effective divisor. It is supported along the zero set of $f$, with multiplicity $\operatorname{ord}_Z(f)$ along each component $Z$.

**Example 16.5.** On $\operatorname{Spec}\mathbb{Z}$, a Weil divisor is an expression of the form

$$D = n_1 V(p_1) + \cdots + n_1 V(p_r)$$

where the $p_i$ are prime numbers. In the function field, $\mathbb{Q}$, the 'rational function' $f = p_1^{n_1} \cdots p_r^{n_r}$ satisfies $\operatorname{div}(f) = D$. Hence every divisor is principal on $\operatorname{Spec}\mathbb{Z}$. △

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**Example 16.6.** Let $k$ be an algebraically closed field, and let $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ be the affine line with fraction field $k(t)$. The prime divisors of $\mathbb{A}^1_k$ correspond to $k$-points $a \in \mathbb{A}^1_k$, each associated to a maximal ideal $(t-a)$ in $k[t]$.

Consider the rational function

$$f = t^2(t-1)^{-1} \in k(t).$$

Note that for $a \in k$, the elements $t$, and $(t-1)$ are invertible in the local ring $\mathcal{O}_{\mathbb{A}^1_k,a} = k[t]_{(t-a)}$ except when $a = 0$ or $a = 1$.

When $a = 0$, then $t$ is a local parameter in $\mathcal{O}_{\mathbb{A}^1_k,a} = k[t]_{(t)}$, and we may write $f = t^2(\text{unit})$. Hence the order of vanishing of $f$ is equal to 2 at the point $a = 0$. Similarly, $\operatorname{ord}_a(f) = -1$ at $a = -1$. Hence

$$\operatorname{div}(f) = 2V(t) - V(t-1).$$

More generally, if $f(t) = (t-a_1)^{n_1} \cdots (t-a_r)^{n_r}$, then

$$\operatorname{div}(f) = n_1 V(t-a_1) + \cdots + n_r V(t-a_r). \tag{16.2}$$

In particular, every divisor $D$ is the divisor of some nonzero rational function. △

**Example 16.7.** The same rational function $f = t^2(t-1)^{-1}$ may be considered as a rational on $\mathbb{P}^1_k$, when $\mathbb{A}^1_k$ is embedded as the open set $U_0$ with coordinate $t = x_1/x_0$. We have already computed the values $\operatorname{ord}_P(f)$ for $P \in U_0$. For the remaining point, $P = (0:1)$, we use the coordinate $s = x_0/x_1 = t^{-1}$ and rewrite $f$ as

$$(s^{-1})^2(s^{-1}-1)^{-1} = s^{-1}(1-s)^{-1}.$$

From this, we see that $\operatorname{ord}_{(0:1)}(f) = -1$ and $\operatorname{ord}_{(1:1)}(f) = -1$ (which was already computed in $U_0$). Therefore, the divisor of $f$ on $\mathbb{P}^1_k$ is given by

$$\operatorname{div}(f) = 2(1:0) - (1:1) - (0:1).$$ △

**Example 16.8.** Let $X$ be the curve $V(x^3-y^3+y) \subset \mathbb{A}^2_k$. Then $x$, $y$ and $y/x^2$ define rational functions on $X$. Here $x$ and $y$ are regular, so they have nonnegative orders everywhere. Let us find the points $P \in X$ where $\operatorname{ord}_P(x) > 0$.

The function $x$ vanishes exactly at the points in $V(x, x^3-y^3+y) \subset \mathbb{A}^2$, i.e., the three points $(0,0)$, $(0,1)$, $(0,-1)$. The local ring at the origin $(0,0)$ is isomorphic to

$$\mathcal{O}_{X,(0,0)} = \big(k[x,y]/(x^3-y^3+y)\big)_{(x,y)}$$

In this ring we have $x^3 = y(y^2-1)$, and so $y = x^3(\text{unit})$. Hence $x$ is a local parameter. In particular, $\operatorname{ord}_{(0,0)}(x) = 1$ and $\operatorname{ord}_{(0,0)}(y) = 3$. Similar computations show that

$$\operatorname{div}(x) = (0,0) + (0,-1) + (0,1)$$

As for $y$, this can only have non-zero orders of vanishing at the points in $V(x^3-y^3, y) = V(x,y)$, i.e., at the origin $(0,0)$. We have already computed that $\operatorname{ord}_{(0,0)}(y) = 3$, so $\operatorname{div}(y) = 3(0,0)$. Putting everything together, we obtain

$$\operatorname{div}(y/x^2) = 3(0,0) - 2\left((0,0) + (0,-1) + (0,1)\right) = (0,0) - 2(0,-1) - 2(0,1).$$

△

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## 16.2 The class group

One of the fundamental invariants of a scheme (or of a ring) is the class group, along with its close relative, the Picard group. The class group can be interpreted as the group that captures the obstructions to a divisor being the divisor of a rational function.

**Definition 16.9.** The *class group of* $X$ is defined as the group of Weil divisors modulo the principal divisors, i.e.,

$$\mathrm{Cl}(X) = \mathrm{Div}(X)/\langle \mathrm{div}(f) \mid f \in K(X)^\times \rangle.$$

Two Weil divisors $D$ and $D'$ are said to be *linearly equivalent* (written $D \sim D'$) if they have the same image in $\mathrm{Cl}(X)$, or equivalently, that $D - D'$ is principal.

For an affine scheme $X = \operatorname{Spec} A$, the divisor of a rational function $f \in K(X)$ is equal to

$$\mathrm{div}(f) = \sum_{\mathrm{ht}\,\mathfrak{p}=1} \mathrm{ord}_{\mathfrak{p}}(f) V(\mathfrak{p}).$$

Hence $\mathrm{div}(f) = 0$ if and only if $\mathrm{ord}_{\mathfrak{p}}(f) = 0$ for all height 1 primes, which by Proposition 16.3 happens if and only if $f \in A^\times$. Hence the kernel of the map $\mathrm{div}\colon K(X)^\times \to \mathrm{Div}(\operatorname{Spec} A)$ equals $A^\times$, and the cokernel is by definition the class group $\mathrm{Cl}(\operatorname{Spec} A)$. Hence we have the exact sequence

$$0 \to A^\times \to K(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\operatorname{Spec} A) \to \mathrm{Cl}(\operatorname{Spec} A) \to 0. \tag{16.3}$$

**Example 16.10.** The class group of $\operatorname{Spec} \mathbb{Z}$ is trivial, as $\mathbb{Z}$ is a principal ideal domain, so every prime ideal is principal, not just the ones of height 1. (See also Example 16.5.) △

**Example 16.11.** The class group of $\operatorname{Spec} \mathbb{Z}[i]$ is also trivial. This follows by the analysis of Example 2.41, which showed that every prime ideal of $\mathbb{Z}[i]$ is principal. △

**Example 16.12.** Let $A$ be a discrete valuation ring and let $X = \operatorname{Spec} A$. In $A$, the only nonzero prime ideal is the maximal ideal $\mathfrak{m}$. Therefore, if $x \in X$ denotes the closed point, we have $\mathrm{Div}(X) = \mathbb{Z} \cdot x$. Any Weil divisor on $X$ is principal: if $t$ is a generator for $\mathfrak{m}$, then, then $\mathrm{div}(t^n) = n \cdot x$ for each $n \in \mathbb{Z}$. Hence $\mathrm{Cl}(X) = 0$. △

Both the terms 'divisor' and 'class group' originate from algebraic number theory and their origins can be traced back to Kummer's work on Fermat's Last Theorem. In this setting, the class groups are designed to measure how far the base ring is from being a unique factorization domain.

**Proposition 16.13.** Let $A$ be a normal Noetherian integral domain. Then the following are equivalent:

(i) $\mathrm{Cl}(\operatorname{Spec} A) = 0$.
(ii) Every height 1 prime ideal in $A$ is a principal ideal.
(iii) $A$ is a unique factorization domain.

*Proof* The equivalence of (ii) and (iii) is a fact from commutative algebra (see Proposition A.59). It remains to show the equivalence (i) ⇔ (ii).

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(ii) $\Rightarrow$ (i): Write $X = \operatorname{Spec} A$. If $Z$ is a prime divisor in $X$, then $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \subset A$, and as $Z$ has codimension 1, $\mathfrak{p}$ is of height 1. By assumption (ii), $\mathfrak{p} = (f)$ for an element $f \in A$, that is, $Z = \operatorname{div}(f)$. This shows that $\operatorname{Cl}(\operatorname{Spec} A) = 0$.

(i) $\Rightarrow$ (ii): Assume that $\operatorname{Cl}(\operatorname{Spec} A) = 0$. Let $\mathfrak{p}$ be a prime of height 1, and let $Z = V(\mathfrak{p}) \subset X$. By assumption, there is an $f \in K(X)^\times$ so that $\operatorname{div}(f) = Z$. We want to show that in fact $f \in A$ and that $\mathfrak{p} = (f)$. For the first statement: as $\operatorname{div}(f) = Z$, the valuations of $f$ are given by $\operatorname{ord}_\mathfrak{q}(f) = 0$ for $\mathfrak{q} \neq \mathfrak{p}$ and $\operatorname{ord}_\mathfrak{p}(f) = 1$, and so $f$ lies in $\mathcal{O}_X(X) = A$ by Proposition 16.3.

Secondly, to prove that $f$ generates $\mathfrak{p}$, consider any nonzero element $g \in \mathfrak{p}$. Then $\operatorname{ord}_\mathfrak{p}(g) \geqslant 1$ and $\operatorname{ord}_\mathfrak{q}(g) \geqslant 0$ for all $\mathfrak{q} \neq \mathfrak{p}$. It follows that $\operatorname{ord}_\mathfrak{q}(g/f) = \operatorname{ord}_\mathfrak{q}(g) - \operatorname{ord}_\mathfrak{q}(f) \geqslant 0$ for *every* prime ideal $\mathfrak{q} \in \operatorname{Spec} A$. Hence $g/f \in A_\mathfrak{q}$ for every prime $\mathfrak{q}$ of height 1, and hence $g/f \in A$, by Proposition 16.3. It follows that $g \in fA$, and so $\mathfrak{p} = (f)A$ is principal. □

**Example 16.14.** The class group of affine $n$-space $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ is trivial, as the polynomial ring is a unique factorization domain. △

**Theorem 16.15.** Let $k$ be a field. The class group of projective $n$-space is

$$\operatorname{Cl}(\mathbb{P}^n_k) = \mathbb{Z},$$

generated by the class of a hyperplane.

*Proof* Write $\mathbb{P}^n_k = \operatorname{Proj} R$ where $R = k[x_0, \ldots, x_n]$ and let $H = V_+(x_0)$ be the hyperplane defined by $x_0$.

First we show that $H$ generates $\operatorname{Cl}(\mathbb{P}^n_k)$. Let $Y \subset \mathbb{P}^n_k$ be any prime divisor. Then $Y = V_+(\mathfrak{a})$ for some homogeneous prime ideal of height 1 in $R$. As the polynomial ring is a UFD, we may write $\mathfrak{a} = (F)$ for some irreducible homogeneous polynomial $F$. Letting $d = \deg F$, the fraction $f = F/x_0^d$ defines a regular function on $U_0$, hence a rational function on $\mathbb{P}^n_k$. Note that $f$ can only have nonzero orders of vanishing along $Y$ and $H$. Since $F$ is irreducible, we have $\operatorname{div}(F/x_0^d) = Y - dH$. Therefore, $[Y] = dH$ in $\operatorname{Cl}(\mathbb{P}^n_k)$.

To conclude, we show that no nonzero multiple $[H]$ is zero $\operatorname{Cl}(\mathbb{P}^n_k)$. If $m[H] = 0$ for some $m > 0$, then there would exist an $f \in k(\mathbb{P}^n_k)$ such that $mH = \operatorname{div}(f)$. Note that $\operatorname{div}(f)$ is effective, meaning $\operatorname{ord}_Z(f) \geqslant 0$ for every prime divisor $Z$. By Proposition 16.3, $f$ is regular on $\mathbb{P}^n_k$, hence a constant, which implies that $m = 0$. □

The isomorphism $\operatorname{Cl}(X) \simeq \mathbb{Z}$ is better defined by the *degree map*. For a divisor $D = \sum n_i V_+(G_i)$, where $G_i \in k[x_0, \ldots, x_n]$ are irreducible homogeneous polynomials, we define

$$\deg D = \sum_i n_i \deg G_i.$$

As rational functions on $\mathbb{P}^n_k$ have the form $F/G$ where $F, G \in k[x_0, \ldots, x_n]$ of the same degree, this descends to a map $\operatorname{Cl}(\mathbb{P}^n_k) \to \mathbb{Z}$.

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## 16.3 The class group of an open set

If $U \subset X$ is an open set and $Z \subset X$ is a prime divisor, the intersection $Z \cap U$ is either empty (if $Z \subset X - U$) or a prime divisor on $U$. This allows us to define the restriction of a divisor $D = \sum n_Z Z$ to $U$ by the formula

$$D|_U = \sum_{Z \cap U \neq \varnothing} n_Z \cdot Z \cap U. \tag{16.4}$$

This defines a map of groups $\mathrm{Div}(X) \to \mathrm{Div}(U)$. Note that any divisor with support in $X - U$ is sent to 0 via this map.

If $f$ is a rational function on $X$, the restriction $f|_U$ is a rational function on $U$, and it holds that $\mathrm{ord}_{Z\cap U}(f|_Z) = \mathrm{ord}_Z(f)$ whenever $Z \cap U \neq \varnothing$ (the two valuation rings are the same). Consequently the divisor $\mathrm{div}(f)$ restricts to the divisor $\mathrm{div}(f|_U)$, and the restriction map sends principal divisors to principal divisors.

In this setting, it is natural to ask how the class group of $X$ and $U$ are related. The answer is given by the theorem below.

**Theorem 16.16.** Let $X$ be a normal integral Noetherian scheme. Let $W \subset X$ be a closed subscheme and let $U = X - W$. If $Z_1, \ldots, Z_r$ are the prime divisors corresponding to the codimension 1 components of $W$, there is an exact sequence

$$\bigoplus_{i=1}^{r} \mathbb{Z} Z_i \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0, \tag{16.5}$$

where the map $\mathrm{Cl}(X) \to \mathrm{Cl}(U)$ is defined by $[Z] \mapsto [Z \cap U]$.

*Proof* If $Z$ is a prime divisor on $U$, the closure in $X$ is a prime divisor in $X$, so the right-most map in (16.5) is surjective, and we just need to check exactness in the middle.

Suppose $Z$ is a prime divisor which is principal on $U$. Then $Z|_U = \mathrm{div}(f)$ for some $f \in K(U) = K(X)$. Now $D = \mathrm{div}(f)$ is a divisor on $X$ such that $D|_U = \mathrm{div}(f)|_U$. Hence $D - Z$ is a Weil divisor supported in $X - U$, and hence it must be a linear combination of the $Z_i$'s. Therefore $D - Z$ is in the image of the left-most map, and we are done. □

**Corollary 16.17.** If $Z \subset X$ is a closed subset of codimension at least 2, then

$$\mathrm{Cl}(X - Z) = \mathrm{Cl}(X)$$

**Example 16.18.** If $P$ denotes the origin in $\mathbb{A}^n_k$, then $\mathrm{Cl}(\mathbb{A}^n_k - P) = \mathrm{Cl}(\mathbb{A}^n_k) = 0$ for $n \geqslant 2$.

If $n = 1$, we have $\mathrm{Cl}(\mathbb{A}^1_k - P) = \mathrm{Cl}(\mathrm{Spec}\, k[x, x^{-1}]) = 0$ because $k[x, x^{-1}]$ is a UFD. △

**Example 16.19.** Consider the projective line $\mathbb{P}^1_k$ over a field $k$, and let $P$ be a $k$-point. We have the exact sequence

$$\mathbb{Z} \cdot [P] \longrightarrow \mathrm{Cl}(\mathbb{P}^1_k) \longrightarrow \mathrm{Cl}(\mathbb{A}^1_k) \longrightarrow 0.$$

We saw that $\mathrm{Cl}(\mathbb{A}^1_k) = 0$, so the map $\mathbb{Z} \to \mathrm{Cl}(\mathbb{P}^1_k)$ is surjective. It is also injective: If $[nP] = 0$ in $\mathrm{Cl}(\mathbb{P}^1_k)$ for some $n$, then $nP = \mathrm{div}(f)$ for some $f \in K(\mathbb{P}^1)$. We have

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$nP|_{\mathbb{A}^1_k} = 0$, so we must have $\operatorname{div}(f)|_{\mathbb{A}^1_k} = 0$. Therefore, $f$ must have neither zeros nor poles on $\mathbb{A}^1_k$, implying that $f$ is a constant, and hence $n = 0$.

The same proof works in higher dimensions, and gives a new proof of the formula $\operatorname{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$. △

## 16.4 Cartier divisors

If $f \in \mathcal{O}_X(X)$ is a nonzero regular function, then $\operatorname{div}(f)$ is an effective divisor with support along the zero set of $f$, and 'multiplicity' $\operatorname{ord}_Z(f)$ along each component. Generally, we expect that a codimension 1 closed subscheme can be defined by a single equation, at least locally. As we will see this intuition is often correct, at least when $X$ has mild singularities.

**Definition 16.20.** A divisor $D$ is called *locally principal* or a *Cartier divisor* if there is an open cover $\{U_i\}_{i\in I}$ of $X$ and rational functions $f_i \in K(X)^\times$ such that for each $i$,

$$D|_{U_i} = \operatorname{div}(f_i)$$

If a divisor $D$ is Cartier, then it is completely determined by the collection of rational functions $f_i$. This is sometimes a more useful description than the underlying prime divisors and their multiplicities, especially in local computations.

To formalize this, let us define a set of *Cartier data*, to be the following data:

(i) an open cover $\{U_i\}_{i\in I}$ of $X$,

(ii) elements $f_i \in K(X)$ satisfying $f_i f_j^{-1} \in \mathcal{O}_X(U_i \cap U_j)^\times$ for every $i, j \in I$.

By Proposition 16.3, the second condition is equivalent to

$$\operatorname{ord}_Z(f_i) = \operatorname{ord}_Z(f_j)$$

for any prime divisor $Z$ intersecting both $U_i$ and $U_j$. This ensures that the coefficient $n_Z$ of any prime divisor in $D$ is uniquely determined by $n_Z = \operatorname{ord}_Z(f_i)$ where $i$ is any index such that $Z \cap U_i \neq \varnothing$.

The Cartier data is however not unique, as $D$ may be locally principal both with respect to several open coverings and different choices of rational functions. The ambiguity is resolved if we consider two defining Cartier data $\{(U_i, f_i)\}_{i\in I}$ and $\{(V_j, g_j)\}_{j\in J}$ to be the same if $f_i g_j^{-1} \in \mathcal{O}_X(U_i \cap V_j)^\times$ for all $i$ and $j$.

The set of Cartier divisors forms a subgroup of $\operatorname{Div}(X)$. In terms of Cartier data, the identity element is represented by $(X, 1)$, that is, the trivial covering consisting of the open set $X$ and the rational function '1'. If $D$ and $E$ are represented by the data $(U_i, f_i)$ and $(V_j, g_j)$, then $D + E$ is represented by $(U_i \cap V_j, f_i g_j)$ and $-D$ is represented by $(U_i, f_i^{-1})$. Furthermore, $(U_i, f_i)$ represents a principal divisor if and only if it is equivalent to $(X, f)$ for some $f \in K(X)^\times$.

**Example 16.21.** Consider the projective $n$-space $\mathbb{P}^n_k$ over a field $k$. Write $\mathbb{P}^n_k = \operatorname{Proj} R$ where $R = k[x_0, \ldots, x_n]$. Any homogeneous polynomial of degree $d$, $F(x_0, \ldots, x_n) \in R_d$ defines a closed subscheme of $\mathbb{P}^n_k$ of codimension 1. The corresponding Weil divisor $D$ is Cartier. Concretely, we can write down the Cartier data with respect to the standard covering $U_i = D_+(x_i)$ of $\mathbb{P}^n_k$. Note that $F(x/x_i) = F(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i})$ defines a

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non-zero regular function on $U_i$, and the collection

$$(U_i, F(x/x_i))$$

forms a Cartier divisor $D$ on $X$. Indeed, on the overlap $U_i \cap U_j$, we have the relation

$$F(x/x_i) = (x_j/x_i)^d F(x/x_j) \tag{16.6}$$

and $x_j/x_i$ is a regular and invertible function on $U_i \cap U_j$. Two homogeneous polynomials $F, G$ of the same degree $d$ give linearly equivalent divisors, because the quotient $F(x)/G(x)$ is a global rational function on $\mathbb{P}^n_k$. the coefficient $n_Z$ of $Z$ in $D$ is determined by $n_Z = \operatorname{ord}_Z(f_i)$ where $i$ is any index such that $Z \cap U_i \neq \varnothing$. △

It is not true in general that every divisor is Cartier. We will see some basic counterexamples in Section 16.8. However, if $X$ is *locally factorial*, meaning that all the local rings $\mathcal{O}_{X,x}$ are UFD's, then the two notions are the same:

**Theorem 16.22.** Let $X$ be a Noetherian normal scheme such that every local ring $\mathcal{O}_{X,x}$ is a UFD. Then every Weil divisor is Cartier.

*Proof* 'Being Cartier' is a local condition, so we may assume $X = \operatorname{Spec} A$ is affine.

Let $Z = V(\mathfrak{p})$ be a prime divisor on $X$, where $\mathfrak{p}$ is a height 1 prime ideal. Note that $Z$ is automatically principal over the open set $U = X - Z$, because $Z|_U = 0$. If $x \in Z$, then $x$ corresponds to a prime ideal $\mathfrak{q}$ containing $\mathfrak{p}$. As $\mathfrak{p}$ has height 1 in $A$, then $\mathfrak{p}A_\mathfrak{q}$ remains of height 1 (there is a one-to-one correspondence between primes in $A$ lying in $\mathfrak{q}$ and primes in $A_\mathfrak{q}$). As $A_\mathfrak{q} = \mathcal{O}_{X,x}$ is assumed to be a UFD, $\mathfrak{p}A_\mathfrak{q}$ is a principal ideal, by Proposition A.59. Let $a, b \in A$ be elements such that $a/b \in A_\mathfrak{q}$ is a generator for $\mathfrak{p}A_\mathfrak{q}$. As $b \notin \mathfrak{q}$, we see that in fact $\mathfrak{p} = A_\mathfrak{q} = (a)A_\mathfrak{q}$.

If $\mathfrak{p} = (f_1, \dots, f_r)$, then $f_i A_\mathfrak{q} \in (a)A_\mathfrak{q}$ for each $i = 1, \dots, r$, so we may write $f_i/1 = a_i/b_i \cdot a$. Setting $b = b_1 \cdots b_r$, we have $\mathfrak{p}A_b = (a)A_b$. Therefore, $U = D(b) \subset X$ is an open neighbourhood of $x$ such that $Y \cap U$ is defined by a principal ideal in $\mathcal{O}_X(U)$. In other words, $Y|_U = V(\mathfrak{p})|_U = \operatorname{div}(a)$ is principal over $U$. □

**Corollary 16.23.** Let $X$ be a Noetherian, nonsingular scheme. Then every Weil divisor is Cartier.

*Proof* If $X$ is nonsingular, then each local ring $\mathcal{O}_{X,x}$ is a regular local ring, hence a UFD by the Auslander–Buchsbaum theorem (see Theorem 10.5). □

Cartier divisors have better formal properties than general Weil divisors because of their close links to invertible sheaves. We will explore this in Section 16.5.

## 16.5 The sheaf associated to a Weil divisor

Let $X$ be an integral, Noetherian, normal scheme. For a Weil divisor $D = \sum n_Z Z$, we will construct a sheaf, denoted by $\mathcal{O}_X(D)$, whose sections are the rational functions with 'poles at worst along $D$'. There are several ways to express this condition. The simplest way is to require that a rational function $f \in K(X)$ satisfies $\operatorname{ord}_Z(f) \geqslant -n_Z$ for all $Z$, meaning that

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the order of the pole of $f$ along $Z$ is at most $n_Z$. Equivalently, we can formulate it by saying that $\operatorname{div}(f) + D$ is an effective Weil divisor; in other words, $\operatorname{div}(f) + D \geqslant 0$. Informally, this means that the 'pole part' of $\operatorname{div}(f)$ is canceled out by adding $D$.

**Definition 16.24.** Let $X$ be a Noetherian integral normal scheme and let $D$ be a Weil divisor on $X$. We define the sheaf $\mathcal{O}_X(D)$ by letting

$$\mathcal{O}_X(D)(U) = \{\, f \in K(X) \mid \operatorname{ord}_Z(f) \geqslant -n_Z \text{ for all } Z \text{ with } Z \cap U \neq \varnothing \,\}$$

for each open subset $U \subset X$.

The rational function $0 \in K(X)$ has by convention infinite order of vanishing at every prime divisor, so every $\mathcal{O}_X(U)$ contains 0.

The condition in the bracket is less restrictive when applied to a smaller subset $U' \subset U$, so there is an inclusion $\mathcal{O}_X(D)(U) \subset \mathcal{O}_X(D)(U')$. Hence $\mathcal{O}_X(D)$ becomes a presheaf with inclusion maps as the restriction maps. It is not hard to see that the sheaf axioms are satisfied, so $\mathcal{O}_X(D)$ is a sheaf. As such, it is a subsheaf of the constant sheaf $\mathbf{K}(X)$ on the function field $K(X)$. It is moreover an $\mathcal{O}_X$-module, because if $a \in \mathcal{O}_X(U)$ is a regular function $U$, we have $\operatorname{ord}_Y(af) = \operatorname{ord}_Y(a) + \operatorname{ord}_Y(f) \geqslant -n_Z$ for all $Z$ and all $f \in \mathcal{O}_X(D)(U)$, so $\mathcal{O}_X(D)$ is stable under multiplication by elements of $\mathcal{O}_X$.

**Example 16.25.** If $D = 0$, then $\mathcal{O}_X(D) = \mathcal{O}_X$. Indeed, if $U$ is any open in $X$, then $\Gamma(U, \mathcal{O}_X(D))$ consists of the rational functions $g \in K(X)$ such that $\operatorname{ord}_Z(g) \geqslant 0$ for every prime divisor $Z$. Any such must be regular, by Proposition 16.3, so $\Gamma(U, \mathcal{O}_X(D)) = \Gamma(U, \mathcal{O}_X)$. △

**Example 16.26.** If $Z$ is a prime divisor, then

$$\mathcal{O}_X(-Z) \simeq \mathcal{I}_Z$$

is the ideal sheaf defining $Z$. Indeed, over an open set $U$, $\Gamma(U, \mathcal{O}_X(D)(U))$ consists of the rational functions $g \in K(X)$ so that $\operatorname{ord}_Z(g) \geqslant 1$ and $\operatorname{ord}_{Z'}(g) \geqslant 0$ for every prime divisor $Z' \neq Z$. In other words, $g \in \mathcal{O}_X(U)$ is regular, and vanishes along $Z$. △

**Example 16.27.** Let $X$ be the projective line $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$ over $k$ and consider the divisor $D = V(x_1) = (1:0)$. We will compute $\Gamma(X, \mathcal{O}_X(D))$ using the standard covering of $\mathbb{P}^1_k$ by the open sets $U_0 = \operatorname{Spec} k[x_1/x_0] = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[x_0/x_1] = \operatorname{Spec} k[s]$ (so $s = t^{-1}$ on $U_0 \cap U_1$).

Note that the point $(1:0)$ does not lie in $U_1$. This means that a rational function $f \in K(X)$ such that $\operatorname{div}(f) + D \geqslant 0$ if and only if $f$ is regular on $U_1$. This means that

$$\Gamma(U_1, \mathcal{O}_X(D)) = k[s].$$

Over the open set $U_0$, we are looking at elements $f \in k(t)$ a pole of order at most 1 at $t = 0$, and no other poles. This implies that

$$\Gamma(U_0, \mathcal{O}_X(D)) = \{\, \alpha t^{-1} + p(t) \mid \alpha \in k, p(t) \in k[t] \,\}.$$

A section of $\Gamma(X, \mathcal{O}_X(D))$ corresponds to a pair $(f(t), g(s))$ with $f(t) = \alpha t^{-1} + p(t)$ in

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$\Gamma(U_0, \mathcal{O}_X(D))$ and $g(s) \in k[s] = \Gamma(U_1, \mathcal{O}_X(D))$ which agree over the overlap, meaning

$$f(t) = \alpha t^{-1} + p(t) = \alpha s + p(s^{-1}) = g(s).$$

For equality to hold in $k[t, t^{-1}] = k[s, s^{-1}]$, it is clear that $p$ must be a constant, so we obtain

$$\Gamma(X, \mathcal{O}_X(D)) = k \oplus k\, t^{-1}.$$

In fact, we will see in a bit that $\mathcal{O}_X(D) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. △

**Proposition 16.28.** Let $X$ be Noetherian integral normal scheme and let $D$ be a Weil divisor. Then $\mathcal{O}_X(D)$ is a coherent sheaf.

*Proof* We may assume that $X = \operatorname{Spec} A$ is affine. We first show that $\mathcal{O}_X(D)$ is quasi-coherent by showing that for any $g \in A$, there is an equality of subgroups of $K(X)$:

$$\Gamma(X, \mathcal{O}_X(D))_g = \Gamma(D(g), \mathcal{O}_X(D)).$$

Given this, it follows that $\mathcal{O}_X(D)$ is the tilde of the $A$-module $M = \Gamma(X, \mathcal{O}_X(D))$.

The inclusion '$\subset$' is clear: if $h \in \Gamma(X, \mathcal{O}_X(D))$, then $\operatorname{div}(h) + D \geqslant 0$ on $X$, and since $g$ is invertible over $D(g)$, $\operatorname{ord}_Z(h/g^m) = \operatorname{ord}_Z(h)$ for any prime divisor $Z$ which intersects $D(g)$.

For '$\supset$', take any $f \in \Gamma(D(g), \mathcal{O}_X(D))$ such that $\operatorname{div}(f) + D \geqslant 0$ on $D(g)$. Then the negative part of the divisor $\operatorname{div}(f) + D$ must be supported in the closed set $V(g) \subset \operatorname{Spec} A$. However, as $V(g)$ has only finitely many components, there exists an $m > 0$ such that

$$\operatorname{div}(g^m f) + D \geqslant 0$$

over all of $X$. Then $g^m f \in \Gamma(X, \mathcal{O}_X(D))$ and $f$ is the image of $(g^m f)/g^m$. This shows that $\mathcal{O}_X(D)$ is quasi-coherent.

To prove that $\mathcal{O}_X(D)$ is coherent, it suffices to show that $\Gamma(X, \mathcal{O}_X(D))$ is a finitely generated $A$-module. Write $D = \sum n_i Z_i$, and let $g \in A$ be a nonzero element which vanishes on all the $Z_i$, so that $\operatorname{ord}_{Z_i}(g) > 0$. Then we may choose an $m$ so that $\operatorname{ord}_{Z_i}(g^m)$ is greater than $-n_i$ for all $i$. This implies that for any $f \in \Gamma(X, \mathcal{O}_X(D))$ we have $\operatorname{ord}_{Z_i}(g^m f) = \operatorname{ord}_{Z_i}(g^m) + \operatorname{div}(f) \geqslant 0$, and hence $g^m f$ is contained in $\Gamma(X, \mathcal{O}_X) = A$. Hence $g^m \Gamma(X, \mathcal{O}_X(D))$ is an ideal of $A$. As we assume that $A$ is Noetherian, this ideal is finitely generated, and we deduce that $\Gamma(X, \mathcal{O}_X(D))$ is finitely generated as well. □

**Lemma 16.29.** Let $D$ and $E$ be two Weil divisors on $X$. Then $\mathcal{O}_X(D) = \mathcal{O}_X(E)$ as subsheaves of $\mathbf{K}(X)$ if and only if $D = E$.

*Proof* Write $D = \sum m_Z Z$ and $E = \sum n_Z Z$. If $D \neq E$, there is a prime divisor $Z$ such that $m_Z \neq n_Z$. We may assume that $m_Z > n_Z$.

Choose an affine open $U = \operatorname{Spec} A \subset X$ containing the generic point $\zeta$ of $Z$. Assuming that $\mathcal{O}_X(D) = \mathcal{O}_X(E)$, we get that $\mathcal{O}_X(D)_\zeta = \mathcal{O}_X(E)_\zeta$. However,

$$\mathcal{O}_X(D)_\zeta = \{\, f \in K(X) \mid \operatorname{ord}_{V(\mathfrak{p})}(f) \geqslant -m_Z \,\} \tag{16.7}$$

$$\mathcal{O}_X(E)_\zeta = \{\, f \in K(X) \mid \operatorname{ord}_{V(\mathfrak{p})}(f) \geqslant -n_Z \,\}. \tag{16.8}$$

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Let $t \in A_{\mathfrak{p}}$ be a generator for the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$. If $m_Z > n_Z$, then $t^{-m_Z}$ lies in (16.7), but not in (16.8). This is a contradiction, so $\mathcal{O}_X(D) \neq \mathcal{O}_X(E)$. □

The sheaf $\mathcal{O}_X(D)$ is best behaved when $D$ is a Cartier divisor. In that case, $\mathcal{O}_X(D)$ is an invertible sheaf. This is essentially a consequence of the following lemma.

**Lemma 16.30.** For a nonzero rational function $h \in K(X)$, we have

$$\mathcal{O}_X(\operatorname{div}(h)) = \frac{1}{h} \cdot \mathcal{O}_X \tag{16.9}$$

Moreover, a Weil divisor $D$ is principal if and only if $\mathcal{O}_X(D) \simeq \mathcal{O}_X$.

*Proof* Let $U \subset X$ be an open set. We claim that

$$\Gamma(U, \mathcal{O}_X(\operatorname{div} h)) = \frac{1}{h} \cdot \Gamma(U, \mathcal{O}_X) \tag{16.10}$$

as subgroups of $K(X)$. Note that a rational function $f \in K(X)$ belongs to $\Gamma(U, \mathcal{O}_X(D))$ if and only if $\operatorname{div}(f) + \operatorname{div}(h) \geqslant 0$, or equivalently, $\operatorname{div}(fh) \geqslant 0$. In other words, $r = fh$ is a rational function which has non-negative order of vanishing at every prime divisor $Z$ which intersects $U$. Therefore $r \in \mathcal{O}_V(U)$ is regular, by Proposition 16.3, and $g = r/f$ belongs to the right-hand side. The opposite containment is clear.

For the last statement: If $D = \operatorname{div}(h)$ is principal, then (16.10) shows that multiplcation by $h$ induces an isomorphism of $\mathcal{O}_X(U)$-modules

$$\Gamma(U, \mathcal{O}_X(D)) \longrightarrow \Gamma(U, \mathcal{O}_X)$$

This is compatible with the restriction maps, and so $\mathcal{O}_X(D) \simeq \mathcal{O}_X$.

Conversely, suppose that $\mathcal{O}_X(D) \simeq \mathcal{O}_X$. As $\mathcal{O}_X(D)$ is an $\mathcal{O}_X$-submodule of $\mathbf{K}(X)$ which is free of rank 1, there is a nonzero $g \in K(X)$ so that $\mathcal{O}_X(D) = g \cdot \mathcal{O}_X$. Define $h = 1/g$. Then

$$\mathcal{O}_X(\operatorname{div}(h)) = \frac{1}{h}\mathcal{O}_X = g \cdot \mathcal{O}_X = \mathcal{O}_X(D)$$

as subsheaves of $\mathbf{K}(X)$, and hence $D = \operatorname{div}(h)$ by Lemma 16.29. □

**Proposition 16.31.** Let $X$ be a Noetherian integral normal scheme and let $D$ be a Weil divisor on $X$.

(i) $D$ is Cartier if and only if $\mathcal{O}_X(D)$ is an invertible sheaf.

(ii) If $D$ is given by Cartier data $\{(U_i, f_i)\}$, then

$$\mathcal{O}_X(D)|_{U_i} = \frac{1}{f_i} \cdot \mathcal{O}_{U_i} \qquad \text{for each } i \in I$$

*Proof* (i): First of all, if $\mathcal{O}_X(D)$ is locally free, then it must have rank 1. This is because over an open set $V \subset X - \operatorname{Supp}(D)$, the group $\mathcal{O}_X(D)(V)$ consists of the rational functions such that $\operatorname{ord}_Z(f) \geqslant 0$, for every prime divisor $Z$ that intersects $V$, and hence $f \in \mathcal{O}_X(V)$. This means that $\mathcal{O}_X(D)$ is isomorphic to $\mathcal{O}_X$ over an open set, so it has rank 1.

Suppose first that $D$ is Cartier, say given by $D|_{U_i} = \operatorname{div}(f_i)$ for rational functions $f_i$

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and an open cover $\{U_i\}$. Over each $U_i$, Lemma 16.30 shows that $\mathcal{O}_X(D)|_{U_i} \simeq \mathcal{O}_X|_{U_i}$, so $\mathcal{O}_X(D)$ is invertible.

Conversely, if $\mathcal{O}_X(D)$ is an invertible subsheaf of $\mathbf{K}(X)$, we can define $f_i \in K(X)$ by choosing local generators so that $\mathcal{O}_X(D)(U_i) = \frac{1}{f_i}\mathcal{O}_X(U_i) \subset K(X)$. Then it follows that $D|_{U_i} = \operatorname{div}(f_i)$ by Lemma 16.30.

The item (ii) follows from Lemma 16.30. □

Two different Cartier data $(U_i, f_i)$ and $(V_j, g_j)$ for the same divisor $D$ give rise to the *same* invertible sheaf. This is because over $U_i \cap V_j$, the rational functions $1/f_i$ and $1/g_j$ are related by a unit in $\mathcal{O}_X(U_i \cap V_j)^\times$, and so they generate the same $\mathcal{O}_X$-submodule of $\mathbf{K}(X)$.

Note that the units $g_{ij} = f_i f_j^{-1}$ automatically satisfy the cocycle condition, because $f_k f_j^{-1} \cdot f_j f_i^{-1} = f_k f_i^{-1}$. In the notation of Section 14.1, $\mathcal{O}_X(D)$ is the invertible sheaf determined by the isomorphisms $\tau_{ij} \colon \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ given by multiplication by $g_{ij}$.

**Proposition 16.32.** Let $X$ be a Noetherian integral normal scheme and let $D$ and $E$ be two Cartier divisors. Then:

(i) $\mathcal{O}_X(D + E) \simeq \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E)$

(ii) $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ if and only if $D$ and $E$ are linearly equivalent.

*Proof* We may pick a common affine covering $U_i$ so that both $D$ and $E$ are both represented by data $(U_i, f_i)$, $(U_i, g_i)$. Then $D + E$ is determined by the Cartier data $(U_i, f_i g_i)$. Locally, over $U_i$ the sheaf $\mathcal{O}_X(D + E)$ is defined as the subsheaf of $\mathbf{K}(X)$ given by $(f_i g_i)^{-1}\mathcal{O}_{U_i} = f_i^{-1} g_i^{-1}\mathcal{O}_{U_i}$. As $U_i$ is affine, the tensor product is locally given by $f_i^{-1}\mathcal{O}_{U_i} \otimes g_i{}^{-1}\mathcal{O}_{U_i}$, which is clearly isomorphic to $f_i^{-1} g_i{}^{-1}\mathcal{O}_{U_i}$ via the map $a f_i^{-1} \otimes b g_i{}^{-1} \mapsto ab f_i{}^{-1} g_i{}^{-1}$.

For the second claim, it suffices (i) to show that $\mathcal{O}_X(D) \simeq \mathcal{O}_X$ if and only if $D$ is a principal Cartier divisor. But this is a consequence of Lemma 16.30.

□

**Example 16.33.** Consider again the example of projective space $\mathbb{P}^n_k$. In the notation of Example 16.21, if $F$ is a homogeneous polynomial of degree $d$ on $\mathbb{P}^n_k$, the sheaf $\mathcal{O}_{\mathbb{P}^n_k}(D)$ determined by $(D_+(x_i), F(x/x_i))$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$. This follows from (16.6), which shows that $\mathcal{O}_X(D)$ has exactly the same gluing functions as $\mathcal{O}_{\mathbb{P}^n_k}(d)$. In particular, this shows that two divisors of different degrees are not linearly equivalent. This, together with the fact that any Weil divisor is Cartier, gives a new proof of the equality $\operatorname{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$. △

## 16.6 The divisor associated to a section of an invertible sheaf

A prototype example of a Cartier divisor is the divisor associated to a section of an invertible sheaf. The construction pararells the definition of a principal divisor.

One of the benefits of using zero schemes of sections of invertible sheaves is that they can be defined on any scheme; we do not need to assume that $X$ is normal, integral or Noetherian to be able to work with them. (See Exercise 16.10.14.)

Let $L$ be an invertible sheaf on $X$ and let $s$ be a nonzero global section of $L$. Let $U_i$ be an open cover of $X$ such that $L$ is trivial over $U_i$. Let $s_i \in \Gamma(U_i, L)$ be the local generators of

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$L$. Over each $U_i$, we may express

$$s|_{U_i} = f_i s_i$$

for some $f_i \in \mathcal{O}_X(U_i)$. Equialently, $f_i$ is the element that maps to $s|_{U_i}$ via the local isomorphisms $\phi_i \colon \mathcal{O}_X|_{U_i} \to L|_{U_i}$. As $X$ is integral, we may regard $f_i$ as a nonzero element of the function field $K(X)$. Then the data $(U_i, f_i)$ defines a Cartier divisor on $X$, which we denote by $\operatorname{div}(s)$.

Explicitly, $\operatorname{div}(s)$ has the following description. For a prime divisor $Z \subset X$ define $\operatorname{ord}_Z(s) = \operatorname{ord}_Z(f_i)$, where $i$ is any index such that $U_i$ intersects $Z$. Then

$$\operatorname{div}(s) = \sum_Z \operatorname{ord}_Z(s) \cdot Z.$$

The divisor $\operatorname{div}(s)$ is independent of the choice of local isomorphisms $\phi_i$. If $\psi_\alpha \colon \mathcal{O}_{V_\alpha} \to L|_{V_\alpha}$ is another set of isomorphisms, with local generators $t_\alpha$, then over the overlaps $U \subset U_i \cap V_\alpha$, the restrictions $s|_U = f_i s_i|_U = g_\alpha t_\alpha|_U$ and $f_i$ and $g_\alpha$ are equal up to multiplication by an invertible element in $\mathcal{O}_X(U \cap U_i \cap V_\alpha)$, and so the orders of vanishing along prime divisors are the same.

Geometrically, $D = \operatorname{div}(s)$ is an effective divisor on $X$ which is supported on the zero scheme $Y = V(s)$ of $s$, with 'multiplicity' $\operatorname{ord}_Z(s)$ along each component. Note that $\operatorname{ord}_Z(s)$ is nonnegative for each $Z$, because the $f_i$ are regular functions.

More precisely, the ideal sheaf of $D$ is defined as the image of the map

$$s^\vee \colon L^\vee \longrightarrow \mathcal{O}_X$$

given by multiplication by $s$. Over each $U_i$, the map $s^\vee|_{U_i}$ can be identified with the map $\mathcal{O}_{U_i} \to \mathcal{O}_{U_i}$ which sends 1 to $f_i$. Hence $\mathcal{I}$ is locally generated by the local equations $f_i \in \mathcal{O}_X(U_i)$. Hence

$$\mathcal{I} = \mathcal{O}_X(-D) \simeq L^\vee$$

, and the ideal sheaf sequence takes the form

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \iota_* \mathcal{O}_D \longrightarrow 0$$

where $\iota \colon D \to X$ is the inclusion.

**Example 16.34.** On $X = \mathbb{P}^1_k$, the polynomial $s = x_0 x_1^3 - 2x_0^2 x_1^2$ defines a global section $s$ of the invertible sheaf $L = \mathcal{O}_{\mathbb{P}^1}(4)$. Over $U_0 = \operatorname{Spec} k[x_1/x_0]$, we have $L|_{U_0} = \mathcal{O}_{U_0} x_0^4$ and so $s = f_0 x_0^4$ where $f_0 = (x_1/x_0)^3 - 2(x_1/x_0)^2$. This rational function has divisor

$$\operatorname{div}(s)|_{U_0} = \operatorname{div}(f_0) = 2(1:0) + (1:2)$$

Similarly, over $U_1$, we have $s = f_1 \cdot x_1^4$ where $f_1 = (x_0/x_1) - 2(x_0/x_1)^2$. This gives

$$\operatorname{div}(s)|_{U_1} = \operatorname{div}(f_0) = (0:1) + (1:2).$$

Hence

$$\operatorname{div}(s) = 2(1:0) + (0:1) + (1:2).$$

△

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**Proposition 16.35.** Let $L$ be an invertible sheaf, $s \in \Gamma(X, L)$ a global section. Then there is an isomorphism

$$\mathcal{O}_X(\operatorname{div}(s)) \simeq L. \tag{16.11}$$

Two sections $s, t$ give rise to the same divisor if and only if $t = \lambda \cdot s$ for some unit $\lambda \in \mathcal{O}_X(X)^\times$.

*Proof* Let $\{U_i\}$ be a trivializing cover of $L$ and let $s_i$ denote local generators for $L$. Let $g_{ij} \in \mathcal{O}_X(U_i \cap U_j)^\times$ be the transition functions, satsifying $s_i = g_{ij}s_j$ on the overlaps. We may express $s|_{U_i} = f_i s_i$ where $f_i \in K(X)$. Over $U_i \cap U_j$ we have

$$f_j s_j = f_i s_i = f_i g_{ij} s_j$$

which shows that $g_{ij} = f_j/f_i$.

Now consider the sheaf $\mathcal{O}_X(\operatorname{div}(s))$, which is locally given by $\frac{1}{f_i}\mathcal{O}_{U_i}$. Passing from $\frac{1}{f_j}\mathcal{O}_X|_{U_i\cap U_j}$ to $\frac{1}{f_i}\mathcal{O}_X|_{U_i\cap U_j}$ corresponds to multiplying by $f_j/f_i$. This means tat $L$ and $\mathcal{O}_X(\operatorname{div}(s))$ are defined by the same transition functions, and so they are isomorphic.

For the second statement, write $t|_{U_i} = h_i s_i$ where $h_i \in K(X)$. Since $\operatorname{div}(s) = \operatorname{div}(t)$, we see that the rational function $\lambda_i = h_i/f_i$ has no zeroes nor poles in $U_i$ and hence $\lambda_i \in \mathcal{O}_X(U_i)^\times$. Over $U_i \cap U_j$, we have $f_i = g_{ij}f_j$ and $h_i = g_{ij}h_j$, and so

$$\lambda_i = \frac{h_i}{f_i} = \frac{g_{ij}h_j}{g_{ij}f_j} = \frac{h_j}{f_j} = \lambda_j$$

Hence the $\lambda_i$ glue to a global unit $\lambda \in \mathcal{O}_X(X)^\times$, satisfying $t = \lambda s$. □

The above constuction can in fact be carried out for a section $s$ of $L$ defined over any nonempty subset $V \subset X$. We call such a section a *rational section*. If $s \in L(V)$, the trivializations of $L$ still give rational functions $f_i$ (working over the open sets $U_i \cap V$) and we have well-defined orders of vanishing $\operatorname{ord}_Z(s)$ for any prime divisor $Z \subset X$.

Unlike the previous construction, when $s$ was a global section, the divisor $\operatorname{div}(s)$ of a rational section may no longer be effective. Here is a typical example:

**Example 16.36.** Continuing the example of $X = \mathbb{P}^1_k$, consider the quotient $s = \frac{x_0^3}{x_1}$ which defines a section of $L = \mathcal{O}_{\mathbb{P}^1_k}(2)$ over $D_+(x_1)$, hence a rational section on $X$. Let us compute the divisor associated to $s$: Let $t = \frac{x_0}{x_1}$ be the coordinate on $U = D_+(x_1) = \operatorname{Spec} k[t]$.

$$\mathcal{O}_X(2)(U) = k[x_0/x_1]x_1^2 = k[t]x_1^2.$$

So the rational function $f = \phi(s)$ is given by $\frac{x_0^3}{x_1^3} = t^3$ which has non-zero order of vanishing only at the point $t = 0 \in U$, where we have $\operatorname{ord}_t(f) = 3$. To compute $\operatorname{div}(s)$, we must also consider the point outside $D_+(x_1)$. On $U = D_+(x_0)$, we use the coordinate $u = \frac{x_1}{x_0}$, and we have

$$\mathcal{O}_X(2)(U) = k[u]x_0^2.$$

So the rational function $\phi(s)$ is given by $f = \frac{x_0}{x_1} = u^{-1}$. This has order of vanishing $\operatorname{ord}_t(f) = -1$ at $t = 0$ (and $\operatorname{ord}_Z(f) = 0$ at all other points). Hence we obtain

$$\operatorname{div}(s) = 3(0:1) - (1:0).$$

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**Proposition 16.37.** Let $X$ be a Noetherian integral normal scheme.
(i) For any invertible sheaf $L$ and any nonzero section $s$ of $L$, the divisor of $s$ is a Cartier divisor.
(ii) Every Cartier divisor $D$ can be realized as $\mathrm{div}(s)$ for some rational section $s$ of an invertible sheaf $L$ (specifically $\mathcal{O}_X(D)$)

*Proof* The divisor $\mathrm{div}(s)$ is Cartier, by definition. For the second statement, if $D$ is a Cartier divisor, given by the data $(U_i, f_i)$, then the sheaf $L = \mathcal{O}_X(D)$ is invertible. Consider the open set $V = X - \mathrm{Supp}(D)$. Then the element $1 \in K(X)$ belongs to $\Gamma(V, \mathcal{O}_X(D))$ and defines a distinguished rational section $s$ of $\mathcal{O}_X(D)$. It follows from the definitions that $\mathrm{div}(s) = D$: over $U_i$, the local trivialization $\mathcal{O}_{U_i} \to \mathcal{O}_X(D)|_{U_i}$ is given by multiplication by $1/f_i$, so the section $1 \in \Gamma(V, \mathcal{O}_X(D))$ to the rational function $f_i$, and hence the associated Cartier divisor is exactly $D$. □

**Corollary 16.38.** Let $X$ be a Noetherian integral normal scheme. Then the map $D \mapsto \mathcal{O}_X(D)$ induces an isomorphism between $\mathrm{Pic}(X)$ and the subgroup of $\mathrm{Cl}(X)$ generated by Cartier divisors.

*Proof* By the item (i) and (ii) in Proposition 16.32, the assignment $D \mapsto \mathcal{O}_X(D)$ is additive, and has the subgroup of principal divisors as its kernel. Also, by Proposition 16.37, any invertible sheaf $L$ is isomorphic to one of the form $\mathcal{O}_X(\mathrm{div}(s))$. □

**Corollary 16.39.** On a nonsingular variety $X$, then every Weil divisor is Cartier, and there are natural bijections between
(i) Weil divisors (up to linear equivalence)
(ii) Cartier divisors (up to linear equivalence)
(iii) Invertible sheaves (up to isomorphism)

From our previous computation of $\mathrm{Cl}(\mathbb{A}^n_k)$ and $\mathrm{Cl}(\mathbb{P}^n_k)$, we get the following theorem:

**Theorem 16.40.** Let $k$ be a field. Then
(i) $\mathrm{Pic}(\mathbb{A}^n_k) = \mathrm{Cl}(\mathbb{A}^n_k) = 0$.
(ii) $\mathrm{Pic}(\mathbb{P}^n_k) = \mathrm{Cl}(\mathbb{P}^n_k) = \mathbb{Z}$.

In particular, any invertible sheaf on $\mathbb{P}^n_k$ is isomorphic to some $\mathcal{O}_{\mathbb{P}^n_k}(m)$.

## 16.7 Pullbacks of divisors

Given a morphism $\phi\colon X \to Y$ and a divisor $D$ on $Y$, we can ask whether we can define a Weil divisor on $X$ supported on $\phi^{-1}D$. In general, this is not possible. Consider for instance, the case where $\phi$ is the inclusion of a closed subscheme, say $Y = \mathbb{P}^2_k$ and $f\colon X \to Y$ is the inclusion of a line $X = V(x_0)$. Then of course $D = V(x_0)$ defines a Weil divisor on $Y$, but there is no reasonable definition of $\phi^{-1}D$ that defines a codimension 1 subscheme of $X$.

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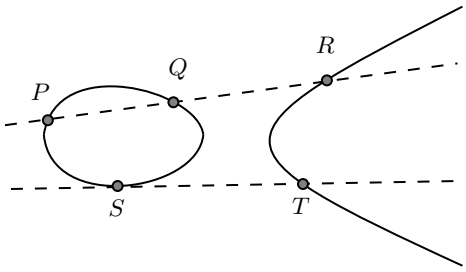


**Figure 16.1** $P + Q + R \sim 2S + T$

There is a situation where we can always define the pullback of a divisor $D$. This is when $\phi\colon X \to Y$ is a dominant morphism and $D$ is a Cartier divisor. In that case, there is a covering $U_i$ such that $D|_{U_i}$ is given by $\operatorname{div}(f_i)$ over $U_i$. The fact that $f$ is dominant, means that there is an induced map on function fields $\phi^\sharp\colon K(Y) \to K(X)$. We can therefore define a divisor $\phi^*D$ by

$$\phi^*D = \sum_{Z \subset X} \operatorname{ord}_Z(\phi^\sharp f_i)Z$$

where $Z$ runs over all the prime divisors in $X$.

As before, this is well-defined, as the rational functions $\phi^\sharp(f_i)$ are related by invertible elements in $\mathcal{O}_X(\phi^{-1}U_i \cap \phi^{-1}U_j)$.

On the level of invertible sheaves, we have

$$\phi^*\mathcal{O}_Y(D) = \mathcal{O}_X(\phi^*D).$$

Note that there is always a pullback map for invertible sheaves, even if $f$ is not dominant. Furthermore, if $s \in \Gamma(Y, L)$ is a global section, there is also a well-defined pullback $f^*s \in \Gamma(X, f^*L)$. However, as $f^*s$ may be zero section, it may not define a divisor $\operatorname{div}(s)$.

**Example 16.41.** Consider the plane curve

$$X = V(x_0^3 - x_0x_1^2 + x_1x_2^2) \subset \mathbb{P}^2_k.$$

Then $\mathcal{O}_{\mathbb{P}^2_k}(1)$ restricts to an invertible sheaf on $X$ and the sections $x_0, x_1, x_2$ restricts to nonzero sections. The corresponding divisors are computed as follows:

$$\begin{aligned}
\operatorname{div}(x_0) &= (0:0:1) + 2(0:1:0)\\
\operatorname{div}(x_1) &= 3(0:0:1)\\
\operatorname{div}(x_2) &= (0:1:0) + (1:1:0) + (1:-1:0)
\end{aligned}$$

From these, we see that $2(0:0:1) \sim 2(0:1:0)$, as the their difference is the principal divisor $\operatorname{div}(x_1/x_0)$. It actually turns out that $(0:0:1) \not\sim (0:1:0)$ in the class group of $X$, that is, $(0:0:1) - (0:1:0)$ is not the divisor of a rational function on $X$. This is in fact not so easy to prove (it will be a consequence of Proposition 20.13).

On the other hand, the section $s = x_0^3 - x_0x_1^2 + x_1x_2^2$ of $\mathcal{O}_{\mathbb{P}^2_k}(3)$ does not define a divisor on $X$,, as it restricts to the zero section in $\mathcal{O}_X(3)$.

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## 16.8 Extended examples

### *The circle*

**Proposition 16.42.** Let $X = \operatorname{Spec} A$, where $A = \mathbb{R}[x,y]/(x^2+y^2-1)$. Then

$$\operatorname{Cl}(X) \simeq \mathbb{Z}/2.$$

*Proof* Consider the point $P = (0,1)$ corresponding to the maximal ideal $\mathfrak{m} = (x, y-1)$. The first step will be to show that the complement $X - P$ has trivial class group. Note that $X - P = D(y-1)$ is the spectrum of the ring

$$A_{y-1} = \big(\mathbb{R}[x,y]/(x^2+y^2-1)\big)_{y-1}.$$

By the formulas for stereographic projection from $P$, there is an isomorphism of rings

$$A_{y-1} \simeq \mathbb{R}[t]_{1+t^2}, \qquad (x,y) \mapsto \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$$

with inverse sending $t$ to $x/(y-1)$. As $\mathbb{R}[t]_{1+t^2}$ is a localization of a UFD, we have $\operatorname{Cl}(X-P) = 0$.

The exact sequence (16.5) applied to the open set $X - P$ shows that $P$ generates $\operatorname{Cl}(X)$. To determine the order of $P$, note that $\operatorname{div}(y-1) = 2P$, because $x$ is a local parameter at $P$ and $y - 1 = (y+1)^{-1}x^2$ in $\mathcal{O}_{X,P}$. Hence $2P = 0$ in $\operatorname{Cl}(X)$.

To see that $P$ is nonzero in $\operatorname{Cl}(X)$, it suffices to show that $\mathcal{O}_X(P) \neq \mathcal{O}_X$. We do this by computing the global sections. Consider the covering of $X$ consisting of the two open sets $U_0 = D(y-1)$ and $U_1 = D(y+1)$. Then

$$\Gamma(U_0, \mathcal{O}_X(P)) = A_{y-1}, \qquad \Gamma(U_1, \mathcal{O}_X(P)) = x^{-1} \cdot A_{y+1}$$

and the two local sections $s_0 = x/(y-1)$ and $s_1 = -(y+1)/x$ glue to a global section of $\mathcal{O}_X(P)$, which is not constant. This shows that $\mathcal{O}_X(P) \not\simeq \mathcal{O}_X$. □

The non-triviality of the class group $\operatorname{Cl}(X)$ reflects the arithmetic properties of $A$: the equation $x^2 = (y-1)(y+1)$ shows that $A$ is not a UFD, as $x$, $y-1$, and $y+1$ are irreducible but not prime. In contrast, for the complexified circle $X_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x,y]/(x^2+y^2-1)$, the class group is trivial, as $\mathbb{C}[x,y]/(x^2+y^2-1) \simeq \mathbb{C}[u,v]/(uv-1) \simeq \mathbb{C}[t,t^{-1}]$ is a UFD.

### *The quadric cone*

**Proposition 16.43.** Let $k$ be a field of characteristic $\neq 2$ and let $X = \operatorname{Spec} A$ where $A = k[x,y,z]/(xy - z^2)$. Then

$$\operatorname{Cl}(X) = \mathbb{Z}/2$$

generated by the prime divisor $Z = V(y,z)$.

*Proof* Note that $Z$ corresponds to the line $\{y = z = 0\}$, which is indeed an integral subscheme of $X$ of codimension 1.

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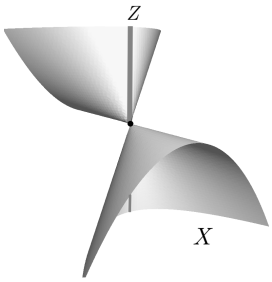


**Figure 16.2** A singular quadric surface

The complement $X - Z = X - V(y) = D(y)$ can be identified with $\operatorname{Spec} k[y, y^{-1}, u]$ via the isomorphism

$$k[x, y, y^{-1}, z]/(xy - z^2) \simeq k[y, y^{-1}][t, u]/(t - u^2) \simeq k[y, y^{-1}, u]$$

where $u = y^{-1}z$. As this is a UFD, we have $\operatorname{Cl}(X - Z) = 0$. By the exact sequence for class groups, we see that $\operatorname{Cl}(X)$ is generated by $Z$.

We next claim that $\operatorname{div}(y) = 2Z$, so $2[Z] = 0$ in $\operatorname{Cl}(X)$. Because of the equation $xy = z^2$, we have $V(y) = V(y, z) = Z$, as sets. Therefore, the only prime divisor where $y$ has non-zero order of vanishing is $Z$. The local ring at the generic point of $Z$ is given by

$$A_{(y,z)} = (k[x, y, z]/(xy - z^2))_{(y,z)}$$

Since $x$ is invertible in this ring, we see that $y \in (z^2)$ and that $z$ is a local parameter. This shows that $\operatorname{ord}_Z(y) = 2$, so $2Z = \operatorname{div}(y)$.

To see that $Z$ itself not a principal divisor, we investigate the local ring at the origin $P = (0, 0, 0)$:

$$\mathcal{O}_{X,P} = (k[x, y, z]/(xy - z^2))_{(x,y,z)}$$

In this ring, $\mathfrak{p} = (y, z)$ is a height 1 prime ideal, but it cannot be principal. Indeed, the quotient $\mathfrak{m}/\mathfrak{m}^2$ (that is, the Zariski cotangent space at $P$) is 3-dimensional as a vector space over $k$, spanned by $\{\overline{x}, \overline{y}, \overline{z}\}$ and $\overline{y}, \overline{z}$ define a 2-dimensional subspace of $\mathfrak{m}/\mathfrak{m}^2$. If $\mathfrak{p}$ were principal, the span of $\bar{y}$ and $\bar{z}$ would be contained in a 1-dimensional subspace of $\mathfrak{m}/\mathfrak{m}^2$, a contradiction.

This means that $[Z] \neq 0$ in $\operatorname{Cl}(X)$ and hence $\operatorname{Cl}(X) = \mathbb{Z}/2$. □

Note that the open subscheme $U = X - \{(0, 0, 0)\}$ is nonsingular, and the divisor $Z$ becomes Cartier on $U$. This example shows that removing a codimension 2 subset has no effect on Weil divisors, as $\operatorname{Cl}(U) = \operatorname{Cl}(X) = \mathbb{Z}/2$, but the subgroup of Cartier divisors might change.

Here is an alternative, direct way to see that $\operatorname{Cl}(X) = 0$. The key point is that there is a natural action of $k^\times$ on $X$ defined by $t \cdot (x, y, z) \mapsto (tx, ty, tz)$. More formally, there is a morphism of schemes $\mathbb{G}_m \times X \to X$ induced by the ring map $A \mapsto A[t, t^{-1}]$ defined by $x \mapsto tx$, $y \mapsto ty$, and $z \mapsto tz$. This extends to a morphism

$$\varphi\colon \mathbb{A}^1_k \times_k X \longrightarrow X$$

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Let $\iota_0$ and $\iota_1$ be the closed embeddings $X \to \mathbb{A}^1_k \times X$ defined by the fibers over $t = 0$ and $t = 1$ respectively. Note that $\varphi \circ \iota_1 = \mathrm{id}_X$, so the composition

$$\mathrm{Cl}(X) \xrightarrow{\varphi^*} \mathrm{Cl}(\mathbb{A}^1 \times X) \xrightarrow{\iota_1^*} \mathrm{Cl}(X)$$

is equal to the identity. As $\iota_1^*$ is an isomorphism, we conclude that $\varphi^*$ is an isomorphism.

On the other hand, $\varphi \circ \iota_0$ equals the zero map $(x, y, z) \mapsto (0, 0, 0)$ and so $\varphi^* \circ \iota_0^*$ is the zero map $\mathrm{Cl}(X) \to \mathrm{Cl}(X)$. As both $\iota_0^*$ and $\varphi^*$ are isomorphisms, this shows that $\mathrm{Cl}(X) = 0$ must be 0.

### *Nonsingular quadric surfaces*

**Proposition 16.44.** Let $k$ be an field, and let $X = \mathbb{P}^1_k \times_k \mathbb{P}^1_k$. Then

$$\mathrm{Cl}(X) = \mathbb{Z}[F_1] \oplus \mathbb{Z}[F_2]$$

where $F_1 = (0 : 1) \times \mathbb{P}^1_k$ and $F_2 = \mathbb{P}^1_k \times (0 : 1)$.

*Proof* Note that $X - F_1 - F_2$ is isomorphic to $\mathbb{A}^2_k$, so it has trivial class group. From the exact sequence

$$\mathbb{Z}F_1 \oplus \mathbb{Z}F_2 \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X - F_1 - F_2) \longrightarrow 0$$

we deduce that $\mathrm{Cl}(X)$ is generated by $[F_1]$ and $[F_2]$. We claim that the first map is also injective, so that in fact

$$\mathrm{Cl}(X) = \mathbb{Z}[F_1] \oplus \mathbb{Z}[F_2].$$

If $F_1$ and $F_2$ are related by $aF_1 \sim bF_2$ for some integers $a, b$, then

$$\mathcal{O}_X(aF_1) \simeq \mathcal{O}_X(bF_2). \tag{16.12}$$

Let us compute the restrictions of both sides of (16.12) to $F_1$ and $F_2$. First, since $F_1 \sim F_1'$ where $F_1 = (1 : 0) \times \mathbb{P}^1_k$ and $F_1 \cap F_2 = \varnothing$, we have $\mathcal{O}_X(F_1)|_{F_1} \simeq \mathcal{O}_{F_1}$.

For the second restriction, observe that that $F_2$ is the pullback of $(0 : 1) \in \mathbb{P}^1_k$ under the second projection $\mathrm{pr}_2 \colon X \to \mathbb{P}^1_k$. Therefore,

$$\mathcal{O}_X(F_2)|_{F_1} \simeq \mathrm{pr}_2^*\, \mathcal{O}_{\mathbb{P}^1_k}(1)|_{F_1} \simeq \mathcal{O}_{\mathbb{P}^1_k}(1)$$

since $\mathrm{pr}_2$ restricts to the identity on $F_1$.

Finally, restricting both sides of (16.12) to $F_1$, the two statements imply that $a = b = 0$, so $F_1$ and $F_2$ are independent. □

We say that a divisor $D \sim aF_1 + bF_2$ has 'type' $(a, b)$ the 'type' of $D$. For instance, a divisor of type $(1, 0)$ or $(0, 1)$ is a line on the quadric surface $X \subset \mathbb{P}^3$. We have $\mathcal{O}_{\mathbb{P}^3}(1)|_X \simeq \mathcal{O}_X(F_1 + F_2)$, so a $(1, 1)$-divisor is represented by a hyperplane section of $X$ (a conic).

**Exercise 16.8.1.** Let $X = \mathbb{P}^1_k \times_k \mathbb{P}^1_k$ and consider $X$ as a the quadric surface $X = V_+(x_0x_3 - x_1x_2)$.

a) Show that $X$ contains the twisted cubic curve $C$ of Example 5.28 and that $C$ defines a prime divisor of type $(1, 2)$ or $(2, 1)$.

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b) Conversely, show that any prime divisor $(1,2)$ or $(2,1)$ is projectively equivalent to the twisted cubic curve.

### *The 3-dimensional quadratic cone*

**Proposition 16.45.** Let $k$ be a field and let $X = \operatorname{Spec} A$, where $A = k[x,y,z,w]/(xw-yz)$. Then

$$\operatorname{Cl}(X) = \mathbb{Z},$$

generated by the class of the line $V(x,y)$.

Note that the ring $A$ is a normal integral domain, but it is not a UFD, as $xw = yz$.

Consider the principal divisor $H = \operatorname{div}(y) = V(y)$, which decomposes as $Z + Z'$ where $Z = V(x,y)$ and $Z' = V(y,w)$. The complement $X - H = \operatorname{Spec}(A_y) = \operatorname{Spec}(k[x,y,y^{-1},w])$ has trivial class group, as $A_y$ is a UFD. From the exact sequence (16.5), we see that $\operatorname{Cl}(X)$ is generated by $Z$ and $Z'$. As $H$ is principal, we have $Z + Z' \sim 0$ in $\operatorname{Cl}(X)$, so $\operatorname{Cl}(X)$ is in fact generated by $Z$.

To prove that $\operatorname{Cl}(X) \simeq \mathbb{Z}.$, we will show that no nonzero multiple of $Z$ is principal. In fact, $Z$ is a Weil divisor which has no multiple which is Cartier!

The key observation is that the open subset $X - Z$ is not affine (as shown in Exercise 4.11.50). Note that the complement $X - Z$ does not see the scheme structure on $Z$. In particular, if $nZ$ were Cartier for some $n$, then $X - Z$ would be affine by the folowing general fact:

**Proposition 16.46.** Let $X$ be an affine Noetherian normal scheme, and let $V(s)$ be the zero scheme of a section $s$ of an invertible sheaf $L$. Then $X - V(s)$ is also affine.

*Proof* Let $X = \operatorname{Spec} A$. Let $V_i = \operatorname{Spec} B_i$ be a finite affine cover of $X$ that trivializes $L$ with isomorphisms $\phi_i \colon L|_{V_i} \to \mathcal{O}_{V_i}$. Under these isomorphisms, the section $s|_{V_i}$ corresponds to some $f_i \in B_i$. Then $V(s) \cap V_i = \operatorname{Spec}(B_i/(f_i))$ and $V_i - V(s) = \operatorname{Spec}((B_i)_{f_i})$.

Consider the inclusion $\iota \colon U \to X$ of the open set $U = X - V(s)$. As $\iota^{-1}(V_i) = V_i - V(s)$ is affine, and the open sets $\iota^{-1}(V_i)$ form a cover of $U$, we see that $\iota$ is an affine morphism. But then $U = \iota^{-1}(X)$ is affine as well. □

### *The projective quadric cone*

**Proposition 16.47.** Let $X = V_+(xy - z^2) \subset \mathbb{P}^3_k$ be the *projective quadratic cone*. Then

$$\operatorname{Cl}(X) = \mathbb{Z}$$

generated by the hyperplane section $H = V_+(w)$.

The exact sequence of class groups takes the form

$$0 \longrightarrow \mathbb{Z}H \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(X - H) \longrightarrow 0 \tag{16.13}$$

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Injectivity on the left follows because $H$ corresponds to the invertible sheaf $\mathcal{O}_{\mathbb{P}^3_k}(1)|_X$ which has infinite order in $\mathrm{Cl}(X)$ since $\mathcal{O}_{\mathbb{P}^3_k}(d)|_X \neq \mathcal{O}_X$ for every $d > 0$ (to see this, consider the space of global sections).

The complement $X - H$ is isomorphic to the affine quadric cone $\operatorname{Spec} k[x, y, z]/(xy - z^2)$, hence $\mathrm{Cl}(X - H) = \mathbb{Z}/2$. Hence (16.13) shows that $\mathrm{Cl}(X)$ is either $\mathbb{Z}$, generated by a divisor $D$ such that $2D = H$ or $\mathbb{Z} \oplus \mathbb{Z}/2$. The correct answer is $\mathrm{Cl}(X) = \mathbb{Z}$, generated by the divisor $D = V_+(x, z)$. This can be seen from the relation $\operatorname{div}(x) = 2D$, where $D$ is the prime divisor corresponding to the line $\ell = \{x = z = 0\}$.

The Weil divisor $D$ is not Cartier: being Cartier is a local condition, so this follows from the example of the affine quadric cone above.

Here is an alternative way to see it: if $D = V_+(x, z)$ were Cartier, the sheaf $L = \mathcal{O}_X(D)$ would be invertible, and the same would be true for for the restriction to the line $\ell$. The Picard group of $\ell \simeq \mathbb{P}^1_k$ is $\mathbb{Z}$, generated by $\mathcal{O}_{\mathbb{P}^1_k}(1)$, so we would have $L|_\ell \simeq \mathcal{O}_{\mathbb{P}^1_k}(a)$ for some $a \in \mathbb{Z}$. On the other hand, we know that the divisor $H = 2D$ is Cartier, and

$$\mathcal{O}_X(H)|_\ell \simeq \mathcal{O}_{\mathbb{P}^3_k}(1)|_\ell \simeq \mathcal{O}_{\mathbb{P}^1_k}(1),$$

as the divisor of $w$ is just one reduced point on $\ell$. This implies that $2a = 1$, which is not possible. Hence $D$ is not Cartier.

## 16.9 Extended example: $\mathbb{Z}[\sqrt{-5}]$

The ring $A = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[x]/(x^2 + 5)$ has already appeared a few places throughout the book. As we have seen, $A$ is not a UFD, so $\mathrm{Cl}(\operatorname{Spec} A) \neq 0$. Now we claim the following:

**Proposition 16.48.** The class group of $A = \mathbb{Z}[\sqrt{-5}]$ is given by $\mathrm{Cl}(\operatorname{Spec} A) = \mathbb{Z}/2$, and it is generated by the class of the prime divisor $Y = V(2, 1 + x)$.

*Proof* First, note that $(2, 1 + x)$ is prime, because

$$A/(2, 1 + x) \simeq \mathbb{Z}[x]/(2, 1 + x) \simeq \mathbb{Z}/2.$$

Let $\mathfrak{p}$ be a height 1 prime ideal in $A$. We will show that $V(\mathfrak{p})$ is either principal or equivalent to $Y$ in $\mathrm{Cl}(\operatorname{Spec} A)$.

The idea is to study the morphism $\pi\colon \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ induced by the integral extension $\mathbb{Z} \subset A$. Note that $\pi$ is surjective, by the Going-Up Theorem. The point $\mathfrak{p}$ lies over a nonzero prime $(p) \in \operatorname{Spec} \mathbb{Z}$. There are three cases to consider:

(i) $(p)$ remains prime in $A$.
(ii) $(p)$ is a square of a prime ideal in $A$.
(iii) $(p)$ decomposes as

$$(p) = (p, x - a) \cap (p, x + a) \tag{16.14}$$

where $a \in \mathbb{Z}$ is an integer such that $a^2 + 5 \equiv 0 \pmod{p}$.

In case (i), $pA$ is prime in $A$, and so $V(\mathfrak{p}) = V(pA)$ is principal. The case (ii) happens if and only if the minimal polynomial $x^2 + 5 = 0$ has a multiple root modulo $p$. This happens

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only for the primes $p = 2$ and $p = 5$. For $p = 2$, we have $(2) = (1 + x)^2$. Therefore, $\operatorname{div}(2) = 2V(2, 1 + x) = 2Y$, and hence

$$2Y = 0 \text{ in } \operatorname{Cl}(\operatorname{Spec} A).$$

For $p = 5$, we have $(5) = (x)^2$, and so $V(\mathfrak{p}) = V(5, x) = V(x)$, which is principal.

We therefore reduce to the case (iii). By adjusting $a$ by a multiple of $p$, we may assume $0 \leqslant a < p$. From (16.14),

$$\operatorname{div}(p) = V(p, x + a) + V(p, x - a), \tag{16.15}$$

Which shows that $V(p, x + a) = -V(p, x - a)$ in $\operatorname{Cl}(\operatorname{Spec} A)$. Hence it suffices to analyze $V(p, x + a)$.

Consider more generally ideals of the form $(n, x + a)$ where $n \geqslant 2$ and $0 \leqslant a < n$ and $a^2 + 5 \equiv 0 \pmod{n}$.

If $n = 2$, we recover $Y = V(2, 1 + x)$.

If $n = 3$, then we must have $a = 1$. As $(1 + x, x^2 + 5) = (1 + x, 6)$, we get

$$\operatorname{div}(1 + x) = V(3, 1 + x) + V(2, 1 + x).$$

Hence $V(3, 1 + x) = -Y$ in $\operatorname{Cl}(\operatorname{Spec} A)$.

If $n = 4$, no solutions exist, as $a^2 + 1 \equiv 0 \pmod{4}$ has no integer solutions.

If $n = 5$, then $a = 0$, and $(5, x) = (x)$, which is principal.

If $n \geqslant 6$, write $a^2 + 5 = bn$. Then we get

$$\begin{aligned} \operatorname{div}(x + a) &= V(x + a, x^2 + 5) \\ &= V(a + x, bn) = V(a + x, b) + V(a + x, n) \end{aligned}$$

But as $a^2 + 5 = bn$ and $n \geqslant 6$, we must have $b < n$. Therefore, by induction on $n$, the class of $V(b, a + x)$ can be written as a multiple of $Y$ in the class group. This completes the proof. □

More generally, consider the quadratic number field $K = \mathbb{Q}(\sqrt{d})$, where $d$ is a square-free integer. The ring of integers $\mathcal{O}_K$, that is, the integral closure of $\mathbb{Z}$ in $K$, is a normal integral domain of dimension 1 (see Exercise 10.6.8 for more details). When $d < 0$, it is known that the class group $\operatorname{Cl}(\mathcal{O}_K)$ is trivial if and only if

$$d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}.$$

It is currently unknown whether there exist infinitely many positive values of $d$ for which $\operatorname{Cl}(\mathcal{O}_K) \neq 0$.

## 16.10 Exercises

**Exercise 16.10.1.** Verify that the set of principal divisors form a subgroup of $\operatorname{Div}(X)$.

**Exercise 16.10.2.** Consider the curve $y^2 = x^3 - 1$ in $\mathbb{A}^2_k$ where $k$ is algebraically closed of characteristic different from 2 and 3. If $(a, b) \in X$ we let $\sigma(a, p) = (a, -b)$, which also lies in $X$.

a) Show that for any $P \in X$, it holds that $P + \sigma(P) \sim 0$.

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b) Show that if $P, Q$ and $R$ are three collinear points on $X$, then $P + Q + R \sim 0$.
c) Show that any Weil divisor on $X$ is linearly equivalent to a prime divisor.

**Exercise 16.10.3.** Let $X$ be an integral normal scheme and let $\mathscr{K}_X$ denote the constant sheaf on $K = K(X)$. Note that the sheaf $\mathcal{O}_X^\times$ of invertible sections of $\mathcal{O}_X$ embeds as a subsheaf of $\mathscr{K}_X$. Show that a Cartier divisor is the same thing as a global section of the sheaf $\mathscr{K}_X^\times/\mathcal{O}_X^\times$.

**Exercise 16.10.4.** Check that the inverse of a Cartier divisors and the sum of two are well-defined; that is, that all cocycle conditions are fulfilled and that the inverse, respectively the sum, is independent of choices of representatives.

**Exercise 16.10.5.** Check that the ideal sheaf $\mathcal{I}_{nP}$ of the divisor $nP$ in Example **??** is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n)$.

**Exercise 16.10.6.** Describe Cartier data that defines the hyperplane $V(x_i)$ in $\mathbb{P}_k^n$.

**Exercise 16.10.7.** Show that all the local rings $\mathcal{O}_{X,p}$ of the curve $X$ given by $y^2 = x^3 - 1$ in $\mathbb{A}_k^2$ are discrete valuation rings, and hence $X$ is a normal variety. We assume that $k$ is algebraically closed and of characteristic different from three and two. More precisely, if $(a, b)$ is a point on $X$ show that $x - a$ is a parameter if $b \neq 0$ and that $y$ is one when $b = 0$. HINT: Note that $y^2 - b^2 = x^3 - a^3$.

**Exercise 16.10.8.** Let $X = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3 - x)$. Compute the divisors of the rational functions $x, y$ and $x^2/y$.

**Exercise 16.10.9.** Consider the curve $Z = V(F) \subset \mathbb{P}_k^2$ defined by an irreducible homogeneous polynomial $F$ of degree $d \geqslant 1$. Consider the open set $U = \mathbb{P}_k^2 - Z$. Show that the exact sequence (16.5) takes the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot d} \mathbb{Z} \longrightarrow \operatorname{Cl}(U) \longrightarrow 0.$$

Deduce that $\operatorname{Cl}(U) \simeq \mathbb{Z}/d$.

**Exercise 16.10.10.** Show that for the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, d)$ we have $\operatorname{Cl}(\mathbb{P}) = \mathbb{Z}D$ and $\operatorname{CaCl}(\mathbb{P}) = \mathbb{Z}H$ where $H = dD$.

**Exercise 16.10.11.** The same reasoning as for $\mathbb{P}_k^1$ can be applied to the affine line $X$ with two origins. Compute $\operatorname{Pic}(X)$ for this example.

**Exercise 16.10.12.** The aim of this exercise is to prove the following statement, known as *Nagata's lemma*: Let $A$ be a Noetherian integral domain, and let $x \in A - 0$. Suppose that $(x)$ is prime, and that $A_x$ is a UFD. Then $A$ is a UFD.

a) Show that $A_x$ is normal.
b) Show that $A$ is normal. HINT: If $t \in K(A)$ is integral over $A$, then $t \in A_x$.
c) Show that there is an exact sequence

$$\mathbb{Z}D \to \operatorname{Cl}(\operatorname{Spec} A) \to \operatorname{Cl}(\operatorname{Spec} A_x) = 0.$$

d) Use the above sequence to show that $\operatorname{Cl}(\operatorname{Spec} A) = 0$, and conclude that $A$ is a UFD.

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**Exercise 16.10.13.** Let $d$ be a square free integer and assume that $d \not\equiv 1 \mod 4$ so that $\mathbb{Z}[\sqrt{d}]$ is a Dedekind ring. Show that the class group of $\mathbb{Z}[\sqrt{d}]$ is finite. HINT: Imitate the technique used for the example with $\mathbb{Z}[\sqrt{-5}]$.

**Exercise 16.10.14.** Let $X$ be a scheme and let $D \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I}$. Show that the following statements are equivalent:

(i) $\mathcal{I}$ is an invertible sheaf.
(ii) For every $x \in X$, the ideal $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ is principal and generated by a nonzerodivisor.
(iii) There is an open covering $U_i$ of $X$ and nonzerodivisors $f_i \in \mathcal{O}_X(U_i)$ such that $f_i$ generates $\mathcal{I}(U_i)$.
(iv) For every $x \in X$, there is an open affine neighbourhood $U = \operatorname{Spec} A$ of $x$ such that $U \cap D = \operatorname{Spec} A/(f)$ where $f \in A$ is a nonzerodivisor.
(v) $D$ is the zero scheme of a global section $s$ of an invertible sheaf $L$.

When $X$ is normal integral Noetherian, these are also equivalent to

(vi) $D$ is an effective Cartier divisor.

**Exercise 16.10.15.** Show that the Picard group of the general linear group $\mathrm{GL}_n$ is trivial. HINT: Use an exact sequence.

**Exercise 16.10.16.** Let $X$ be a normal integral scheme and let $\pi\colon X \times \mathbb{A}^1 \to X$. We will consider the map

$$\pi^*\colon \operatorname{Pic}(X) \longrightarrow \operatorname{Pic}(X \times \mathbb{A}^1)$$

We will also need a section $\iota\colon X \to X \times \mathbb{A}^1$ defined by $x \mapsto (x, 0)$.

a) Show that the composition $\iota^* \circ \pi^*$ is the identity. Deduce that $\pi^*$ is injective.
b) Let $L$ be an invertible sheaf on $X \times \mathbb{A}^1$ and assume that there is an affine covering $U_i$ of $X$ such that $L$ is trivial when restricted to each $\pi^{-1}(U_i)$. Show that $L \simeq \pi^* A$ for some $A$ invertible sheaf on $X$.
c) Using *b*), deduce that $\pi^*$ is surjective.

**Exercise 16.10.17.** Let $k$ be a field. Use the fact that $\mathbb{A}^2_k$ has trivial Picard group to show that the morphism $\mathbb{A}^2_k - \{(0,0)\} \to \mathbb{P}^1_k$ does not extend to a morphism $\mathbb{A}^2_k \to \mathbb{P}^1_k$.

**Exercise 16.10.18** (Norms)**.** Let $f\colon X \to Y$ be a finite morphism of integral Noetherian normal schemes. Then $f$ induces an inclusion of function fields $K(Y) \to K(X)$, making $K(X)$ into a finite extension of $K(Y)$.

a) Show that the multiplication $g \in K(Y)$ induces a $K(X)$-linear map of $K(X)$-vector spaces $m_g\colon K(Y) \to K(Y)$.
b) We define the *norm* to be the map $N\colon K(Y) \to K(X)$ to be the determinant of $m_g$. Show that $N$ is multiplicative, i.e., $N(gh) = N(g)N(h)$ for each $g, h \in K(Y)$.
c) Use the norm to define a morphism $f_*\colon \operatorname{Div}(X) \to \operatorname{Div}(Y)$.
d) Show that if $h \in K(X)$, then $f_* f^* \operatorname{div}(h) = (\deg f) \cdot \operatorname{div}(h)$.

**Exercise 16.10.19.** Let $X = \mathbb{A}^2_k - (0,0)$ and let $f\colon X \to \mathbb{P}^1_k$ be the quotient morphism. Show that $f^*\mathcal{O}_{\mathbb{P}^1_k}(1) \simeq \mathcal{O}_X$. Deduce that the induced map $f^*\colon \operatorname{Pic}(\mathbb{P}^1_k) \to \operatorname{Pic}(X)$ may fail to be injective.

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Find a dominant morphism $f\colon X \to Y$ so that $\operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is not surjective.

**Exercise 16.10.20.** Show that the rings $\mathbb{Z}[\sqrt{-5}]_2$ and $\mathbb{Z}[\sqrt{-5}]_3$ are unique factorization domains.

**Exercise 16.10.21.** For non-Noetherian schemes one quickly runs into difficulties when defining the divisor of a rational function.

a) Imitating the construction of the affine line with two origin, one can construct the *affine line $X$ with infinitely many origins*.
b) Show that $X$ is integral, normal, with fraction field $k(t)$, but there are infinitely many closed points $p \in X$ for which $\operatorname{ord}_p(t) = 1$.

**Exercise 16.10.22** (Quadric hypersurfaces in higher dimension)**.** Let $A = k[x_1, \dots, x_n, y, z]/(x_1^2 + \cdots + x_n^2 - yz)$. We will prove that $A$ is a UFD for $n \geqslant 3$.

a) Show that $A$ is an integral domain.
b) Show that $A$ is normal. HINT: Show first that $A_y$ is a UFD, hence normal. $(y)$ is prime for $n \geqslant 3$.
c) Show that $\operatorname{Cl}(\operatorname{Spec} A) = 0$ and conclude that $A$ is a UFD. HINT: Consider $Z = \operatorname{div} y$
d) Suppose that $k$ contains a square root of $-1$ and let $X = V(x_0^2 + \cdots + x_n^2) \subset \mathbb{A}_k^{n+1} = \operatorname{Spec} k[x_0, \dots, x_n]$. Show that
   (a) $n = 2$, $\operatorname{Cl}(X) = \mathbb{Z}/2$
   (b) $n = 3$, $\operatorname{Cl}(X) = \mathbb{Z}$
   (c) $n \geqslant 4$, $\operatorname{Cl}(X) = 0$

**Exercise 16.10.23.** Consider a quadric $X = V(x_0^2 + \cdots + x_{n-2}^2 + x_n x_{n-1}) \subset \mathbb{P}_k^n$. Show that $\operatorname{Cl}(X)$ is

(i) If $n = 2$, then $\operatorname{Cl}(X) = \mathbb{Z}$.
(ii) If $n = 3$, then $\operatorname{Cl}(X) = \mathbb{Z}^2$.
(iii) If $n \geqslant 4$, then $\operatorname{Cl}(X) = \mathbb{Z}$.

**Exercise 16.10.24.** Show that if two Weil divisors $D$ and $E$ are linearly equivalent then $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E)$ as $\mathcal{O}_X$-modules.

**Exercise 16.10.25.** Consider the rational function $f$ from Example **??**. Regarding $f$ as a rational function on $\mathbb{P}_k^1$, compute $\operatorname{div}(f)$.

**Exercise 16.10.26.** Let $A$ be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and let $X = \operatorname{Spec} A$. Fix a local parameter $t \in A$, such that $\mathfrak{m} = (t)$ and let $x \in X$ be the closed point corresponding to $\mathfrak{m}$.

a) Show that $\mathcal{O}_X(nx) = t^{-n} \cdot \mathcal{O}_X$, as subsheaves of $\mathbf{K}(X)$.
b) Show that if $f \in K(X)^\times$, then $\operatorname{div}(f) = \operatorname{ord}_x f \cdot x$ and $\mathcal{O}_X(\operatorname{div}(f)) = t^{-\operatorname{ord}_x f}\mathcal{O}_X$.
c) Show that $\operatorname{Cl}(X) = \mathbb{Z}$.

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# 17

# Differentials

## 17.1 Derivations

In this section, we discuss *derivations* of a $k$-algebra $A$. Here we allow '$k$' to be any base ring, but of course the main example we have in mind is when $k$ is a field.

**Definition 17.1.** Let $k$ be a ring and let $A$ be a $k$-algebra. An *$k$-derivation* from $A$ with values in $M$ is a $k$-linear map $D\colon A \to M$ satisfying the Leibniz rule

$$D(fg) = g \cdot D(f) + f \cdot D(g)$$

for every $f, g \in A$.

Note that even though $M$ is an $A$-module, a derivation $D$ is usually not a map of $A$-modules; we only require it to be linear over $k$.

If $D$ is a derivation, then $D(a) = 0$ for every element $a \in k$. This follows because $D(a \cdot a) = a \cdot D(a)$ and $D(a \cdot a) = a \cdot D(a) + a \cdot D(a) = 2D(a)$. We think of the elements in $A$ coming from $k$ as the 'constants'. However, a derivation can be zero on other elements in $A$ as well (a silly example is the zero map, which is a derivation).

**Example 17.2.** If $A = k[x]$, then we can define a derivation $D\colon A \to A$ by sending a polynomial $f(x) = \sum_i a_i x^i$ to $f'(t)$. Here $f'(x)$ is the formal derivative defined by

$$f'(x) = \sum_i i a_i x^{i-1}. \tag{17.1}$$

More generally, the partial differential operators $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, as well as their $k$-linear combinations, are $k$-linear derivations on the polynomial ring $k[x_1, \ldots, x_n]$. △

Many of the familiar formulas for derivatives from calculus hold also for general derivations. For instance, an easy induction shows that $D(x^n) = nx^{n-1}D(x)$ for every $x \in A$, and in case $g$ is invertible in $A$, $D(f/g) = (D(g)f - D(f))/g^2$. Moreover, if $f(x)$ is a polynomial, one has the chain rule $D(f(g)) = f'(x)D(g)$.

The set of $k$-derivations $D\colon A \to M$ is denoted by $\mathrm{Der}_k(A, M)$. While derivations need not be $A$-linear, the set $\mathrm{Der}_k(A, M)$ has an $A$-module structure, where an element $a \in A$ acts by $D \mapsto a \cdot D$.

If $\rho\colon M \to N$ is a map of $A$-modules, and $D\colon A \to M$ is a $k$-derivation, then the composition $\rho \circ D\colon A \to N$ is again a $k$-derivation with values in $N$. This means that $\mathrm{Der}_k(A, -)$ is a functor from $\mathsf{Mod}_A$ to itself.



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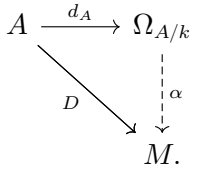
The set of derivations $\mathrm{Der}_k(A, M)$ is also functorial in the base ring $k$ and the $k$-algebra $A$. If $k \to k'$ is a ring map, any $k'$-derivation $A \to M$ can be regarded as a $k$-derivation, so we obtain an inclusion $\mathrm{Der}_{k'}(A, M) \subset \mathrm{Der}_k(A, M)$. Likewise, if $A \to A'$ is a map of $k$-algebras, then a derivation $A' \to M$ induces a derivation $A \to M$ by composition.

## 17.2 Kähler differentials

For a $k$-algebra $A$, there exists an $A$-module $\Omega_{A/k}$ and a derivation $d_A \colon A \to \Omega_{A/k}$ which is *universal* among derivations from $A$ into modules $M$. This means that for any $k$-derivation $D \colon A \to M$ there exists a unique $A$-linear map $\alpha \colon \Omega_{A/k} \to M$ such that $D = \alpha \circ d_A$:




In other words, the map sending $\alpha \mapsto \alpha \circ d_A$ is a bijection

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \longrightarrow \mathrm{Der}_k(A, M). \tag{17.2}$$

We can construct the $A$-module $\Omega_{A/k}$ explicitly as follows. For each element $a \in A$ introduce a symbol $da$ and consider the free $A$-module $G = \bigoplus_{a \in A} A\, da$. Inside $G$, we have the submodule $H$ generated by the expressions

$$d(a + b) - da - db, \quad d(ab) - a \cdot db - b \cdot da, \quad dc$$

for $a, b \in A$ and $c \in k$. We then define $\Omega_{A/k} = G/H$, and the map $d_A \colon A \to \Omega_{A/k}$ by $d_A(a) = da$. Note that $d_A$ is additive, because any $\mathbb{Z}$-linear relation among the $da$'s maps to zero in $G/H$, and it is a derivation because the Leibniz rule $d(ab) = a\, db + b\, da$ is imposed to hold in $G/H$. Finally, it will be $k$-linear because $d(c \cdot a) = c \cdot da + 0 = c \cdot da$ in $G/H$.

The module $G/H$ also satisfies the universal property above: given any $k$-derivation $D \colon A \to M$, we define the $A$-linear map $\alpha \colon \Omega_{A/k} \to M$ by $\alpha(da) = D(a)$, which is well-defined precisely because $D$ is a derivation.

**Definition 17.3.** The $A$-module $\Omega_{A/k}$ is called the *module of differentials* of $A$ over $k$.

Here is an important example, which will serve as a basis for essentially all explicit computations involving $\Omega_{A/k}$.

**Proposition 17.4 (Polynomial rings).** Let $k$ be a ring and let $A = k[x_1, \ldots, x_n]$. Then $\Omega_{A/k}$ is the free $A$-module with basis $dx_1, \ldots, dx_n$:

$$\Omega_{A/k} = \bigoplus_{i=1}^{n} A\, dx_i.$$

The universal derivation is defined by $d_A(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i$.

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*Proof* The universal property follows from the general chain rule: for any $k$-derivation $D\colon A \to M$ into a $A$-module $M$, we have

$$D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} D(x_i). \tag{17.3}$$

Indeed, an easy induction using the Leibniz rule, shows that (17.3) holds when $f$ is a monomial, and then we conclude using $k$-linearity.

Given a derivation $D$, we can define the $A$-linear map $\alpha\colon \bigoplus_{i=1}^{n} A\, dx_i \to M$ by sending $dx_i$ to $D(x_i)$. Then (17.3) implies that we have the equality $D = \alpha \circ d_A$, so the required universal property holds. □

### *Differentials and the diagonal*

While the construction in the previous section is useful for efficiently defining $\Omega_{A/k}$, there is an alternative description of $\Omega_{A/k}$, which is a bit more geometric. This description will be useful when we construct the *sheaf* of differentials, as the previous construction does not behave well with respect to gluing.

Let $A$ be a $k$-algebra, let $\mu\colon A \otimes_k A \to A$ be the multiplication map $\mu(x \otimes y) = xy$ and let $I = \operatorname{Ker}(\mu) \subset A \otimes_k A$. Then as we saw in Chapter **??**, $I$ is the ideal describing the diagonal embedding

$$\Delta\colon \operatorname{Spec} A \longrightarrow \operatorname{Spec}(A \otimes_k A).$$

The next result shows that the module of differentials $\Omega_{A/k}$ can be identified with the quotient $I/I^2$. Just to clarify, there are a priori *two* $A$-module structures on $I \subset A \otimes_k A$, one given by $A \to A \otimes_k A$ via $a \mapsto a \otimes 1$ and the other defined by $a \mapsto 1 \otimes a$. But these two coincide when passing to the quotient $I/I^2$, because for any $x \in I$,

$$(a \otimes 1)x - (1 \otimes a)x = (a \otimes 1 - 1 \otimes a) \cdot x \in I^2.$$

Another useful fact is that $I$ is generated (as an '$A \otimes_k 1$-module') by the elements of the form $a \otimes 1 - 1 \otimes a$ where $a \in A$. Indeed, for a given element $\sum_i a_i \otimes b_i \in I$, we have $\sum_i a_i b_i = 0$ in $A$, hence $\sum_i 1 \otimes a_i b_i = 0$ in $A \otimes_k A$, and this implies that

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes 1)(b_i \otimes 1 - 1 \otimes b_i).$$

**Proposition 17.5.** There is an isomorphism of $A$-modules

$$\Omega_{A/k} \simeq I/I^2. \tag{17.4}$$

The universal derivation $\delta\colon A \to I/I^2$ is defined by $\delta(a) = a \otimes 1 - 1 \otimes a$.

*Proof* Let us first check that $\delta$ is a derivation. Note that $\delta$ is $k$-linear, because for $a, b \in k$ and $x, y \in A$, we have

$$\begin{aligned} \delta(ax + by) &= (ax + by) \otimes 1 - 1 \otimes (ax + by) \\ &= ax \otimes 1 - 1 \otimes ax + by \otimes 1 - 1 \otimes by \\ &= a \cdot \delta(x) + b \cdot \delta(y). \end{aligned}$$

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Moreover, the Leibniz rule holds:

$$\begin{aligned}\delta(ab) &= ab \otimes 1 - 1 \otimes ab \\ &= (a \otimes 1)(b \otimes 1 - 1 \otimes b) + (1 \otimes b)(a \otimes 1 - 1 \otimes a) \\ &= a \cdot \delta(b) + b \cdot \delta(a).\end{aligned}$$

Next, we check that $I/I^2$ and $\delta$ satisfy the universal property. Let $D\colon A \to M$ be a $k$-linear derivation. This defines an '$A \otimes_k 1$'-linear map

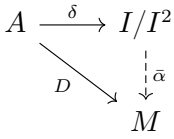
$$\alpha\colon A \otimes_k A \longrightarrow M, \quad a \otimes x \mapsto a \cdot D(x)$$

We have $\alpha(I^2) = 0$, because

$$\begin{aligned}\alpha((x \otimes 1 - 1 \otimes x)(y \otimes 1 - 1 \otimes y)) &= \alpha(xy \otimes 1) - \alpha(x \otimes y) - \alpha(y \otimes x) + \alpha(1 \otimes xy) \\ &= xy \cdot D(1) - x \cdot D(y) - y \cdot D(x) + D(xy) = 0\end{aligned}$$

The last expression is zero because $D$ is a derivation.

Therefore we get an induced map $\bar{\alpha}\colon I/I^2 \to M$ so that $d = \bar{\alpha} \circ \delta$:




The map $\delta$ is surjective, because $I$ is generated by the expressions $x \otimes 1 - 1 \otimes x$. This implies that the induced map $\bar{\alpha}$ is unique. Finally, $\bar{\alpha}$ is $A$-linear because $\alpha$ is $A \otimes 1$-linear. □

## 17.3 Properties of differentials

Let $k$ be a ring and let $\rho\colon A \to B$ be a map of $k$-algebras. There are two important exact sequences that relate the three modules $\Omega_{A/k}, \Omega_{B/k}$ and $\Omega_{B/A}$. The maps involved are defined as follows. Note first that there is a natural $A$-linear map $\Omega_{A/k} \to \Omega_{B/k}$ defined by $da \mapsto d(\rho(a))$. Taking the tensor product, we obtain a natural map of $B$-modules

$$\rho^*\colon \Omega_{A/k} \otimes_A B \to \Omega_{B/k}; \qquad d_A a \otimes b \mapsto b \cdot d_B \rho(a). \tag{17.5}$$

We also have the 'change of constants' map

$$\beta\colon \Omega_{B/k} \longrightarrow \Omega_{B/A}; \qquad d_B b \mapsto d_B b.$$

Note that the elements of the form $d_B\rho(x)$ all map to zero in $\Omega_{B/A}$. The next proposition shows that these elements in fact generate the kernel of $\beta$:

**Proposition 17.6 (Cotangent sequence).** There is an exact sequence of $B$-modules

$$\Omega_{A/k} \otimes_A B \xrightarrow{\rho^*} \Omega_{B/k} \xrightarrow{\beta} \Omega_{B/A} \longrightarrow 0 \tag{17.6}$$

*Proof* The surjectivity of $\beta$ is easy, because both $B$-modules are generated by symbols of

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the form $d_B b$, but there are more relations imposed in $\Omega_{B/A}$. In fact, from the description of $\Omega_{B/A}$ in terms of generators and relations, we see that

$$\Omega_{B/A} = \Omega_{B/k} / \langle B\, d_B a : a \in A \rangle$$

Moreover, the submodule $\langle B\, d_B a : a \in A \rangle$ is exactly the image of the map $\rho^*$, so we get also exactness in the middle. □

In general, the map $\rho^*$ needs not be injective nor surjective. There is one important case where it is surjective, and where one can describe the kernel explicitly, namely when $B = A/I$ and $\rho$ is the quotient map. In this case $\Omega_{B/A} = 0$, because for each $a \in A$, we have $d_{B/A}(\overline{a}) = a \cdot d_{B/A}(1) = 0$ in $\Omega_{B/A}$, by $A$-linearity. Therefore, the left-most map appearing in the cotangent sequence (17.6), $\rho^*$, is surjective. By the description of $\Omega_{B/k}$ in terms of generators and relations, we see that the kernel of $\rho^*$ is generated by elements of the form $da \otimes 1$ where $a \in A$. Note that this is also the image of the map

$$\delta \colon I/I^2 \longrightarrow \Omega_{A/k} \otimes_k B; \qquad \overline{f} \mapsto df \otimes 1.$$

This map is well-defined, because if $f, g \in I$, then $d(fg) \otimes 1 = (g\, df + f\, dg) \otimes 1 = 0$ by the Leibniz rule. Moreover, it is $B$-linear, because

$$\delta(\overline{a \cdot f}) = d(a \cdot f) \otimes 1 = (f\, da + a\, df) \otimes 1 = a\, df \otimes 1 = a \cdot \delta(\overline{f}).$$

We have proved the following proposition.

**Proposition 17.7 (Conormal sequence).** Let $A$ be a $k$-algebra and let $B = A/I$ for some ideal $I \subset A$. Then there is an exact sequence of $B$-modules

$$I/I^2 \xrightarrow{\delta} \Omega_{A/k} \otimes_A B \xrightarrow{\rho^*} \Omega_{B/k} \longrightarrow 0. \tag{17.7}$$

The conormal sequence allows us to find explicit presentations of modules of differentials. If $A$ is a finitely generated $k$-algebra, say $A = k[x_1, \ldots, x_n]/I$ where $I = (f_1, \ldots, f_r)$, then we have

$$\Omega_{k[x_1,\ldots,x_r]/k} \otimes_k A \simeq \textstyle\bigoplus_{i=1}^n A\, dx_i.$$

and the sequence (17.7) takes the form

$$I/I^2 \xrightarrow{\delta} \textstyle\bigoplus_{i=1}^n A\, dx_i \longrightarrow \Omega_{A/k} \longrightarrow 0.$$

As $I/I^2$ is generated as a $A$-module by the classes of the $f_1, \ldots, f_r$ modulo $I^2$, there is a surjection $A^r \to I/I^2$ which gives the exact sequence

$$A^r \longrightarrow \textstyle\bigoplus_{i=1}^n A\, dx_i \longrightarrow \Omega_{A/k} \longrightarrow 0. \tag{17.8}$$

Here the left-most map sends the $i$-th basis vector $e_i$ to $df_i = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\, dx_j$. This means that it can be identified with the $A$-linear map $J^t \colon A^r \to A^n$ defined by multiplication by the transpose of the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n}. \end{pmatrix}$$

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We summarize the discussion in the following theorem:

**Theorem 17.8.** Let $k$ be a ring and let $A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. Then

$$\Omega_{A/k} = \frac{\bigoplus_{i=1}^n A\, dx_i}{\left\langle A\, df_1 + \cdots + A\, df_r \right\rangle} = \operatorname{Coker} J^t \tag{17.9}$$

and the universal $k$-derivation is given by $d_A(f) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i$.

The description in (17.9) allows for easy computations of differentials for a large class of rings. We will see many examples in Section 17.5.

Another nice property of differentials is that they behave well under base change and localization:

**Proposition 17.9.** Let $k$ be a ring and let $A$ and $B$ be $k$-algebras.

(i) (Base change) For the $k$-algebra $C = A \otimes_k B$, we have

$$\Omega_{C/B} = \Omega_{B/A} \otimes_B C.$$

(ii) (Localization): For a multiplicative set $S \subset A$, we have an isomorphism of $S^{-1}A$-modules

$$\Omega_{S^{-1}A/k} = S^{-1}(\Omega_{A/k})$$

*Proof* See Exercises 17.11.3 and 17.11.4. □

Combining Theorem 17.8 and Proposition 17.9, we get the following:

**Corollary 17.10.** Let $k$ be a ring and let $A$ be a finitely generated $k$-algebra, or a localization of such. Then $\Omega_{A/k}$ is a finitely generated $A$-module.

## 17.4 The sheaf of differentials

The primary motivation for studying the module of differentials $\Omega_{B/A}$ is that it provides an intrinsic sheaf $\widetilde{\Omega_{B/A}}$ on $\operatorname{Spec} B$ associated to a morphism $\operatorname{Spec} B \to \operatorname{Spec} A$. In this section, we globalize this construction to an arbitrary separated morphism $X \to S$.

**Definition 17.11.** Let $X$ be a separated scheme over a scheme $S$. Let $\mathcal{I}$ be the ideal sheaf corresponding to the diagonal embedding $\Delta\colon X \to X \times_S X$. We define the *sheaf of Kähler differentials* to be the sheaf

$$\Omega_{X/S} = \Delta^*(\mathcal{I}/\mathcal{I}^2).$$

When $S = \operatorname{Spec} A$, we write $\Omega_{X/A} = \Omega_{X/\operatorname{Spec} A}$.

Concretely, if $U = \operatorname{Spec} B \subset X$ is an affine open that maps into $V = \operatorname{Spec} A \subset S$, then the restriction $\Delta|_U\colon U \to U \times_S U = U \times_V U = \operatorname{Spec}(B \otimes_A B)$ is a closed embedding

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corresponding to the ideal $I = \mathcal{I}(U)$ in $B \otimes_A B$. Moreover, $\mathcal{I}/\mathcal{I}^2|_{U\times_V U} = \widetilde{I/I^2}$. Therefore,

$$\Omega_{X/S}|_U \simeq \widetilde{\Omega_{B/A}}.$$

Hence the sheaf $\Omega_{X/S}$ is built out of the various $\Omega_{B/A}$ over affine opens. In particular, $\Omega_{X/S}$ is a quasi-coherent sheaf. If $X$ is of finite type over a ring $k$, then $\Omega_{X/k}$ is also of finite type, by Corollary 17.10.

The algebraic properties of the Kähler differentials $\Omega_{B/k}$ translate into the following statements for $\Omega_{X/S}$.

**Proposition 17.12.**

(i) (Base change): Let $X \to S$, and $S' \to S$ be separated morphisms and let $X' = X \times_S S'$ with projection $p\colon X' \to X$. Then

$$\Omega_{X'/S'} = p^*\Omega_{X/S}.$$

(ii) (Cotangent sequence): Let $S$ be a scheme and $f\colon X \to Y$ a morphism of separated schemes over $S$. Then there is an exact sequence of $\mathcal{O}_X$-modules

$$f^*(\Omega_{Y/S}) \to \Omega_{X/S} \to \Omega_{X/Y} \to 0. \tag{17.10}$$

(iii) (Conormal sequence): Let $Y$ be a closed subscheme of a scheme $X$ over $S$. Let $\mathcal{I}_Y$ be the ideal sheaf of $Y$. Then there is an exact sequence of $\mathcal{O}_Y$-modules

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega_{X/S}|_Y \to \Omega_{Y/S} \to 0. \tag{17.11}$$

## 17.5 Examples

**Example 17.13** (Affine space)**.** For the affine $n$-space $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$ over a ring $k$, we have

$$\Omega_{\mathbb{A}^n_k/k} = \widetilde{\Omega_{k[x_1,\dots,x_n]/k}} = \bigoplus_{i=1}^n \mathcal{O}_X\, dx_i.$$

△

**Example 17.14.** Let $k$ be a field and let $h(x) \in k[x]$. Then for $A = k[x]/(h)$, we have

$$\Omega_{A/k} = \frac{A\,dx}{h'(x)dx}.$$

Consider the ring map $\rho\colon k[u] \to k[x]$ sending $u$ to $h(x)$. Writing $k[x] = k[u][x]/(h(x) - u)$, we compute

$$\begin{aligned} \Omega_{k[x]/k[u]} &= (k[x]dx)/(h'(x)dx) \\ \Omega_{k[u]/k} \otimes_{k[u]} k[x] &= k[x]du = k[x]h'(x)dx \end{aligned}$$

and the cotangent sequence (17.6) takes the form

$$0 \longrightarrow k[x]\,du \longrightarrow k[x]\,dx \longrightarrow k[x]\,dx/(h'(x)dx) \longrightarrow 0$$

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(Here the left-most map sends $du$ to $h'(x)dx$, so the sequence is in fact short exact.)

On the geometric side, assuming $h$ is non constant, $\rho$ defines a finite morphism $f\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$. The module $\Omega_{k[x]/k[u]}$ is supported on the points where $h$ has multiple roots, or in other words, where $f$ is not $(\deg h)$-to-1. △

**Example 17.15** (Plane curves)**.** Let $k$ be a field and let $X = \operatorname{Spec} A$ where $A = k[x,y]/(f)$. By Theorem 17.8, the module of differentials is given by

$$\Omega_{A/k} = (A\,dx \oplus A\,dy) \,/\, \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\right).$$

Write $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ for the partial derivatives. If $X$ is smooth, that is, $V(f, f_x, f_y) = \varnothing$, then $X$ is covered by the two open sets $D(f_x)$ and $D(f_y)$. The module of differentials $\Omega_{A/k}$ will be a locally free $A$-module of rank 1. Over the open set $D(f_x)$, we have $(\Omega_{A/k})_{f_x} = \Omega_{A_{f_x}/k}$ and we may eliminate $dx$ using the relation $f_x\,dx + f_y\,dy = 0$, and $(\Omega_{A/k})_{f_x}$ is generated by the element $dy/f_x$. Likewise, over $D(f_y)$, the module is generated by $-dx/f_y$. Note that on the overlap $D(f_x) \cap D(f_y)$, these two generators coincide, as $f_x dx + f_y dy = 0$ in $\Omega_{A/k}$. This means that we have defined a global section of $\Omega_{X/k}$. △

**Example 17.16** (The nodal cubic)**.** Let $k$ be a field and let $A = k[x,y]/(y^2 - x^2(x+1))$. Then $X = \operatorname{Spec} A$ is a plane curve which has a singular point at the origin $P$ corresponding to $\mathfrak{m} = (x,y)$, and nonsingular at all other points. The module of differentials is given by

$$\Omega_{A/k} = \frac{A dx \oplus A dy}{(2y\,dy - (3x^2 + 2x)dx)}.$$

In this case $\Omega_{A/k}$ has rank 1 for every point $(x,y) \neq (0,0)$. Indeed, $dx$ will generate $\Omega_{A/k}$ over the open set $D(y)$, and $dy$ will generate over $D(x)$ (except where $3x + 2 = 0$, but these points are covered by the first case, as $y \neq 0$ there.)

At the origin, where $x = y = 0$, the relation $2y\,dy - (3x^2 + 2x)dx$ is identically zero, and $\Omega_{A/k}$ is not locally free there. More precisely, $\Omega_{A/k} \otimes_A k(P)$ is isomorphic to

$$\Omega_{A/k} \otimes_A A/(x,y) = \frac{A\,dx \oplus A\,dy}{(x, y, 2ydy - (3x^2 + 2x)dx)} = k\,dx \oplus k\,dy$$

which is a $k$-vector space of dimension 2. △

**Example 17.17** (The cuspidal cubic)**.** Let $k$ be a field of characteristic $\neq 2, 3$ and let $A = k[x,y]/(y^2 - x^3)$. The module of differentials is given by

$$\Omega_{A/k} = \frac{A\,dx \oplus A\,dy}{(2y\,dy - 3x^2\,dx)})$$

As in the previous example, $\Omega_{A/k}$ is locally free of rank 1 outside the origin $(x,y)$.

In fact, the module $\Omega_{A/k}$ contains non-zero torsion elements. Consider for instance the element $\eta = 3ydx - 2xdy$, which is killed by $y$ and $x^2$:

$$x^2\eta = 3x^2 y\,dx - 2x^3 dy = y(3x^2\,dx - 2y\,dy) = 0$$
$$y\eta = 3y^2\,dx - 2xy\,dy = x(3x^2 dx - 2ydy) = 0.$$

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To see that $\eta$ is non-zero, note that we can view $A$ as an algebra over the polynomial ring $k[x]$, and

$$\Omega_{A/k[x]} = (A\,dy)/(2y\,dy).$$

The 'change of constants map' $\Omega_{A/k} \to \Omega_{A/k[x]}$, sends $dx \mapsto 0$ and $dy \mapsto dy$. Therefore, $\eta = 3ydx - 2xdy$ maps to $2x\,dy$, which is non-zero, and so $\eta$ must be non-trivial. △

**Example 17.18.** Let $k$ be a ring and consider the projective line $\mathbb{P}^1_k$ over $k$. Then

$$\Omega_{\mathbb{P}^1_k/k} \simeq \mathcal{O}_{\mathbb{P}^1_k}(-2).$$

To see this, we use the standard covering of $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$, given by $U_i = D_+(x_i)$. Over $U_0$, Example 17.13 shows that

$$\Omega_{U_0/k} \simeq k\left[\frac{x_1}{x_0}\right] d\left(\frac{x_1}{x_0}\right)$$

and similarly on $U_1$. On the intersection $D_+(x_0) \cap D_+(x_1)$, the two generators are related by

$$d\left(\frac{x_1}{x_0}\right) = -\left(\frac{x_1}{x_0}\right)^2 d\left(\frac{x_0}{x_1}\right).$$

This shows that, up to a sign, $\Omega_{\mathbb{P}^1_k/k}$ is constructed using the same gluing function as $\mathcal{O}_{\mathbb{P}^1_k}(-2)$, and hence $\Omega_{X/k} \simeq \mathcal{O}_X(-2)$. △

**Example 17.19.** In this example we will consider the sheaf of differentials of the hyperelliptic curves of Example B.4. Let $p(x)$ be a polynomial of degree $d = 2g+1$ or $d = 2g+2$ and assume that $p$ has distinct roots. In the affine chart $U = \operatorname{Spec} A$, where $A = k[x,y]/(y^2 - p(x))$, we have

$$\Omega_{U/k} = A\,dx \oplus A\,dy/(2y\,dy - p'(x)\,dx). \tag{17.12}$$

This is locally free of rank 1: it is generated by $dx$ over the open set $D(y)$ and by $dy$ over $D(p'(x))$, and these two open sets cover $U$ as $V(p'(x), y) = V(p'(x), p(x)) = \varnothing$ by assumption. Over the intersection $D(y) \cap D(p'(x))$, these generators are related by $dy = p'(x)/y\,dx$ and $p'(x)/y$ is a unit.

In the other chart, $V = \operatorname{Spec} B$, where $B = k[u,v]/(v^2 - u^{2g+2}p(u^{-1}))$, we have

$$\Omega_{V/k} = B\,du \oplus B\,dv/(2v\,dv - ((2g+2)u^{2g+1}p(u^{-1}) + u^{2g+2}p'(u^{-1}))\,du). \tag{17.13}$$

A similar analysis as above shows that also $\Omega_{V/k}$ is locally free of rank 1 over $V$.

Over the intersection $U \cap V = D(x) = D(u)$, the module of differentials is given by the localization of (17.12) and (17.13) in $x$ and $u$ respectively. Here the generators are related by $x = u^{-1}$ and $y = u^{-g-1}v$, which gives

$$dx = -u^{-2}\,du, \quad dy = -(g+1)u^{-g-2}v\,dx + u^{-g-1}\,dv.$$

Consider the element $\omega = y^{-1}\,dx \in \Omega_{K/k}$. Over the open sets $D(y), D(p'(x))$, $D(v)$, $D(p'(u))$, this can be represented

$$\omega = \frac{dx}{y} = \frac{2\,dy}{p'(x)} = -\frac{u^{g-1}\,du}{v} = -\frac{2\,u^{g-1}\,dv}{p'(u)}.$$

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From this, it follows that $\omega$ in fact defines a global section of the sheaf $\Omega_{X/k}$. The same holds for the $g$ differentials $y^{-1}\,dx, \ldots, x^{g-1}y^{-1}\,dx$. We will see later, in Chapter 18, that $\Gamma(X, \Omega_{X/k})$ is $g$-dimensional, generated by these expressions. $\triangle$

## 17.6 The Euler sequence and differentials of $\mathbb{P}^n_k$

In this section, we will give a concrete description of the sheaf of differentials on projective space, suitable for explicit computations. We continue to work over a general ring $k$.

The description is in terms of the so-called 'Euler sequence' in (17.14) below. The name comes from *Euler's formula* which states that if $f$ is a homogeneous polynomial in $x_0, \ldots, x_n$ of degree $m$, then $\sum_{i=0}^{m} x_i \frac{\partial f}{\partial x_i} = m \cdot f$. (See Exercise 17.11.8). This formula will be used in the proof below.

**Theorem 17.20 (The Euler Sequence).** Then there is an exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n_k/k} \longrightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n_k}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow 0. \tag{17.14}$$

*Proof* Write $\mathbb{P}^n_k = \operatorname{Proj} R$, where $R = k[x_0, \ldots, x_n]$. The module $\Omega_{R/k} = \bigoplus_{i=0}^{n} R\,dx_i$ is naturally a graded $R$-module if we give each $dx_i$ degree 1.

Next consider the 'Euler map' $e\colon \Omega_{R/k} \to R$, defined by $dx_i \mapsto x_i$. This is a map of graded $R$-modules. Let $M = \operatorname{Ker} e$ and consider the exact sequence

$$0 \longrightarrow M \longrightarrow \Omega_{R/k} \xrightarrow{e} R.$$

If we apply the tilde functor, we obtain the sequence of sheaves

$$0 \longrightarrow \widetilde{M} \longrightarrow \widetilde{\Omega_{R/k}} \xrightarrow{\tilde{e}} \mathcal{O}_{\mathbb{P}^n_k} \tag{17.15}$$

Note that with the above grading, we have $\widetilde{\Omega_{R/k}} \simeq \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n_k}(-1)$. This will be the middle sheaf in the sequence (17.14). While $e$ is not surjective, we will show that the map $\tilde{e}$ *is* surjective, so that (17.15) is a short exact sequence. To prove this, it suffices to show this over each distinguished open $D_+(x_i)$, and for simplicity, we assume $i = 0$.

Over $D_+(x_0)$, the $(R_{x_0})_0$-module $\Gamma(D_+(x_0), \widetilde{\Omega_{R/k}}) = \big((\Omega_{R/k})_{x_0}\big)_0$ is free of rank $n+1$, with basis

$$\frac{dx_0}{x_0}, \frac{dx_1}{x_0}, \ldots, \frac{dx_n}{x_0}. \tag{17.16}$$

We then have

$$e\left(\frac{dx_0}{x_0}\right) = 1, \quad e\left(\frac{dx_j}{x_0}\right) = \frac{x_j}{x_0} \quad \text{for } j = 1, \ldots, n$$

This means that $\tilde{e}$ is surjective over $D_+(x_0)$. Moreover, as the modules involved are free $(R_{x_0})_0$-modules, the kernel $M_{x_0}$ is also free; it is generated by the $n$ elements

$$\frac{x_j}{x_0^2}dx_0 - \frac{1}{x_0}dx_j = d\left(\frac{x_j}{x_0}\right) \quad \text{for } j = 1, \ldots, n. \tag{17.17}$$

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Next, we show that $\widetilde{M} \simeq \Omega_{\mathbb{P}^n_k/k}$. Note that the universal derivation $d\colon R \to \Omega_{R/k}$ preserves the grading, and so it induces a morphism of $\mathcal{O}_{\mathbb{P}^n_k}$-modules

$$\tilde{d}\colon \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow \widetilde{\Omega_{R/k}}.$$

Over $D_+(x_0)$, it is given by

$$\tilde{d}\left(\frac{f}{x_0^m}\right) = \frac{x_0^m\, df - fmx_0^{m-1}dx_0}{x_0^{2m}}$$

Note that if $f \in R_m$ is a homogeneous polynomial of degree $m$, Euler's formula implies that $df = \sum_{i=0}^n \frac{\partial f}{\partial x_i} dx_i$ is sent to $m \cdot f$ by the map $e$. This means that

$$e\left(\tilde{d}\left(\frac{f}{x_0^m}\right)\right) = e\left(\frac{x_0^m\, df - fmx_0^{m-1}dx_0}{x_0^{2m}}\right) = \frac{x_0^m m \cdot f - fmx_0^{m-1}x_0}{x_0^{2m}} = 0 \quad (17.18)$$

Note that over $D_+(x_0)$, the map $\tilde{d}$ induces the universal derivation $(R_{x_0})_0 \to \Omega_{(R_{x_0})_0/k}$. By (17.18), this takes values inside $(M_{x_0})_0$, and in fact $(M_{x_0})_0 = \Omega_{(R_{x_0})_0/k}$ as both modules are generated by the expressions in (17.17). This happens over the other distinguished opens $D_+(x_i)$ as well, so we get isomorphisms $\widetilde{M}|_{D_+(x_i)} \simeq \Omega_{\mathbb{P}^n_k/k}|_{D_+(x_i)}$. As these are induced by the universal derivation, they are forced to agree over the overlaps $D_+(x_ix_j)$ as well. This shows that $\widetilde{M} \simeq \Omega_{\mathbb{P}^n_k/}$ and we get the exact sequence (17.14). □

Since $\Omega_{\mathbb{P}^n_k/k}$ injects into $\mathcal{O}_{\mathbb{P}^n_k}(-1)^{n+1}$, which has no global sections, we see that

$$\Gamma(\mathbb{P}^n_k, \Omega_{\mathbb{P}^n_k/k}) = 0 \quad (17.19)$$

**Example 17.21.** On $\mathbb{P}^1_k$, the Euler sequence takes the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k}(-1) \oplus \mathcal{O}_{\mathbb{P}^1_k}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k} \longrightarrow 0.$$

In particular, the exact sequence is not split exact. △

## 17.7 Nonsingularity and smoothness

Recall that a scheme $X$ is said to be nonsingular at a point $P \in X$, if $\mathcal{O}_{X,P}$ is a regular local ring. Moreover, if $X$ is of finite type over a field $k$, we defined $X$ to be *smooth* at $P$ if there is an affine open $U = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ containing $P$, so that the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_r}{\partial x_1} & \cdots & \frac{\partial f_r}{\partial x_n} \end{pmatrix}. \quad (17.20)$$

has maximal rank at $P$, that is $\operatorname{rank} J(P) = n - \dim_P X$.

**Proposition 17.22.** Let $X$ be an integral scheme of finite type over a field $k$. Then $X$ is smooth over $k$ if and only if $\Omega_{X/k}$ is locally free of rank $\dim X$.

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*Proof* Let us first assume that $P \in X$ is a closed point. The statements involved are local in nature, so we reduce to the case where $X = \operatorname{Spec} A$, where

$$A = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r).$$

The conormal sequence takes the form

$$A^r \xrightarrow{J^t} A^n \longrightarrow \Omega_{A/k} \longrightarrow 0$$

Tensoring this by $\kappa(P)$, we obtain

$$\kappa(P)^r \xrightarrow{J(P)^t} \kappa(P)^n \longrightarrow \Omega_{A/k} \otimes_A \kappa(P) \longrightarrow 0. \tag{17.21}$$

In other words, $\Omega_{A/k} \otimes_A \kappa(P)$ equals the cokernel of the transpose of the Jacobian matrix $J(P)$. This shows that

$$\dim_{\kappa(P)} \big(\Omega_{A/k} \otimes_A \kappa(P)\big) = n - \operatorname{rank}(J(P)) \tag{17.22}$$

Now, $X$ is smooth at $P$ if and only if $\operatorname{rank}(J(P)) = n - \dim X$ which by the formula happens if and only if $\dim_{\kappa(P)}(\Omega_{A/k} \otimes_A \kappa(P)) = \dim X$. Therefore, if $\Omega_{X/k}$ is locally free of rank $\dim X$ in a neighbourhood of $P$, then $X$ is smooth at $P$.

For the converse direction, note that $\Omega_{X/k}$ is an $\mathcal{O}_X$-module of finite type, so there is an open subset $W \subset X$ for which $\Omega_{X/k}|_W$ is locally free (see Section 14.8). The rank is in fact equal to $\dim X$ here, because $X$ contains a smooth point, by Corollary 10.22. If we assume that $J(P)$ has rank $n - \dim X$ for every closed point $P \in X$. Then $\dim_{\kappa(P)}(\Omega_{A/k} \otimes_A \kappa(P)) = \dim X$ for every closed point $P$. Recall that the function $P \mapsto \dim_{\kappa(P)} \Omega_{X/k}(P)$ can only increase upon specialization (???). As this takes the value $\dim X$ at the generic point, and every closed point, Corollary 14.36 shows that $\Omega_{X/k}$ is locally free of rank $\dim X$. □

**Corollary 17.23.** If $X$ is a variety over an algebraically closed field $k$, then the following are equivalent:

(i) $X$ is nonsingular
(ii) $X$ is smooth
(iii) $\Omega_{X/k}$ is locally free of rank $\dim X$.

**Example 17.24.** For the curve $X \subset \mathbb{A}^2_k$ defined by $f = y^2 + x^p + t$ over $k = \mathbb{F}_p(t)$ from Example **??**, the module of differentials is given by $\Omega_{A/k} = A\,dx \oplus A\,dy/(2y)$. While $X$ is nonsingular, $\Omega_{X/k} = \widetilde{\Omega_{A/k}}$ is not locally free in a neighbourhood of the point $\mathfrak{m} = (x^p + y, y)$. △

**Example 17.25.** $\Omega_{X/k}$ can be locally free (of the wrong rank) even if $X$ is not smooth. For instance, the scheme $X = \operatorname{Spec} \mathbb{F}_p[\epsilon]/(\epsilon^p)$ is not smooth over $k = \mathbb{F}_p(t)$, but $\Omega_{X/k} = \widetilde{k[\epsilon]/\epsilon^p\, d\epsilon}$ is locally free of rank 1. △

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## 17.8 The Tangent sheaf

For a scheme $X$ over a field $k$, we define the *tangent sheaf* $\mathcal{T}_X$, as the dual of the sheaf of differentials,

$$\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X).$$

This is a quasi-coherent sheaf on $X$, and it coherent if $X$ is of finite type over $k$. To see this, note that if $U = \operatorname{Spec} A$ is an affine open set of $X$, then $\mathcal{T}_X(U) = \operatorname{Hom}_A(\Omega_{A/k}, A) = \operatorname{Der}_k(A, A)$ is the $A$-module of $k$-linear derivations of $A$.

The name comes from the following. If $P \in X$ is a $k$-point, then the sequence (17.21), and the Jacobian Criterion, shows that there is an isomorphism of $k$-vector spaces

$$\mathfrak{m}_P/\mathfrak{m}_P^2 \simeq \Omega_{\mathcal{O}_{X,P}/k} \otimes_{\mathcal{O}_{X,P}} \kappa(P)$$

Taking duals, we obtain the following isomorphism

$$(\mathcal{T}_X)_P \otimes_{\mathcal{O}_{X,P}} \kappa(P) \simeq \operatorname{Hom}_{\kappa(P)}(\mathfrak{m}_P/\mathfrak{m}_P^2, \kappa(P)) = T_P X.$$

Thus $\mathcal{T}_X$ is a sheaf which collects all the tangent spaces as its fibers.

**Example 17.26.** For $X = \mathbb{A}_k^n$, we get $\Omega_{\mathbb{A}_k^n} = \mathcal{T}_{\mathbb{A}_k^n} = \mathcal{O}_{\mathbb{A}_k^n}^n$. △

**Example 17.27** (The tangent sheaf of $\mathbb{P}_k^n$)**.** Taking the dual of the Euler sequence, we get the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n} \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(1)^{n+1} \longrightarrow \mathcal{T}_{\mathbb{P}_k^n} \longrightarrow 0. \tag{17.23}$$

Taking the long exact sequence in cohomology, find that $\Gamma(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}_k^n})$ is a $k$-vector space of dimension $(n+1)^2 - 1$, and $H^i(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}_k^n}) = 0$ for all $i > 0$. △

## 17.9 The sheaf of $p$-forms

If $X$ is a variety of dimension $n$ over a field $k$, and $p$ is an integer we define the *sheaf of $p$-forms* as the $\mathcal{O}_X$-module

$$\Omega_{X/k}^p = \bigwedge^p \Omega_{X/k}.$$

On affine space $\mathbb{A}_k^n$, the sheaf of $p$-forms on affine space is not particularly interesting, as $\Omega_{\mathbb{A}_k^n/k}^p$ is free of rank $\binom{n}{p}$. However, for projective varieties, the sheaves $\Omega_{X/k}^p$ are of significant interest because they allow us to define important invariants of the variety.

The most important of these is the sheaf $\Omega_{X/k}^n$, which is usually called the *canonical sheaf* or the *canonical bundle* of $X$. If $X$ is smooth over $k$, then $\Omega_{X/k}$ is locally free of rank $n$, and so $\Omega_{X/k}^n$ is an invertible sheaf. The canonical sheaf plays a fundamental role in the classification of varieties. We will demonstrate this in the setting of curves in Chapter 18.

The following gives a computation of the canonical sheaf of projective space.

**Proposition 17.28.** $\Omega_{\mathbb{P}_k^n}^n = \mathcal{O}_{\mathbb{P}_k^n}(-n-1)$.

*Proof* Consider the Euler sequence for the cotangent bundle of $\mathbb{P}^n$

$$0 \longrightarrow \Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-1)^{\oplus(n+1)} \longrightarrow \mathcal{O}_{\mathbb{P}_k^n} \longrightarrow 0$$

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Taking determinants, we get

$$\det\left(\mathcal{O}_{\mathbb{P}^n_k}(-1)^{\oplus(n+1)}\right) = \mathcal{O}_{\mathbb{P}^n_k}(-n-1) = \det(\Omega_{\mathbb{P}^n_k}) \otimes \det(\mathcal{O}_{\mathbb{P}^n_k}) = \Omega^n_{\mathbb{P}^n_k}.$$

□

We can use the Euler sequence to compute the cohomology groups of the sheaves of $p$-forms on $\mathbb{P}^n_k$ as follows. By Exercise 17.11.20, there are exact sequences

$$\begin{aligned}
0 &\longrightarrow \Omega^1_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-1)^{\binom{n+1}{1}} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k} \longrightarrow 0. \\
0 &\longrightarrow \Omega^2_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-2)^{\binom{n+1}{2}} \longrightarrow \Omega^1_{\mathbb{P}^n_k} \longrightarrow 0. \\
&\vdots \\
0 &\longrightarrow \Omega^n_{\mathbb{P}^n_k} \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(-n)^{\binom{n+1}{n}} \longrightarrow \Omega^{n-1}_{\mathbb{P}^n_k} \longrightarrow 0.
\end{aligned} \tag{17.24}$$

**Proposition 17.29.** Let $k$ be a field and let $n \in \mathbb{N}$. Then

$$H^q(\mathbb{P}^n_k, \Omega^p_{\mathbb{P}^n_k}) = \begin{cases} k & \text{if } p = q \text{ and } 1 \leqslant p \leqslant n \\ 0 & \text{otherwise} \end{cases} \tag{17.25}$$

*Proof* Recall that the sheaves $\mathcal{O}_{\mathbb{P}^n_k}(-d)$ have no higher cohomology on $\mathbb{P}^n_k$ for $d = 0, \ldots, n$. Therefore, the long exact sequence applied to (17.24) shows that for $p = 1, \ldots n$

$$H^q(X, \Omega^p_{\mathbb{P}^n_k}) = H^{q-1}(X, \Omega^{p-1}_{\mathbb{P}^n_k}).$$

The result then follows by induction on $p$, starting with $p = 0$, where the it follows from Theorem 15.24. □

Using the conormal sequence, we can also study differentials on other projective varieties.

**Example 17.30** (Hypersurfaces)**.** Let $X \subset \mathbb{P}^n_k$ be a smooth hypersurface of degree $d$. The conormal sequence takes the form

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathbb{P}^n_k/k}|_X \longrightarrow \Omega_{X/k} \longrightarrow 0. \tag{17.26}$$

Note that $\mathcal{I}/\mathcal{I}^2 \simeq \mathcal{O}_X(-d)$ is an invertible sheaf. Therefore, the left-most map in the sequence (17.26) is in fact injective (the map is nonzero because the term in the middle has rank $n+1$). Hence we get the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \Omega_{\mathbb{P}^n_k/k}|_X \longrightarrow \Omega_{X/k} \longrightarrow 0. \tag{17.27}$$

Taking determinants, we get that $\Omega^n_{\mathbb{P}^n_k}|_X \simeq \Omega^{n-1}_X \otimes \mathcal{O}_X(-d)$, and hence

$$\Omega^{n-1}_X \simeq \mathcal{O}_X(d-n-1).$$

△

**Example 17.31** (Quartic surfaces)**.** Let $X \subset \mathbb{P}^3_k$ be a smooth surface defined by a homogeneous polynomial of degree 4. We can compute the cohomology groups of $\Omega^1_X$ and

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$\Omega_X^2 \simeq \mathcal{O}_X$ using the following three exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathbb{P}^3_k}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3_k} \longrightarrow \iota_* \mathcal{O}_X \longrightarrow 0 \\ 0 &\longrightarrow \mathcal{O}_X(-4) \longrightarrow \Omega^1_{\mathbb{P}^3_k/k}|_X \longrightarrow \Omega^1_{X/k} \longrightarrow 0 \\ 0 &\longrightarrow \Omega_{\mathbb{P}^3_k/k}(-4) \longrightarrow \Omega^1_{\mathbb{P}^3_k/k} \longrightarrow \Omega^1_{\mathbb{P}^3_k/k}|_X \longrightarrow 0 \end{aligned}$$

Here the first sequence is the ideal sheaf sequence of $X$, the second is the conormal sequence, and the third is obtained by tensoring the first by $\Omega_{\mathbb{P}^3_k/k}$. From the corresponding long exact sequences in cohomology, we compute the various $h^{p,q} = \dim_k H^q(X, \Omega_X^p)$, shown in the following table (the 'Hodge diamond' of $X$):

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{0,1} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{2,1} & & h^{1,2} & \\ & & h^{2,2} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

△

## 17.10 Application: irrationality of hypersurfaces

One of the main reasons for introducing the sheaves $\Omega_X$ and $\Omega_X^p$ is to define geometric invariants. As these are intristically defined, they provide a useful method to distinguish varieties, i.e., show that two varieties are not isomorphic.

In some cases, these invariants are strong enough to even distinguish varieties up to *birational equivalence*. The aim of this section is to prove the following theorem.

**Theorem 17.32.** Let $k$ be a field and let $X \subset \mathbb{P}^n_k$ be a nonsingular hypersurface of degree $d$. If $d \geqslant n+1$, then $X$ is not rational.

In the theorem, it is essential that the hypersurface is nonsingular. For instance, the curve defined by

$$x_0 x_1^{d-1} - x_2^d = 0$$

is rational: it is the image of the morphism $\mathbb{P}^1_k \to \mathbb{P}^2_k$ $(s:t) \mapsto (st^{d-1} : t^d : s^d)$

**Proposition 17.33.** Let $f\colon X \dashrightarrow Y$ be a birational map of varieties over a field $k$ and assume that $X$ is normal. Then the locus of points $x \in X$ where $f$ is not defined has codimension at least 2 in $X$.

**Theorem 17.34.** Let $f\colon X \dashrightarrow Y$ be a birational map between smooth projective varieties over an algebraically closed field. Then, for any integer $p \geqslant 0$, there is an isomorphism of global sections of the sheaves of differential forms:

$$H^0(X, \Omega_X^p) \simeq H^0(Y, \Omega_Y^p).$$

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*Proof* Since $f\colon X \dashrightarrow Y$ is a birational map, there exists a non-empty open subset $U \subset X$ such that the map $f\colon U \to Y$ is a morphism and induces an isomorphism between $U$ and its image $f(U) \subset Y$. The complement $X - U$ has codimension at least 2 in $X$, because $X$ is nonsingular, and birational maps between normal varieties are isomorphisms outside subsets of codimension at least 2.

The morphism $f\colon U \to Y$ induces a pullback map:

$$f^*\colon \Omega^1_Y \longrightarrow \Omega^1_U.$$

Since $f$ is an isomorphism on $U$, the pullback map $f^*$ is an isomorphism over $U$. Taking the exterior powers, this induces an isomorphism:

$$f^*\colon \Omega^p_Y \longrightarrow \Omega^p_U.$$

Since $X$ is nonsingular, and $k$ is algebraically closed, the sheaf $\Omega^p_X$ is locally free and the sections of $\Omega^p_X$ over $U$ extend uniquely to sections over $X$ by the Algebraic Hartogs' theorem (see Exercise 17.11.24, which applies because $X - U$ has codimension at least 2). Thus, the restriction map:

$$H^0(X, \Omega^p_X) \longrightarrow H^0(U, \Omega^p_U)$$

is an isomorphism.

The morphism $f\colon U \longrightarrow Y$ is dominant, so the pullback map on global sections

$$f^*\colon H^0(Y, \Omega^p_Y) \longrightarrow H^0(U, \Omega^p_U)$$

is injective (see Exercise 17.11.23). Combining this with the isomorphism $H^0(U, \Omega^p_U) \simeq H^0(X, \Omega^p_X)$, we obtain an injective map:

$$H^0(Y, \Omega^p_Y) \longrightarrow H^0(X, \Omega^p_X).$$

By symmetry, the same argument applies to the inverse birational map $Y \dashrightarrow X$, which gives an injective map $H^0(X, \Omega^p_X) \to H^0(Y, \Omega^p_Y)$. Combining these, we conclude that

$$H^0(X, \Omega^p_X) \simeq H^0(Y, \Omega^p_Y). \qquad \square$$

We are now ready to prove the main theorem of this section.

*Proof of Theorem 17.32* We apply the previous theorem to the case $p = \dim X = n - 1$. Then

$$\Omega_X \simeq \mathcal{O}_X(n - d - 1).$$

When $n \geqslant d + 1$, we have $H^0(X, \Omega^{n-1}_X) \neq 0$, by the following exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(n - 2d - 1) \longrightarrow \mathcal{O}_{\mathbb{P}^n_k}(n - d - 1) \longrightarrow \mathcal{O}_X(n - d - 1) \longrightarrow 0.$$

Note that $n \geqslant d + 1 \geqslant 2$ implies that $H^1(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(m)) = 0$ for all $m \in \mathbb{Z}$. $\square$

## 17.11 Exercises

**Exercise 17.11.1.** Let $k$ be a field. Compute $\Omega_{A/k}$ for the following $k$-algebras:

a) $A = k[x, y]/(x^2 + y^2)$. HINT: This will depend on the characteristic of $k$.

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b) $A = k[x, y](xy)$
c) $A = k[x, y, z]/(y^2 + x^3 - x^2z^2)$

**Exercise 17.11.2** (Tangent vectors and derivations)**.** Recall that for a scheme $X$ over a field $k$, the tangent space $T_pX$ at a $k$-point $p \in X$ is defined as the $k$-vector space of linear functionals $\ell\colon \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$. There is a natural way to identify $T_pX$ with the vector space of derivations of $\mathcal{O}_{X,p}$ into $k$, that is, $\mathrm{Der}_k(\mathcal{O}_{X,p}, k)$. Write $A = \mathcal{O}_{X,p}$ and $\mathfrak{m} = \mathfrak{m}_p$.

a) Show that the structure map $k \to A/\mathfrak{m}^2$ induces a splitting $A/\mathfrak{m}^2 = k \oplus \mathfrak{m}/\mathfrak{m}^2$.
b) For $f \in A$, let $Df$ denote the image of $f$ in $\mathfrak{m}/\mathfrak{m}^2$ via the composition $A \to A/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$, i.e., $Df = f - f(p) \mod \mathfrak{m}^2$. Show that $D$ is $k$-linear and satisfies the Leibniz rule.
c) Show that for each $k$-linear map $\ell\colon \mathfrak{m}/\mathfrak{m}^2 \to k$, the composition $D \circ \ell$ is a $k$-linear derivation $A \to k$.
d) Show that this defines an isomorphism $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k) \simeq \mathrm{Der}_k(A, k)$.

**Exercise 17.11.3** (Base change)**.** Show that for the $k$-algebra $C = A \otimes_k B$, there is an isomorphism

$$\Omega_{C/B} = \Omega_{B/A} \otimes_B C.$$

HINT: Show that $d \otimes \mathrm{id}_C$ is the universal derivation.

**Exercise 17.11.4** (Localization)**.** For a multiplicative set $S \subset A$, show that there is an isomorphism of $S^{-1}A$-modules $\Omega_{S^{-1}A/k} = S^{-1}(\Omega_{A/k})$. HINT: Define the universal derivation by $d_{S^{-1}A}(a/s) = (sd_Aa - ad_As)/s^2$.

**Exercise 17.11.5** (Products)**.** Let $A$ and $B$ be $k$-algebras.

a) Show that $\Omega_{(A\times B)/k} = \Omega_{A/k} \otimes \Omega_{B/k}$, with universal derivation $d_{A\times B} = d_A \times d_B$.
b) Show that $\Omega_{A\otimes_k B/k} = (\Omega_{A/k} \otimes_k B) \oplus (\Omega_{B/k} \otimes_k A)$ with universal derivation $d(a \otimes b) = (d_Aa) \otimes b + a \otimes (d_Bb)$ for $a \in A$, $b \in B$.

**Exercise 17.11.6** (The differentials of a tensor product)**.** Let $B$ and $C$ be two $A$-algebras. Consider the map

$$d\colon B \otimes_A C \to (\Omega_{B/A} \otimes_A C) \oplus (B \otimes_A \Omega_{C/A})$$

given by $b \otimes c \mapsto b \otimes d_Cc + d_Bb \otimes c$. Show that $d$ is an $A$-derivation, and show that

$$\Omega_{B\otimes_A C/A} \simeq B \otimes_A C \to (\Omega_{B/A} \otimes_A C) \oplus (B \otimes_A \Omega_{C/A}).$$

**Exercise 17.11.7.** Prove the properties of the sheaf of differentials listed in (17.12).

**Exercise 17.11.8.** Prove Euler's formula, that $\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = d \cdot f$ whenever $f$ is a homogeneous polynomial of degree $d$ in $x_0, \ldots, x_n$. HINT: Prove it first for monomials.

**Exercise 17.11.9.** Let $A \subset B$ be a ring extension. Show that

$$\Omega_{B[t]/A} = B[t]dt \oplus (\Omega_{B/A} \otimes_B B[t]).$$

**Exercise 17.11.10.** Let $B = k[x, y]/(x^2 + y^2)$. Show that if $k$ has characteristic $\neq 2$, then

$$\Omega_{B/k} = (B\,dx \oplus B\,dy)\,/(x\,dx + y\,dy)$$

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and if $k$ has characteristic 2, then $\Omega_{B/k} = B\,dx \oplus B\,dy$.

**Exercise 17.11.11.** Show that for a point $x \in X$, the stalks of $\Omega_{X/S}$ are given by

$$(\Omega_{X/S})_x \simeq \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,f(x)}}. \tag{17.28}$$

**Exercise 17.11.12.** Let $\mathcal{F}$ be an $\mathcal{O}_X$ module. A morphism $D\colon \mathcal{O}_X \to \mathcal{F}$ of $\mathcal{O}_X$-modules is an *$S$-derivation* if for all open affine subsets $V \subset S$ and $U \subset X$ with $f(U) \subset V$, the map $D|_U$ is an $\mathcal{O}_S(V)$-derivation of $\mathcal{O}_X(U)$ with values in $\mathcal{F}$. The set of all such $S$-derivations is denoted by $\mathrm{Der}_S(\mathcal{O}_X, \mathcal{F})$. Show that the sheaf $\Omega_{X/S}$ represents the functor of $S$-derivations, in the sense that there is a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, -) \simeq \mathrm{Der}_S(\mathcal{O}_X, -).$$

Show that this property defines $\Omega_{X/S}$ up to isomorphism.

**Exercise 17.11.13.** In this chapter, $\Omega_{X/S}$ was defined for a separated morphism $X \to S$. Show that $\Omega_{X/S}$ can be defined in general, without the separatedness assumption. HINT: Show directly that the sheaves $\widetilde{\Omega_{B/A}}$ glue for each affine $\operatorname{Spec} A \subset S$ and $\operatorname{Spec} B \subset X$ with $\operatorname{Spec} B$ mapping into $\operatorname{Spec} A$.

**Exercise 17.11.14** (Smooth morphisms)**.** Let $f\colon X \to S$ be a morphism and let $p \in X$ be a point. We say that $f$ is *smooth* of relative dimension $r$ at $p$ if there exist affine open sets $U = \operatorname{Spec} B \subset X$, containing $p$, and $V = \operatorname{Spec} A \subset S$ with $f(U) \subset V$ and an open embedding $\iota\colon U \to \operatorname{Spec} A[t_1, \ldots, t_n]/(f_1, \ldots, f_{n-r})$ such $f|_U$ factors as

$$U \xhookrightarrow{\iota} \operatorname{Spec} A[t_1, \ldots, t_n]/(f_1, \ldots, f_{n-r}) \longrightarrow \operatorname{Spec} A$$

and such that the Jacobian matrix $J(p)$ of the $f_i$ has rank $n - r$ at $\iota(p)$.

a) Show that $\mathbb{A}^n_S \to S$ and $\mathbb{P}^n_S \to S$ are smooth for any scheme $S$.
b) Show that the property of smoothness is stable under base change.
c) Show that if $f\colon X \to S$ is smooth of relative dimension $r$, then $\Omega_{X/S}$ is locally free of rank $r$.

**Exercise 17.11.15.** Consider the morphism $f\colon \operatorname{Spec} \mathbb{Z}[x,y]/(xy-3) \to \operatorname{Spec} \mathbb{Z}$. Compute the *smooth locus* of $f$, that is, the set of points $p \in \operatorname{Spec} \mathbb{Z}[x,y]/(xy-3)$ so that $f$ is smooth at $p$. Find a maximal open set $U \subset \operatorname{Spec} \mathbb{Z}$ for which $f^{-1}(U) \to U$ is smooth.

**Exercise 17.11.16** (The conormal sheaf)**.** Let $X$ be a smooth scheme over a field and let $Y$ be a smooth subscheme defined by an ideal sheaf $\mathcal{I}$. The sheaf $\mathcal{I}/\mathcal{I}^2$ is naturally an $\mathcal{O}_Y$-module via $\mathcal{I}/\mathcal{I}^2 = \mathcal{I} \otimes \mathcal{O}_X/\mathcal{I} = \mathcal{I} \otimes \mathcal{O}_Y$. We call $\mathcal{I}/\mathcal{I}^2$ the *conormal sheaf* of $Y$. Its dual, $N_Y = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is the *normal sheaf* of $Y$ in $X$.

a) Show that the sheaves $\mathcal{I}/\mathcal{I}^2$ and $N_Y$ are locally free of rank $r = \operatorname{codim}(Y, X)$.
b) Show that the conormal sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X|k}|_Y \to \Omega_{Y|k} \to 0$$

is exact.
c) Show that

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to N_Y \to 0$$

is exact.

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**Exercise 17.11.17.** Let $k$ be an algebraically closed field, and let $X$ and $Y$ be two smooth hypersurfaces in $\mathbb{P}^n_k$ of different degrees $d$ and $e$. Show that $X$ and $Y$ are not isomorphic unless $n = 2$ and either $(d, e) = (1, 2)$ or $(d, e) = (2, 1)$.

**Exercise 17.11.18.** Let $k$ be a field and let $X \subset \mathbb{A}^2_k$ be the subscheme defined by the ideal $I = (x^2, y^2)$. Show that $I/I^2 \to \Omega_{\mathbb{A}^2_k/k}|_X$ is *not* surjective. What is the kernel?

**Exercise 17.11.19.** Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of locally free sheaves.

a) If $\mathcal{E}$ is an invertible sheaf, show that for any $p \geqslant 1$, there is an exact sequence

$$0 \to \mathcal{E} \otimes \operatorname{Sym}^{p-1} \mathcal{F} \to \operatorname{Sym}^p \mathcal{F} \to \operatorname{Sym}^p \mathcal{G} \to 0.$$

b) If $\mathcal{G}$ is an invertible sheaf, show that for any $p \geqslant 1$, there is an exact sequence

$$0 \to \operatorname{Sym}^p \mathcal{E} \to \operatorname{Sym}^p \mathcal{F} \to \operatorname{Sym}^{p-1} \mathcal{E} \otimes \mathcal{G} \to 0.$$

**Exercise 17.11.20.** Let $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of locally free sheaves.

a) If $\mathcal{E}$ is an invertible sheaf, show that for any $p \geqslant 1$, there is an exact sequence

$$0 \to \mathcal{E} \otimes \wedge^{p-1}\mathcal{F} \to \wedge^p \mathcal{F} \to \wedge^p \mathcal{G} \to 0.$$

b) If $\mathcal{G}$ is an invertible sheaf, show that for any $p \geqslant 1$, there is an exact sequence

$$0 \to \wedge^p \mathcal{E} \to \wedge^p \mathcal{F} \to \wedge^{p-1} \mathcal{E} \otimes \mathcal{G} \to 0.$$

**Exercise 17.11.21.** Use Exercise 17.11.20 to show that the sequences (17.24) are exact.

**Exercise 17.11.22** (Bott vanishing)**.** Let $k$ be a field. Show that $H^q(\mathbb{P}^n_k, \Omega^p_{\mathbb{P}^n_k}(d)) = 0$ except when:

(i) $p = q$ and $d = 0$.
(ii) $q = 0$ and $d > p$.
(iii) $q = n$ and $d > p - n$.

**Exercise 17.11.23.** Let $f\colon X \to Y$ be a dominant morphism of schemes, and let $\mathcal{E}$ be a locally free sheaf on $Y$.

a) Show that if $Y$ is integral, then the pullback map

$$\Gamma(Y, \mathcal{E}) \longrightarrow \Gamma(X, f^*\mathcal{E}) \tag{17.29}$$

is injective.

b) Show that (17.29) is not injective for the morphism $f\colon \operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/(\epsilon^2)$ when $\mathcal{E}$ is the structure sheaf.

**Exercise 17.11.24** (Algebraic Hartogs for locally free sheaves)**.** Let $X$ be a Noetherian normal scheme and let $\mathcal{E}$ be a locally free sheaf of finite rank on $X$. Show that if $U \subset X$ is an open set so that $X - U$ has codimension at least 2, then the restriction map

$$\Gamma(X, \mathcal{E}) \longrightarrow \Gamma(U, \mathcal{E}|_U)$$

is an isomorphism. HINT: This may be reduced to the usual statement for $\mathcal{E} = \mathcal{O}_X$.

**Exercise 17.11.25.** Let $X \subset \mathbb{P}^1_k \times \mathbb{P}^2_k$ be a nonsingular surface defined by a bihomogeneous equation $f$ of bidegree $(d, e)$.

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a) For $d = 1$, describe $X$ and the two morphisms to $\mathbb{P}^1$ and $\mathbb{P}^2$. In particular, show that $X$ is rational.
b) For $e = 1$, show that $X$ is rational.
c) Show that $X$ is not rational if $d \geqslant 2$ and $e \geqslant 3$.
d) * Show that $X$ is rational if $e = 2$. HINT: One way is to apply *Tsen's theorem* on quadrics over $k(t)$.

**Exercise 17.11.26.** Let $X \subset \mathbb{P}^2_k \times \mathbb{P}^2_k$ be a nonsingular 3-fold defined by a bihomogeneous equation $f$ of bidegree $(d, e)$.

a) For $d = 1$ or $e = 1$, describe $X$ and the two morphisms to $\mathbb{P}^2$. In particular, show that $X$ is rational.
b) Show that $X$ is not rational if $d, e \geqslant 3$.

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# Part III

## Curves



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# 18

# Curves

In this chapter, we will study curves. Recall that a *curve* is a variety of dimension 1 over a field $k$. We will mostly be interested in the case when the field $k$ is algebraically closed, although most of the results in this chapter remain valid over more general fields, for instance, perfect fields.

We will also focus on nonsingular curves. In other words, each local ring $\mathcal{O}_{X,P}$ is a regular local ring of Krull dimension 1. Equivalently, for each $k$-point $x$, the cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ is 1-dimensional as a vector space over $k$.

**Proposition 18.1.** Let $X$ be a curve over an algebraically closed field $k$. Then the following are equivalent

(i) $X$ is nonsingular
(ii) $X$ is smooth over $k$
(iii) $\Omega_{X/k}$ is an invertible sheaf
(iv) $X$ is normal.

*Proof* As $k$ is algebraically closed, the equivalence of (i) $\Leftrightarrow$ (ii) holds in any dimension, see Proposition 10.18. The equivalence (i) $\Leftrightarrow$ (iv) holds by Corollary 10.48 as each local ring $\mathcal{O}_{X,P}$ has dimension 1. □

What makes the study of curves special is that the local rings $\mathcal{O}_{X,P}$ at (closed) points are discrete valuation rings, so they satisfy the statements in Proposition 10.47. In particular, this means that the maximal ideal $\mathfrak{m}_P \subset \mathcal{O}_{X,P}$ is a principal ideal. A generator $t \in \mathfrak{m}_P$ for $\mathfrak{m}_P$ is called a *local parameter* at $P$.

Any non-zero rational function $f \in K(X)$ can be expressed uniquely $f = u \cdot t^n$ where $n \in \mathbb{Z}$ and $u$ is a unit in $\mathcal{O}_{X,x}$. The corresponding valuation map

$$\operatorname{ord}_P \colon K(X)^\times \longrightarrow \mathbb{Z}$$

is defined by setting $\operatorname{ord}_P(f) = n$. Note that $\operatorname{ord}_P(\mathbb{Z}_{\geqslant 0}) = \mathcal{O}_{X,P}$ and $\operatorname{ord}_P(\mathbb{Z}_{\geqslant 1}) = \mathfrak{m}_P$.

This fact will be important later on, when we study morphisms from curves to other varieties and when we study divisors.

**Example 18.2.** If $X$ is a curve over $k$, then the normalization $\overline{X}$ is a nonsingular curve. The normalization map $\pi \colon \overline{X} \to X$ is finite and birational. △

**Example 18.3** (Local parameters for plane curves)**.** Let $X \subset \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ be the affine curve defined by an equation $f(x,y) = 0$ and let $P = (a,b)$ be a $k$-point of $X$. Then



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$\mathfrak{m}_P/\mathfrak{m}_P^2$ is generated by $x-a$ and $y-b$. If we assume that $X$ is nonsingular at $P$, the tangent space is defined by the linear equation

$$\frac{\partial f}{\partial x}(P)\cdot(x-a)+\frac{\partial f}{\partial y}(P)\cdot(y-b)=0. \tag{18.1}$$

This means that if $\frac{\partial f}{\partial y}(P)\neq 0$, then we can express $y-b$ in terms of $x-a$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$. In other words: *if* $\frac{\partial f}{\partial y}(P)\neq 0$*, then* $x-a$ *is a local parameter at* $P$.

For a concrete example, consider the curve $X$ defined by the equation $x^2+y^2=1$ and let $P=(1,0)$. Then the rational function $x-1$ has order of vanishing 2 at $P$. To see this, note that $y$ is a local parameter at $P$, because $\frac{\partial}{\partial x}(x^2+y^2-1)=2x$ is nonzero at $P$. Moreover, in $\mathcal{O}_{X,P}$, we have the relation $(x-1)(x+1)=-y^2$, and $x+1$ is a unit in $\mathcal{O}_{X,P}$, and hence $\operatorname{ord}_P(x-1)=2$. △

## 18.1 Morphisms between curves

Morphisms between curves $f\colon X\to Y$ are rather simple. There are two cases: either $f$ is not dominant, in which case the image $f(X)$ is a point (being a proper, closed and irreducible subset of $X$); or it is dominant, in which case $f(X)$ is dense in $Y$, hence equal to an open set of $Y$. In the case $X$ and $Y$ are also projective, we can say more.

**Proposition 18.4.** Let $f\colon X\to Y$ be a dominant morphism of curves over $k$.
(i) $K(X)$ is a finite field extension of $K(Y)$.
(ii) If $X$ is projective over $k$, then $f$ is surjective.
(iii) If both $X$ and $Y$ are projective, then $f$ is finite.

*Proof* (i): As $f$ is dominant, $f^\sharp$ induces a map of function fields $K(Y)\to K(X)$. The two function fields $K(X)$ and $K(Y)$ are both of transcendence degree 1 over $k$, and so $K(X)$ is algebraic over $K(Y)$. Note that $X$ is of finite type over $Y$, since it is of finite type over $k$. Therefore $K(X)$ is a finite extension of $K(Y)$.

(ii): If $X$ is projective, then the image $f(X)$ is closed in $Y$ (by Corollary 13.21), hence equal to $Y$, as $f$ is dominant.

(iii): By the assumption, the morphism $f\colon X\to Y$ is projective, and so Proposition 13.24 shows that $f$ is finite. □

**Example 18.5.** It is essential that $X$ is projective for the items (ii) and (iii) to hold: the morphism $\operatorname{Spec}k[x,y]/(xy-1)\to\operatorname{Spec}k[x]$ is dominant, but it is neither finite nor surjective. △

This leads to the notion of the degree of a morphism between curves:

**Definition 18.6** (The degree of a finite morphism)**.** Let $f\colon X\to Y$ be a dominant morphism between curves. The degree of the field exension $K(Y)\subset K(X)$ is called the *degree* of $f$ and is denoted $\deg f$.

Note that if $f\colon X\to Y$ and $g\colon Y\to Z$ are dominant morphisms between curves, the

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composition $g \circ f$ is dominant and $\deg(g \circ f) = \deg f \cdot \deg g$. This follows because $[M : K] = [M : L][L : K]$ for a tower of field extensions $K \subset L \subset M$.

**Example 18.7.** The '$n$-th power map' $f\colon \mathbb{P}^1_k \to \mathbb{P}^1_k$ of Example 12.33 has degree $n$. Indeed, locally it is given by the ring map $k[u] \to k[x]$ sending $u$ to $x^n$, and $k(x) = k(u)[x]/(u - x^n)$ has dimension $n$ as a $k(u)$-vector space. △

**Example 18.8.** If $X$ is a hyperelliptic curve, as discussed in Section B.4, then the morphism $X \to \mathbb{P}^1_k$ has degree 2. In this case, $K(X)$ is obtained from $k(x)$ by adjoining a square root $y$ of a polynomial $f(x) \in k[x]$, so it is spanned by 1 and $y$ as a $k(x)$-vector space. △

## 18.2 Extensions of rational maps

This section presents two fundamental results on nonsingular curves. The first basically says that any rational map from a nonsingular curve into a a projective variety can be extended to a morphism. Combining this with Theorem **??** and the fact that every curve is birationally equivalent to a nonsingular curve, we obtain the second result, which says that the category of projective nonsingular curves over $k$ with dominant maps is equivalent to the category of finitely generated field extensions of $k$ of transcendence degree 1.

**Proposition 18.9.** Let $X$ be an irreducible curve over $k$ and let $P \in X(k)$ be a nonsingular point. Then any morphism $f\colon X - \{P\} \to Y$ to a projective variety $Y$ has a unique extension to a morphism $\overline{f}\colon X \to Y$.

*Proof* It suffices to treat the case where $Y = \mathbb{P}^n_k$. Let $t$ be a local parameter of $\mathcal{O}_{X,P}$, and denote by $K$ the function field of $X$. Restricting to the generic point, the morphism $f$ gives a $K$-point $\operatorname{Spec} K \to \mathbb{P}^n_K$. By Proposition **??**, any such a morphism is described by homogenous coordinates $(a_0 t^{\nu_0} : \cdots : a_n t^{\nu_n})$ where the $a_i$ are units in $\mathcal{O}_{X,P}$ and the $\nu_i$'s are integers. After multiplying by $t^{-\min \nu_i}$ we may assume that $\nu_i \geqslant 0$ for all $i$ and at $\nu_{i_0} = 0$ for at least one $i_0$. Now the $a_i$ are induced by non-vanishing sections of $\mathcal{O}_X$ over some some open neighbourhood $U$ of $P$. Also, after shrinking $U$ if neccesary, we may assume that $t$ will also be a section of $\mathcal{O}_X$ over $U$ with $P$ as the only zero. Therefore, the $a_i t^{\nu_i}$ define a map $U \to \mathbb{P}^n_k$. Over the open set $U - \{P\}$, this morphism coincides with $f$, so we have the desired extension.

Finally, the extension is unique because $X$ is separated over $k$. □

**Corollary 18.10.** Any rational map between two nonsingular projective curves can be extended to a morphism. In particular, if two nonsingular projective curves are birational, then they are isomorphic.

*Proof* Any rational map $X \dashrightarrow Y$ is a morphism outside a finite number of points $x_1, \ldots, x_r$. Therefore, the first statement follows by applying Proposition 18.9 finitely many times.

If two curves $X$ and $Y$ are birationally equivalent, then there are open subsets $U \subset X$ and $V \subset Y$ and an isomorphism $f\colon U \to V$. If $X$ and $Y$ are nonsingular, both $f$ and $f^{-1}$ extend respectively to morphisms $g\colon X \to Y$ and $h\colon Y \to X$, and since $h \circ g|_U = \mathrm{id}_U$ and

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$g \circ h|_V = \mathrm{id}_V$, it follows that $h \circ g = \mathrm{id}_X$ and $g \circ h = \mathrm{id}_Y$ (two morphisms that agree on an open dense set of a separated scheme are equal, by Proposition 9.14). □

**Example 18.11.** Let $X \subset \mathbb{P}^2_k$ be the projective curve defined by the equation $x_0x_2^2 = x_1^3 + x_0x_1^2$. The projection $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1)$ induces a morphism

$$f\colon X - \{(0:0:1)\} \to \mathbb{P}^1_k.$$

To see that $f$ extends over $(0 : 0 : 1)$, following Proposition 18.9, consider the affine open set $D_+(x_2) = \mathbb{A}^2_k$ with affine coordinates $u = x_0/x_2$ and $v = x_1/x_2$. In this chart, the equation of $X$ becomes $u = v^3 + uv^2$ and $(0 : 0 : 1)$ corresponds to $(u, v) = (0, 0)$. The morphism $f$ is given by $(u, v) \mapsto (x : y)$.

Since $u = v^3 + uv^2$, we can express $u$ in terms of $v$ as $u = v^3(1 - v^2)^{-1}$ in $\mathcal{O}_{X,(0,0)}$ (note that $1 - v^2$ is a unit in $\mathcal{O}_{X,(0,0)}$). This shows that $t = v$ is a local parameter at $(0, 0)$. In terms of $t$, the morphism takes the form

$$(u, v) \mapsto \left(\frac{t^3}{1 - t^2} : t\right).$$

Dividing by $t$, shows that the extension is given by

$$(u, v) \mapsto \left(\frac{t^2}{1 - t^2} : 1\right) = \left(t^2 : 1 - t^2\right).$$

Hence the extension of $f$ sends $(0 : 0 : 1)$ to $(0 : 1)$.

Note that while $f$ extends over $(0 : 0 : 1)$ the original map $\mathbb{P}^2 - \{(0 : 0 : 1)\} \to \mathbb{P}^1_k$ does not. △

**Proposition 18.12.** Any nonsingular curve is quasi-projective.

*Proof* Let $X$ be a nonsingular curve and let $U_1, \ldots, U_s$ be an affine cover of $X$. For each $i = 1, \ldots, s$, choose an affine embedding $U_i \to \mathbb{A}^{n_i}_k$, and embed $\mathbb{A}^{n_i}_k \subset \mathbb{P}^n_k$ as a distinguished open set. Let $\overline{U_i}$ denote the closure of $U_i$ in $\mathbb{P}^{n_i}_k$. As $X$ is nonsingular and $\overline{U_i}$ is projective, we may extend the embedding $U_i \to \overline{U_i}$ to a morphism $\phi_i\colon X \to \overline{U_i}$.

Next we consider the product of all of the $\phi_i$

$$\phi : X \longrightarrow \overline{U_1} \times \cdots \times \overline{U_s} \subset \mathbb{P}^{n_1}_k \times \cdots \times \mathbb{P}^{n_s}_k.$$

Let $V$ be the closure of $\phi(X)$. Then $V$ is a projective variety (by the Segre embedding).

Note that if we restrict to $U_i \subset X$, the composition of $\phi$ with the projection $p_i : V \to \overline{U_i}$ to the $i$-th factor coincides with the inclusion $U_i \to \overline{U_i}$. Therefore, the induced morphism $\phi\colon X \to V$ is not constant, so $V$ is a projective curve. Moreover, the image $W = \phi(X)$ is an open set of $Y$. We claim that $\phi : X \to W$ is an isomorphism; this will prove the proposition.

To see this, let $x \in X$, and choose an index $i$ such that $x \in U_i$. As the composition $U_i \xrightarrow{\phi_i} W \xrightarrow{p_i} \overline{U_i}$ coincides with the inclusion $U_i \subset \overline{U_i}$, the induced composition of maps of local rings

$$\mathcal{O}_{\overline{U_i},\phi_i(x)} \xrightarrow{p^\sharp_{i,\phi(x)}} \mathcal{O}_{V,\phi(x)} \xrightarrow{\phi^\sharp_{i,x}} \mathcal{O}_{X,x}$$

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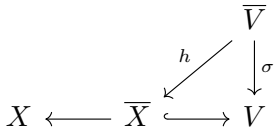
is an isomorphism. Note that both $p_i$ and $\phi_i$ are dominant, so both ring maps here are injective, and hence $\phi_{i,x}^\sharp$ is an isomorphism. Moreover, $V$ is nonsingular, hence normal, at every point of the open set $W = \phi(X)$. As $\phi\colon X \to W$ is a birational morphism of normal curves, it is an isomorphism (by the universal property of normalization). □

**Corollary 18.13.** Let $X$ be a projective curve over a field $k$. Then the normalization $\overline{X}$ is also projective.

*Proof* By the previous proposition, the nonsingular curve $\overline{X}$ is in any case quasi-projective, so we may regard $\overline{X} \subset \mathbb{P}_k^n$ as a locally closed subset of projective space. We claim that $\overline{X}$ is closed in $\mathbb{P}_k^n$. Let $V$ denote the closure of $\overline{X}$ and let $\sigma\colon \overline{V} \to V$ denote the normalization of $V$. As $\overline{V}$ and $X$ are birational, there is a birational map $\overline{V} \dashrightarrow X$. As $X$ is projective, this extends to a morphism $g\colon \overline{V} \to X$. As $\overline{V}$ is normal, this factors via a morphism $h\colon \overline{V} \to \overline{X}$ as in the diagram below:




From the diagram, we have $\sigma(\overline{V}) \subset \overline{X}$. However, $\sigma$ is finite, hence surjective by Proposition 6.38. This gives a contradiction unless $\overline{X} = V$. This completes the proof. □

**Theorem 18.14.** There is an equivalence of categories between the following categories:
(i) The category of nonsingular projective curves over $k$ and dominant morphisms.
(ii) The category of finitely generated field extensions of $k$ of transcendence degree 1 and maps of $k$-algebras (with the arrows reversed).

*Proof* If $X$ and $Y$ are two nonsingular projective curves, any rational map $X \dashrightarrow Y$ extends to a morphism. This, combined with Theorem **??** on page ??, shows that the functor $X \mapsto K(X)$ is fully faithful.

Next, to show that it is essentially surjective, we need to show that every finitely generated field $K$ of transcendence degree 1 over $k$ is of the form $K(X)$ for some nonsingular projective curve $X$. If $a_1, \ldots, a_r$ is a transcendence basis for $K$ over $k$, then the $k$-subalgebra $A = k[a_1, \ldots, a_r]$ will be of dimension 1, by Theorem **??** on page ??. The affine scheme $X = \operatorname{Spec} A$ is naturally a closed subvariety of the affine space $\mathbb{A}_k^r$, and taking the Zariski closure in $\mathbb{P}_k^r$, yields a (possibly singular) projective curve $Y$ which is birational to $X$. Taking the normalization of $Y$, we finally get a nonsingular projective curve which has $K$ as its function field (by Corollary 18.13). □

The above theorem is not true with 'projective' replaced by 'affine'. $\mathbb{A}_k^1$ and $\mathbb{A}_k^1 - \{0\}$ are two affine curves which isomorphic function fields but which are not isomorphic.

**Example 18.15** (Morphisms to $\mathbb{P}_k^1$)**.** If $X$ is a nonsingular curve, then any element $g \in K(X)$,

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defines a morphism to $\mathbb{P}^1_k$,

$$G\colon X \to \mathbb{P}^1_k. \tag{18.2}$$

We define $G$ as follows. Let $U \subset \mathbb{P}^1_k$ be the maximal open set where $g$ is regular, that is, $g \in \mathcal{O}_{\mathbb{P}^1_k}(U)$. Likewise, let $V \subset \mathbb{P}^1_k$ be the maximal open set where $1/g$ is regular. Note that $U \cap V$ is non-empty (it contains the generic point), and $U \cup V = X$ (for each $p \in \mathbb{P}^1_k$, the local ring $\mathcal{O}_{X,P}$ is a DVR, hence either $g$ or $1/g$ belongs to it). Now $g \in \mathcal{O}_X(U)$ defines a morphism $U \to D_+(t_0) \simeq \mathbb{A}^1_k$. Likewise, $1/g$ defines a morphism $U \to D_+(t_1) \simeq \mathbb{A}^1_k$, and these clearly glue to a morphism $G\colon X \to \mathbb{P}^1_k$.

Note that $G$ is a constant map if and only if $g$ is constant. This has the following consequence:

**Proposition 18.16.** For a nonsingular curve $X$ over $k$, there is a one-to-one correspondence between the elements of $K(X) - k$ and dominant morphisms of $k$-varieties $X \to \mathbb{P}^1_k$.

△

## 18.3 Sheaves on curves

Recall that an element of an $A$-module is called a *torsion element* if it is killed by a nonzerodivisor of $A$, and a module is a *torsion module* if all elements are torsion. On the other hand, a module is *torsion free* if no non-zero element is torsion. The sum of two torsion elements is clearly torsion, so the subset of a module $M$ formed by the torsion elements, is a submodule $T$. It has the property that $M/T$ is torsion free. If the base ring $A$ is a PID, a torsion free $A$-module is in fact free. This follows from the Structure Theorem for Modules over a PID, see Corollary A.61.

This generalizes to sheaves on curves as follows. If $X$ is a curve, then any $\mathcal{O}_X$-module of finite type contains a torsion subsheaf $\mathcal{T}$, whose sections over an open set $U \subset X$ equals the subgroup of $\mathcal{F}(U)$ of elements annihilated by some nonzerodivisor of $\mathcal{O}_X(U)$ (see Exercise 12.12.41 on page 249). The quotient $\mathcal{E} = \mathcal{F}/\mathcal{T}$ is torsion free in the sense that on open affine subsets $U$ $\mathcal{E}(U)$ (which equals $\mathcal{F}(U)/\mathcal{T}(U)$) is a torsion free module over $\mathcal{O}_X(U)$. Note that for $x \in X$, the stalk $\mathcal{E}_x$ is a torsion free module over $\mathcal{O}_{X,x}$ which is a PID. Hence every stalk $\mathcal{E}_x$ is free, and so $\mathcal{E}$ is locally free by Proposition **??**.

As $X$ is a curve, the torsion part $\mathcal{T}$ is a sheaf supported on finitely many points $p_1, \ldots, p_r$. Moreover, $\mathcal{T}$ is the direct sum of its stalks at these points: $\mathcal{T} = \bigoplus_{i=1}^r \mathcal{T}_{p_i}$.

**Theorem 18.17.** Let $X$ be a nonsingular curve and let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite type. Then there is a decomposition

$$\mathcal{F} = \mathcal{E} \oplus \mathcal{T}$$

where $\mathcal{T} \subset \mathcal{F}$ is the torsion subsheaf and $\mathcal{E}$ is locally free.

*Proof* What remains to be seen is that the splitting $\mathcal{F} = \mathcal{E} \oplus \mathcal{T}$. We claim that the exact

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sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0 \tag{18.3}$$

is split exact. Let $U$ be an affine neighbourhood about $p_i$. By Exercise **??**, $\mathcal{E}|_U$ is the tilde of a projective module, and so the sequence (18.3) splits when restricted to $U$. Hence there is a map $\phi_i \colon \mathcal{F}|_U \to \mathcal{T}_{p_i}$ splitting the inclusion $\mathcal{T}_{p_i} \to \mathcal{F}$. This map extended by zero gives a map of sheaves $\phi_i \colon \mathcal{F} \to \mathcal{T}_{p_i}$. Then, taking the direct sum of the $\phi_i$, we get a map $\phi_i \colon \mathcal{F} \to \mathcal{T}$ which splits the inclusion $\mathcal{T} \to \mathcal{F}$ for the entire torsion sheaf. □

**Proposition 18.18.** Let $f \colon X \to Y$ be a finite morphism of projective nonsingular curves over a field $k$. Then $f_*\mathcal{O}_X$ is a locally free sheaf of rank $\deg f$. More generally, if $\mathcal{E}$ is a locally free sheaf of finite rank on $X$, then $f_*\mathcal{E}$ is locally free of rank $(\deg f)(\text{rank } \mathcal{E})$.

*Proof* Note first that $f_*\mathcal{O}_X$ and $f_*\mathcal{E}$ are coherent, because $f$ is finite (see Exercise 12.12.36).

Let $U = \operatorname{Spec} B \subset X$ and $V = \operatorname{Spec} A \subset Y$ be affine open subsets so that $f(U) \subset V$ and $f|_U$ is induced by a ring map $\phi \colon A \to B$. Then $\phi$ is injective, because $X$ and $Y$ are integral and $f$ is dominant.

Then $f_*\mathcal{O}_X|_V = f_*\widetilde{A} = \widetilde{B_A}$. Now, as $B$ is an integral domain, and $\phi \colon A \to B$ is injective, $B$ is torsion-free as an $A$-module, and hence $f_*\mathcal{O}_X$ is torsion-free when restricted to $V$. Therefore, $f_*\mathcal{O}_X$ is locally free by Theorem 18.17. To compute the rank, note that the stalk of $f_*\mathcal{O}_X$ at the generic point is given by the fraction field of $B$, i.e., $K(X)$, and this has rank $[K(X) : K(Y)] = \deg f$ as a vector space over $K(Y)$.

The statement for $f_*\mathcal{E}$ follows from this, because as $X$ is a curve, $\mathcal{E}$ is trivial over a covering consisting of open sets of the form $f^{-1}(U)$ where $U$ is affine. □

## 18.4 Divisors on curves

Let $X$ be a nonsingular curve over an algebraically closed field $k$. The codimension 1-subsets of a curve are precisely the closed points, so that a Weil divisor is a finite formal combination

$$D = \sum n_i P_i$$

of closed points $P_1, \ldots, P_r$ in $X$, where the coefficients are integers. We define the *degree* as

$$\deg D = \sum n_i$$

If $D$ is given by the Cartier data $(U_i, f_i)$ (see Definition 16.31), then

$$\deg D = \sum_{P \in X} \operatorname{ord}_P(f_i)$$

where we for each $P$ choose an $i$ so that $P \in U_i$.

### *Pullbacks of divisors*

If $f \colon X \to Y$ is a dominant morphism of curves, we can pull back invertible sheaves from $Y$ to $X$, as well as sections of these. By the correspondence between divisors and invertible

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sheaves, this gives us a way of pulling back divisors from $Y$ to $X$. In the context of projective curves, we can make this a little bit more explicit.

We assume that the morphism $f\colon X \to Y$ is dominant, hence finite, it is then surjective, and $f^\sharp$ induces an inclusion of function fields $K(Y) \subset K(X)$.

If $Q \in Y$ is a closed point, choose a local parameter $t \in \mathcal{O}_{Y,Q}$. Then we define

$$f^*(Q) = \sum_{P \in f^{-1}(Q)} \operatorname{ord}_P(f^\sharp t) P,$$

where as usual $\operatorname{ord}_P$ is the order of vanishing at $P$. The number $\operatorname{ord}_P(f^\sharp t)$ is called the *ramification index* of $f$ at $P$, and it is usually written $e_P$. Multiplying $t$ by a unit in $\mathcal{O}_{Y,Q}$ does not alter $\operatorname{ord}_P(f^\sharp t)$ because a unit in $\mathcal{O}_{Y,Q}$ stays a unit in $\mathcal{O}_{X,P}$. Therefore, the ramification index $e_P$ is independent of the choice of local parameter $t$. Extending $f^*$ by linearity, we obtain a well-defined map of groups

$$f^*\colon \operatorname{Div} X \longrightarrow \operatorname{Div} Y.$$

We can also understand this map on the level of Cartier divisors: if $D$ is a Cartier divisor on $Y$ given by the data $\{(U_i, g_i)\}$, where $g_i \in K(Y)^\times$, we can consider the data $\{(f^{-1}U_i, f^\sharp g_i)\}$, which defines a Cartier divisor on $X$.

Note that $f^*(Q)$ is a divisor supported on the inverse image $f^{-1}(Q)$. In fact, divisor $f^*(Q)$ has the same multiplicities as the scheme-theoretic fiber $X_Q$. This follows because $f^*(Q)$ is locally defined by the equations $f^\sharp(g_i)$ and these form local generators for the ideal sheaf of the closed subscheme $X_Q$.

**Example 18.19** (Principal divisors)**.** Let $g \in K(X)^\times$ be a rational function. Then if $G\colon X \to \mathbb{P}^1_k$ denote the induced morphism, (from Example 18.15 on page 373), we have

$$\operatorname{div} g = G^*((0:1) - (1:0)).$$

This follows because $G^\sharp(t_0/t_1) = g$ and $G^\sharp(t_1/t_0) = 1/g$ and and $t_0/t_1$ and $t_1/t_0$ are local parameters at $(0:1)$ and $(1:0)$ respectively. △

**Lemma 18.20.** If $f\colon X \to Y$ is finite and $D$ is a divisor on $Y$, we have

$$\deg f^*D = \deg f \cdot \deg D.$$

*Proof* It suffices to treat the case when $D = Q$ is a point. Let $\operatorname{Spec} A$ be an affine neighbourhood of $Q$ and $\operatorname{Spec} B$ the inverse image of $\operatorname{Spec} A$. As we saw in Proposition 18.18, $B$ is a torsion free $A$-algebra which is locally free of rank equal to $[K(X) : K(Y)] = \deg f$. If $t$ is a local parameter at $y$ the value $\operatorname{ord}_P(f^\sharp t)$ is the ramification index of $f$ at $P$.

Then

$$\deg f^*Q = \sum_{f(P)=Q} \operatorname{ord}_P(f^\sharp t) = \sum_{f(P)=Q} e_P = \deg f.$$

□

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**Lemma 18.21.** For a non-zero $g \in K(Y)$ and a morphism $f\colon X \to Y$, we have

$$f^* \operatorname{div} g = \operatorname{div} g \circ f.$$

*Proof* Let $G\colon Y \to \mathbb{P}^1$ be the extension of $g$ then $G \circ f$ is the extension of $g \circ f$ and so according to Example 18.19 above, it holds that

$$f^* \operatorname{div} g = f^*(G^*((0:1) - (1:0)) = (G \circ f)^*((0:1) - (1:0)) = \operatorname{div} g \circ f.$$

□

**Corollary 18.22.** For a non-zero rational function $g \in K(X)$, we have $\deg \operatorname{div} g = 0$. Hence the degree map descends to a well-defined map

$$\deg\colon \operatorname{Cl}(X) \to \mathbb{Z}.$$

In other words, linearly equivalent divisors have the same degree.

*Proof* This is clear if $g$ is a constant. If not, $g$ defines a dominant morphism $G\colon X \to \mathbb{P}^1_k$ so that

$$\operatorname{div} g = G^*((1:0) - (0:1)).$$

and we are done by the above lemma. □

**Corollary 18.23.** If $D$ is a divisor with $\deg(D) < 0$, then $\Gamma(X, \mathcal{O}_X(D)) = 0$.

*Proof* The group $\Gamma(X, \mathcal{O}_X(D))$ consists of rational functions $f \in K(X)$ so that $\operatorname{div}(f) + D$ is effective. However, if $f$ is non-zero, then $\operatorname{div}(f) + D$ has negative degree, whereas effective divisors have non-negative degree. □

**Example 18.24.** Assume that $k$ is a field of characteristic 0. and consider the curve $X \subset \mathbb{A}^2_k = \operatorname{Spec} k[u, v]$ given by the equation

$$v^2 = u^3 + u^2 + 1 \tag{18.4}$$

which is a nonsingular curve. Consider the rational function $g = v + 1$ on $X$. What is $\operatorname{div} g$? Note that $g$ is regular, so there are no points $P$ for which $\operatorname{ord}_P(g) < 0$. Rewriting (18.4) as

$$(v - 1)(v + 1) = u^2(u + 1), \tag{18.5}$$

we see that the zeros of $v+1$ are the points $x = (0, -1)$ and $y = (-1, -1)$. Near $x = (0, 1)$ both $(v + 1)$ and $u + 1$ are invertible, and the equality

$$v - 1 = u^2(u + 1)(v + 1)^{-1} \tag{18.6}$$

shows that $u$ is local parameter there (the maximal ideal $\mathfrak{m}_P$ is generated by $v - 1$ and $u$). In the same vein, near $y = (-1, -1)$ both $v + 1$ and $u$ are invertible, and we infer from (18.6) that $u + 1$ is a local parameter. It follows that

$$\operatorname{div} g = \operatorname{ord}_P(u^2)P + \operatorname{ord}_y(u + 1)Q = 2P + Q. \tag{18.7}$$

△

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**Example 18.25.** Consider the curve $Y \subset \mathbb{P}^2_k = \operatorname{Proj} t_0, t_1, t_2$ given by the equation

$$t_2^2 t_0 = t_1^3 + t_1^2 t_0 + t_0^3$$

Note that the curve in the previous example equals $X \cap D(t_0)$, where we use coordinates $u = t_1/t_0, v = t_2/t_0$. Let us compute $\operatorname{div} g$ for the same rational function $g = t_2/t_0 + 1$ as before, but this time on $Y$. For this, we only need to consider the points where $t_0 = 0$. From the equation, we see that there is a single point in $Y \cap V(t_0)$, namely the point $z = (0 : 0 : 1)$. To compute $\operatorname{ord}_P(g)$ here, we use the chart $D(t_1)$. Then $Y \cap D(x_2)$ is isomorphic to the plane curve given by the equation

$$u = v^3 + v^2 u + u^3 \tag{18.8}$$

where now $u = t_0/t_2$ and $v = t_1/t_2$. The point $z$ is then the origin $(u, v) = (0, 0)$ in $D_+(t_2)$. Note that $g = u^{-1} + 1$. Rewriting (18.8) as

$$v^3 = u(1 - v^2 u - u^3),$$

we see that $v$ is a uniformzer at $z$ and that $\operatorname{ord}_P(u) = 3$. Hence we find we also see that $\operatorname{ord}_z(u) = 3$, and so

$$\operatorname{ord}_P(g) = \operatorname{ord}_P(u^{-1} + 1) = \operatorname{ord}_P(u + 1)/u) = \operatorname{ord}_P(u + 1) - \operatorname{ord}_P u = -3$$

Finally, we concude thats

$$\operatorname{div} g = 2(1 : 0 : -1) + (1 : -1 : -1) - 3(0 : 0 : 1). \tag{18.9}$$

Note that, since $Y$ is projective we may use Corollary 18.22 and conclude that $\deg \operatorname{div} g = 0$, which immediately yields 18.9. △

### *The canonical divisor*

Also in this section, we work with a curve $X$ defined over an algebraically closed field $k$. As $k$ is algebraically closed, the module of differentials $\Omega_{X/k}$ is an invertible sheaf.

**Definition 18.26** (The canonical divisor)**.** Let $X$ be a nonsingular curve over $k$. A *canonical divisor* is a divisor $K_X$ such that

$$\mathcal{O}_X(K_X) \simeq \Omega_{X/k}. \tag{18.10}$$

Because of (18.10), any two canonical divisors will be linearly equivalent. It is therefore the class of $K_X$ in the class group $\operatorname{Cl}(X)$ which is 'canonical', rather than the divisor itself.

In our setting, one can describe the set of canonical divisors as follows. We will consider $\Omega_{K(X)/k}$, which is are called *rational differential forms*. Note that $\Omega_{K(X)/k}$ is 1-dimensional as a $K(X)$-vector space, being the stalk of the invertible sheaf $\Omega_{X/k}$ at the generic point.

If $\eta$ is a local generator $\Omega_{X/k}$ at a point $P \in X$, it defines a generator for $\Omega_{K(X)/k}$ as a $K(X)$-vector space as well, and every rational differential form $\omega$ is of the form $\omega = g\eta$ for some rational function $g \in K(X)$. In fact, $\Omega_{K(X)/k} = \Omega_{\mathcal{O}_{X,P}/k} \otimes_{\mathcal{O}_{X,P}} K(X)$.

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Now let $\omega \in \Omega_{K(X)/k}$ be a nonzero element. For each point $P \in X$ choose a generator $\eta_P$ for $\Omega_{\mathcal{O}_{X,P}/k}$ and write $\omega = g_P \eta_P$ with $g_P \in K(X)$. Then define

$$\operatorname{div} \omega = \sum_{P \in X} \operatorname{ord}_P(g_P) P. \tag{18.11}$$

The expression on the right in (18.11) is independent of the choice of local generators. This is because two generators $\eta'_P$ and $\eta_P$ will be related by $\eta_P = u\eta'_P$ with $u$ a unit in $\mathcal{O}_{X,P}$, and hence $\omega = g_P\eta_P = ug_P\eta'_P$, and $\operatorname{ord}_P(ug_P) = \operatorname{ord}_P(g_P)$. Note also that the sum in fact is finite, because any local generator $\eta_P$ extends to a generator for $\Omega_{U/k}$ in some neighbourhood $U$ of $P$, and the corresponding rational function $g_P$ has only finitely many zeroes and poles (and $X - U$ is a finite set as well).

If $\omega$ and $\omega'$ are two rational differentials, then $\omega' = h\omega$ for some $h \in K(X)$. Therefore, for each $P \in X$ we have $\operatorname{ord}_P(hg_P) = \operatorname{ord}_P(h) + \operatorname{ord}_P(g_P)$, and by the formula (18.11), we get

$$\operatorname{div}(h\omega) = \operatorname{div} h + \operatorname{div} \omega.$$

What we have done so far is valid over any field as long as $\Omega_{X/k}$ is invertible. When the ground field is algebraically closed, there is a local description of the rational differentials in terms of local parameters that make calculations easier.

**Lemma 18.27.** Assume that $X$ is nonsingular at the closed point $P \in X$ and that $t$ is local parameter at $P$. Then each element of $\Omega_{K(X)/k}$ is of the form $g\,dt$ with $g \in K(X)$; in other words, $\Omega_{K(X)/k}$ is of rank 1 over $K(X)$ with $dt$ as a basis.

*Proof* The Zariski cotangent space $\mathfrak{m}_P/\mathfrak{m}_P^2$ at $P$ is always generated by the class of a local parameter, and by Proposition **??**, it follows that $dt$ generates $\Omega_{\mathcal{O}_{X,P}/k}$ when $X$ is nonsingular at $P$. □

Note that $g dt$ defines a section of $\Omega_{U/k}$ over some open set $U \subset X$. In other words, $\omega$ is a rational section of $\Omega_{X/k}$. Using Proposition 16.35, we obtain

**Proposition 18.28.** The invertible sheaf associated to $\operatorname{div} \omega$ is isomorphic to $\Omega_X$.

**Example 18.29.** Let us find the canonical divisor of $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. Write $x$ for the affine coordinate $t = x_1/x_0$ on $D_0(x_0)$. We will consider the differential

$$\omega = dt$$

which is an element in $\Omega_{k(t)/t}$. Note that $\omega = 1 \cdot dt = 1 \cdot d(t-a)$ for any $a \in k$. Thererefore, as $t - a$ forms a local parameter at $a$, we see that the divisor of $\omega$ is 0 on $D_+(x_0)$. It remains to see what happens at the point $(0:1)$. Here $u = x^{-1}$ is a local parameter, and

$$dx = u^{-2} du$$

Therefore, $\operatorname{div}(\omega) = -2(0:1)$. As this divisor has negative degree, we again compute that $\Gamma(\mathbb{P}^1_k, \Omega_{\mathbb{P}^1_k}) = 0$. △

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**Example 18.30.** Assume that $k$ is of characteristic different from 2. Let $X \subset \mathbb{A}^2_k$ be the elliptic curve given by the equation

$$v^2 = u^3 - u,$$

and consider the differential $\omega = du$. At a point $p = (a, b)$ where $b \neq 0$, the coordinate $u$ is a local parameter, and so $du = d(u - a)$ has zero valuation at $p$. When $b = 0$, the curve has three points: $p_1 = (0, 0)$, $p_2 = (-1, 0)$, and $p_3 = (1, 0)$.

At these points, $v$ will be a local parameter, and since $2vdv = (3u^2 - 1)du$, it holds that

$$du = 2v/(3u^2 - 1)dv.$$

Hence $\operatorname{ord}_{p_i}(du) = 1$ for all three. Summing up, we conclude that

$$\operatorname{div} \omega = p_1 + p_2 + p_3.$$

△

**Example 18.31.** We consider the projectivization $X \subset \mathbb{P}^2_k$ of the previous example, i.e. the curve whose homogeneous equation is

$$x_1^2 x_2 = x_0^3 - x_0 x_2^2.$$

Consider again the rational differential $\omega = d(x_0/x_2)$. We know the behaviour of $\omega$ on the distinguished open set $D_+(x_2)$, so what remains to compute the divisor of $\omega$, is the valuation $\operatorname{ord}_P(\omega)$ for each point in $X \cap V(x_2)$, but this intersection has just one single point $x = (0 : 1 : 0)$.

Dehomogenizing the chart $D_+(x_1)$ by setting $u = x_0/x_1$ and $v = x_2/x_1$, the equation of $X$ in $D_+(x_1)$ becomes

$$v = u^3 - uv^2.$$

Since $1 + uv$ is invertible near $x$, this shows that $u$ is a local parameter at $x$ and that $\operatorname{ord}_P(v) = 3$. Our differential $\omega$ takes the form $\omega = d(x_0/x_1 \cdot x_1/x_2) = d(u/v) = (udv - vdu)/v^2$. We find

$$dv = 3u^2 - v^2 - 2uvv')dv,$$

which yields

$$\begin{aligned} udv - vdu &= (3u^3 - uv^2 - 2u^2vv' - v)du \\ &= (2u^3 - 2uv^2 - 2u^2vv')du \end{aligned} \tag{18.12}$$

The terms $uv^2$, $2u^2vv'$ vanish to order at least 5 at $x$, and the dominating term in (18.12) is $2u^3$, which means that $\operatorname{ord}_P(\omega) = \operatorname{ord}_P(u^3) - 2\operatorname{ord}_P(v) = -3$. We conclude that

$$\operatorname{div} \omega = (0 : 0 : 1) + (-1 : 0 : 0) + (1 : 0 : 0) - 3(0 : 1 : 0)$$

△

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## 18.5 The genus of a curve

Let $X$ be a projective nonsingular curve over a field $k$. We define the *arithmetic genus* of $X$ is defined as the integer

$$p_a(X) = 1 - \chi(\mathcal{O}_X) = 1 - \dim_k H^0(X, \mathcal{O}_X) + \dim_k H^1(X, \mathcal{O}_X).$$

If we assume that $X$ is geometrically irreducible, that is, $X_{\bar{k}}$ is irreducible, then $H^0(X, \mathcal{O}_X) = k$, and so $p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$.

The *geometric genus* of $X$ is defined as

$$p_g(X) = \dim_k H^0(X, \Omega_X).$$

Note that $\chi(\mathcal{O}_X) = \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X)$, so we get

$$p_a(X) = 1 - \chi(\mathcal{O}_X).$$

These numbers are defined using different sheaves, and there is no a priori reason to expect that they should have anything to do with each other. However, we shall see later in the chapter that there is a strong relation between them: they are equal whenever $X$ is nonsingular. For the time being we will still refer to the arithmetic genus $p_a$ as the *genus* of $X$.

**Example 18.32.** When $X = \mathbb{P}^1$, we have $H^1(\mathbb{P}^1, \mathcal{O}_X) = 0$ so the arithmetic genus is 0. Likewise, we have that $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$, so also $p_g = 0$. △

**Example 18.33.** Let $X \subset \mathbb{P}^2$ be a plane curve, defined by a homogeneous polynomial $f(x_{0,1}, x_2)$ of degree $d$. In Chapter **??**, we computed that $H^1(X, \mathcal{O}_X) \simeq k^{\binom{d-1}{2}}$. Hence the genus of $X$ is equal to $\frac{(d-1)(d-2)}{2}$. △

**Example 18.34.** Let $X = \mathbb{P}^1_k$ with the usual covering $U_0 = \operatorname{Spec} k[t]$ and $U_1 = \operatorname{Spec} k[t^{-1}]$. The differential form $dt$ is an element of $\Omega_{K(X)/k}$, which generates $\Omega_{X/k}|_{U_0}$. This means that $\operatorname{ord}_x(dt) = 0$ for every $x \in U_0$. For the remaining point $(1 : 0)$ at infinity, note that $t^{-1}$ is the local parameter there, $(1 : 0)$ corresponding to the origin in $U_1$. We have $d(t^{-1}) = -t^{-2}dt$, and hence $dt = -(t^{-1})^{-2}d(t^{-1})$. This means that $\operatorname{ord}_{(1:0)} dt = -2$, so that $\operatorname{div} dt = -2(1 : 0)$.

As a Cartier divisor, the corresponding divisor is given by $(U_0, 1), (U_1, t^2)$. This shows that $\Omega_X \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$. △

**Example 18.35.** The arithmetic genus can show unexpected behaviour over non-algebraically closed fields. For instance, one can regard $X = \mathbb{P}^1_{\mathbb{C}}$ as a nonsingular curve $\operatorname{Spec} \mathbb{R}$. Then $\dim_{\mathbb{R}} H^0(X, \mathcal{O}_X) = 2$, and $\dim_{\mathbb{R}} H^1(X, \mathcal{O}_X) = 0$, and hence $X$ has arithmetic genus $p_a(X) = -1$! The issue is that $X$ is not *geometrically irreducible*, as the base change $X_{\mathbb{C}}$ has two components. △

## 18.6 Hyperelliptic curves

Let $X$ be a hyperelliptic curve defined by the affine equation

$$y^2 = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

Let $\lambda_1, \ldots, \lambda_n$ be the roots of $p(x)$. We assume that $X$ is smooth, so the $\lambda_i$ are distinct.

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**Example 18.36** (Canonical divisors)**.** We can compute a canonical divisor of $X$ by choosing $K_X = \operatorname{div}(\omega)$, where $\omega = y^{-1}dx$ (the appearance of the term $y^{-1}$ will be explained later.)

In the affine chart $U = \operatorname{Spec} k[x,y]/(y^2 - p(x))$, with coordinates $x$ and $y$, we have

$$K_X|_U = 0. \tag{18.13}$$

Indeed, for any point $(a,b) \in U$, where $b \neq 0$, $x - a$ defines a local parameter, as $\frac{\partial}{\partial y}(y^2 - p(x)) = 2y$ is nonzero at $P$. Moreover,

$$\omega = y^{-1}dx = y^{-1}d(x-a).$$

so $\operatorname{ord}_P(\omega) = 0$ in this case, because $y$ is a unit in $\mathcal{O}_{X,P}$.

For the points of the form $(\lambda_i, 0)$, $y$ forms a local parameter (as $y^2$ equals $(x - \lambda_i)$ times a unit in the local ring). Here we have

$$\frac{dx}{y} = \frac{2\,dy}{p'(x)}$$

As $p$ as no repeated roots, $p'(x)$ is a unit in $\mathcal{O}_{X,P}$, so $\operatorname{ord}_P(\omega) = 0$ also in this case. This means that $K_X|_U = 0$.

The remaining points lie in the closed set $V(u)$ in $\operatorname{Spec} k[u,v]/(v^2 - u^{2g+2}p(u^{-1}))$. If we assume $n$ is even, then these are two points $P_1 = (0,1)$ and $P_2 = (0,-1)$. For these points, $u$ defines a local parameter. Moreover,

$$\omega = \frac{dx}{y} = \frac{2\,dy}{p'(x)} = -\frac{u^{g-1}\,du}{v}.$$

As $v$ is invertible at $P_1$ and $P_2$, we get

$$\operatorname{div}(\omega) = (g-1)P_1 + (g-1)P_2. \tag{18.14}$$

If $n$ is odd, then $V(u)$ consists of a single point, $P = (0,0)$, and $v^2 = u\cdot$(unit) in $\mathcal{O}_{X,P}$. Therefore, $v$ is a local parameter at $P$ and $\operatorname{ord}_P(u) = 2$. We furthermore get that $2vdv = du\cdot$(unit), and hence by (18.14), we get

$$\operatorname{div}(\omega) = (g-1)\operatorname{ord}_P(u)P = (2g-2)P.$$

△

**Proposition 18.37.** Let $X$ be a hyperelliptic curve of genus $g$ defined by the affine equation $y^2 = p(x)$ where $p$ has degree $2g+1$ or $2g+2$. Then

$$H^0(X, \Omega_X) = k\frac{dx}{y} \oplus k\frac{x\,dx}{y} \oplus \cdots \oplus k\frac{x^{g-1}\,dx}{y}$$

Let us consider some examples of divisors on these hyperelliptic curves. For simplicitly, we will assume that $n = 2g+1$ is odd, and $X$ is defined using the affine equation

$$y^2 = p(x) = x^{2g+1} + x$$

In other words, $X$ is glued together by the affine schemes $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$,

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where

$$A = \frac{k[x,y]}{(-y^2 + x^{2g+1} + x)} \text{ and } B = \frac{k[u,v]}{(-v^2 + u + u^{2g+1})}$$

As before, we glue $D(x)$ to $D(u)$ using the identifications $u = x^{-1}$ and $v = x^{-g-1}y$.

We will compute the groups $\Gamma(X, \mathcal{O}_X(mP))$, where $P$ is the 'point at infinity' of $X$, that is, the point corresponding to the maximal ideal $\mathfrak{m} = (u, v)$ in $V$. In other words, we look for rational functions $f \in K(X)$ which are regular on $X - P = U$ and have a pole of order at most $m$ at $p$.

To control the pole of $f$ at $P$, we first compute the orders of vanishing of $x, y, u, v$ at $P$. In the local ring $\mathcal{O}_{X,P} = B_{\mathfrak{m}}$, we can use the defining relation to express $u$ as

$$u = v^2(1 + u^{2g})^{-1} = v^2(\text{unit}). \tag{18.15}$$

Therefore, $v$ is a local parameter at $P$. Using this, the orders of $v, u, x, y$ are given by

$$\begin{aligned} \operatorname{ord}_P(v) = 1, \quad \operatorname{ord}_P(u) = 2 \quad \operatorname{ord}_P(x) = \operatorname{ord}_P(u^{-1}) = -2 \\ \operatorname{ord}_P(y) = v(x^{g+1}v) = (g+1)(-2) + 1 = -2g - 1. \end{aligned} \tag{18.16}$$

Let us identify $K(X)$ with the fraction field of $A$, so that $f$ is expressed in terms of $x$ and $y$. Note $P$ does not lie in $U$, and in fact $X - P = U$, as $V(u) = V(u, v) = \{P\}$. As $f$ is regular on $U = \operatorname{Spec} A$, this means that $f$ can be identified with a *polynomial* in $x$ and $y$.

If we view $A$ as a module over $k[x]$, we can write $A = k[x] \oplus k[x]y$ and express $f$ as $a(x) + b(x)y$ where $a(x)$ and $b(x)$ are polynomials. By (18.16), we have

$$\begin{aligned} \operatorname{ord}_P(a(x)) = \operatorname{ord}_P(a(u^{-1})) = -2 \deg a \\ \operatorname{ord}_P(b(x)y) = \operatorname{ord}_P(b(u^{-1})y) = -\deg b - 2g - 1. \end{aligned}$$

Therefore, as we assume $g \geqslant 1$, we have $\operatorname{ord}_P(f) \leqslant -2$ for any non-constant rational function with a pole at $P$. Hence $\Gamma(X, \mathcal{O}_X(P)) = k$ consists of only the constants.

On the other hand, for the divisor $2P$, we obtain a new section, namely the rational function $x$. As $-\deg b - 2g - 1 \leqslant -3$, we find that there are additional functions with $b(x) \neq 0$, and hence

$$\Gamma(X, \mathcal{O}_X(2P)) = k \oplus k\,x.$$

The invertible sheaf $L = \mathcal{O}_X(2P)$ is in fact globally generated by these two sections: the section '1' generates $L$ over the open set $X - p$, and $x$ generates $L$ in a neighbourhood of $p$. This is of course no big surprise: the morphism defined by $1, x$ is exactly the double cover morphism $X \to \mathbb{P}^1_k$.

For the divisor $D = 3P$, we allow poles of order 3 at $P$. Note that we have $\operatorname{ord}_P(a(x) + b(x)y) \leqslant -4$ unless $g = 1$ and $b$ is constant. This means that $\Gamma(X, \mathcal{O}_X(3p)) = k \oplus kx \oplus ky$ if $g = 1$, and $\Gamma(X, \mathcal{O}_X(3p)) = k \oplus kx$ for $g \geqslant 2$.

Let us investigate the two cases $g = 1$ and $g = 2$ in more detail.

**Example 18.38** ($g = 1$)**.** When $g = 1$, the sections $s_0 = 1, s_1 = x, s_2 = y$ define a closed embedding

$$\phi : X \longrightarrow \mathbb{P}^2_k.$$

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To see this, write $t_0, t_1, t_2$ for the homogeneous coordinates on $\mathbb{P}^2_k$, so that $\phi^* t_i = s_i$ for $i = 0, 1, 2$. Over $U = D(s_0) = \phi^{-1} D_+(t_0)$, the morphism $\phi$ is given by

$$\phi|_U \colon U \longrightarrow D_+(t_0) \simeq \mathbb{A}^2_k; \qquad (x, y) \mapsto (1 : x : y) \tag{18.17}$$

which can be identified with the closed embedding

$$\operatorname{Spec} k[x, y]/(-y^2 + x^3 + x) \longrightarrow \operatorname{Spec} k[x, y].$$

In particular, $\phi$ is a closed embedding over $\phi^{-1} D_+(t_0)$. Likewise, over $V = D(x) = \phi^{-1} D_+(t_1)$, we rewrite the sections in terms of $u$ and $v$:

$$(1 : x : y) = (x^{-1} : 1 : x^{-1} y) = (u : 1 : v),$$

so the morphism takes the form

$$\phi|_V \colon V \longrightarrow D(t_1) \subset \mathbb{P}^2_k; \qquad (u, v) \mapsto (u : 1 : v). \tag{18.18}$$

This shows that $\phi$ is a closed embedding over $\phi^{-1} D_+(t_1)$ as well. As these two open sets cover the image $\phi(X) \subset \mathbb{P}^2_k$, the morphism $\phi$ is a closed embedding. The relation $y^2 = x^3 + x$ gives the following defining equation for $X$ in $\mathbb{P}^2_k$:

$$t_2^2 t_0 = t_1^3 + t_0^2 t_1.$$

That is, $X$ embeds as a plane cubic curve. △

**Example 18.39** ($g = 2$)**.** When $g = 2$, the divisor $3P$ does not give a projective embedding, as there is only the two global sections 1 and $x$. For the divisor $4P$, we have $\Gamma(X, \mathcal{O}_X(4P))$ is spanned by the three sections $1, x, x^2$. Geometrically, the induced morphism $X \to \mathbb{P}^2_k$ maps onto the degree 2 curve $t_1^2 = t_0 t_2$. It is not a projective embedding because degree 2 curves have genus 0. More directly, $X \to \mathbb{P}^2_k$ factors as $X \to \mathbb{P}^1_k \to \mathbb{P}^2_k$, where the first map is the double cover and $\mathbb{P}^1_k \to \mathbb{P}^2_k$ is the second Veronese embedding.

On the other hand, for the divisor $5P$, we obtain

$$\Gamma(X, \mathcal{O}_X(5P)) = k \oplus kx \oplus kx^2 \oplus ky.$$

Moreover, the sections $s_0 = 1, s_1 = x, s_2 = x^2$, and $s_3 = y$ globally generate $\mathcal{O}_X(5p)$, and define a morphism

$$\phi \colon X \longrightarrow \mathbb{P}^3_k$$

Over the open set $U = D(s_0)$, $\phi$ is given by the map

$$\phi|_U \colon U \longrightarrow \mathbb{A}^3_k; \quad (x, y) \mapsto (1 : x : x^2 : y)$$

Over $V = D(s_1)$, we rewrite the sections in terms of $u, v$ as $1, u^{-1}, u^{-2}, u^{-3} v$, and so multiplying by the unit $u^3$, the morphism is given by

$$\phi|_V \colon V \longrightarrow \mathbb{A}^3_k; \quad (u, v) \mapsto (u^3 : u^2 : u : v)$$

This shows that $\phi$ is a closed embedding.

From the defining sections, we see that $s_1^2 - s_0 s_2 = 0$, so the image of $X$ lies on the quadric surface $t_1^2 - t_0 t_2 = 0$. There are additional relations among the sections coming from the relation $y^2 = x^5 + x$, for instance, $t_3^2 t_0 - t_1 t_2^2 - t_1 t_0^2 = 0$. Computing the primary

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decomposition of the ideal generated by these two polynomials, we find that $X$ is embedded in $\mathbb{P}^3_k$ as the curve defined by the prime ideal

$$I = (t_1^2 - t_0t_2, t_0^2t_2 + t_2^3 - t_1t_3^2, t_0^2t_1 + t_1t_2^2 - t_0t_3^2).$$

Let us use the ideal $I$ to verify that the genus of $X$ is equal to 2. A free resolution of $I$ is given by

$$0 \to R(-4)^2 \oplus \xrightarrow{T} R(-2) \oplus R(-3)^2 \to I \to 0$$

where $T$ is the matrix

$$T = \begin{pmatrix} t_0^2 + t_2^2 & -t_3^2 \\ -t_1 & t_2 \\ t_0 & -t_1 \end{pmatrix}$$

Applying tilde, we get the following sequence of sheaves on $\mathbb{P}^3_k$

$$0 \to \mathcal{O}_{\mathbb{P}^3_k}(-4)^2 \xrightarrow{T} \mathcal{O}_{\mathbb{P}^3_k}(-2) \oplus \mathcal{O}_{\mathbb{P}^3_k}(-3)^2 \to \mathcal{I} \to 0$$

Taking Euler characteristics, we find that $\chi(X, \mathcal{O}_X)$ equals

$$\chi(\mathcal{O}_{\mathbb{P}^3}) - \chi(\mathcal{I}) = 1 - \left(-2\binom{-4+3}{3} + \binom{-2+3}{3} + 2\binom{-3+3}{3}\right) = -1.$$

As $h^0(X, \mathcal{O}_X) = 1$, we find that $h^1(X, \mathcal{O}_X) = 2$, as expected. △

## 18.7 Exercises

**Exercise 18.7.1.** Let $k$ be an algebraically closed field. Show that any two curves over $k$ are homeomorphic (!).

**Exercise 18.7.2.** Let $X$ be a variety over an algebraically closed field $k$ and let $x, y \in X$ be two $k$-points. Show that if $\mathcal{O}_{X,x} \subset \mathcal{O}_{X,y}$ and $\mathfrak{m}_x \subset \mathfrak{m}_y$, then $x = y$.

**Exercise 18.7.3.** Let $f\colon X \to Y$ be a birational morphism between curves, where $Y$ is nonsingular. Then $f(X)$ is open in $Y$, and $f$ induces an isomorphism $X \simeq f(X)$. HINT: The complement of $f(X)$ is a finite set of points. For $x \in X$, show that $\mathcal{O}_{Y,f(x)}$ contains $\mathcal{O}_{X,x}$ inside $K(X)$ and deduce that $\mathcal{O}_{Y,f(x)} = \mathcal{O}_{X,x}$ by the properties of discrete valuation rings.

**Exercise 18.7.4.** Find the singularities of the curve in $\mathbb{P}^2_k$ whose equation is $x^2y^2 + x^2z^2 + y^2z^2 = 0$.

**Exercise 18.7.5.**
a) Show that the pushforward of a torsion sheaf is a torsion sheaf.
b) Show that a sheaf $\mathcal{F}$ is torsion iff it is supported on a proper closed subset
c) Show that if $\mathcal{F}$ is a torsion sheaf on a curve $X$ then $H^0(X, \mathcal{F}) = 0$ if and only if $\mathcal{F} = 0$.

**Exercise 18.7.6.** Let $R = k[x_0, \ldots, x_n]/\mathfrak{a}$ and suppose that each of the ideals $x_i$ are prime. Consider $X = \operatorname{Proj} R$ as a closed subscheme of $\mathbb{P}^n_k$ with ideal sheaf $\mathcal{I}_X$. Show that $H^1(\mathbb{P}^n_k, \mathcal{I}_X(d)) = 0$ for every $d \in \mathbb{Z}$. HINT: Use Proposition **??**.

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# 19

# The Riemann–Roch theorem

A fundamental problem in algebraic geometry, is how to compute the spaces of global sections $H^0(X, \mathcal{O}_X(D))$ for some divisor $D$. The sections of $\mathcal{O}_X(D)$ allow us to define morphisms and rational maps to projective space $\mathbb{P}^n_k$, and in good cases, we even get a projective embedding. Subsequently, we we can ask about geometric information coming from the embedding, e.g., the degree, or the equations defining the image.

In this chapter we will develop the most powerful tool to attack these questions, the Riemann–Roch theorem. This has many important consequences for the classification of algebraic curves. We will study a few of these in Chapter XXX.

## 19.1 The Riemann–Roch formula

In this chapter, $X$ will denote a nonsingular projective curve over an algebraically closed field $k$. When we say 'point', we will always mean a closed point.

The cohomology groups $H^i(X, \mathcal{F})$ are finite-dimensional $k$-vector spaces. For notational purposes, we define

$$h^i(X, \mathcal{F}) := \dim_k H^i(X, \mathcal{F})$$

Since $X$ is a curve, the interesting cohomology groups are $H^0(X, \mathcal{F})$ and $H^1(X, \mathcal{F})$; the other cohomology groups are zero by Theorem **??**. We will mostly be interested in the case when $\mathcal{F} = \mathcal{O}_X(D)$ for some divisor $D$. In this case, we will occasionally write $h^i(D)$ for $h^i(X, \mathcal{O}_X(D))$, provided the context is clear.

Our most basic tool for studying the cohomology groups $H^0(X, \mathcal{O}_X(D))$ is the ideal sheaf sequence of a point $P \in X$, which takes the form

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \kappa(P) \longrightarrow 0 \tag{19.1}$$

The first map is the inclusion and the second is evaluation at $P$. Here we have identified the ideal sheaf $\mathfrak{m}_P \subset \mathcal{O}_X$ by the invertible sheaf $\mathcal{O}_X(-P)$, and the sheaf $i_*\mathcal{O}_P$ with the skyscraper sheaf with value $\kappa(P)$ at $P$. If $L$ is an invertible sheaf, we can tensor (19.1) by $L$ and get the exact sequence

$$0 \longrightarrow L(-P) \longrightarrow L \longrightarrow \kappa(P) \longrightarrow 0, \tag{19.2}$$

where $L(-P)$ is the invertible sheaf of sections of $L$ vanishing at $P$, and we also identify $L \otimes \kappa(P) \simeq \kappa(P)$ (note that $L \otimes \kappa(P)$ is also the constant sheaf with value $k$ at $P$). In particular, taking $L = \mathcal{O}_X(D + P)$ in (19.2) we get

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D + P) \longrightarrow \kappa(P) \longrightarrow 0 \tag{19.3}$$



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This leads to the following basic bound:

**Lemma 19.1.** Let $X$ be a nonsingular projective curve, and let $D$ be a divisor on $X$. Then
(i) $h^0(X, \mathcal{O}_X(D+P)) \leqslant h^0(X, \mathcal{O}_X(D)) + 1$ for each $P \in X$.
(ii) $h^0(X, \mathcal{O}_X(D)) \leqslant \deg D + 1$.

*Proof* The first item follows by taking global sections in the sequence (19.3). To see the second inequality, it suffices to consider the case when $D = \sum n_P P$ is effective (otherwise the left-hand side is 0). In that case, the inequality follows by applying the first inequality $\deg D$ times. □

Recall that the Euler characteristic $\chi(\mathcal{F})$ is defined as the alternating sum of the dimensions $h^i(X, \mathcal{F})$. Applying $\chi$ to (19.3), and recalling that $\chi$ is additive on exact sequences, we get

$$\chi(\mathcal{O}_X(D+P)) = \chi(\mathcal{O}_X(D)) + \chi(\kappa(P)) = \chi(\mathcal{O}_X(D)) + 1. \tag{19.4}$$

The next result is a very useful formula to compute $\chi(\mathcal{O}_X(D))$ for any divisor $D$:

**Theorem 19.2 (Easy Riemann–Roch).** Let $X$ be a smooth projective curve of genus $g = h^1(X, \mathcal{O}_X)$ and let $D$ be a divisor on $X$. Then

$$\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) = \deg D + 1 - g \tag{19.5}$$

*Proof* Let $P \in X$ be a point. Consider the formula (19.5). By (19.4), the left-hand side increases by 1 if we replace $D$ by $D + P$. On the other hand, also the right-hand side of the equation above increases by 1 by adding $P$ to $D$, because $\deg(D+P) = \deg D + 1$. This means that the theorem holds for a divisor $D$ if and only if it holds for $D + P$ for any closed point $P$. So by adding and subtracting points, we can reduce to the case when $D = 0$. But in that case the left-hand side of the formula is simply

$$\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = 1 - g$$

which equals the right-hand side, by the definition of $g$. □

As a first basic corollary, we prove that the divisor of a rational function always has degree 0. This explains the observations that the 'number of zeros' and 'number of poles' matched up in the examples in Chapter 18.

**Corollary 19.3.** If $f \in K(X) - 0$, then $\deg(\operatorname{div}(f)) = 0$.

*Proof* Let $D = \operatorname{div}(f)$, then $\mathcal{O}_X(D) \simeq \mathcal{O}_X$ (Proposition 16.32), so $\deg(\operatorname{div}(f)) = 0$ by the formula (19.5). □

Note that the right-hand side of 19.5 is easy to compute, as it only involves the degree of $D$ and the genus. Of course, the number we are really after is the number $h^0(X, \mathcal{O}_X(D))$, since this is the dimension of global sections of $\mathcal{O}_X(D)$. So if we, for some reason, could argue that $H^1(X, \mathcal{O}_X(D)) = 0$ we would have a formula for the dimension of the space of global sections of $\mathcal{O}_X(D)$.

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In any case, we can certainly say that $h^1(X, \mathcal{O}_X(D)) \geqslant 0$, and this leads to the following *lower bound* on $h^0(X, \mathcal{O}_X(D))$, which is often enough in applications.

**Corollary 19.4.** $h^0(X, \mathcal{O}_X(D)) \geqslant \deg D + 1 - g$.

**Example 19.5.** That the Riemann–Roch formula holds for $X = \mathbb{P}^1$ follows from the computations in Chapter 15. Indeed, as $\mathrm{Cl}(\mathbb{P}^1_k) = \mathbb{Z}$, it suffices to check that the formula holds for divisors of the form $D = dP$ where $P \in \mathbb{P}^1$ is a point and $d \in \mathbb{Z}$. In this case, the right-hand-side of the fomula equals $\deg D + 1 - 0 = d + 1$.

If $d \geqslant 0$, $H^0(\mathbb{P}^1, \mathcal{O}_X(dP))$ is identified with the space of homogenous degree $d$ polynomials in $x_0, x_1$, while $H^1(\mathbb{P}^1, \mathcal{O}_X(dP)) = 0$. Hence $h^0(X, D) = d + 1$. If $d < 0$, we have $h^0(X, D) = 0$ and $h^1(X, D) = -d - 1$. △

**Example 19.6.** Consider again the case where $X$ is a hyperelliptic curve of genus 2, as in Example 18.39. We have the following table of the various cohomology groups $H^i(X, \mathcal{O}_X(nP))$ for the point $P = (u, v)$:

| $D$ | 0 | $1P$ | $2P$ | $3P$ | $4P$ | $5P$ | $6P$ | $7P$ |
|---|---|---|---|---|---|---|---|---|
| $H^0(X, \mathcal{O}_X(D))$ | 1 | 1 | 2 | 2 | 3 | 4 | 5 | 6 |
| $H^1(X, \mathcal{O}_X(D))$ | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| $\chi(\mathcal{O}_X(D))$ | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

△

As the example shows, the cohomology groups $H^1(X, \mathcal{O}_X(D))$ vanish provided that the degree $\deg D$ is large enough. This is a general fact for all divisors on all projective curves, and it will be a consequence of the following theorem, known as 'Serre duality'. Although we will only prove this when $X$ is a curve, we include the general statement, which holds in any dimension:

**Theorem 19.7 (Serre duality).** Let $X$ be a smooth projective variety of dimension $n$ over an algebraically closed field $k$. Then there is an isomorphism $H^n(X, \Omega^n_X) \simeq k$, and for any locally free sheaf $\mathcal{E}$ on $X$, a perfect pairing

$$H^0(X, \mathcal{E}) \times H^n(X, \mathcal{E}^\vee \otimes \Omega^n_X) \longrightarrow H^n(X, \Omega^n_X) \simeq k \tag{19.6}$$

The theorem allows us to view the space of global sections $H^0(X, \mathcal{E})$ as the dual of the top cohomology group $H^n(X, \mathcal{E}^\vee \otimes \Omega^n_X)$ via a certain $k$-linear pairing. The important consequence of this is that $H^0(X, \mathcal{E})$ and $H^n(X, \mathcal{E}^\vee \otimes \Omega^n_X)$ have the same dimension.

In the case where $X$ is a curve, we choose a canonical divisor $K_X$ so that $\Omega_X = \mathcal{O}_X(K_X)$. Applying the theorem to $\mathcal{E} = \mathcal{O}_X(K_X - D)$, we get an equality

$$h^1(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(K_X - D)),$$

and the previous Riemann–Roch formula takes the following form:

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**Theorem 19.8 (Riemann–Roch).** Let $X$ be a nonsingular projective curve of genus $g$ and let $D$ be a divisor on $X$. Then

$$h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X(K_X - D)) = \deg D + 1 - g$$

This result is significantly more powerful than the Riemann–Roch formula we saw earlier. It gives stronger conclusions, because the group $H^0(X, \mathcal{O}_X(K_X - D))$ is easier to interpret: it is the space of global sections of the sheaf associated to the divisor $K_X - D$. Often, we can readily argue that no such global sections exist, due to simple reasons. For instance, in the case $\deg D > \dim K_X$ then $K_X - D$ cannot be effective, as effective divisors have non-negative degree.

In fact, we can use the Riemann–Roch formula to compute the degree of the canonical divisor $K_X$:

**Corollary 19.9.** Let $X$ be a nonsingular projective curve of genus $g = h^1(\mathcal{O}_X)$. Then
(i) $\deg K_X = 2g - 2$
(ii) $h^0(X, \mathcal{O}_X(K_X)) = g$.

*Proof* From Serre duality, we get that $H^0(X, \mathcal{O}_X(K_X))$ and $H^1(X, \mathcal{O}_X)$ have the same dimension. Therefore, applying the Riemann–Roch formula to $D = K_X$, we get

$$g - 1 = \dim_k H^0(X, \mathcal{O}_X(K_X)) - \dim_k H^0(X, \mathcal{O}_X(K_X - K_X)) = \deg K + 1 - g$$

which gives that $\deg K_X = 2g - 2$. □

Combining this with the previous paragraph, we obtain:

**Corollary 19.10.** Let $D$ be a divisor of degree $\geqslant 2g - 1$. Then
(i) $H^1(X, \mathcal{O}_X(D)) = 0$.
(ii) $h^0(X, \mathcal{O}_X(D)) = \deg D + 1 - g$.
Moreover, if $\deg D = 2g - 2$, then $H^1(X, \mathcal{O}_X(D)) \neq 0$ if and only if $D \sim K_X$.

*Proof* For the last part, note that $h^0(X, \mathcal{O}_X(K_X - D)) \neq 0$ if and only if $K_X - D$ is effective. But the only effective divisor of degree 0 is the divisor 0, so this happens if and only if $K_X$ and $D$ are linearly equivalent. □

Therefore, in our path to computing $h^0(X, \mathcal{O}_X(D))$ for a divisor $D$, we are left with the 'intermediate cases' where $D$ has degree between 0 and $2g - 2$. In this region, the computations become more subtle, and in particular, the rank of $H^0(X, \mathcal{O}_X(D))$ does not depend on $\deg D$ and $g$ alone.

| $\deg D$ | $< 0$ | $0$ | $1$ | $\dots$ | $2g-2$ | $\geqslant 2g-1$ |
|---|---|---|---|---|---|---|
| $h^0(X, D)$ | ? | ? | ? | $\dots$ | ? | $\deg D + 1 - g$ |
| $h^1(X, D)$ | $-(\deg D + 1 - g)$ | ? | ? | $\dots$ | ? | $0$ |

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## 19.2 Proof of Serre duality

The aim of the next few sections is to prove the following:

**Theorem 19.11 (Serre duality).** Let $X$ be a projective curve over an algebraically closed field $k$. Then there is an $\mathcal{O}_X$-module $\omega_X$ of finite type, together with an isomorphism $H^1(X, \omega_X) \simeq k$, such that for any locally free sheaf $\mathcal{F}$ on $X$, there is a perfect pairing

$$H^0(X, \mathcal{F}) \times H^1(X, \omega_X \otimes \mathcal{F}^\vee) \longrightarrow H^1(X, \omega_X) \simeq k \tag{19.7}$$

In particular, $H^0(X, \mathcal{F})$ and $H^1(X, \omega_X \otimes \mathcal{F}^\vee)$ have the same dimension.

The sheaf $\omega_X$ is called a *dualizing sheaf*. The existence of $\omega_X$ is usually not enough for applications or explicit computations. The cruicial point is that, when $X$ is nonsingular, the dualizing sheaf is isomorphic to the cotangent sheaf:

**Theorem 19.12.** If $X$ is a nonsingular, projective curve, the dualizing sheaf $\omega_X$ is isomorphic to the cotangent sheaf $\Omega_X$.

These results are fundamental in algebraic geometry and one can find several proofs in the literature (see for instance, Hartshorne (2013), Liu (2002), Serre (1955),Serre (2013), Kempf (1993), Vakil (2024)). The proof presented here uses very little machinery, and it is inspired by the proofs found in Kempf (1993) and Vakil (2024). The approach here is however more ad hoc and much less conceptual than the standard proofs, and it gives essentially no information about the isomorphism $H^1(X, \Omega_X) \simeq k$.

We will prove the two theorems in three steps:

(i) We note that both theorems hold for $X = \mathbb{P}^1_k$, where $\omega_X = \mathcal{O}_{\mathbb{P}^1_k}(-2)$ serves as a dualizing sheaf (and we know this coincides with $\Omega_{\mathbb{P}^1_k}$).

(ii) Then we prove the existence of $\omega_X$ for a general curve, by using a finite map $f\colon X \to \mathbb{P}^1_k$. The sheaf $\omega_X$ is constructed just to satisfy the formal properties of Serre duality.

(iii) We finally prove that $\omega_X \simeq \Omega_X$.

The first step, Serre duality for $\mathbb{P}^1_k$, is the easiest:

**Lemma 19.13 (Serre duality for $\mathbb{P}^1_k$).** Serre duality holds for $\mathbb{P}^1_k$ with $\omega_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$.

*Proof* We showed in Chapter 15 that Serre duality holds for any projective space for the sheaves $\mathcal{F}$ of the form $\mathcal{O}_{\mathbb{P}^1_k}(d)$. For $\mathbb{P}^1_k$ specifically, we identify $H^0(X, \mathcal{O}_{\mathbb{P}^1_k}(d))$ with the $k$-vector space of homogeneous polynomials of degree $d$ and $H^1(X, \mathcal{O}_{\mathbb{P}^1_k}(-2-d))$ with the $k$-vector space spanned by Laurent monomials $x_0^{-u}x_1^{-v}$ with $u + v = d + 2$, $u, v \geqslant 1$. The multiplication map

$$x_0^a x_1^b \times x_0^{-u}x_1^{-v} := \begin{cases} x_0^{-1}x_1^{-1} & \text{if } (u, v) = (a-1, b-1) \\ 0 & \text{otherwise} \end{cases}$$

defines a perfect pairing, which induces (19.7) for $\mathcal{F} = \mathcal{O}(d)$ with $d \geqslant 0$. For $d < 0$, all groups are zero.

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Recalling that any locally free sheaf $\mathcal{F}$ on $\mathbb{P}^1_k$ is a direct sum of invertible sheaves, we get the pairing (19.7) in general. □

## 19.3 The dualizing sheaf

Let $\phi\colon A \to B$ be a map of rings making $B$ into a finite $A$-module. For an $A$-module $N$, we will consider the 'dual module' $\mathrm{Hom}_A(B, N)$, which is a priori an $A$-module. The crucial observation is that this can also be viewed as a *$B$-module*, via the rule

$$b \cdot \phi(y) := \phi(b \cdot y), \quad y \in B$$

for each $A$-linear map $\phi : B \to N$. Note that $\mathrm{Hom}_A(B, N)$ is finitely generated as a $B$-module if $N$ is finitely generated as an $A$-module.

The main property we need is the following:

**Lemma 19.14.** For a $B$-module $L$, there is a natural isomorphism of $A$-modules

$$\mathrm{Hom}_A(L_A, N) \longrightarrow \mathrm{Hom}_B(L, \mathrm{Hom}_A(B, N))_A \tag{19.8}$$

*Proof* The map (19.8) is defined by sending $\psi : L \to N$ to $\phi : L \to \mathrm{Hom}_A(B, N)$ defined by $\phi(\ell) = (b \mapsto \psi(b{\cdot}\ell))$. The inverse is defined by the map sending $\phi : L \to \mathrm{Hom}_A(B, N)$ to $\ell \mapsto \phi(\ell)(1)$.

□

The operation $N \mapsto \mathrm{Hom}_A(B, N)$ defines a functor from $A$-modules to $B$-modules. It behaves well with respect to localization: if $S \subset A$ is a multiplicative set, then as $B$ is finitely generated, there is a natural isomorphism of $S^{-1}B$-modules

$$S^{-1}\,\mathrm{Hom}_A(B, N) = \mathrm{Hom}_{S^{-1}B}(S^{-1}B, S^{-1}N). \tag{19.9}$$

Because of this, the construction generalizes well to a version for sheaves. For a finite morphism $f\colon X \to Y$ of Noetherian schemes and a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{G}$, we define $f^!\mathcal{G}$ by the formula

$$\Gamma(f^{-1}U, f^!\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_Y(U)}(\mathcal{O}_X(f^{-1}U), \mathcal{G}(U)) \tag{19.10}$$

for each affine $U = \mathrm{Spec}\, A$. Note that $f^!\mathcal{G}$ is quasi-coherent, as it coincides with the tilde of $\mathrm{Hom}_A(B, N)$ over affines. As $f$ is finite, $f^!\mathcal{G}$ is moreover of finite type if $\mathcal{G}$ is.

By (19.10), we have $f_*f^!\mathcal{G} = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{G})$. More generally, if $\mathcal{F}$ is an $\mathcal{O}_X$-module of finite type, we have a natural isomorphism

$$f_*\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, f^!\mathcal{G}) \simeq \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^1_k}}(f_*\mathcal{F}, \mathcal{G}). \tag{19.11}$$

This follows because the isomorphisms in the map (19.8) are compatible with localizations.

**Definition 19.15.** Let $X$ be a nonsingular projective curve over $k$, and let $f : X \to \mathbb{P}^1_k$ be a finite morphism. We define the *dualizing sheaf* of $X$ to be the $\mathcal{O}_X$-module

$$\omega_X = f^!\omega_{\mathbb{P}^1_k}.$$

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So far we haven't used the fact that $X$ is nonsingular; any projective curve admits a dualizing sheaf $\omega_X$. In the nonsingular case, we will prove in Section 19.4 that $\omega_X \simeq \Omega_X$. In particular, the sheaf is independent of the chosen finite map to $\mathbb{P}^1_k$. As a first step towards this, we next show that $\omega_X$ is invertible.

**Proposition 19.16.** Let $X$ be a nonsingular projective curve. Then $\omega_X$ is an invertible sheaf.

*Proof* Since $X$ is a nonsingular curve, $\omega_X$ is locally free if and only if it is torsion free. Let $\mathcal{T}$ denote the torsion subsheaf and $\mathcal{E}$ is the torsion free part, so that $\omega_X = \mathcal{E} \oplus \mathcal{T}$ by Theorem 18.17. Applying $f_*$, we get

$$f_*\omega_X = f_*\mathcal{E} \oplus f_*\mathcal{T}.$$

Applying the formula (19.11), shows that $f_*\omega_X = \mathcal{H}om(f_*\mathcal{O}_X, \omega_{\mathbb{P}^1_k})$. As $f$ is finite and surjective, $f_*\mathcal{O}_X$ is locally free (Proposition 18.18). Since $\omega_{\mathbb{P}^1_k} = \mathcal{O}_{\mathbb{P}^1_k}(-2)$ is also invertible, we find that $f_*\omega_X$ is also locally free. Note that $f_*\mathcal{T}$ is again a torsion sheaf on $\mathbb{P}^1_k$. As $f_*\mathcal{T}$ is a direct summand of a locally free sheaf, we must have $f_*\mathcal{T} = 0$. This implies that $\Gamma(X, \mathcal{T}) = \Gamma(\mathbb{P}^1_k, f_*\mathcal{T}) = 0$. On a curve, the only torsion sheaf with no global sections is the zero sheaf, so $\mathcal{T} = 0$, and hence $\omega_X$ is locally free.

We next show that $\omega_X$ has rank 1. To show this, it suffices to compute the stalk at the generic point $\xi = \operatorname{Spec} K(X)$. We can compute this using the affine open sets $\operatorname{Spec} B \subset X$ and $\operatorname{Spec} A \subset \mathbb{P}^1_k$. Then with $S = B - 0$, the stalk $\omega_{X,\xi}$ is given by

$$S^{-1}\operatorname{Hom}_A(B, L) = \operatorname{Hom}_{K(\mathbb{P}^1_k)}(K(X), K(\mathbb{P}^1_k))$$

which is a $K(\mathbb{P}^1_k)$-vector space of dimension equal to the degree of the field extension $K(X)/K(\mathbb{P}^1_k)$, i.e., the degree of $f$. Hence, as a $K(X)$-vector space it has dimension 1. Therefore, $\omega_X$ is an invertible sheaf. □

Having established this fact, we can finish the proof of Serre duality on $X$. For any locally free sheaf $\mathcal{F}$ on $X$, we have a a chain of natural bijections:

$$\begin{aligned}
H^1(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X) &= H^1(\mathbb{P}^1_k, f_*(\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X)) && (19.12)\\
&= H^1(\mathbb{P}^1_k, f_*\,\mathcal{H}om(\mathcal{F}, \omega_X)) && (19.13)\\
&= H^1(\mathbb{P}^1_k, \mathcal{H}om(f_*\mathcal{F}, \omega_{\mathbb{P}^1_k})) && (19.14)\\
&= H^1(\mathbb{P}^1_k, (f_*\mathcal{F})^\vee \otimes_{\mathcal{O}_{\mathbb{P}^1_k}} \omega_{\mathbb{P}^1_k}) && (19.15)\\
&= H^0(\mathbb{P}^1_k, f_*\mathcal{F})^\vee && (19.16)\\
&= H^0(X, \mathcal{F})^\vee. && (19.17)
\end{aligned}$$

Many of the steps here are non-trivial: (19.12) and (19.17) use Lemma 15.20. (19.13) and (19.15) follow by Proposition 14.3 and the fact that $\mathcal{F}$, $f_*\mathcal{F}$, $\omega_X$ and $\omega_{\mathbb{P}1}$ are locally free. (19.14) uses the adjoint property (19.11) and finally (19.16) is Serre duality on $\mathbb{P}^1_k$.

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## 19.4 The dualizing sheaf is the canonical sheaf

The goal of this section is to show that the dualizing sheaf $\omega_X$ is isomorphic to the cotangent sheaf $\Omega_X$. Both of these sheaves are locally free: the first by Corollary 19.16, and $\Omega_X$ because $X$ is nonsingular. However, at this point, we do not even have a map between them.

The proof makes use of the following two basic properties of the dualizing sheaf:

(i) $H^1(X, \omega_X) \neq 0$.

(ii) $H^1(X, \omega_X \otimes \mathcal{O}_X(P)) = 0$ for every point $P \in X$.

In fact, we will see that these properties characterize $\omega_X$ uniquely, and it is this which in turn allows us to conclude that $\Omega_X = \omega_X$.

*Proof of Theorem 19.12* We will work with the fiber product $X \times X$ (taken over $k$) with the two projections $p, q\colon X \times X \to X$. Consider the diagonal $\Delta \subset X \times X$ (which we view as a closed subscheme, because $X$ is separated over $k$). Since $X \times X$ is a nonsingular surface, $\Delta$ is a divisor on $X \times X$, and the corresponding ideal sheaf $\mathcal{I}$ is the invertible sheaf $\mathcal{O}_{X\times X}(-\Delta)$. Consider the ideal sheaf sequence

$$0 \longrightarrow \mathcal{O}_{X\times X}(-\Delta) \longrightarrow \mathcal{O}_{X\times X} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$$

where $i\colon X \to X \times X$ is the diagonal embedding. Tensoring the sequence by $\mathcal{O}_{X\times X}(\Delta)$, we obtain

$$0 \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{X\times X}(\Delta) \longrightarrow \mathcal{O}_{X\times X}(\Delta) \otimes_{\mathcal{O}_{X\times X}} i_*\mathcal{O}_X \longrightarrow 0 \tag{19.18}$$

By the projection formula, the sheaf on the right is isomorphic to $i_*(i^*\mathcal{O}_{X\times X}(\Delta))$. On the other hand, by the definition of the cotangent sheaf, the pullback $i^*\mathcal{O}_{X\times X}(-\Delta)$ is given by $\Omega_X$. Dually, this means that the sheaf on the right-hand side in (19.18) is isomorphic to $i_*\mathcal{T}_X$, and the sequence takes the form

$$0 \longrightarrow \mathcal{O}_{X\times X} \longrightarrow \mathcal{O}_{X\times X}(\Delta) \longrightarrow i_*\mathcal{T}_X \longrightarrow 0. \tag{19.19}$$

We now tensor this sequence by $q^*\omega_X$, to get a sequence

$$0 \longrightarrow q^*\omega_X \longrightarrow q^*\omega_X(\Delta) \longrightarrow i_*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{T}_X) \longrightarrow 0 \tag{19.20}$$

Here we have used the projection formula again:

$$i_*\mathcal{T}_X \otimes_{\mathcal{O}_{X\times X}} q^*\omega_X = i_*(\mathcal{T}_X \otimes_{\mathcal{O}_X} i^*q^*\omega_X) = i_*(\mathcal{T}_X \otimes_{\mathcal{O}_X} \omega_X).$$

If we restrict the sequence (19.20) to the open set $V \times X$, where $V$ is affine, and take the long exact sequence in cohomology, we get an exact sequence

$$\Gamma(V \times X, i_*(\omega_X \otimes_{\mathcal{O}_X} \mathcal{T}_X)) \longrightarrow H^1(V \times X, q^*\omega_X) \longrightarrow H^1(V \times X, q^*\omega_X(\Delta)) \tag{19.21}$$

By Lemma 15.20, we may identify the first group with $\Gamma(V, \omega_X \otimes_{\mathcal{O}_X} \mathcal{T}_X)$. The next lemma identifies the middle group.

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**Lemma 19.17.** Let $V$ and $X$ be varieties over $k$ with $V$ affine. Let $q\colon V \times X \to X$ denote the second projection. Then for any quasi-coherent sheaf $\mathcal{F}$ on $X$, there are natural isomorphisms

$$\begin{aligned} H^0(V \times X, q^*\mathcal{F}) &= \mathcal{O}_V(V) \otimes_k H^0(X, \mathcal{F}) \\ H^1(V \times X, q^*\mathcal{F}) &= \mathcal{O}_V(V) \otimes_k H^1(X, \mathcal{F}). \end{aligned} \tag{19.22}$$

*Proof* Recall the exact sequence

$$0 \to H^0(X, \mathcal{F}) \to \mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{d^0} \mathcal{F}(U_0 \cap U_1) \to H^1(X, \mathcal{F}) \to 0. \tag{19.23}$$

Tensoring by $\mathcal{O}_V(V)$ over $k$ preserves exactness, so we get an exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_V(V) \otimes_k H^0(X, \mathcal{F}) \to \mathcal{O}_V(V) \otimes_k \mathcal{F}(U_0) \times \mathcal{O}_V(V) \otimes_k \mathcal{F}(U_1) \\ \to \mathcal{O}_V(V) \otimes_k \mathcal{F}(U_0 \cap U_1) \to \mathcal{O}_V(V) \otimes_k H^1(X, \mathcal{F}) \to 0. \end{aligned}$$

Note that the map in the middle is exactly the Cech complex with of the sheaf $q^*\mathcal{F}$ with respect to the two affine open sets $V \times U_0$ and $V \times U_1$. By exactness, we get the desired isomorphisms. □

By the lemma, we may identity the middle cohomology group in (19.21) with $\Gamma(V, \mathcal{O}_X) \otimes_k H^1(X, \omega_X) \simeq \Gamma(V, \mathcal{O}_X)$. Below, we will show that

$$H^1(V \times X, q^*\omega_X(\Delta)) = 0. \tag{19.24}$$

Given this, the sequence (19.21) can be written

$$\Gamma(V, \omega_X \otimes \mathcal{T}_X) \longrightarrow H^1(X, \omega_X) \otimes_k \mathcal{O}_X(V) \longrightarrow 0. \tag{19.25}$$

These identifications are compatible with restriction maps, i.e., if $V' \subset V$ is a smaller affine open set, the following diagram commutes

$$\begin{array}{ccccc} \Gamma(V, \omega_X \otimes \mathcal{T}_X) & \xrightarrow{\delta} & H^1(V \times X, q^*\omega_X) & = & H^1(X, \omega_X) \otimes_k \mathcal{O}_X(V) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma(V', \omega_X \otimes \mathcal{T}_X) & \xrightarrow{\delta} & H^1(V' \times X, q^*\omega_X) & = & H^1(X, \omega_X) \otimes_k \mathcal{O}_X(V') \end{array}$$

Therefore, these maps glue to define a map of *sheaves*

$$\omega_X \otimes \mathcal{T}_X \longrightarrow \mathcal{O}_X.$$

By (19.25), this map is surjective. As both sheaves are locally free, this must be an isomorphism (the kernel is locally free of rank 0), so $\omega_X \otimes \mathcal{T}_X \simeq \mathcal{O}_X$. But this means that $\omega_X \simeq \Omega_X$.

To conclude, we need to show that (19.25) holds. Let $y \in V$ be a $k$-point and consider the subscheme $\iota\colon y \times X \to V \times X$. Identifying $y \times X$ with $X$, we have

$$\iota^*(q^*\omega_X) = \omega_X$$

and

$$\iota^*(\mathcal{O}_{X\times X}(\Delta)|_{V\times X}) = \mathcal{O}_X(y)$$

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Hence $\iota^*(q^*\omega_X(\Delta)) = \omega_X(y)$. This sheaf has an $H^1$ which is zero, by the dualizing property:

$$H^1(X, \omega_X(y)) = H^0(X, \mathcal{O}_X(-y))^\vee = 0 \tag{19.26}$$

Indeed, $\Gamma(X, \mathcal{O}_X(-y))$ is the group of global regular functions vanishing at $y$, but all regular functions are constants, as $X$ is projective.

Now we will apply the following lemma.

**Lemma 19.18 ($H^1$ and base change).** Let $f\colon Y \to S$ be a a projective morphism of Noetherian schemes, where $S = \operatorname{Spec} A$ is affine, and assume that $Y$ can be covered by two affine open sets $U_0$ and $U_1$. Let $h\colon Y_B \to Y$ be the morphism defined by a base change $\operatorname{Spec} B \to \operatorname{Spec} A$. Then for any $\mathcal{O}_Y$-module of finite type $\mathcal{F}$, we have a natural isomorphism of $B$-modules

$$H^1(Y \times_A B, h^*\mathcal{F}) = H^1(Y, \mathcal{F}) \otimes_A B. \tag{19.27}$$

*Proof* Consider the exact sequence

$$\mathcal{F}(U_0) \times \mathcal{F}(U_1) \xrightarrow{d} \mathcal{F}(U_0 \cap U_1) \to H^1(Y, \mathcal{F}) \to 0. \tag{19.28}$$

coming from the Cech complex on $U_0$ and $U_1$. Tensoring this complex by $B$ gives the Cech complex of the pullback $h^*\mathcal{F}$. Therefore, $H^1(Y \times_A B, h^*\mathcal{F})$ is the cokernel of $d \otimes \mathrm{id}_B$ and so since the tensor product is right exact, we get the isomorphism (19.27). □

We apply the lemma to $Y = V \times X$, $S = V$, $f = p$, $B = A/\mathfrak{m}$ and $\mathcal{F} = q^*\omega_X(\Delta)$. Since $H^1(X, \iota^*(q^*\omega_X)) = 0$, we find that the $A$-module $M = H^1(V \times X, q^*\omega_X(\Delta))$ satisfies $M \otimes_A A/\mathfrak{m} = 0$. Since this happens for every maximal ideal $\mathfrak{m}$ in $A$, we get that $M = 0$ by Nakayama's lemma. This concludes the proof. □

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# 20

# Applications of the Riemann–Roch theorem

## 20.1 A criterion for being a closed embedding

If $f\colon X \to Y$ is a morphism of schemes, it can easily happen that $f$ is a homeomorphism without being an isomorphism, even when $X$ and $Y$ are varieties. (See Example 20.3 below). In light of this, it is natural to ask if there is a sufficient extra condition to guarrantee that $f$ is a closed embedding. As it turns out, it turns out that it is enough that $f$ induces an isomorphism on cotangent spaces:

**Proposition 20.1.** Let $f\colon X \to Y$ be a finite map of schemes of finite type over an algebraically closed field $k$. Then $f$ is a closed embedding if and only if

(i) $f$ is injective on $k$-points; and

(ii) for each $k$-point $P \in X(k)$, the induced map on cotangent spaces

$$f^*\colon \mathfrak{m}_{f(P)}/\mathfrak{m}_{f(P)}^2 \longrightarrow \mathfrak{m}_P/\mathfrak{m}_P^2 \tag{20.1}$$

is surjective.

*Proof* The conditions (i) and (ii) are easily seen to be satisfied for a closed embedding, so we need to prove the converse direction.

The question is local on $Y$, so we may assume that $Y$ is affine, say $Y = \operatorname{Spec} A$. As $f$ is finite, $X$ is also affine, say $X = \operatorname{Spec} B$, for some finite $A$-module $B$, and $f$ is induced by a ring map $\phi\colon A \to B$. We need to show that $\phi$ is surjective.

Viewing $\phi$ as a map of $A$-modules, it is enough to show that $\phi_{\mathfrak{m}} : A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is surjective for every maximal ideal $\mathfrak{m}$ in $A$. Let $Q \in Y$ be the $k$-point corresponding to $\mathfrak{m}$. Recall that the scheme-theoretic fiber $X_Q$ of $Q$ is given by $\operatorname{Spec}(B/\mathfrak{m}B)$. If $Q$ is not in the image of $f$, then $B/\mathfrak{m}B = 0$. In this case $B_{\mathfrak{m}} = 0$ because $\mathfrak{m}B$ is the unit ideal in $B$, and $\phi_{\mathfrak{m}}$ is trivially surjective. If $Q$ is in the image, then assumption (i) shows that $Q = f(P)$ for a unique $k$-point $P \in X$, corresponding to a maximal ideal $\mathfrak{n}$ in $B$.

We claim that $\mathfrak{n} = \mathfrak{m}B$. Consider the ideal $\mathfrak{b} = \phi(\mathfrak{m})B$ in $B$. As $P$ maps to $Q$, we have in any case an inclusion $\mathfrak{b} \subset \mathfrak{n}$. By the hypothesis (ii), the composition $\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2$ is surjective, so $\mathfrak{b} \to \mathfrak{n}/\mathfrak{n}^2$ is surjective, and hence $\mathfrak{b} + \mathfrak{n}^2 = \mathfrak{n}$. Consider the $B$-module $M = \mathfrak{n}/\mathfrak{b}$. Then $\mathfrak{b} + \mathfrak{n}^2 = \mathfrak{n}$ implies that $\mathfrak{n}M = M$ and so $M = 0$ by Nakayama's lemma. Therefore $\mathfrak{b} = \mathfrak{n}$.

From this it follows that $A/\mathfrak{m} = B/\phi(\mathfrak{m})B = B/\mathfrak{n} = k$. Moreover, the map $\phi \otimes A/\mathfrak{m}$, which is the map $A/\mathfrak{m} \to B/\mathfrak{n}$, is surjective. As $B$ is a finite $A$-module, it follows that $\phi$ is surjective as well (by Nakayama's lemma again). □



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Note that the condition 20.1 is equivalent to the induced map on tangent spaces

$$df_P \colon T_P X \longrightarrow T_{f(P)} Y \tag{20.2}$$

is injective. Yet another way of rephrasing it is that $f$ is a closed embedding if and only if the induced maps $X(k) \to Y(k)$ and $X(R) \to Y(R)$ are injective, where $R = k[\epsilon]/\epsilon^2$ is the ring of dual numbers.

**Example 20.2.** The assumption that $k$ is algebraically closed is important in the above proposition. For instance, the scheme map $f \colon \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{R}$ satisfies (i) and (ii), but it is not a closed embedding. △

**Example 20.3.** We saw in Example **??** that the morphism

$$f \colon \operatorname{Spec} \mathbb{C}[x] \longrightarrow \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3)$$

is a homeomorphism. $f$ satisfies the condition (i), but not (ii), as the map on tangent spaces at the origin is the zero map. △

## 20.2 Ample invertible sheaves and Serre's theorems

If $L$ is a globally generated invertible sheaf, and $X$ is Noetherian, we can find finitely many sections $s_0, \ldots, s_n$ that generate $L$ and define a morphism to projective space $\phi : X \to \mathbb{P}^n$ so that $L = \phi^* \mathcal{O}_{\mathbb{P}^n}(1)$. It is natural to ask when $\phi$ is a closed embedding. If this is the case, then we can view $X$ as a closed subscheme of $\mathbb{P}^n$ and $L$ is simply the restriction $\mathcal{O}_{\mathbb{P}^n}(1)|_X$ to $X$. Hence the closed embedding is completely determined by the sections $s_0, \ldots, s_n$ and the polynomial relations between them.

**Definition 20.4.** Let $X$ be a scheme over a Noetherian ring $A$. We say that an invertible sheaf $L$ is *very ample* if there is a closed embedding $\iota \colon X \to \mathbb{P}^n_A$ such that $L \simeq \iota^* \mathcal{O}_{\mathbb{P}^n_S}(1)$. $L$ is said to be *ample* if $L^{\otimes N}$ is very ample for some $N > 0$.

**Example 20.5.** If $X = \operatorname{Spec} A$ is affine, then $\mathcal{O}_X$ is very ample. The closed embedding in question is simply the identity morphism $\operatorname{Spec} A \to \mathbb{P}^0_A = \operatorname{Spec} A$. △

**Example 20.6.** If $R$ is a graded ring which is finitely generated in degree 1 over a field $k$, then $\mathcal{O}_{\operatorname{Proj} R}(d)$ is very ample for every $d > 0$. If $x_0, \ldots, x_n$ are generators for $R$, then the homogeneous polynomials in $x_0, \ldots, x_n$ of degree $d$ define a morphism to projective space $\operatorname{Proj} R \to \mathbb{P}^N_k$ which is a composition of the closed embedding $X \to \mathbb{P}^n_k$ and the Veronese embedding $v_d \colon \mathbb{P}^n_k \to \mathbb{P}^N_k$. △

We will see several examples of ample, but non-very ample invertible sheaves in Chapter 18 (Example 18.39).

In this chapter, we give a few of the (many) consequences of the Riemann–Roch formula. We begin by translating the results of Chapter **??** into concrete numerical criteria for a divisor $D$ to be base point free or very ample. Then we use these results to classify all curves of genus $\leqslant 4$.

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### 20.3 Base-point freeness and very ampleness on curves

Let $X$ be a nonsingular projective curve over an algebraically closed field $k$ and let $D$ be a divisor on $X$. We say that a point $P \in X$ is a *base point* of $D$ (or perhaps more precisely, the linear system $|D|$) if every section of $H^0(X, \mathcal{O}_X(D))$ vanishes at $P$. If there are no base points, we say that $D$ is base-point free.

In light of the isomorphism $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathfrak{m}_P \simeq \mathcal{O}_X(D-P)$, we see that $P$ is a base-point if and only if the inclusion $H^0(X, \mathcal{O}_X(D-P)) \subset H^0(X, \mathcal{O}_X(D))$ is an isomorphism. In general, the dimension of $H^0(X, \mathcal{O}_X(D-P))$ can be at most 1 less than $H^0(X, \mathcal{O}_X(D))$, by the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X(D-P)) \longrightarrow H^0(X, \mathcal{O}_X(D)) \xrightarrow{\mathrm{ev}_P} \kappa(p) \tag{20.3}$$

Therefore, $D$ is base-point free if and only if for every point $P \in X$,

$$h^0(D-P) = h^0(D) - 1.$$

If this condition is satisfied, we obtain a morphism $f\colon X \to \mathbb{P}^n_k$, by Theorem 14.26.

We now turn to the question whether $f$ is a closed embedding. For this, we use the two conditions appearing in Proposition 20.1. That is, $f$ should be injective on $k$-points, and the map $f^*$ on cotangent spaces should be surjective.

Let us choose a basis $s_0, \ldots, s_n$ for $H^0(X, \mathcal{O}_X(D))$. Then, over the open set $D(s_0)$, $f$ is induced by the ring map

$$k\left[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right] \longrightarrow \Gamma(D(s_0), \mathcal{O}_X) \tag{20.4}$$

sending $x_i/x_0$ to $s_i/s_0$. Here we recall that $s_i/s_0$ is viewed as regular function on $D(s_0)$ as in Section 14.7.

To show that $f$ is injective, we want to show that for any two points $P, Q \in X(k)$, we have $f(P) \neq f(Q)$. This happens if we are able to find a section $s = \sum a_i s_i$ such that $s(P) = 0$ and $s(Q) \neq 0$. Because in that case, $f(P)$ lies in the hyperplane $V(\sum a_i x_i) \subset \mathbb{P}^n_k$, whereas $f(Q)$ does not.

Therefore, we want to show that there is a section $s$ which lies in $H^0(X, \mathcal{O}_X(D-P))$, but not in $H^0(X, \mathcal{O}_X(D-P-Q))$. It is therefore enough to show that for any $P$ and $Q$, we have a strict inequality $h^0(D-P-Q) < h^0(D-P)$. As $h^0(D-P) = h^0(D) - 1$, by the above, this is equivalent to the condition

$$h^0(D-P-Q) = h^0(D) - 2 \quad \text{for every pair } P, Q \in X. \tag{20.5}$$

Next we would like to know whether the pullback map on cotangent spaces

$$f^*\colon \mathfrak{m}_{f(P)}/\mathfrak{m}^2_{f(P)} \longrightarrow \mathfrak{m}_P/\mathfrak{m}^2_P$$

is surjective. As $X$ is a nonsingular curve, $\mathfrak{m}_P/\mathfrak{m}^2_P$ is 1-dimensional vector space, so we just need $f^*$ to be non-zero.

Let us for simplicitly assume that $P$ maps to the point $(1 : 0 : \cdots : 0) \in \mathbb{P}^n_k$ (we may acheive this by a linear change of coordinates). This means that the morphism is given locally by $D(s_0) \to D(x_0)$ as in (20.4). More precisely, if $y_i = x_i/x_0$ $i = 1, \ldots, n$ denote the affine coordinates on $D(x_0)$, we have $f^\sharp(y_i) = s_i/s_0 \in \Gamma(D(s_0), \mathcal{O}_X)$. In these coordinates,

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$\mathfrak{m}_{f(P)}/\mathfrak{m}_{f(P)}^2$ is the $k$-vector space generated by the classes $\overline{y_1}, \dots, \overline{y_n}$, and $f^*$ maps $\overline{y_i}$ to $\overline{s_i/s_0}$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$. It is therefore enough to show that some linear combination of the $y_i$ pulls back to an element in $\mathfrak{m}_P$ which vanishes to order 1 at $P$. Note that a linear combination $\sum a_i \frac{s_i}{s_0}$ vanishes to order 1 at $P$ if and only if $\sum a_i s_i$ vanishes to order 1 at $P$, that is, lies in $H^0(X, \mathcal{O}_X(D-P))$ but not in $H^0(X, \mathcal{O}_X(D-2P))$.

Therefore, to show that such linear forms exist, we need to show that $H^0(X, \mathcal{O}_X(D-2P))$ is strictly smaller than $H^0(X, \mathcal{O}_X(D-P))$. As above, the dimension can drop by at most 1, so this happens if and only if

$$h^0(D-2P) = h^0(D-P) - 1 = h^0(D) - 2$$

To summarize, we have proved the first two items in the following theorem:

**Theorem 20.7 (Base-point free and very ample divisors on curves).**
Let $X$ be a nonsingular projective curve over an algebraically closed field $k$ and let $D$ be a divisor on $X$. Then

(i) $D$ is base point free if and only if

$$h^0(D-P) = h^0(D) - 1 \qquad \text{for every point } P \in X(k).$$

(ii) $D$ is very ample if and only if

$$h^0(D-P-Q) = h^0(D) - 2 \qquad \text{for every two points } P, Q \in X(k)$$

(including the case $P = Q$)

(iii) If $\deg D \geqslant 2g$, then $D$ is base point free and $h^1(X, D) = 0$.

(iv) If $\deg D \geqslant 2g+1$, then $D$ is very ample.

As $\mathcal{O}_X(D)|_U \simeq \mathcal{O}_U$ over $U = X - \operatorname{Supp}(D)$, it is enough to check the condition (i) for all $P$ in the support of $D$.

*Proof* The items (i) and (i) follow from Proposition 20.1 and the discussion above.

For (iii), note that $H^1(D-P)$ and $H^1(D)$ are both zero, by Corollary 19.10. Therefore, the exact sequence (20.3) is in fact a short exact sequence, and hence $h^0(D-p) = h^0(D)-1$. Since this holds for any $P \in X$, we conclude by (i)

Similarly, for (iv), we get $h^1(D-P-Q) = 0$, which implies $h^0(D-P-Q) = h^0(D)-2$, and this allows us to apply (ii). □

In particular, the theorem implies that a divisor $D$ on a curve is ample if and only if $\deg D > 0$.

**Example 20.8.** On $X = \mathbb{P}^1$ a divisor $D$ is base point free if and only if $\deg \geqslant 0$ and very ample if and only if $\deg D > 0$. △

**Example 20.9.** If $X$ is a curve of genus 1, a divisor $D$ is base point free if $\deg D \geqslant 2$. We will see later that, if $D = P$ for a $k$-point $P$, we have $h^0(X, \mathcal{O}_X(D)) = 1$, so $D$ cannot be base point free (a generator $s \in H^0(X, \mathcal{O}_X(D))$ satisfies $\operatorname{div}(s) = P$, so it vanishes at $P$).

A divisor $D$ on $X$ is very ample if $\deg D \geqslant 3$. △

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## 20.4 Curves of genus 0

The results of the previous sections are particularly strong when the genus is small. In the case $X$ is a curve of genus 0, a divisor $D$ is base-point free if and only if $\deg D \geqslant 0$ and very ample if and only if $\deg D > 0$.

If we apply this observation to a divisor of the form $D = P$ for a $k$-point $P$, we see that $h^0(D) = 1 + 1 - 0 = 2$ and $D$ defines a closed embedding $X \to \mathbb{P}^1_k$. Therefore, we have proved the following remarkable theorem:

**Theorem 20.10.** Let $X$ be a nonsingular curve of genus 0 over an algebraically closed field $k$. Then $X$ is isomorphic to $\mathbb{P}^1_k$.

**Example 20.11.** Consider the plane curve $X \subset \mathbb{P}^2_k$ defined by the homogeneous equation

$$x_0^2 + x_1^2 - x_2^2 = 0.$$

Then $X$ as a $k$-point $P = (1 : 0 : 1)$. Let us find a basis for $H^0(X, \mathcal{O}_X(P))$.

We are looking for rational functions $g$ which have a pole of order 1 at $P$ and no poles at any other point. The linear form $x_0 - x_2$ vanishes at $P$, but it has multiplicity 2 there. To make up for this, we can consider a quotient $\ell/(x_0 - x_2)$ where $\ell$ is a linear form in $x_0, x_1, x_2$ vanishing with order 1 at $P$. For instance, we can take $\ell = x_1$. And indeed, the quotient

$$g = \frac{x_1}{x_0 - x_1}$$

satisfies

$$\operatorname{div}(g) = (1 : 0 : 1) + (1 : 0 : -1) - 2(1 : 0 : 1) = (1 : 0 : -1) - (1 : 0 : 1).$$

and so $g \in H^0(X, \mathcal{O}_X(P))$. As $P$ has degree 1, we see that $H^0(X, \mathcal{O}_X(P)) = k \oplus k\, g$.

The corresponding morphism

$$\begin{aligned} X &\longrightarrow \mathbb{P}^1_k \\ (x_0 : x_1 : x_2) &\mapsto (x_1 : x_0 - x_2) \end{aligned}$$

can be viewed as the projection from $P$. △

*Remark* 20.12 (Curves of genus 0 over general fields). For non-algebraically closed fields $k$, there exist curves of genus 0 which are not isomorphic to $\mathbb{P}^1_k$. The isomorphism to $\mathbb{P}^1_k$ is closely linked to the existence of a $k$-point on $X$.

For instance, the conic curve $X = V_+(x_0^2 + x_1^2 + x_2^2) \subset \mathbb{P}^2_{\mathbb{R}}$ has genus 0, but it has no $\mathbb{R}$-points at all. Similarly, the conic $X = V_+(x_0^2 + x_1^2 - 3x_2^2)$ has infinitely many $\mathbb{R}$-points, but no $\mathbb{Q}$-points. In these examples, $X$ cannot be isomorphic to $\mathbb{P}^1_k$, but the base change to the algebraic closure $X_{\bar{k}} = X \times_k \overline{k}$ becomes isomorphic to $\mathbb{P}^1_{\overline{k}}$.

That being said, curves of genus 0 over arbitrary feidls are well understood. Here is a brief explaination. If we assume that $H^0(X, \mathcal{O}_X) = k$ and $H^1(X, \mathcal{O}_X) = 0$ and $X(k) \neq \varnothing$, then $X \simeq \mathbb{P}^1_{\mathbb{Q}}$. Indeed, if $X$ admits a $k$-point $P$, then one shows that the invertible sheaf $\mathcal{O}_X(P)$ is very ample by pulling back to $X_{\bar{k}}$, where one can apply Theorem 20.10. Therefore, we again find that $X \simeq \mathbb{P}^1_k$.

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A nice and surprising fact, which holds for any field $k$, is that any nonsingular projective curve of genus 0 is isomorphic to plane curve of degree 2, $X = V_+(Q) \subset \mathbb{P}^2_k$. To prove this, one considers the anticanonical divisor $D = -K_X$ (which is always defined over $k$, regardless of whether $X$ has a $k$-point or not). By working on $X_{\bar{k}}$, one shows that this is very ample, and embeds $X$ as a degree 2 curve in $\mathbb{P}^2_k$.

We conclude by another characterisation of $\mathbb{P}^1$, which will be important for studying curves of higher genus as well.

**Proposition 20.13.** Let $X$ be a nonsingular projective curve over an algebraically closed field $k$ and $D$ any divisor on $X$. Then

$$\dim_k H^0(X, \mathcal{O}_X(D)) \leqslant \deg D + 1$$

with equality if and only if $X \simeq \mathbb{P}^1_k$.

*Proof* We may assume that $D$ is effective, i.e., $D = P_1 + \cdots + P_d$ for some $k$-points $P_1, \ldots, P_k \in X$ (possibly equal) (otherwise replace $D$ by some other effective divisor $D' \sim D$). The proof is by induction on $\deg D = d$.

First suppose $d = 1$. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(P) \longrightarrow \kappa(P) \longrightarrow 0.$$

Now $h^0(X, \mathcal{O}_X) = 1$ and $h^0(\kappa(P)) = 1$ therefore $h^0(\mathcal{O}_X(P)) \leqslant 2$. If $h^0(\mathcal{O}_X(P)) = 2$, then $|P|$ cannot have any base points, so we obtain a finite morphism $X \to \mathbb{P}^1_k$. As $X$ is nonsingular and $\deg P = 1$, this must be an isomorphism, and so $X \simeq \mathbb{P}^1_k$.

Next suppose $D = P_1 + \cdots + P_d$. Let $D' = P_1 + \cdots + P_{d-1}$. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(D') \longrightarrow \mathcal{O}_X(D) \longrightarrow \kappa(P_d) \longrightarrow 0.$$

Now $h^0(\mathcal{O}_X(D')) \leqslant d$ by induction and $h^0(\kappa(P_d)) = 1$ so $h^0(\mathcal{O}_X(D)) \leqslant d+1$. Moreover, $h^0(\mathcal{O}_X(D)) \leqslant d + 1$ with equality if and only if $h^0(\mathcal{O}_X(D')) = d$ which by induction happens if and only if $X \simeq \mathbb{P}^1_k$. □

**Corollary 20.14.** A nonsingular curve $X$ over an algebraically closed field is isomorphic to $\mathbb{P}^1_k$ if and only if $\mathrm{Cl}(X) \simeq \mathbb{Z}$.

*Proof* By Proposition **??**, the Picard group of any projective space $\mathbb{P}^n_k$ is isomorphic to $\mathbb{Z}$. Conversely, suppose $X$ is a curve with $\mathrm{Cl}(X) \simeq \mathbb{Z}$. Let $p, q$ be two distinct points on $X$. By assumption, the two divisors $p$ and $q$ are linearly equivalent, so the invertible sheaf $L = \mathcal{O}_X(p)$ has at least 2 linearly independent global sections (one vanishing at $p$ and one vanishing at $q$). Therefore, $X \simeq \mathbb{P}^1_k$ by Proposition 20.13. □

## 20.5 Curves of genus 1

A plane curve $X \subset \mathbb{P}^2_k$ of degree 3 has genus 1. This follows by taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2_k}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2_k} \longrightarrow \iota_* \mathcal{O}_X \longrightarrow 0.$$

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In this section, we show that in fact *every* curve of genus 1 arises this way:

**Theorem 20.15.** Any projective curve $X$ of genus 1 over an algebraically closed field can be embedded as a plane cubic curve in $\mathbb{P}^2_k$.

*Proof* Pick a point $P \in X$ and consider the divisor $D = 3P$. Then $D$ has degree $3 \geqslant 2g + 1$, so it is very ample. Furthermore, by Riemann–Roch, $h^0(3P) = 3$, so there is a projective embedding $\phi : X \to \mathbb{P}^2_k$. The image $\phi(X)$ is a smooth curve of degree equal to $\deg \phi^* \mathcal{O}_{\mathbb{P}^2_k}(1) = \deg D = 3$. □

We can make this a little bit more explicit, as follows. For a point $P \in X$, consider the multiples $D = mP$ where $m = 1, \ldots, 6$. By the Riemann–Roch formula, we can compute the spaces of global sections as follows

| | $h^0$ | basis |
|---|---|---|
| $\mathcal{O}_X(P)$ | 1 | $1$ |
| $\mathcal{O}_X(2P)$ | 2 | $1, x$ |
| $\mathcal{O}_X(3P)$ | 3 | $1, x, y$ |
| $\mathcal{O}_X(4P)$ | 4 | $1, x, y, x^2$ |
| $\mathcal{O}_X(5P)$ | 5 | $1, x, y, x^2, xy$ |
| $\mathcal{O}_X(6P)$ | 6 | $1, x, y, x^2, xy, x^3, y^2$ |

As $H^0(X, \mathcal{O}_X(6P))$ is 6-dimensional, there must be a linear relation between the sections $1, x, y, x^2, xy, x^3, y^2$. It is not hard to see that, after a coordinate change, we can take it to be the following cubic

$$y^2 + a_1xy + a_2y = x^3 + a_3x^2 + a_4x + a_5$$

where $a_1, a_2, a_3, a_4, a_5 \in k$. When the characteristic of $k$ is not 2 or 3, the equation can be further simplified, to

$$y^2 = x^3 + ax + b.$$

This is known as the *Weierstrass form* of the curve $X$. Any such equation defines a nonsingular genus 1 curve, provided that $4a^3 + 27b^2 \neq 0$ (Exercise 10.6.4).

In contrast to the genus 0 case, there are many non-isomorphic genus 1 curves. For instance, among the curves $X_t = V(x_1^2x_2 + x_0^3 + x_0x_2^2 + tx_2^3) \subset \mathbb{P}^2_k$, where $t \in k$, a given curve $X_a$ is isomorphic to at most a finite number of other $X_t$'s.

### *The group law on $X$*

One of the most important facts about genus 1 curves is that the set of $k$-points $X(k)$ form an abelian group. That is, there is a natural way of 'adding' $k$-points on $X$. This only happens for curves of genus 1.

To make the discussion a bit more concrete, we consider the curve $X \subset \mathbb{P}^2_k$ defined by the equation

$$y^2z = x^3 - xz^2$$

The key point in understanding the group structure is to consider divisors on $X$.

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If $L \subset \mathbb{P}^2_k$ is a line, we get a divisor $L|_X$ on $X$, supported on the finite set $L \cap X$. More formally, we take a section $s \in \mathcal{O}_{\mathbb{P}^2_k}(1)$ defining $L$ and restrict it to $X$. The restricted divisor on $X$ consists of three points $P, Q, R$ (counted with multiplicity). In particular, since any two lines are linearly equivalent on $\mathbb{P}^2_k$, we get for every pair of lines $L, L'$ and corresponding triples $P, Q, R$, a relation

$$P + Q + R \sim P' + Q' + R'$$

where $\sim$ denotes linear equivalence.

Let us consider the point $O = (0 : 1 : 0)$ on $X$. This is a special point on $X$: it is an *inflection point*, in the sense that there is a line $L = V(z) \subset \mathbb{P}^2_k$ which intersects $X$ with multiplicity three at $O$. That means that $L$ restricts to $3O$ on $X$. The point $O$ will serve as the identity in the group $X(k)$. It has the property that any three collinear points $P, Q, R$ in $X$ satisfy

$$P + Q + R \sim 3O$$

Next, consider the subgroup $\mathrm{Cl}^0(X) \subset \mathrm{Cl}(X)$ consisting of degree 0. This fits into the exact sequence

$$0 \longrightarrow \mathrm{Cl}^0(X) \longrightarrow \mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

Let us define the following map:

$$\begin{aligned} \xi \colon X(k) &\longrightarrow \mathrm{Cl}^0(X) \\ P &\mapsto [P - O] \end{aligned}$$

**Lemma 20.16.** $\xi$ is a bijection.

*Proof* $\xi$ is injective: $\xi(P) = \xi(Q)$ implies that $P \sim Q$. Then $P = Q$ (otherwise $X$ would be rational by Proposition 20.14). (Alternatively, it follows because $h^0(X, \mathcal{O}_X(P)) = 1$ by Riemann–Roch).

$\xi$ is surjective: Take a divisor $D = \sum n_i P_i \in \mathrm{Div}(X)$ of degree 0. Then $D' = D + O$ has degree 1, so by Riemann–Roch, $H^0(X, \mathcal{O}_C(D'))$ is 1-dimensional. Hence there exists an effective divisor of degree 1 in $|D'|$, which must then be of the form $D' = Q$ for some $k$-point $Q$. But this means that $D + O \sim Q$, or in other words, $D \sim Q - O = \xi(Q)$, as we wanted to show. □

Using this bijection, we can put a group structure on the set $X(k)$ by transporting the group structure of $\mathrm{Cl}^0(X)$. We have shown:

**Theorem 20.17.** The set of $k$-points $X(k)$ on a genus 1 form an abelian group.

The group law has the following famous geometric interpretation. Given two points $P, Q \in X$, we draw the line $L$ they span (see Figure 20.1). This intersects $X$ in one more point, say $R$. In the group $\mathrm{Cl}^0(X)$ we have

$$P + Q + R = 3O$$

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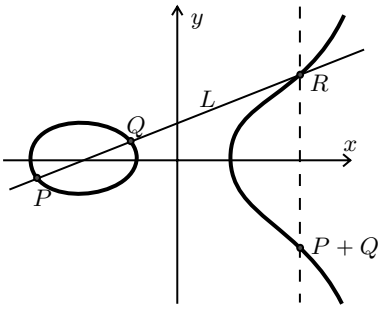


**Figure 20.1** The group law on $X$

To define the 'sum' $P + Q$ (which should be a new $k$-point of $X$), we then look for a point $S$ so that

$$S - O = (P - O) + (Q - O)$$

or in other words, $S + O = P + Q$. In light of the above, this becomes $S + O = 3O - R$ or, $R + S + O = 3O$. This tells us that we should define $S$ as follows: We draw the line $L'$ from $O$ to $R$ (shown as the dotted line in the figure), and define $S$ to be the third intersection point of $L'$ with $X$. By construction, we get $(P - O) + (Q - O) = (S - O)$ in $\mathrm{Cl}^0(X)$. We then define $P + Q$ to be the $k$-point $S$.

Given the equation of $X$ in $\mathbb{P}^2_k$, and coordinates for the points $P$ and $Q$, we can of course write down explicit formulas for the coordinates of $S$, and they are rational functions in the coordinates of $P$ and $Q$. This construction has many applications in cryptography.

## 20.6 Hyperelliptic curves

A nonsingular curve $X$ of genus $g \geqslant 2$ is said to be *hyperelliptic* if there is a finite degree 2 map $X \to \mathbb{P}^1_k$. There was already a notion of 'hyperelliptic curves' in Section **??**, so we should start by showing that the two definitions are equivalent.

Let $X$ be curve admitting a degree 2 morphism $f\colon X \to \mathbb{P}^1_k$. Then the function field $K(X)$ is a degree 2 extension of $K(\mathbb{P}^1_k) = k(x)$. Let $1, y$ be a basis for $K(X)$ as a $k(x)$-vector space. Then the elements $1, y, y^2$ are $k(x)$-linearly dependent, so there is a relation of the form $Ay^2 + By + C = 0$ for some elements $A, B, C \in k(x)$. Multiplying this relation by a suitable polynomial, we may assume that $A, B, C \in k[x]$. Furthermore, multiplying the relation by $A$, and replacing $y$ by $Ay$, we see that we may take the relation to be of the form $y^2 + By + C$ where $B, C \in k[x]$. If the characteristic of $k$ is not equal to 2, we may furthermore complete the square to reduce to $y^2 = p(x)$ for some $p(x) \in k[x]$. Finally, by performing variable changes of the form $y = (x - a) \cdot v$, and we may assume that $p(x)$ has no repeated roots.

This shows that $X$ has the same function field as the nonsingular curve defined by $y^2 = p(x)$ in $\mathbb{A}^2_k$. Hence $X$ is birational, hence isomorphic, to one of the hyperelliptic curves of Section **??**.

If the field $k$ has characteristic 2, most of the steps in the above parapgraph remain valid,

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except that the defining equation has the form $y^2 + q(x)y = p(x)$ for some $q(x) \in k[x]$ of degree $\leqslant g$.

The construction in Section **??** shows that there are hyperelliptic curves of any genus $g$. Here is another construction:

**Example 20.18.** Let $X \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ be a smooth divisor of bidegree $(2, g+1)$. Then by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^1_k}(-2, -g-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1_k \times \mathbb{P}^1_k} \longrightarrow \iota_* \mathcal{O}_X \longrightarrow 0$$

and by Exercise 15.18.5,

$$H^2(\mathbb{P}^1_k \times \mathbb{P}^1_k, \mathcal{O}(-2, -g-1)) = H^1(\mathbb{P}^1_k, \mathcal{O}(-2)) \otimes H^1(\mathbb{P}^1_k, \mathcal{O}(-g-1)) \simeq k^g,$$

so $X$ has genus $g$.

Explicitly, $X$ is defined by a bihomogeneous equation of the form

$$f(x_0, x_1; y_0, y_1) = A(y_0, y_1)x_0^2 + B(y_0, y_1)x_0x_1 + C(y_0, y_1)x_1^2$$

where $A, B, C \in k[y_0, y_1]$ are homogeneous of degree $g+1$. Therefore, the second projection $p_2 \colon X \to \mathbb{P}^1_k$ is finite of degree 2, so $X$ is hyperelliptic of genus $g$.

In fact, any hyperelliptic curve can be embedded in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ as a divisor of bidegree $(2, g+1)$, at least when char $k \neq 2$. To see this, note that we may write $p(x) = q(x)^2 + r(x)$ where $q$ and $r$ are polynomials of degree $\leqslant g+1$. Then, if we make the substitution $u = x$ and $v = y + q(x)$, we see that the curve is birational to the plane curve with affine equation $v^2 - 2vq(u) - r(u) = 0$ in $\operatorname{Spec} k[u, v]$. A projective model in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ is given by the bihomogeneous equation

$$x_0^2 y_0^{g+1} - 2x_0x_1 q(y_1/y_0) y_0^{g+1} - r(y_1/y_0) y_0^{g+1} x_1^2 = 0.$$

△

The following theorem summarizes the various ways to define a hyperelliptic curve.

**Theorem 20.19.** Let $X$ be nonsingular curve of genus $g \geqslant 2$ over an algebraically closed field with char $k \neq 2$. Then the following are equivalent:

(i) $X$ admits a finite degree 2 map $f \colon X \to \mathbb{P}^1_k$.
(ii) $X$ is birational to a plane curve of the form $y^2 = p(x)$, where $p$ is a polynomial with distinct roots.
(iii) $X$ admits a divisor $D$ of degree 2 and $h^0(D) = 2$.
(iv) $X$ is isomorphic to a divisor in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ of bidegree $(2, g+1)$.

*Proof* The discussion in the beginning of this section showed that (i)⇔ (ii). Example 20.18 showed that (iv)⇔(i). Hence we need only address the item (iii).

(i)⇒(iii): Take $D = f^*Q$ for a point $Q \in \mathbb{P}^1(k)$. Then $D$ has degree 2, and two sections, because the morphism is non-constant.

(iii)⇒(i): Given such a divisor $D$, let us show that $D$ is base-point free. By Theorem 20.7, it suffices to show that $h^0(D - P) = 1$ for every point $P \in X$. Since $h^0(D) = 2$, $h^0(D - P)$ can drop by at most 1 by the exact sequence (20.3), so $h^0(D) \leqslant 1$. On the other hand, $D - P$ is a degree 1 divisor, hence $h^0(D - P) \leqslant 1$ by Proposition 20.13. □

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## 20.7 Curves of genus 2

Let $X$ be a nonsingular projective curve of genus 2. We have seen examples of such curves already, namely the hyperelliptic curve obtained by gluing two copies of the affine curve $y^2 = p(x)$ where $p(x)$ is a polynomial of degree 5 or 6. However, we show the following, which tells us that any genus 2 curve arisises from the construction in Section B.4.

In fact, if $X$ is a genus 2 curve, the canonical divisor $K_X$ is base-point free, and gives a morphism

$$f\colon X \longrightarrow \mathbb{P}^1_k$$

As $K_X$ has degree $2g - 2 = 2$, this proves the following:

**Proposition 20.20.** Any curve of genus 2 is isomorphic to a hyperelliptic curve.

We saw in Proposition 15.36 that a genus 2 curve cannot be embedded in the projective plane $\mathbb{P}^2_k$. However, in light of Theorem 20.19, it can be embedded as a $(2,3)$-divisor in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

## 20.8 Curves of genus 3

We next classify curves of genus 3. We have seen a few examples of such curves already:

**Example 20.21.** A plane curve $X \subset \mathbb{P}^2$ of degree $d = 4$ has genus $\frac{1}{2}(d-1)(d-2) = 3$. Note that

$$\Omega_X = \mathcal{O}_{\mathbb{P}^2}(d-3)|_X = \mathcal{O}_X(1)$$

so $K_X$ is very ample, since it is the restriction of the very ample invertible sheaf $\mathcal{O}_{\mathbb{P}^2}(1)$. △

**Example 20.22.** The $X$ on the quadric surface $Q \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$ in $\mathbb{P}^3_k$ of type (2,4) is hyperelliptic. It is a curve of degree 6 and genus 3 in $\mathbb{P}^3_k$. As we showed in Section 20.7, the canonical divisor $K_X$ is not very ample (in this case, it defines a degree 2 morphism $X \to \mathbb{P}^2_k$ onto a nonsingular conic). △

These two examples are a bit different. The curves in the first example have very ample canonical divisor $K_X$, whereas in the second case $K_X$ defines the degree 2 map $X \to \mathbb{P}^1_k$. We show that this distinction is a general phenomenon for curves of genus 3:

**Proposition 20.23.** Let $X$ be a curve of genus 3. Then there are two possibilities:
(i) $K_X$ is very ample. Then $X$ embeds as a plane curve of degree 4.
(ii) $K_X$ is not very ample. Then $X$ is a hyperelliptic curve, and it embeds as a $(2,4)$ divisor in $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

*Proof of Proposition 20.23* We still need to check the last part of the above theorem, namely that every hyperelliptic curve arises as a curve of type $(2,4)$ on the quadric surface $\mathbb{P}^1_k \times \mathbb{P}^1_k \subset \mathbb{P}^3_k$.

Write $F$ for the divisor inducing the degree 2 morphism $X \to \mathbb{P}^1_k$. We first claim that $K_X \sim 2F$. Since both divisors have degree 4, it suffices to show that $K_X - 2F$ is effective. Note that in any case $h^0(X, 2F) \geqslant 3$, as the sections $x_0^2, x_0x_1, x_1^2$ are linearly independent

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(they are pullbacks of independent sections from $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(2))$ and the pullback map is injective by Exercise **??**). Then applying Riemann–Roch to the divisor $D = 2F$, we get

$$h^0(2F) - h^0(K_X - 2F) = 4 + 1 - 3 = 2,$$

from which we find that $h^0(K_X - 2F) \neq 0$, as desired.

We next look for a base-point divisor $D$ of degree 4. This is the divisor which will induce the 'first projection' $X \to \mathbb{P}^1_k$, once we have established that $X$ is a $(2,4)$-divisor. We will find points $P_1, \dots, P_4$ so that the divisor $D = P_1 + \cdots + P_4$ satisfies

(i) $h^0(D) = 2$
(ii) $D$ is base-point free.
(iii) $D - F$ is not effective.

In fact, there is a Zariski open set $U \subset X(k) \times X(k) \times X(k) \times X(k)$ so that these conditions are satisfied for every $(P_1, P_2, P_3, P_4) \in U$.

Let us start with the condition (i). Note that by Riemann–Roch, we have

$$h^0(D) - h^0(K_X - D) = 4 + 1 - 3 = 2$$

Note that $K_X - D$ has degree $2g - 2 - 4 = 0$, so $K_X - D$ is a divisor of degree 0. This is effective if and only if $K_X \sim D$. Note that in selecting the points $P_1, P_2, P_3, P_4$, we have 4 dimensions of freedom, whereas there is only a 2-dimensional space of canonical divisors $|K_X|$, therefore most divisors $P_1 + P_2 + P_3 + P_4$ will not be a canonical divisor.

To argue a little bit more rigorously, note that $K_X$ is base-point free, so $h^0(K_X - P_1)$ will be be equal to $3 - 1 = 2$ for all points $P_1$. The divisor $L_X - P_1$ has at most finitely many base points, so $h^0(K_X - P_1 - P_2) = 1$ for most points $P_2$. Continuing subtracting points like this, we see that $K_X - P_1 - P_2 - P_3 - P_4$ is not effective, and hence the claim (i) holds.

We can use a similar argument to show (ii). We need to show that

$$h^0(D - P) = h^0(D) - 1 = \deg D + 1 - 3 - 1 = 1$$

for every point $P$. Suppose this is not the case, and let $P$ be a base point of $D$. Since $D = P_1 + P_2 + P_3 + P_4$, we may suppose that $P = P_4$. By Riemann–Roch, we are done if we can show that $h^0(K_X - D + P) = 0$. Note that $K_X - D + P = K_X - P_1 - P_2 - P_3$. There is a 3-dimensional space of effective divisors of the form $P_1 + P_2 + P_3$ for points $P_i \in X$, but only a 2-dimensional linear system of effective canonical divisors $|K_X|$. Hence $K_X - D + P$ is not effective.

Finally, to see (iii), write $F = Q_1 + Q_2$ for the degree 2 divisor on $X$. When defining the divisor $D$, we may assume that the points $P_i$ are choosen so that $Q_1$ will not be a base-point of $D$ and $Q_2$ will not be a base-point of $D - Q_1$. Therefore $h^0(D - Q_1) = 1$ and $h^0(D - Q_1 - Q_2) = 0$, and so $D - F$ is not effective.

We therefore have two morphisms: $f\colon X \to \mathbb{P}^1_k$ and $g\colon X \to \mathbb{P}^1_k$ (induced by $D$). By the universal property of the fiber product, this gives a morphism

$$\phi = (f \times g)\colon X \longrightarrow \mathbb{P}^1_k \times_k \mathbb{P}^1_k$$

We claim that this is a closed embedding.

Let $F = P + Q$ be the divisor which induces $f$.

Next, we claim that $D + F$ is very ample. This divisor is base-point free of degree $2+4 = 6$ and $h^0(D + F) = 6 + 1 - 3 = 4$ (there is no $h^1$ because $6 > 2g - 2$).



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We can see the sections of $H^0(X, \mathcal{O}_X(D+F))$ as follows. Write $H^0(F) = \langle x_0, x_1 \rangle$ and $H^0(D) = \langle y_0, y_1 \rangle$. As $F$ is base-point free, the Koszul complex on $x_0, x_1$ gives the following exact sequence:

$$0 \to \mathcal{O}_X(D-F) \to \mathcal{O}_X(D)^2 \to \mathcal{O}_X(D+F) \to 0 \tag{20.6}$$

Note that $H^0(D-F) = 0$ by item (iii). Then $H^1(D-F) = 0$ as well, by Riemann–Roch. This implies that the multiplication map $H^0(\mathcal{O}_X(D))^2 \to H^0(\mathcal{O}_X(D+F))$ which sends $(s,t)$ to $x_0 s + x_1 t$, is an isomorphism. Hence

$$H^0(D+F) = \langle x_0y_0, x_0y_1, x_1y_0, x_1y_1 \rangle. \tag{20.7}$$

Now, to show that $D+F$ is very ample, we need to show that

$$h^0(X, D+F-p-q) = h^0(D+F) - 2 = 2$$

for any pair of points $p, q \in X$. By Riemann–Roch again, we can conclude if we know that $h^0(K_X - D - F + p + q) = 0$. But since $K_X \sim 2F$, we have

$$K_X - D - F + p + q \sim F - D + p + q$$

These are divisors of degree 0, so if this is effective, we must have $D - F \sim p + q$. But this contradicts item (iii) above.

Therefore, $D+F$ is very ample, and embeds $X$ into $\mathbb{P}^3_k$. By (20.7), the image of $X$ lies on the quadric surface $u_0u_3 - u_1u_2 = 0$. Hence $X \to \mathbb{P}^3_k$ factors via the Segre embedding $X \to \mathbb{P}^1_k \times_k \mathbb{P}^1_k \to \mathbb{P}^3_k$. Hence we may view $X$ as a divisor $X \subset \mathbb{P}^1_k \times_k \mathbb{P}^1_k$. By construction, the two projections to $\mathbb{P}^1_k$ are given by the sections of $D$ and $F$. This means that $X$ has bidegree $(2,4)$ as a divisor on $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

□

## 20.9 Curves of genus 4

Recall that curves of genus $g \geqslant 2$ split up into two disjoint classes, the hyperelliptic curves, which admit a degree 2 morphism to $\mathbb{P}^1_k$; and the canonical curves, where $K_X$ is very ample.

Here are two examples of genus 4 curves in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

**Example 20.24.** Consider a type $(2,5)$ curve $C$ on $Q \subset \mathbb{P}^3_k$. Then $C$ has degree $7 = 2+5$ and $C$ is hyperelliptic (because of the degree 2 map coming from projection onto the first fact $p_1 : Q \to \mathbb{P}^1$). A type $(3,3)$ curve on $Q$ is also of genus 4. It is a degree 6 complete intersection of $Q$ and a cubic surface. △

Let us try to classify curves $X$ of genus 4. If $X$ is hyperelliptic, then a very similar argument to the one in the previous section shows that $X$ embeds as a $(2,5)$-divisor in $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

So we may assume that $K_X$ is very ample. This means that $K_X$ defines an embedding $\phi\colon X \to \mathbb{P}^{g-1}_k = \mathbb{P}^3_k$ and the degree of the embedded curve is $\deg K_X = 2g-2 = 6$.

What are the equations of $X$ in $\mathbb{P}^3_k$? To answer this, we use a very powerful technique in curve theory, namely combining Riemann–Roch with the sheaf cohomology of projective space.

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Twisting the ideal sheaf sequence of $X$ by $\mathcal{O}_{\mathbb{P}^3_k}(2)$ and taking cohomology gives the exact sequence

$$0 \to H^0(\mathbb{P}^3_k, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(2)) \to H^0(X, \mathcal{O}_X(2)) \to \cdots$$

Note that $\mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}_X(K_X)$. Applying Riemann-Roch to the divisor $2K_X$, we get

$$h^0(\mathcal{O}_{\mathbb{P}^3}(2)|_X) = \deg 2K_X + 1 - g + h^1(\mathcal{O}_X(2K_X)) = 12 + 1 - 4 = 9.$$

(Note that $h^1(\mathcal{O}_{\mathbb{P}^3}(2)|_X) = h^1(2K_X) = h^0(-K_X) = 0$, as $-K_X$ has negative degree). Since $h^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(2)) = 10$, it follows that the map $H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_X(2))$ must have a nontrivial kernel, and so $h^0(\mathbb{P}^3_k, \mathcal{I}_X(2)) > 0$.

The conclusion is that $X$ lies in some surface of degree 2, say $Q \subset \mathbb{P}^3_k$ is defined by $q \in k[x_0, x_1, x_2, x_3]$. Note that $X$ does not lie in a plane (then $X$ could not have genus 4). This implies that $q$ must be irreducible. So $X$ lies on either a singular quadric cone $V(x_0x_2 - x_1^2)$ or the nonsingular quadric surface $V(x_0x_3 - x_1x_2)$. The quadric surface $Q$ is moreover unique, as if $X \subset Q \cap Q'$, and so $X$ would have degree $\leqslant 4$.

What about cubics in $I_X$? Of course $x_0q, \ldots, x_3q$ give a 4-dimensional space in $I_X = H^0(\mathbb{P}^3_k, \mathcal{I}_X(3))$. We next consider the sequence

$$0 \to H^0(\mathcal{O}_X(3)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathcal{O}_X(3)) \to \cdots$$

We have $h^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3}(3)) = 20$, and $h^0(X, \mathcal{O}_X(3)) = 15$ by Riemann–Roch. Hence $h^0(\mathcal{O}_X(3)) \geqslant 5$, and we find that there is a cubic $c \in k[x_0, x_1, x_2, x_3]$ which is not in the span of $x_0q, \ldots, x_3q$, hence $c \notin (q)$. Note that $V(q, c)$ is a 1-dimensional subscheme of degree 6 containing $X$ as a component. As $X$ is also of degree 6, we must have $X = V(q, c)$.

This proves the following theorem:

**Theorem 20.25.** Let $X$ be a nonsingular curve of genus 4. Then either

(i) $X$ is hyperelliptic (in which case $X$ embeds as a $(2, 5)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^1$); or

(ii) $X = V(Q) \cap V(C)$ is the complete intersection of a quadric surface and a cubic surface in $\mathbb{P}^3_k$.

There are further classical results about the canonical embedding for genus $\geqslant 3$.

**Theorem 20.26 (Enriques–Petri).** Let $X$ be a nonsingular non-hyperelliptic curve of genus $\geqslant 3$. Then the image of the canonical emedding $X \to \mathbb{P}^{g-1}$ is defined by quadratic equations unless:

(i) $X$ is trigonal, i.e., birational to an affine curve of the form $y^3 = p(x)$.

(ii) $X$ is isomorphic to a curve of degree 5 in $\mathbb{P}^2_k$.

## 20.10 Exercises

**Exercise 20.10.1.** Let $X$ be a nonsingular quasi-projective curve over an algebraically closed field $k$. Show that if $\Gamma(X, \mathcal{O}_X) \neq k$, then $X$ is affine. HINT: Embed $X$ in a projective nonsingular curve $\overline{X}$ and consider divisors supported on the closed subset $\overline{X} - X$.

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**Exercise 20.10.2.** Consider the relation $D \leqslant D'$ on divisors such that $D - D'$ is effective. Let $D$ be a divisor of maximal degree such that $H^1(X, \mathcal{O}_X(D)) \neq 0$. Show that $D$ is a canonical divisor, i.e., $D \sim K_X$.

**Exercise 20.10.3.** Let $f\colon X \to Y$ be a non-constant morphism of curves. If $D$ is a divisor on $Y$, show that $h^0(X, f^*\mathcal{O}_Y(D)) \geqslant h^0(Y, \mathcal{O}_Y(D))$.

**Exercise 20.10.4** (Riemann–Roch for higher rank)**.** Let $X$ be a nonsingular projective curve and let $\mathcal{E}$ be a locally free sheaf of rank $r$. Show that

$$\chi(X, \mathcal{E}) = \deg(\det \mathcal{E}) + r\chi(\mathcal{O}_X) \tag{20.8}$$

**Exercise 20.10.5** (Lüroth's theorem)**.** Let $X$ be a nonsingular proper curve over an algebraically closed field $k$ and let $f\colon \mathbb{P}^1_k \to X$ be a non-constant morphism. Show that $X$ is isomorphic to $\mathbb{P}^1_k$.

**Exercise 20.10.6.** Find an example of an $\mathcal{O}_X$-module $\mathcal{F}$ of finite type so that the equality in (19.27) is not true if we replace $H^1$ by $H^0$.

**Exercise 20.10.7.** Let $X \subset \mathbb{P}^2_k$ be a plane curve of degree $\leqslant 3$. Show that the automorphism group of $X$ is infinite.

**Exercise 20.10.8.** Let $X \subset \mathbb{P}^2_k$ be a plane curve of degree 4. Show that any automorphism $\phi : X \to X$ is induced by a linear transformation $\mathbb{P}^2_k \to \mathbb{P}^2_k$. Deduce that the general plane quartic has trivial automorphism group.

**Exercise 20.10.9** (Curves of genus 5)**.**

a) Let $X$ be a complete intersection of 3 quadric hypersurfaces in $\mathbb{P}^4_k$. Show that $X$ is non-hyperelliptic of genus 5.

b) Conversely, show that any non-hyperelliptic curve of genus 5 is contained in the intersection of 3 quadric hypersurfaces $Q_0 \cap Q_1 \cap Q_2$ in $\mathbb{P}^4_k$.

c) * Show that there exist non-hyperelliptic genus 5 curves which are not complete intersections of 3 quadrics (it can happen that $Q_0 \cap Q_1 \cap Q_2$ has dimension $\geqslant 2$).

**Exercise 20.10.10** (Curves of genus 6)**.** Show that the following varieties define genus 6 curves:

a) A degree 5 curve in $\mathbb{P}^2_k$

b) A $(2, 7)$-divisor in $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

c) A $(3, 4)$-divisor in $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

d) * Show that the classes of curves constructed in a), b), c) are disjoint.

**Exercise 20.10.11.** Let $X$ be a smooth hyperelliptic curve of genus $g$. Show that if $f\colon X \to \mathbb{P}^1_k$ is a finite morphism of degree at most $g$, then $f$ factors via the degree 2 covering $\sigma\colon X \to \mathbb{P}^1_k$. HINT: Consider a map $X \to \mathbb{P}^1_k \times_k \mathbb{P}^1_k$.

**Exercise 20.10.12** (Curves of degree 3)**.** Let $X \subset \mathbb{P}^n_k$ be a nonsingular curve of degree 3. Show that $X$ has genus $\leqslant 1$ and is either a twisted cubic or a plane cubic.

**Exercise 20.10.13** (Curves of degree 4)**.** Let $X \subset \mathbb{P}^n_k$ be a nonsingular curve of degree 4. Show that $X$ has genus $0, 1$ or 3. Find a projective models in each case.

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**Exercise 20.10.14** (Hyperelliptic curves)**.** Let $X$ be a hyperelliptic curve of genus $g$, that is, admitting a base-point free divisor $D$ of degree 2.

a) Show that if $g \geqslant 2$, then $D$ is unique up to linear equivalence, and that $K_X \sim (g-1)D$. POSSIBLE HINT: If $f, g\colon X \to \mathbb{P}^1_k$ are two degree 2 morphisms, consider a point $P \in X$ such that $f^{-1}(f(P)) = P + Q$ and $g^{-1}(g(P)) = P + Q$ for $P, Q, R \in X$ distinct. Then consider the divisor $P + Q + R$.

b) Show that if $g = 1$, then $X$ contains infinitely many such divisors $D$ which are non-linearly equivalent.

**Exercise 20.10.15.** Let $X$ be a curve over a field $k$ and let $\mathcal{F}$ be a sheaf on $X$. Show that $H^i(X_{\overline{k}}, q^*\mathcal{F}) \simeq H^i(X, \mathcal{F}) \otimes_k \overline{k}$ for all $i$, where $q\colon X_{\overline{k}} \to X$ is the base change morphism.

**Exercise 20.10.16** (Trace map)**.** Let $f\colon X \to Y$ be a finite morphism of nonsingular projective curves. Show that there is a natural map $\mathrm{Tr}\colon f_*\mathcal{O}_X \to \mathcal{O}_Y$ which splits the usual map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$. Deduce that

$$f_*\mathcal{O}_X \simeq \mathcal{O}_Y \oplus \mathcal{E}$$

for some locally free sheaf $\mathcal{E}$.

**Exercise 20.10.17.**

a) Let $f\colon X \to Y$ be a domninant morphism of nonsingular projective curves. Show that $g(X) \geqslant g(Y)$ HINT: Apply Exercise 20.10.16.

b) (Lüroth's theorem): Let $f\colon \mathbb{P}^1_k \to X$ be a dominant morphism of nonsingular projective curves. Show that $X \simeq \mathbb{P}^1_k$.

**Exercise 20.10.18** (Fermat's Last Theorem for polynomials)**.** As an application of Exercise 20.10.17, show that if $n \geqslant 3$, there are no non-constant polynomials $P, Q, R \in k[t]$ such that

$$P^n + Q^n = R^n.$$

What happens when $n = 1$ or 2?

HINT: Construct a morphism to a Fermat curve in $\mathbb{P}^2_k$.

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# Appendix A

## Some results from Commutative Algebra



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## A.1 Operations on ideals

If $A$ is a ring and $S \subset A$ is subset, the ideal generated by $S$ is defined as the ideal

$$\{a_1 s_1 + \cdots + a_r s_r \mid r \in \mathbb{N}, a_i \in A, s_i \in S\}$$

For a collection of ideals $\{\mathfrak{a}_i\}_{i\in I}$, the sum $\sum_{i\in I} \mathfrak{a}_i$ is defined as the ideal generated by finite sums $a_{i_1} + \cdots + a_{i_r}$ where $a_{i_j} \in \mathfrak{a}_{i_j}$. This is the smallest ideal containing all the $\mathfrak{a}_i$. In particular, for two ideals $\mathfrak{a}$ and $\mathfrak{b}$, the sum $\mathfrak{a} + \mathfrak{b}$ consists of sums $a + b$ where $a \in \mathfrak{a}$, $b \in \mathfrak{b}$.

If $\mathfrak{a}_1, \ldots, \mathfrak{a}_r$ are ideals, the product ideal $\mathfrak{a}_1 \cdots \mathfrak{a}_r$ is the ideal generated by products $a_1 \cdots a_r$, where $a_i \in \mathfrak{a}_i$. There is always an inclusion

$$\mathfrak{a}_1 \cdots \mathfrak{a}_r \subset \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r.$$

By the next result, this is an equality if the ideals are pairwise coprime, that is, $\mathfrak{a}_i + \mathfrak{a}_j = (1)$ for each $i \neq j$.

**Theorem A.1 (Chinese remainder theorem).** Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subset A$ be a collection of ideals which are pairwise coprime. Then the homomorphism $\varphi\colon A \to \prod_{i=1}^n A/\mathfrak{a}_i$ is surjective, and

$$\operatorname{Ker}\varphi = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n,$$

giving an isomorphism

$$A/\bigcap \mathfrak{a}_i \simeq \prod A/\mathfrak{a}_i.$$

The *radical* of an ideal $\mathfrak{a}$ is defined as

$$\sqrt{\mathfrak{a}} = \{\, a \mid a^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N} \,\}$$

If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, we define the ideal quotient as the ideal

$$(\mathfrak{a} : \mathfrak{b}) = \{\, a \in A \mid a\mathfrak{b} \subset \mathfrak{a} \,\}.$$

If $x \in A$, the ideal $(0 : x)$ is sometimes called the annihilator of $x$.

An element $a \in A$ is called a *zero divisor* if there exists a nonzero $b \in A$ such that $ab = 0$. It is a *unit* if there exists a $b$ with $ab = 1$. $a$ is nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$. $a$ is *irreducible* if it is not a unit and $a = xy$ implies that $x$ or $y$ is a unit.

An *$A$-algebra* is a ring $B$ equipped with a ring map $\phi\colon A \to B$. For instance, $B = A[x_1, \ldots, x_n]$ is an $A$-algebra. $B$ is said to be *finitely generated* if it is isomorphic to one of the form $A[x_1, \ldots, x_n]/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset A[x_1, \ldots, x_n]$.

The ring map $\phi$ allow us to multiply elements of $B$ with elements of $A$, by defining $ab = \phi(a)b$. For this reason, it is common to suppress $\phi$ from the notation.

### A.1.1 Prime ideals.

An ideal $\mathfrak{p}$ is prime if $xy \in \mathfrak{p}$ implies either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. This is equivalent to saying that $A/\mathfrak{p}$ is an integral domain.

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**A.1.2 Ideals in quotients.** For an ideal $\mathfrak{a}$ of $A$, there is a one-to-one correspondence

$$\{\text{Ideals in } A/\mathfrak{a}\} \longleftrightarrow \{\text{Ideals in } A \text{ containing } \mathfrak{a}\} \tag{A.1}$$

Let $\pi\colon A \to A/\mathfrak{a}$ be the quotient map. If $\mathfrak{b} \subset A$ is an ideal containing $\mathfrak{a}$, then $\pi(\mathfrak{b}) = \mathfrak{b}/\mathfrak{a}$ is an ideal of $A/\mathfrak{a}$. Conversely, if $J$ is an ideal of $A/\mathfrak{a}$, then $\pi^{-1}(J)$ is an ideal containing $\mathfrak{a}$. These operations are inverse to each other, as $\pi(\pi^{-1}(J)) = J$ and $\pi^{-1}(\pi(\mathfrak{b})) = \{a \in A\}\pi(a) \in \pi(\mathfrak{b}) = \mathfrak{b} + \mathfrak{a} = \mathfrak{b}$.

This correspondence preserves inclusions: if $\mathfrak{a} \subset \mathfrak{b} \subset \mathfrak{c}$, then $\mathfrak{b}/\mathfrak{a} \subset \mathfrak{c}/\mathfrak{a}$, and if $I \subset J \subset A/\mathfrak{a}$, then $\pi^{-1}(I) \subset \pi^{-1}(J) \subset A$.

**A.1.3 Prime ideals in quotients.** The correspondence (A.1) also gives a bijection between the prime ideals of $A/\mathfrak{a}$ and the prime ideals containing $\mathfrak{a}$. This follows because $\mathfrak{b}$ is prime if and only if $A/\mathfrak{a}$ is an integral domain, and $A/\mathfrak{b} = (A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$ is an integral domain if and only if $\mathfrak{b}/\mathfrak{a}$ is prime in $A/\mathfrak{a}$.

## A.2 Some field theory

A *field extension* $K/k$ is a ring map $k \to K$ between two fields. The *degree* of the field extension is defined as the dimension of $K$ as a $k$-vector space; this number is denoted by $[K : k]$. If $k \to K \to L$ is a sequence of field extensions, then $[L : k] = [K : k][L : K]$.

We say $K/k$ is called *finite* if $K$ is finite-dimensional as a $k$-vector space. It is called *fintely generated* if there exist elements $a_1, \dots, a_n \in K$ so that $K = k(a_1, \dots, a_n)$. Here $k(a_1, \dots, a_n)$ denotes the field extension generated by the $a_i$, i.e., the field of fraction of the ring $k[a_1, \dots, a_n]$.

An element $a \in K$ is called *algebraic* over $k$ if it satisafies a polynomial equation $P(a) = 0$, where $P$ has coefficients in $k$. The field extension $K/k$ is said to be *algebraic*, if every element of $K$ is algebraic over $k$. $K/k$ is called *transcendental* if it is not algebraic. $K/k$ is *purely transcendental* if there exists an algebraically independent collection of elements $\{x_i\}_{i\in I}$ such that $K = k((x_i)_{i\in I})$. An extension is finite if and only if it is algebraic and finitely generated.

A field $k$ is called *algebraically closed* if $k \subset L$ algebraic implies that $k = L$. If $k$ is a field, we call a field extension $K/k$ an *algebraic closure* if it is algebraically closed, and for any other extension $k \to L$ where $L$ is algebraically closed, there is a map of $k$-algebras $K \to L$. The algebraic closure is unique up to isomorphism (of $k$-algebras).

**A.2.1 Transcendence degree.** If $K/k$ is a field extension, a set of elements $a_1, \dots, a_r$ from $K$ is said to *algebraically independent* over $k$ if there are no polynomial relations between them, i.e., $P(a_1, \dots, a_r) \neq 0$ for every polynomial $P \in k[t_1, \dots, t_n]$. Equivalently, the map $t_i \mapsto a_i$ defines a $k$-algebra isomorphism $k[t_1, \dots, t_r] \simeq k[a_1, \dots, a_r]$. Likewise, one says that a possibly infinite subset $S \subset K$ is *algebraically independent* if any finite collection of (distinct) elements of $S$ is algebraically independent.

A *transcendence basis for $K$ over $k$* is a maximal algebraically independent set $S \subset K$. If $a_1, \dots, a_n$ are algebraically independent, they form a transcendence basis if and only if the field extension $k(a_1, \dots, a_n) \subset K$ is algebraic. One can prove that all transcendence bases have the same cardinality. This common cardinality is called the *transcendence degree of $K$*

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*over* $k$ and is denoted by $\operatorname{trdeg}_k K$. In general, the transcendence degree may be infinite, but for finitely generated field extensions, it will always be finite. Note that if $A \subset B$ is an extension of integral domains with $B$ of finite type over $A$, then the associated extension $K(A) \subset K(B)$ will be a finitely generated field extension with a finite transcendence degree.

If $k \subset K \subset L$ are fields, and $\{a_i\}_{i\in I}$ is a transcendence basis for $K/k$ and $\{b_j\}_{j\in J}$ is a transcendence basis for $L/K$, then $\{a_i, b_j\}_{i\in I, j\in J}$ is a transcendence basis for $L/k$. In particular,

$$\operatorname{trdeg}_k(L) = \operatorname{trdeg}_k(K) + \operatorname{trdeg}_K(L).$$

**A.2.2 Separable extensions.** A field extension $K/k$ is called *separable* if every algebraic element $K$ over $k$ is a root of a separable polynomial over $k$, that is, a polynomial whose roots are distinct in an algebraic closure of $k$. Equivalently, $K/k$ is separated if for every field extension $k \to L$, the tensor product $K \otimes_k L$ is reduced.

- A field extension of characteristic 0 is always separable.
- In characteristic $p > 0$, an extension is separable if and only if the formal derivative of every irreducible polynomial is non-zero.
- Every purely transcendental extension is separable.

A field $k$ is said to be *perfect* if every irreducible polynomial over $k$ has no repeated root in any field extension $K/k$. Any field of characteristic 0, such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, is perfect. If $k$ has characteristic $p$, $k$ is perfect if and only if every element of $k$ is a $p$-th power.

**Example A.2.** Examples of perfect fields include: Any field of characteristic 0, such as $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$; finite fields, since any finite field of characteristic $p$ is closed under taking $p$-th powers; and algebraically closed fields, by virtue of having no proper algebraic extensions. △

As in the definition of an algebraic closure, we define the separable closure and perfect closure of a field. These are again unique up to isomorphism.

## A.3 Modules

An $A$-module $M$ is said to be *free* if $M \simeq A^I$, where $I$ is an index set and $A^I = \bigoplus_{i\in I} A$. The *rank* of $M$ is defined as the cardinality of $I$.

**A.3.1 Finiteness conditions.** $M$ is said to be *finite* if $M \simeq A^n$ for some $n \geqslant 0$.

$M$ is *finitely generated*, or *of finite type* if there there are finitely many elements $m_1, \ldots, m_n$ which generate it as an $A$-module, that is, there is a surjective map of $A$-modules $A^n \to M$.

$M$ is of *finite presentation* if there is an exact sequence

$$A^m \to A^n \to M \to 0$$

for some integers $m, n \geqslant 0$.

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**A.3.2 Composition series.** An $A$-module $M$ is called *simple* if it is nonzero and its only submodules are 0 and $M$ itself. Equivalently, $M \simeq A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

If $M$ is an $A$-module, then a *compostion series* is a chain of submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_s = M$ such that $M_{i+1}/M_i$ is simple for any $i = 0, \ldots, s-1$. Given a compsition series, the quotients $M_{i+1}/M_i$ are uniquely determined up to isomorphism. We define the*length* of $M$, $\operatorname{length}(M)$ to be the length $s$ of a composition series.

The length is additive on exact sequences.

**A.3.3 Torsion.** If $M$ is an $A$-module, we define the *torsion submodule* of $M$ as the submodule $M_{\text{tors}} = \{\, m \in M \mid am = 0 \text{ for some a} \in A - 0 \,\}$. $M$ is said to be *torsion free* if the torsion submodule is zero.

## A.4 Localization

A nonempty subset $S$ of a commutative ring $A$ is called a *multiplicative set* if it is closed under multiplication and contains the identity element of $A$.

The *localization* of $A$ with respect to a multiplicative set $S$, denoted $S^{-1}A$, is the set of fractions $a/s$ with $a \in A$ and $s \in S$. There is a well-defined addition and multiplication making $S^{-1}A$ into a ring. Formally, $S^{-1}A$ is constructed by definiing an equivalence relation on $A \times S$ by $(a, s) \sim (a', s')$ if there exists an element $t \in S$ such that $t(as' - a's) = 0$ in $A$. The elements of $S^{-1}A$ are denoted by $a/s$ or $\frac{a}{s}$.

There is a canonical localization map

$$\rho\colon A \to S^{-1}A, x \mapsto x/1$$

which makes $S^{-1}A$ into an $A$-module. The kernel of $\rho$ consists of those elements which are annihilated by some element of $S$, i.e., $sx = 0$ for some $s \in S$. In particular, the map $\rho$ is injective if $A$ contains no zerodivisors.

The localization $S^{-1}A$ is the zero ring if and only if $0 \in S$ (if $0 \in S$, then $a/s = 0/1$ by definition).

If $M$ is an $A$-module, one also defines a localization $S^{-1}M$ as the set of fractions $m/s$, for $m \in M$, $s \in S$, using the equivalence relation $(m, s) \sim (m', s')$ if $t(ms' - m's) = 0$ in $M$. As above, there is a canonical localization map $\rho\colon M \to S^{-1}M$. Also, $S^{-1}M$ is naturally an $S^{-1}A$-module.

The localization $S^{-1}A$ and the map $\rho$ satisfies the following universal property: For every $A$-module $N$ and any ring map $f\colon A \to B$ such that $f(s) \in B$ is invertible for all $s \in S$, there exists a *unique* map of rings $g\colon S^{-1}A \to B$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle\varphi}\downarrow & \nearrow_{\exists! g} & \\ S^{-1}A & & \end{array} \tag{A.2}$$

The map $g$ is uniquely determined by $g(m/s) = f(m) \cdot f(s)^{-1}$ for all $m \in M$ and $s \in S$.

**Example A.3.** The first prototype example is when $S = A - \mathfrak{p}$ for a prime ideal $\mathfrak{p}$. In this case the localization $S^{-1}A$ is denoted $A_{\mathfrak{p}}$. The ring $A_{\mathfrak{p}}$ is a local ring with maximal ideal

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$\mathfrak{p}A_\mathfrak{p}$. The elements in $A - \mathfrak{p}$ become units in $A_\mathfrak{p}$, and every non-unit in $A_\mathfrak{p}$ is in the maximal ideal $\mathfrak{p}A_\mathfrak{p}$. △

**Example A.4.** The second prototype example is when $S = \{1, f, f^2, \dots\}$ for some $f \in A$. In this case the localization $S^{-1}A$ is denoted $A_f$. Elements of $A_f$ are of the form $a/f^n$ where $a \in A$ and $n > 0$. The map $\sum a_n t^n \mapsto \sum a_n f^{-n}$ induces an isomorphism $A[t]/(1 - ft) \simeq A_f$. △

The functor $M \mapsto S^{-1}M$ has excellent properties. It is exact, and commutes with direct sums, direct limits, and tensor products.

If $\mathfrak{p}$ is a prime ideal, and $f \notin \mathfrak{p}$, then there is a canonical localization map $A_f \to A_\mathfrak{p}$. The localizations $A_f$ form a directed system as $f$ runs over the set of elements in $A - \mathfrak{p}$ and there is an isomorphism

$$\varinjlim_{f \in A - \mathfrak{p}} A_f \longrightarrow A_\mathfrak{p}.$$

**A.4.1 Ideals in localizations.** Let $S \subset A$ be a multiplicative set. If $\mathfrak{a} \subset A$ is an ideal such that $S \cap \mathfrak{a} \neq \varnothing$, then $S^{-1}\mathfrak{a} = (1)$ in $S^{-1}A$. Hence the ideals that intersect $S$ map to non-proper ideals in $S^{-1}A$.
For prime ideals, there is a one-to-one correspondence

$$\{\text{Prime ideals } P \text{ in } S^{-1}A\} \longleftrightarrow \{\text{Prime ideals } \mathfrak{p} \text{ in } A \text{ with } \mathfrak{p} \cap S = \varnothing\} \tag{A.3}$$

The correspondence sends $P \subset S^{-1}A$ to $\rho^{-1}(P)$, and the inverse is given by $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

These maps are mutual inverses. For a prime ideal $P$ in $S^{-1}A$, we have $S^{-1}\rho^{-1}(P) = P$. Indeed, $a/s \in P \Leftrightarrow a/1 \in P \Leftrightarrow a \in \rho^{-1}(P) \Leftrightarrow a/s \in S^{-1}(\rho^{-1}(P))$. On the other hand, if $\mathfrak{p}$ is a prime ideal in $A$ such that $\mathfrak{p} \cap S = \varnothing$, then $\rho^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}$. The '$\supset$'-inclusion is clear. Conversely, if $a \in \rho^{-1}(S^{-1}\mathfrak{p})$, then $a/1 \in S^{-1}\mathfrak{p}$, so there exists $t \in S$ such that $ta \in \mathfrak{p}$. Since $\mathfrak{p} \cap S = \varnothing$, $t \notin \mathfrak{a}$, therefore $a \in \mathfrak{a}$.

**A.4.2 Radicals and intersections.** For an ideal $\mathfrak{a} \subset A$, we have

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subset \mathfrak{p}} \mathfrak{p}. \tag{A.4}$$

Indeed, if $f \in \sqrt{\mathfrak{a}}$, then $f^r \in \mathfrak{a}$ for some $r \in \mathbb{N}$. Therefore, $f^r \in \mathfrak{p}$ for every $\mathfrak{p}$ that contains $\mathfrak{a}$, which implies $f \in \mathfrak{p}$.

Conversely, if $f^r \notin \mathfrak{a}$ for every $r \in \mathbb{N}$, then $A_f \neq 0$ and the localized ideal $\mathfrak{a}_f$ is a proper ideal in $A_f$. Taking the preimage of any maximal ideal in $A_f$ containing $\mathfrak{a}_f$, we get a prime ideal $\mathfrak{p} \subset A$ such that $\mathfrak{p} \supset \mathfrak{a}$ and $\mathfrak{p} \cap \{1, f, f^2, \dots\} = \varnothing$. But then $f$ does not lie in the intersection of the right-hand side of (A.4).

**A.4.3 Local rings.** A ring $A$ is called *local* if there is exactly one maximal ideal in $A$.

If $A$ is a ring and $\mathfrak{m}$ is an ideal, the following are equivalent:

(i) $A$ is a local ring and $\mathfrak{m}$ is the maximal ideal
(ii) All elements in $A - \mathfrak{m}$ are units
(iii) $\mathfrak{m}$ is maximal, and every element of the form $1 + x$ where $x \in \mathfrak{m}$ is a unit.

Fields are local rings (with maximal ideal $(0)$). Other examples include discrete valuation rings and localizations $A_\mathfrak{p}$ (with maximal ideal $\mathfrak{p}A_\mathfrak{p}$.)

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**A.4.4 Integral domains.** If $A$ is an integral domain, we let $K = K(A)$ be the fraction field, i.e., the localization $S^{-1}A$ where $S = A - \{0\}$. Then $A \subset K$ and via the map $a \mapsto a/1$. Moreover, the various localizations $A_f$, $A_{\mathfrak{p}}$ and $A_{\mathfrak{m}}$ lie as subrings of $K$.

**Lemma A.5.** Let $A$ be an integral domain. Then

$$A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$$

where the intersection is taken over all maximal ideals of $A$.

*Proof* The containment $A \subseteq \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ is clear. Conversely, let $f \in K$ be an element which lies in all the localizations $A_{\mathfrak{m}}$. Consider the ideal $I = \{\, a \in A \mid af \in A \,\}$. If $I = (1)$, then $1 \in I$, i.e., $f \in A$, so we are done. If $I$ is a proper ideal, it is contained in a maximal ideal $\mathfrak{m}$. But then $f$ cannot be contained in $A_{\mathfrak{m}}$, because otherwise there would exist some $s \notin \mathfrak{m}$ such that $sf \in A$ and hence $A \in I$, which contradicts the maximality of $\mathfrak{m}$. □

## A.5 Tensor products

Let $M$ and $N$ be two $A$-modles. We define the tensor product $M \otimes_A N$ as the quotient of the free module $A^{M\times N}$ with basis $(e_{m,n})$ modulo the relations

$$\begin{aligned} e_{m_1+m_2,n} - e_{m_1,n} - e_{m_2,n} &= 0 \\ e_{m,n_1+n_2} - e_{m,n_1} - e_{m,n_2} &= 0 \\ e_{am,n} - e_{m,an} = ae_{m,n} - e_{am,n} &= 0 \end{aligned}$$

for all $m \in M$, $n \in N$, $a \in A$. We write $m \otimes n$ for the class of $e_{m,n}$ in $M \otimes_A N$.

The assignment $(m, n) \mapsto m \otimes_A n$ defines map $\gamma\colon M \times N \to M \otimes_A N$ which is bilinar as a map of $A$-modules. It satisfies the following universal property: for any bilinear map $\phi\colon M \times N \to P$, there is a unique map $\overline{\phi}\colon M \otimes_A N \to P$ so that $\phi = \overline{\phi} \circ \gamma$.

**Example A.6.** If $A$ and $B$ are $k$-algebras over some field $k$, then $A \otimes_k B$ is a ring via the multiplication rule $(a \otimes b) \cdot (a' \otimes B') = (aa' \otimes bb')$. As such, it is naturally a $k$-algebra via the map $k \to A \otimes_k B$ sending $x \mapsto x(1 \otimes 1)$. If $R$ is another $k$-algebra, the universal property says that there is a bijection

$$\operatorname{Hom}_k(A \otimes_k B, R) = \operatorname{Hom}_k(A, R) \times \operatorname{Hom}_k(B, R).$$

△

## A.6 Basic formulas

In the formulas below, $M, N, L$ denote $A$-modules; $S \subset A$ is a multiplicative set; and $\mathfrak{p}$ is a prime ideal of $A$. Each equality '=' between two modules means that there is a canonical isomorphism between them.

**A.6.1 Localization identities.**

(i) $S^{-1}(M/N) = S^{-1}M/S^{-1}N$

(ii) If $M$ is finitely presented: $S^{-1}\operatorname{Hom}_A(M, N) = \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$.

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**A.6.2 Tensor product identities.**

(i) $A \otimes_A M = M$
(ii) $M \otimes_A N = N \otimes_A M$
(iii) $M \otimes_A (N \otimes_A P) = (M \otimes_A N) \otimes_A P$
(iv) $(\bigoplus_{i\in I} M_i) \otimes_A N = (\bigoplus_{i\in I} M_i \otimes_A N)$
(v) $M \otimes_A A/\mathfrak{a} = M/\mathfrak{a}M$ for every ideal $\mathfrak{a} \subset A$.
(vi) If $A \to B$ is a ring map, and $P$ is a $B$-module, then there is a canonical isomorphism of $B$-modules

$$M \otimes_A (N \otimes_B P) = (M \otimes_A N) \otimes_B P.$$

(vii) $\mathrm{Hom}_A(M \otimes_A N, P) = \mathrm{Hom}_A(M, \mathrm{Hom}(N, P))$.
(viii) $S^{-1}M = M \otimes_A S^{-1}A$.
(ix) $S^{-1}(M \otimes_A N) = S^{-1}M \otimes_{S^{-1}A} S^{-1}N$.
(x) If $A = k$ is a field, $\dim_k(M \otimes_k N) = (\dim_k M) \cdot (\dim_k N)$.
(xi) If $f\colon A \to B$ is a ring map and $M$, $N$ are $B$-modules, then $M \otimes_B N \equiv M \otimes_A N$.

**A.6.3 Exactness properties.** If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then

$$0 \longrightarrow S^{-1}M' \longrightarrow S^{-1}M \longrightarrow S^{-1}M'' \to 0$$

is exact for every multiplicative set $S$;

$$0 \to \mathrm{Hom}(L, M') \to \mathrm{Hom}(L, M) \to \mathrm{Hom}(L, M'')$$
$$0 \to \mathrm{Hom}(M'', L) \to \mathrm{Hom}(M, L) \to \mathrm{Hom}(M', L)$$

are exact for every $A$-module $L$; and

$$M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$$

is exact for any $A$-module $N$.

**Proposition A.7.**

(i) For an $A$-module $M$, $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$.
(ii) A sequence of $A$-modules $0 \to M' \to M \to M'' \to 0$ is exact if and only if the induced sequence $0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}} \to 0$ is exact for every maximal ideal $\mathfrak{m}$.
(iii) A map of $A$-modules $\phi\colon M \to N$ is injective (resp. surjective) if and only if $\phi_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective (resp. surjective) for every maximal ideal $\mathfrak{m}$.

**A.6.4 Finitely presented modules and Hom.** We say that a module $M$ is *finitely presented* if there is an exact sequence

$$A^q \to A^p \to M \to 0. \tag{A.5}$$

If $A$ is Noetherian, then any finitely generated module finitely presented.

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**Proposition A.8.** If $M$ is finitely presented, then $\operatorname{Hom}_A(M, -)$ commutes with localization. That is, there is an isomorphism of $S^{-1}A$-modules

$$S^{-1}\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N) \tag{A.6}$$

*Proof* The isomorphism is defined as follows: sending $\phi\colon M \to N$ to the $S^{-1}A$-linear map $m/s \mapsto \phi(m)/s$, defines a map of $A$-modules

$$\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

Since multiplication by an element $s \in S$ is an isomorphism on the $A$-module on the right, the map extends to a map (A.6).

For $M = A$, we have $\operatorname{Hom}_A(A, N) = N$ and (A.6) is the isomorphism $S^{-1}\operatorname{Hom}_A(A, N) = \S^{-1}N \to \operatorname{Hom}_{S^{-1}A}(S^{-1}A, S^{-1}N) = S^{-1}N$. More generally, it is an isomorphism for any $M = A^N$, $N \in \mathbb{N}$.

The map (A.6) is functorial in $M$. Applying $\operatorname{Hom}_A(M, -)$ to (A.5), we get the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & S^{-1}\mathrm{Hom}_A(M,N) & \longrightarrow & S^{-1}\mathrm{Hom}_A(A^p,N) & \longrightarrow & S^{-1}\mathrm{Hom}_A(A^q,N) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N) & \longrightarrow & \mathrm{Hom}_{S^{-1}A}(S^{-1}A^p,S^{-1}N) & \longrightarrow & \mathrm{Hom}_{S^{-1}A}(S^{-1}A^q,S^{-1}N)
\end{array}$$

The two right-most vertical maps are isomorphisms (as shown above). By the five lemma, the left-most vertical map is an isomorphism as well. □

## A.7 Change of rings

For a ring map $\phi\colon A \to B$, there are three natural operations on modules. Here $M$ is an $A$-module, and $N$ is a $B$-module:

- $\phi_! M = M \otimes_A B$ is the *induced $B$-module*.
- $\phi_* M = \operatorname{Hom}_A(B, M)$ is the *coinduced module*.
- $\phi^* N = N_A$ is the module formed by *restriction of scalars*, i.e., $N$ but viewed as an $A$-module.

These form adjoint functions, in the sense that there are natural bijections

$$\operatorname{Hom}_A(\phi_! M, P) \to \operatorname{Hom}_B(M, \phi^* P) \tag{A.7}$$

defined by sending $\ell\colon M \otimes_A B \to P$ to $m \mapsto \ell(m \otimes 1)$; and

$$\operatorname{Hom}_A(\phi^* P, M) = \operatorname{Hom}_B(P, \phi_* M) \tag{A.8}$$

sending $\ell\colon P_A \to M$ to $h\colon P \to \operatorname{Hom}_A(B, M)$ defined by $h(p) = (b \mapsto \ell(b \cdot p))$.

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## A.8 Nakayama's Lemma

**Lemma A.9 (Nakayama's lemma, local version).** Let $A$ be a local ring with maximal ideal $\mathfrak{m}$, and let $M$ be a finitely generated module. If $\mathfrak{m}M = M$, then $M = 0$.

*Proof* Assume $M \neq 0$, and let $m_1, \ldots, m_r$, $r \geqslant 1$, be a minimal set of generators for $M$. As $M = \mathfrak{m}M$, we may write $m_r$ as a linear combination of the form $m_r = a_1 m_1 + \cdots + a_r m_r$ for some $a_i \in \mathfrak{m}$. This means that $(1 - a_r)m_r = a_1 m_1 + \cdots + a_{r-1} m_{r-1}$. However, $1 - a_r$ is a unit in $A$, so this relation allows us to express $m_r$ in terms of $m_1, \ldots, m_{r-1}$, contradicting the minimality assumption on $m_1, \ldots, m_r$. □

**Corollary A.10.** Let $A$ be a local ring, and let $\varphi\colon M \to N$ be map of finitely generated $A$-modules, such that $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective. Then $\varphi$ is surjective.

*Proof* If $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective, then $N = \varphi(M) + \mathfrak{m}N$. Hence

$$\mathfrak{m}(N/\varphi(M)) = (\mathfrak{m}N + \varphi(M))/\varphi(M) = N/\varphi(M).$$

Applying Nakayama's Lemma to $N/\varphi(M)$, we get that $N/\varphi(M) = 0$, so $\varphi$ is surjective. □

The above result can be helpful also in the case $A$ is not local, as by Proposition A.7, a map of $A$-modules $\phi\colon M \to N$ is surjective if and only if the localization $\phi_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is surjective for each maximal ideal $\mathfrak{m}$.

## A.9 Cayley–Hamilton Theorem

**Theorem A.11 (Cayley-Hamilton).** Let $A$ be a ring, $\mathfrak{a} \subset A$ an ideal and let $M$ be a finitely generated $A$-module. Let $\varphi\colon M \to M$ be an $A$-module homomorphism with $\varphi(M) \subset \mathfrak{a}M$. Then there exists a monic polynomial

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in A[x]$$

with $a_0, \ldots, a_{n-1} \in \mathfrak{a}$ such that

$$\varphi^n + a_{n-1}\varphi^{n-1} + \cdots + a_0 \mathrm{id} = 0 \in \mathrm{Hom}_A(M, M),$$

Here $\varphi^i$ denotes the composition of $\varphi$ with itself $i$ times.

Let $T = (b_{ij})_{1 \leqslant i,j \leqslant n}$ be an $(n \times n)$-matrix with entries in $A$, and let $T_{ij}$ be the $(n \times n)$-matrix obtained by deleting row $i$ and column $j$. The adjugate of $T$ is the matrix $\mathrm{adj}(T) = (c_{ij})_{1 \leqslant i,j \leqslant n}$, where

$$c_{ij} = (-1)^{i+j} \det(T_{ji}).$$

The adjoint matrix has the property that $T \cdot \mathrm{adj}(T) = \mathrm{adj}(T) \cdot T = \det(T) I_n$, where $I_n$ is the $(n \times n)$ identity matrix. This follows by the cofactor expansion of the determinant.

*Proof* Let $m_1, \ldots, m_n$ be generators for $M$. By the assumption on $\phi$, we have $\varphi(m_i) \in$

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$\mathfrak{a}M$ for all $i = 1, \dots, n$. Therefore, for each $i$, we may express

$$\varphi(m_i) = \sum_{j=1}^{n} b_{ij} m_j$$

for some elements $b_{ij} \in \mathfrak{a}$. Next, consider $M$ as a module over the ring $A[x]$ by setting $x^i \cdot m := \varphi^i(x)$ for all $m \in M$. If $I$ denotes the $n \times n$ identity matrix, and $T = (b_{ij})$, then the above equation can be written as

$$(xI - T) \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$$

If we multiply both sides by the adjoint matrix of $(xI - T)$, we obtain

$$\det(xI - T) \cdot I \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0$$

Let $P(x)$ be the monic polynomial $\det(xI - T)$. Then $P(x)$ anihilates all the $m_i$, and hence $P(\phi)$ acts as the zero endomorphism on $M$. By the Laplace expansion of the determinant, we also see that the coefficient of $x^j$ of $P$ belongs to $\mathfrak{a}^j$. □

## A.10 Integral Extensions

Let $A \subset B$ be rings, and let $b \in B$. We say $b$ is integral over $A$ if it satisfies a monic polynomial relation

$$b^n + a_{n-1}b^{n-1} + \cdots + a_1 b + a_0 = 0, \quad a_i \in A.$$

**Theorem A.12.** Let $A \subset B$, and let $b \in B$. The following are equivalent:

(i) $b$ is integral over $A$.
(ii) $A[b]$ is a finitely generated $A$-module.
(iii) There is a ring $C$ with $A[b] \subset C \subset B$ such that $C$ is a finitely generated $A$-module.

*Proof* (i) ⇒ (ii): If $b$ is integral, we can write $b^k = -(a_{k-1}b^{k-1} + \cdots + a_0)$, and therefore we may express every higher power

$$b^{k+m} = (a_{k-1}b^{k+m-1} + \cdots + a_0 b^m).$$

in terms of $b^i$ for $i < k + m$. This gives that $1, b, \dots, b^{k-1}$ generate $A[b]$.

(ii) ⇒ (iii): Take $C = A[b]$.

(iii) ⇒ (i): Applying the Cayley–Hamilton theorem to $M = C$, $\mathfrak{a} = A$ and $\phi\colon M \to M$ given by multiplication by $b$, we get that $\phi$ satisfies an integral relation with coefficients in $A$. Applying this relation to $1 \in C$, we get that $b$ is integral over $A$. □

**Corollary A.13.** Any finite ring extension $A \subset B$ is integral.

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*Proof* Taking $C = B$ in (iii) in the proposition above, we see that any element $b \in B$ is integral over $A$. □

For a ring extension $A \subset B$, we define integral closure of $A$ in $B$ as the subring

$$C = \{b \in B \mid b \text{ integral over } A\}$$

If $A$ is equal to its integral closure, we say it is integrally closed in $B$.

Using Theorem A.12 on can show that The integral closure $C$ of $A$ in $B$ is itself integrally closed in $B$.

An integral domain $A$ is said to be *normal* if it is integrally closed in its fraction field $K = k(A)$. In other words, any element $z \in K$ which satisfies a monic equation with coefficients in $A$, is already contained in $A$.

**Example A.14.** The ring $\mathbb{Z}$ is normal. More generally, every unique factorisation domain is integrally closed. △

**Proposition A.15.** Let $A$ be an integral domain. Then the following are equivalent:
(i) $A$ is integrally closed.
(ii) $A_{\mathfrak{p}}$ is integrally closed for all prime ideals $\mathfrak{p} \subset A$.
(iii) $A_{\mathfrak{m}}$ is integrally closed for all maximal ideals $\mathfrak{m} \subset A$.

**Proposition A.16.** Let $A \subset B$ be rings, and let $C \subset B$ be the integral closure of $A$ in $B$. Let $S \subset A$ be a multiplicatively closed subset. Then $S^{-1}C \subset S^{-1}B$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.

**Proposition A.17.** Let $A \subset B$ be an integral extension of integral domains. Then $A$ is a field if and only if $B$ is a field.

There is a collection of results, the Cohen–Seidenberg Theorems, about prime ideals in integral extensions. We collect the ones we need in the following theorem.

**Theorem A.18.** Let $A \subset B$ be an integral extension of rings.
(i) (Lying-Over) If $\mathfrak{p}$ prime ideal in $A$, there is prime ideal $\mathfrak{q}$ in $B$ so that $\mathfrak{q} \cap A = \mathfrak{p}$.
(ii) If $\mathfrak{q} \subset \mathfrak{q}'$ are prime ideals in $B$ such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$, then $\mathfrak{q} = \mathfrak{q}'$.
(iii) (Going–Up) If $\mathfrak{p} \subset \mathfrak{p}'$ are two prime ideals in $A$ and $\mathfrak{q} \in \operatorname{Spec} B$ with $\mathfrak{q} \cap A = \mathfrak{p}$, there is a $\mathfrak{q}' \in \operatorname{Spec} B$ with $\mathfrak{q}' \cap A = \mathfrak{q}$.
(iv) (Going–Down) Assume that $A$ is integrally closed and that $\mathfrak{p}' \subset \mathfrak{p}$ are two prime ideals. If $\mathfrak{q} \in \operatorname{Spec} B$ is such that $\mathfrak{q} \cap A = \mathfrak{p}$, then there is a $\mathfrak{q}' \in \operatorname{Spec} B$ such that $\mathfrak{q}' \cap A = \mathfrak{p}'$.

**Proposition A.19.** Let $A \subset B$ be an integral ring extension. Then $\dim A = \dim B$.

*Proof* Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_s$ be a chain of prime ideals in $A$. By the Lying-Over Theorem, there is a prime ideal $\mathfrak{q}_0 \subset B$ such that $\mathfrak{q}_0 \cap A = \mathfrak{p}_0$. Now $B/\mathfrak{q}_0$ is integral over

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$A/\mathfrak{p}_0$ and $\mathfrak{p}_1/\mathfrak{p}_0$ is a prime ideal of $A/\mathfrak{p}_0$. Applying the Going-Up Theorem again, we see that there is a prime ideal $\overline{\mathfrak{q}_1}$ in $B/\mathfrak{q}_0$ such that $\overline{\mathfrak{q}_1} \cap (A/\mathfrak{p}_0) = \mathfrak{p}_1/\mathfrak{p}_0$. This ideal must have the form $\overline{\mathfrak{q}_1} = \mathfrak{q}_1/\mathfrak{q}_0$ for some prime ideal $\mathfrak{q}_1 \supset \mathfrak{q}_0$ in $B$. The inclusion $\mathfrak{q}_1 \supset \mathfrak{q}_0$ must be strict, because otherwise we would have $\mathfrak{p}_1 = \mathfrak{p}_0$. Furthermore, $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. Proceeding in this manner, we can produce a chain of prime ideals in $B$ of the same length $l$, and hence $\dim A \leqslant \dim B$.

For the reverse inequality, suppose that $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_s$ is a chain of prime ideals in $B$. Define $\mathfrak{p}_i = \mathfrak{q}_i \cap A$. Then $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_s$ is a chain of prime ideals in $A$. We claim that each inclusion $\mathfrak{p}_i \subset \mathfrak{p}_{i+1}$ is strict. Given this, we can conclude that $\dim A \leqslant \dim B$, complete the proof. Suppose for a contradiction that $\mathfrak{p}_i = \mathfrak{p}_{i+1}$. It suffices to show that $\mathfrak{q}_i \supseteq \mathfrak{q}_{i+1}$. Let $y \in \mathfrak{q}_{i+1}$. Since $B/\mathfrak{q}_i$ is integral over $A/\mathfrak{p}_i$, we have an integral relation $\overline{y}^n + \overline{a}_{n-1}\overline{y}^{n-1} + \cdots + \overline{a}_1\overline{y} + \overline{a}_0 = 0$, and we may assume that $n$ is minimal with this property. Note that $\overline{a}_0 \in A/\mathfrak{p}_i = A/\mathfrak{p}_{i+1}$ and $\overline{a}_0 \in \mathfrak{q}_{i+1}/\mathfrak{q}_i$, and so $\overline{a}_0 = 0$. But this means that $\overline{y}(\overline{y}^{n-1} + \cdots + \overline{y}_1) = 0$. As $B/\mathfrak{q}_i$ is an integral domain, and $\overline{y}^{n-1} + \cdots + \overline{a}_1$ is non-zero (by the minimality of $n$), we must have $\overline{y} = 0$, and hence $y \in \mathfrak{q}_i$. □

The following is a non-trivial result from commutative algebra about the integral closure:

**Theorem A.20 (Finite generation of integral closure).** Let $A$ be an integral domain, $K = K(A)$ its fraction field, and let $K \subset L$ be a finite separable field extension. Let $B$ be the integral closure of $A$ in $L$ (that is, the elements of $L$ which are integral over $A$). Then

(i) If $A$ is integrally closed, then $B$ is a finitely generated $A$-module
(ii) If $A$ is finitely generated as a $k$-algebra, then $B$ is a finitely generated $A$-module.

The second part does not hold in general: there are non-noetherian rings where the integral closure is not finitely generated.

## A.11 Noetherian rings

A ring $A$ is called *Noetherian* if every ascending chain of ideals $I_1 \subset I_2 \subset \cdots$ stabilizes, meaning that there exists $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \cdots$. Equivalently, $A$ is Noetherian if every ideal of $A$ is finitely generated.

**Theorem A.21 (Hilbert's Basis Theorem).** If $A$ is a Noetherian ring, then the polynomial ring $A[x_1, \ldots, x_n]$ is also Noetherian.

The ring $A$ is called *Artinian* if it satisfies the descending chain condition on ideals, which means that every descending chain of ideals $I_1 \supset I_2 \supset \cdots$ eventually stabilizes, with there existing some $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \cdots$.

If $A$ is an Artinian ring, then $A$ is also Noetherian and there are only finitely many prime ideals. In particular, its has Krull dimension zero. Moreover, $A$ is the product of local Artinian rings. More precisely, the natural map $A \simeq \prod_{\mathfrak{p}} A_{\mathfrak{p}}$ is an isomorphism.

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**Theorem A.22 (Krull's Intersection Theorem).** Let $A$ be a Noetherian local integral domain with maximal ideal $\mathfrak{m}$. Then $\bigcap_{j\geqslant 1} \mathfrak{m}^j = (0)$.

*Proof* [1] Write $J$ for the ideal $\bigcap_{j\geqslant 1} \mathfrak{m}^j$, and let $a \in J$. We claim that $a \in \mathfrak{m}J$, so that $J = \mathfrak{m}J$. Given this, Nakayama's Lemma implies that $J = 0$.

Choose generators $x_1, \dots, x_n$ for the maximal ideal $\mathfrak{m}$. Then the ideal $\mathfrak{m}^d$ consists of elements of the form $P(x_1, \dots, x_n)$ where $P \in A[X_1, \dots, X_n]$ is a homogeneous polynomial of degree $d$. As $a \in \mathfrak{m}^d$ for every $d$, we may choose homogeneous polynomials $P_1, P_2, \dots \in A[X_1, \dots, X_n]$ so that $a = P_d(x_1, \dots, x_n)$ for every $d$. Consider the ideal $(P_1, P_2, \dots)$ generated by all the $P_i$. As $A[X_1, \dots, X_n]$ is Noetherian, this ideal is finitely generated, say by homogeneous elements $P_1, \dots, P_r$ of degrees $d_1, \dots, d_r$. In particular, we may express

$$P_{r+1} = c_1 P_1 + \cdots + c_r P_r$$

for some $c_i \in A[X_1, \dots, X_n]$. In this relation, we may choose the $c_i$ to be homogeneous of degree $\deg P_{r+1} - d_i = d + 1 - d_i > 0$. This means that $c_i(x_1, \dots, x_n) \in \mathfrak{m}$ and hence

$$a = P_{r+1}(x_1, \dots, x_r) = \sum_i c_i(x_1, \dots, x_n) P_i(x_1, \dots, x_n) \in \mathfrak{m} \cdot \bigcap_{d\geqslant 1} \mathfrak{m}^d.$$

□

In particular, this implies that in a Noetherian ring no nonzero element $a \in A$ can lie in all powers of all maximal ideals.

## A.12 Primary decomposition

An ideal $\mathfrak{q}$ is called *primary* if $xy \in \mathfrak{q}$ implies that either $x \in \mathfrak{a}$ or $y^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$. If $\mathfrak{q}$ is a primary ideal, then the radical $\mathfrak{p} = \sqrt{\mathfrak{q}}$ is a prime ideal; it is the smallest prime ideal containing $\mathfrak{q}$. We say that $\mathfrak{q}$ is $\mathfrak{p}$-primary. If $\mathfrak{q}, \mathfrak{q}'$ are $\mathfrak{p}$-primary, then the intersection $\mathfrak{q} \cap \mathfrak{q}'$ is also $\mathfrak{p}$-primary.

If $\mathfrak{p}$ is a prime ideal, then any power $\mathfrak{p}^l$ is primary. $\mathfrak{q} = (x^2, y)$ is a primary ideal which is not the power of any prime ideal.

**Theorem A.23 (Lasker–Noether).** In a Noetherian ring $A$, any ideal $\mathfrak{a}$ can be expressed as a finite intersection of primary ideals:

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{a}_r \tag{A.9}$$

The set of prime ideals $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ are uniquely determined by $\mathfrak{a}$.

A decomposition (A.9) is called *minimal* if the prime ideals $\sqrt{\mathfrak{q}_i}$ are different, and the decomposition is minimal, i.e., $\bigcap_{i\neq j} \mathfrak{q}_i \not\subseteq \mathfrak{q}_j$ for each $j$.

**Example A.24.** Consider the ideal $\mathfrak{a} = (x^2, xz, xy, yz)$ of the polynomial ring $A =$

[1] This proof follows "Hervé Perdry. An elementary proof of Krull's intersection theorem"

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$k[x, y, z]$. Then we have two different minimal primary decompositions

$$\mathfrak{a} = (x, y) \cap (x, z) \cap (x, y, z)^2 = (x, y) \cap (x, z) \cap (x^2, y, z).$$

Note by the way that $\mathfrak{a} = (x, y) \cdot (x, z)$. △

If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal, then we have

- $(\mathfrak{q} : x) = A$ if $x \in \mathfrak{q}$
- $(\mathfrak{q} : x) = \mathfrak{q}$ if $x \notin \mathfrak{p}$
- $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary if $x \in \mathfrak{q}$.

Furthermore, there is an element $x \in A$ such that $\mathfrak{p} = (\mathfrak{q} : x)$. Using this observation, one shows that the set of prime ideals $\mathfrak{p}_i$ appearing in the decomposition of $\mathfrak{a}$ is exactly the prime ideals of the form $(\mathfrak{q} : x)$ where $x \in A$ (in particular, they are uniquely determined by $\mathfrak{q}$). These prime ideals are called the *associated primes* of $\mathfrak{a}$, and the set of these is sometimes denoted by $\mathrm{Ass}(\mathfrak{a})$.

The *minimal* prime ideals $\mathfrak{p}_i$ associated to $\mathfrak{a}$ (minimal with respect to inclusion) play an important role: $V(\mathfrak{p}_i)$ give the irreducible components of the closed set $V(\mathfrak{p})$. The non-minimal associated prime ideals are sometimes called the *embedded components* of $\mathfrak{a}$.

For instance, in the example above, the associated prime ideals are $\mathfrak{p}_1 = (x, y)$ and $\mathfrak{p}_2 = (x, z)$. The prime ideal $\mathfrak{p}_3 = \sqrt{(x^2, y, z)} = (x, y, z)$ is an embedded prime.

We will need two results about primary decompositions in localizations.

**Proposition A.25.** Let $A$ be a Noetherian ring and let $S \subset A$ be a multiplicative subset. Let $\mathfrak{a} \subset A$ be an ideal, and $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a mimimal primary decomposition of $\mathfrak{a}$ with associated prime ideals $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Assume that the $\mathfrak{q}_i$ are ordered such that $\mathfrak{p}_i \cap S = \varnothing$ for every $i = 1, \ldots, s$ and $\mathfrak{p}_i \cap S \neq \varnothing$ for $i = s + 1, \ldots, r$. Then

(i) For each $i = 1, \ldots, s$, the ideal $S^{-1}\mathfrak{q}_i$ is $S^{-1}\mathfrak{q}_i$-primary
(ii) $S^{-1}\mathfrak{a} = \bigcap_{i=1}^{s} S^{-1}\mathfrak{q}_i$
(iii) $S^{-1}\mathfrak{a} \cap A = \bigcap_{i=1}^{s} S^{-1}\mathfrak{q}_i$.

Moreover, the decompositions in (ii) and (iii) are minimal.

Here $S^{-1}\mathfrak{a} \cap A$ denotes the preimage of $S^{-1}\mathfrak{a}$ via the localization map $A \to S^{-1}A$.

*Proof* This is Proposition 4.9 in Atiyah (2018). □

**Proposition A.26.** Let $\mathfrak{a} \subset A$ be an ideal with a minimal primary decomposition $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{q}_i$, where each $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. Then, for each index $i$ such that $\mathfrak{p}_i$ is a minimal prime ideal, we have

$$\mathfrak{q}_i = \mathfrak{a}A_{\mathfrak{p}_i} \cap A.$$

In this case, we call $\mathfrak{q}_i$ the $\mathfrak{p}_i$-primary component of $\mathfrak{a}$.

In particular, the primary components $\mathfrak{q}_i$ corresponding to minimal primes $\mathfrak{p}_i$ are unique and depend only on $\mathfrak{a}$.

Given a Noetherian ring $R$, we can use the uniqueness result mentioned above to define the $n$-th symbolic power of any prime ideal $\mathfrak{p} \subset R$ for $n \geqslant 1$. Consider a primary decomposition

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of $\mathfrak{p}^n$, and note that $\mathfrak{p}$ is the only minimal prime ideal of $\mathfrak{p}^n$. The $\mathfrak{p}$-primary component of $\mathfrak{p}^n$ is unique and given by

$$\mathfrak{p}^{(n)} := \mathfrak{p}^n R_{\mathfrak{p}} \cap R.$$

This is called the $n$-th symbolic power of $\mathfrak{p}$. Clearly, $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ and equality $\mathfrak{p}^n = \mathfrak{p}^{(n)}$ holds if and only if $\mathfrak{p}^n$ is primary.

## A.13 Dimension theory

The *Krull dimension* of a ring $A$, denoted by $\dim(A)$, is the supremum of the lengths $n$ of chains

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

of distinct prime ideals $\mathfrak{p}_i$ in $A$. The *height* of a prime ideal is the supremum of the lengths of chains of prime ideals contained in $\mathfrak{p}$. By the correspondence between primes in $A_{\mathfrak{p}}$ and primes contained in $\mathfrak{p}$, we have $\mathrm{ht}\mathfrak{p} = \dim A_{\mathfrak{p}}$.

**Example A.27.** The Krull dimension of a field is zero. The Krull dimension of $\mathbb{Z}$ equals 1. △

**Example A.28.** If $\mathfrak{p}$ is a prime ideal in a finitely generated $k$-algebra $A$, then

$$\dim(A) = \dim(A/\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}).$$

△

**Theorem A.29 (Krull's Principal Ideal Theorem).** Let $A$ be a Noetherian ring and let $f \in A$ be a nonunit. Let $\mathfrak{p}$ be a minimal prime ideal containing $(f)$. Then either (i) $\mathfrak{p}$ is a minimal prime ideal of $A$; or (ii) the height of $\mathfrak{p}$ is 1, i.e., the maximal chains in $\mathfrak{p}$ have the form $\mathfrak{p}_0 \subset \mathfrak{p}$. If $f$ is a nonzerodivisor, then

**Theorem A.30 (Krull's Principal Ideal Theorem).** Let $A$ be a Noetherian ring and $I = (f_1, \ldots, f_r)$ a proper ideal of $A$. Then each prime ideal that is minimal among those that contain $I$ has height at most $r$.

**Corollary A.31.** Let $A$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and let $f \in \mathfrak{m}$. Then

$$\dim(A/(f)) \geqslant \dim A - 1$$

with equality if $f$ is not contained in any minimal prime ideal of $A$.

**Proposition A.32.** If $A$ is a Noetherian ring, then the Krull dimension of $A[x]$ equals $\dim A + 1$.

In particular, $\dim k[x_1, \ldots, x_n] = n$ and $\dim \mathbb{Z}[x_1, \ldots, x_n] = n + 1$.

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*Proof* Here is a proof in the case $A = k$ is a field. The proof goes by induction on $n$. The case $n = 1$ is clear.

Consider an irreducible nonconstant polynomial $f$ in $A = k[t_1, \ldots, t_n]$. As in Lemma A.36, let $u_i = t_i - t_1^s$ Then

$$f(t_1, \ldots, t_n) = f(t_1, u_2 + t_1^s, \ldots, u_n + t_1^s)$$

is a monic polynomial in $t_1$ with coefficients in $B = k[u_2, \ldots, u_n] \subset A$, and the $u_i$'s are algebraically independent. By induction $\dim B = n - 1$.

Consider now the algebra $A/(f)A$. The algebra $B$ maps injectively into $A/(f)A$: a polynomial in the kernel depends only on the $u_i$'s, but it also is a multiple of $f$ (which depends on $t_1$), hence it must vanish. The extension

$$B \subset A/(f)A$$

is integral because $A/(f)A$ is generated over $B$ by the class of $t_1$, which is integral since $f$ is monic. The Going-Up Theorem then shows that $\dim A/(f)A = n - 1$.

□

**Example A.33.** If $K/k$ and $L/k$ are field extensions, then

$$\dim(K \otimes_k L) = \min(\mathrm{trdeg}_k(K), \mathrm{trdeg}_k(L)).$$

△

## A.14 Noether's Normalization Lemma

We now turn to one of the key results in the theory of varieties, *Noether's Normalization Lemma*, due to Emmy Noether.

**Theorem A.34 (Noether's Normalization Lemma).** Let $k$ be a field and let $B$ be an integral domain which is finitely generated as a $k$-algebra and let $n$ be the transcendence degree of the fraction field $K(B)$ over $k$. Then there are algebraically independent elements $x_1, \ldots, x_n \in B$ such that $B$ is finite as a module over $k[x_1, \ldots, x_n]$.

Noether's Normalization Lemma is a powerful tool, which allows us to reduce many questions about $k$-algebras to statements about the polynomial ring $k[x_1, \ldots, x_n]$. It has many fundamental results as consequences; in particular, we will obtain Hilbert's Nullstellensatz as a corollary.

With only minor modifications, the standard proof of the classical version of the Normalization Lemma yields a more general result, which will be useful for our purposes.

**Theorem A.35.** Let $A \subset B$ be two integral domains with $B$ of finite type over $A$, and let $n$ be the transcendence degree of the quotient field $K(B)$ over $K(A)$. Then there are elements $x_1, \ldots, x_n$ in $B$ which are algebraically independent over $A$ and an element $f \in A$ such that $B_f$ is a finite module over $A_f[x_1, \ldots, x_n]$.

When $A$ is a field, the localization has no effect, and Theorem A.34 follows directly from Theorem A.35.

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We prove the theorem by induction on the minimal number of generators of $A$. The key to the inductive step is the following lemma:

**Lemma A.36.** Let $A$ be a ring and let $P \in A[t_1, \ldots, t_n]$. Then if $s$ is a sufficiently large integer, we have

$$P(t_1 + t_n^s, \ldots, t_{n-1} + t_n^{s^{n-1}}, t_n) = g \cdot t_n^d + (\text{terms of order } < d \text{ in } t_n)$$

where $g \in A$ is nonzero.

*Proof* If we replace $t_i$ with $t_i + t_n^{s^i}$ for $i = 1, \ldots, n-1$ in a monomial $t_1^{e_1} \ldots t_n^{e_n}$, we get a polynomial in $t_n$ of the form $t_n^N +$(lower degree terms), where $N = e_n + e_1 s + e_2 s^2 + \cdots + e_{n-1} s^{n-1}$. The key observation is that for $s$ sufficiently large, the expressions $e_n + e_1 s + e_2 s^2 + \cdots + e_{n-1} s^{n-1}$ will all be different. Indeed, for two $n$-tuples $(e_1, \ldots, e_n)$, $(e_1', \ldots, e_n')$, an equality

$$e_1 + e_2 s + \cdots + e_m s^{m-1} = e_1' + e_2' s + \cdots + e_m' s^{m-1}$$

holds only for finitely many integers $s$ because non-zero polynomials have only finitely many zeros. Since $f$ has only finitely monomials, we can simply choose $s$ so that no such equalities take place.

This means that if we expand $P(t_1 + t_n^s, t_2 + t_n^{s^2}, \ldots, t_{n-1} + t_n^{s^{n-1}})$ into powers of $t_n$, there cannot be any cancellation between the leading terms, and so the leading term is of the desired form $g t_n^N$, where $g$ is a non-zero element of $A$. □

*Proof of Theorem A.35* Choose generators $w_1, \ldots, w_m$ for $B$ as an algebra over $A$. Then $K(B)$ is generated as a field over $K(A)$ by the $w_i$'s as well. It follows that $m \geqslant n = \text{tr.deg}_{K(A)}(K(B))$. In the case $m = n$, the $w_1, \ldots, w_m$ are algebraically independent over $A$, and $B = A[w_1, \ldots, w_m]$ is a polynomial ring, so the theorem holds in this case.

If $m > n$, the elements are not algebraically independent, so there is a non-zero polynomial $P(t_1, \ldots, t_m) \in A[t_1, \ldots, t_m]$ such that $P(w_1, \ldots, w_m) = 0$. Pick an $s \in \mathbb{N}$ as in Lemma A.36. Then the polynomial

$$Q(t_1, t_2, \ldots, t_m) = P(t_1 + t_m^s, t_2 + t_m^{s^2}, \ldots, t_{m-1} + t_n^{s^{m-1}}, t_m)$$

has a leading coefficient $g \in A$ as a polynomial in $t_m$. If we define $z_i = w_i - w_m^{s^{i-1}}$ for $i = 1, \ldots, m-1$, then $Q(z_1, z_2, \ldots, z_{m-1}, w_m) = 0$. The polynomial $g^{-1}Q$ has a leading term equal to 1 in $t_m$, so the localization $B_g$ is an integral extension of the subalgebra $B' = A_g[z_1, \ldots, z_{m-1}]$.

Now, $K(B) = K(B_g)$ is an algebraic field extension of $K(B')$, so the two fields have the same transcendence degree over $K(A) = K(A_g)$. Moreover, by construction, $B'$ is generated by fewer than $m$ elements over $A_g$. By induction, there is an $h \in A_g$ and algebraically independent elements $x_1, \ldots, x_n \in B'$ such that $B'_h$ is finite over $A_{gh}[x_1, \ldots, x_n]$. Now, $g$ is invertible in $B'$, so $B'_h = B_{gh}$, and taking $f = gh$, we are done. □

**Example A.37.** The algebra $k[x, x^{-1}]$ is not finite over $k[x]$, as it requires all the negative powers $x^{-n}$ as generators. However, for any elements $a, b$ of $k$ with $ab \neq 0$, the ring $k[x, x^{-1}]$ is finite over $k[ax + bx^{-1}]$. Indeed, $k[x, x^{-1}]$ is generated by $x$ over $k[ax + bx^{-1}]$,

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and $x$ satisfies the monic equation

$$T^2 - a^{-1}(ax + bx^{-1})T + ba^{-1} = 0.$$

△

**Corollary A.38.** Let $A$ be a finitely generated algebra over a field $k$ and let $\mathfrak{m}$ be a maximal ideal in $A$. Then $A/\mathfrak{m}$ is a finite field extension of $k$.

*Proof* The field $K = A/\mathfrak{m}$ is finitely generated as a $k$-algebra because $A$ is. By Noether's Normalization Lemma, $K$ is an integral extension over some polynomial ring $k[y_1, \ldots, y_r]$. If $r \geqslant 1$, then $K$ has dimension $\geqslant 1$, which is not possible. Therefore, $r = 0$, and $K$ is a finite field extension of $k$. □

## A.15 Prime ideals in $k$-algebras

**Lemma A.39.** Let $A$ be an integral domain which is finitely generated as a $k$-algebra. If $\mathfrak{p} \subset A$ is a prime ideal of height one, then $A/\mathfrak{p}$ has transcendence degree $n - 1$ over $k$.

*Proof* Assume first that $A = k[x_1, \ldots, x_n]$. Then $\mathfrak{p}$ is a principal ideal, say $\mathfrak{p} = (f)$, for some polynomial $f \in k[x_1, \ldots, x_n]$. After a change of coordinates, we may assume that $f$ is of the form

$$f = a_d(x_1, \ldots, x_{n-1})x_n^d + \cdots + a_0(x_1, \ldots, x_{n-1})$$

Consider the polynomial ring $B = k[x_1, \ldots, x_{n-1}] \subset A$. We have $B \cap (f) = (0)$. Therefore, the ring map $B \to A/\mathfrak{p}$ is injective, and the classes of $x_1, \ldots, x_{n-1}$ are algebraically independent in $A/\mathfrak{p}$. On the other hand, by the relation $f$, the element $x_1$ is algebraic over the fraction field $K(B) = k(x_1, \ldots, x_{n-1})$. Hence $\text{tr.deg}_k(A/\mathfrak{p}) = n - 1$.

For the general case, let $B = k[x_1, \ldots, x_n] \subset A$ be a Noether normalization of $A$. Applying the Going-Down theorem, we see that the prime $\mathfrak{q} = \mathfrak{p} \cap B$ has height 1 in $B$. By the previous paragraph, $B/\mathfrak{q}$ has transcendence degree $n - 1$, and so □

## A.16 Graded rings

In this book, a *graded ring* will refer to a ring $R$ which is graded by the non-negative integers, i.e. a ring which admits a decomposition

$$R = \bigoplus_{n \geqslant 0} R_n = R_0 \oplus R_1 \oplus \cdots$$

as an abelian group, such that $R_m \cdot R_n \subset R_{m+n}$ for each $m, n \geqslant 0$. Occasionally, we will also discuss *$\mathbb{Z}$-graded rings*, where we allow negative degrees as well.

A ring map $\phi\colon R \to S$ between two graded rings $R$ and $S$ is said to be a *map of graded rings* if it respects the grading, that is, if $\phi(R_n) \subset S_n$ for all $n$.

Note that $R_0$ is a subring of $R$ and that $R$ is an algebra over $R_0$. Moreover, each $R_n$ is an $R_0$-module. The elements in $R_n$ are said to be *homogeneous of degree* $n$, and one writes $\deg x = n$ when $x \in R_n$. (By convention, 0 is considered homogeneous of any degree.)

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Every non-zero element $x \in R$ can be expressed uniquely as a finite sum $x = \sum_n x_n$ with $x_n \in R_n$, and the non-zero terms in the sum are called the *homogeneous components* of $x$.

**Example A.40.** The simplest examples of graded rings are the polynomial rings $R = A[t_0, \dots, t_r]$, with the *standard grading* where each variable $t_i$ has degree 1 and the elements from $A$ have degree 0. Each graded piece $R_n$ is a free module over $R_0 = A$ with the monomials of degree $n$ serving as a basis. △

An $R$-module $M$ is *graded* if it has a similar decomposition $M = \bigoplus_{n\in\mathbb{Z}} M_n$ as an abelian group and $R_m M_n \subset M_{m+n}$ for all $n$ and $m$. Note that we allow also elements of negative degrees. A *map of graded $R$-modules* is an $R$-linear map $\phi\colon M \to N$ satisfying $\phi(M_n) \subset N_n$ for all $n \in \mathbb{Z}$. With this notion of morphisms, the graded $R$-modules form a category, denoted $\mathsf{GrMod}_R$.

As for graded rings, a non-zero element $x \in M$ is *homogeneous* of degree $n$ if it lies in $M_n$. Any element $x \in M$ may be expressed in a unique way as a finite sum $x = \sum_n x_n$ with each $x_n$ in $M_n$, and the non-zero terms are called the *homogeneous components* of $x$.

Most of the familiar definitions for modules carry over to the graded setting. For instance, the direct sum of a collection of graded modules $\bigoplus_i M_i$ is graded in a natural way such that canoncal inclusions $M_j \hookrightarrow \bigoplus_i M_i$ preserve the grading. Likewise, the kernel and the cokernel of a map of graded modules are also graded in a natural way.

A sequence of graded modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

is exact if it is exact as sequence of ordinary modules. As maps of graded modules preserve the grading, this is equivalent to saying that each of the sequences

$$0 \longrightarrow M'_n \longrightarrow M_n \longrightarrow M''_n \longrightarrow 0.$$

are exact (as a sequence of $R_0$-modules).

An ideal $\mathfrak{a} \subset R$ is called *homogeneous* if the homogeneous components of each element in $\mathfrak{a}$ belongs to $\mathfrak{a}$. In other words, we may write $\mathfrak{a} = \bigoplus_n \mathfrak{a}_n$ with $\mathfrak{a}_n = \mathfrak{a} \cap R_n$. An ideal $\mathfrak{a}$ is homogeneous if and only if it is generated by homogeneous elements (see Exercise 5.4.4). It is readily verified that radicals, intersections, sums and products of homogeneous ideals are homogeneous. If $\mathfrak{a}$ is an homogeneous ideal, the quotient $R/\mathfrak{a}$ inherits a grading from $R$ and $R/\mathfrak{a} = \bigoplus_n R_n/\mathfrak{a}_n$.

A homogenous ideal $\mathfrak{p}$ is prime if and only if for each pair of homogeneous elements $f, g \in R$, $fg \in \mathfrak{p}$ implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$.

We will write $R_+$ for the direct sum $\bigoplus_{n>0} R_n$. This is naturally a homogeneous ideal of $R$, which we call the *irrelevant ideal*.

**Example A.41.** The irrelevant ideal of a polynomial ring $R = A[t_0, \dots, t_r]$ is equal to $R_+ = (t_0, \dots, t_r)$. △

**Example A.42** (Veronese subrings)**.** For a graded ring $R$ and $d \in \mathbb{N}$, we define $R^{(d)}$ to be the subring of $R$ defined by $\bigoplus_{n\geqslant 0} R_{nd}$. As every element of $R^{(d)}$ has degree equal to a multiple of $d$, it is often useful to redefine the grading on $R^{(d)}$ by dividing all degrees by $d$.

For instance, when $R = k[x_0, x_1]$, the Veronese subring $R^{(2)}$ is given by $k[x_0^2, x_0x_1, x_1^2]$.

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We may adjust the grading by defining $\deg x_0 = \deg x_1 = \frac{1}{2}$, the elements of the ring $R^{(2)}$ will have integer degrees. A different viewpoint is to note that $R^{(2)}$ is isomorphic to the graded ring $k[u_0, u_1, u_2]/(u_1^2 - u_0u_2)$, where in the latter ring, the variables have degree 1. △

If $S \subset R$ is a multiplicative system consisting of homogeneous elements, and $M$ is a graded module, the localization $S^{-1}M$ is naturally a graded $R$-module with degree $n$ part equal to

$$(S^{-1}M)_n = \big\{m/s \in S^{-1}M \mid m \in M \text{ homogeneous}, s \in S \text{ and } \deg m - \deg s = n\big\}.$$

In particular, if $f$ is a homogeneous element of positive degree, the localization $R_f$ is a $\mathbb{Z}$-graded ring. As we will see, the degree 0 part $(R_f)_0$ will play a crucial role in the Proj-construction.

**Example A.43.** In the polynomial ring $R = A[t_0, \ldots, t_n]$, with the standard grading, the elements of degree zero in the localization $R_{t_j}$ are polynomials in the ratios $t_0/t_j, \ldots, t_n/t_j$, so the piece of degree zero $(R_{t_j})_0$ is the polynomial ring

$$(R_{t_j})_0 = A\left[\frac{t_0}{t_j}, \ldots, \frac{t_n}{t_j}\right].$$

△

## A.17 Regular local rings

If $A$ is a local Noetherian ring with maximal ideal $\mathfrak{m}$, the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ is naturally a vector space over the residue field $A/\mathfrak{m}$. This is finite dimensional, because $\mathfrak{m}$ is finitely generated. If $x_1, \ldots, x_r$ are elements from $A$ that generate $\mathfrak{m}/\mathfrak{m}^2$ as a $A/\mathfrak{m}$-vector space, then Nakayama's lemma implies that the $x_1, \ldots, x_r$ generate $\mathfrak{m}$. But then Krull's Principal ideal theorem implies $\dim A = \operatorname{ht} \mathfrak{m} \leqslant r = \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$. This gives:

**Proposition A.44.** Let $A$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then

$$\dim A \leqslant \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 < \infty.$$

We say that the ring $A$ is a *regular* if equality holds, that is, if $\mathfrak{m}$ can be generated by $n$ elements, where $n = \dim A$ is the Krull dimension. A general Noetherian ring $A$ is said to be regular if every localization $A_{\mathfrak{p}}$ is a regular local ring.

## A.18 Discrete valuation rings

Let $K$ be a field. A *discrete valuation* is a surjective function $v\colon K - 0 \to \mathbb{Z} \cup \{\infty\}$ satisfying, for each $f, g \in K$:

(i) $\nu(fg) = \nu(f) + \nu(g)$.
(ii) $\nu(f + g) \geqslant \min\{\nu(f), \nu(g)\}$, with equality in the latter when $\nu(f) \neq \nu(g)$.
(iii) $\nu(f) = 0$ if and only if $f = 0$.

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An integral domain $A$ with fraction field $K$ is called a *discrete valuation ring* if there exists a valuation $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$ such that

$$A = \{\, x \in K \mid \nu(x) \geqslant 0 \,\}$$

Given a valuation $\nu$, the set $\{\, x \in K \mid \nu(x) \geqslant 0 \,\}$ is a subring of $K$. Therefore the discrete valuation ring determines and is determined by $\nu$.

**Example A.45.** Let $K = k(x)$ be the field of rational functions in one variable. Then any element $f \in K$ can be written as $f = x^n g(x)/h(x)$ where $n \in \mathbb{Z}$; and $g(x), h(x)$ are polynomials such that $f(0), g(0) \neq 0$. The 'order of vanishing at 0' gives a valuation $v_a\colon K^\times \to \mathbb{Z}$ by setting $v(f) = n$. In this case, the elements of non-negative valuation are exactly the ones of the form $g(x)/h(x)$, where $h(0) \neq 0$. Therefore, the valuation ring is the localization of $k[x]$ at $(x)$:

$$A = k[x]_{(x)}.$$

△

**Example A.46.** Let $K = k(x)$ be the field of rational functions in one variable. Define the valuation $v_\infty\colon K^\times \to \mathbb{Z}$ by setting

$$v_\infty\left(\frac{f}{g}\right) = \deg g - \deg f$$

One can check that this defines a valuation on $k(x)$. The valuation $v_\infty$ is supposed to measure the order of a pole 'at infinity'. The corresponding valuation ring is

$$A = \{f/g \in k(x) | \deg f \leqslant \deg g\}.$$

with maximal ideal $\mathfrak{m} = \{f/g \in k(x) | \deg f < \deg g\}$. △

**Example A.47.** Let $K = \mathbb{Q}$ be the field of rational numbers, and let $p$ be a prime number. Any $y \in \mathbb{Q}$ can be expressed as $y = p^n a/b$ where $n \in \mathbb{Z}$ and $a, b$ are coprime to $p$. We can define the *p-adic* valuation $v_p : \mathbb{Q}^\times \to \mathbb{Z}$ by setting $v(y) = n$. In this case, the valuation ring is the localization of $\mathbb{Z}$ at $(p)$:

$$A = \mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \in \mathbb{Q} \,|\, \gcd(p, n) = 1 \right\}$$

△

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**Proposition A.48.** Let $A$ be a discrete valuation ring with fraction field $K$. Then

(i) $A$ is a local ring with maximal ideal

$$\mathfrak{m} = \{\, f \in K \mid \nu(f) \geqslant 1 \,\}$$

and group of units

$$A^\times = \{\, f \in K \mid \nu(f) = 0 \,\}.$$

(ii) If $t \in A$ is an element with $\nu(t) = 1$, then $\mathfrak{m} = (t)$.

(iii) If $t$ is any element with $\nu(t) = 1$, then every non-zero ideal of $A$ is of the form $(t^n)$ for some $n \geqslant 1$.

(iv) $A$ is Noetherian and of Krull dimension 1.

(v) $A$ is normal.

*Proof* (i): Let $f \in A$ be a nonzero elemement. Then $f$ is a unit if and only if the inverse $1/f \in K$ belongs to $A$, which is if and only if $\nu(1/f) = -\nu(f) \geqslant 0$. But as $f \in A$, $\nu(f) \geqslant 0$, and so $\nu(f) = 0$. Therefore $A^\times = \{\, f \in K \mid \nu(f) = 0 \,\}$ and the non-units is given by $\mathfrak{m} = \{\, f \in K \mid \nu(f) \geqslant 1 \,\}$.

(ii) In general, if $f, g \in A$ and $v(g) \geqslant v(f)$, then $v(f/g) \geqslant 0$, so $f/g \in A$, which means that $g \in (f)$. In particular, $\mathfrak{m} = (t)$ for any element $t \in A$ with $v(t) = 1$. (The discrete valuation $\nu$ is assumed to be surjective, so there exists at least one such element $t$.)

(iii) Let $\mathfrak{a}$ be an ideal, and let $f \in \mathfrak{a}$ be such that $v(f)$ is minimal. Then for any $g \in \mathfrak{a}$, we have $v(g) \geqslant v(f)$, so as above we find $g \in (f)$. Consequently, $\mathfrak{a} \subset (f)$ and hence $\mathfrak{a} = (f)$.

(iv) By the previous two points, we have that the ideals of $A$ are $(1), (x), (x^2), \ldots$, and $(0)$. It is easy to see that $(x)$ and $(0)$ are the only prime ideals of $A$, so $A$ is Noetherian of Krull dimension 1.

(v) Let $x \in K$, and assume that $x$ is integral over $A$. We must show that $x \in A$. Since $x$ is integral over $A$, we can find a relation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0, \quad a_i \in A$$

so

$$x^n = -a_{n-1}x^{n-1} - \cdots - a_0.$$

If $v(x) = d$, we then get

$$v(x^n) = nv(x) = nd = v(-a_{n-1}x^{n-1} - \cdots - a_0) \geqslant \min_i(v(-a_i x^i)).$$

Hence there exists an $i \leqslant n-1$ such that

$$nd = v(x^d) \geqslant v(-a_i x^i) = v(-a_i) + v(x^i) \geqslant id.$$

This gives $(n-i)d \geqslant 0$, and so $d \geqslant 0$. Hence, $x \in A$.

□

An element $t \in A$ with $\nu(t) = 1$ is called a *uniformizing parameter* for $A$. By the above proof, any element $t \in \mathfrak{m} - \mathfrak{m}^2$ is a uniformizing parameter. In fact, we can recover the valuation from the maximal ideal $\mathfrak{m}$: any non-zero element in the fraction field $K$ may be written as $ut^n$ with $u$ a unit in $A$ and $n$ an integer. Indeed, if $f \in A$ and $f \neq 0$, we let $\nu(f)$

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be the unique non-negative integer $n$ such that $f \in \mathfrak{m}^n - \mathfrak{m}^{n+1}$, then $f = ut^{v(f)}$ for a unit $u$. For a general non-zero element $f/g$ in $K$, one has $f/g = ut^{\nu(f)-\nu(g)}$ with $u$ a unit.

**Proposition A.49.** Let $A$ be a Noetherian local domain of dimension 1 with maximal ideal $\mathfrak{m}$. Then the following are equivalent:

(i) $A$ is a discrete valuation ring.
(ii) The maximal ideal $\mathfrak{m}$ is principal.
(iii) Every non-zero ideal is a power of $\mathfrak{m}$.
(iv) $A$ is integrally closed.
(v) $A$ is regular, i.e., $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.

*Proof* (i) $\Rightarrow$ (ii), (iii) and (iv) by Proposition A.48.

(ii) $\Rightarrow$ (i). We will need the fact that

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0) \tag{A.10}$$

Let $t$ be a generator for $\mathfrak{m}$. Suppose $a \in \bigcap_{n=1}^{\infty}(t^n)$. This means that for every $n \in \mathbb{N}$, there is an $a_n \in A$ such that $a = a_n t^n$. As $A$ is an integral domain, we have $a_{n+1} = a_n t$ for every $n \geqslant 1$ and hence there is a chain of ideals $(a_1) \supset (a_2) \supset (a_3) \supset \cdots$. As $A$ is Noetherian, this chain stabilizes, meaning that there is an $N$ such that $(a_N) = (a_{N+1}) = \cdots$, or in other words, $a_{n+1} = u_n a_n$ for $n \geqslant N$ and $u_n$ is a unit. Then $a = a_{n+1}t^{n+1} = a_n t^n$, implies $(u_n t - 1)a_n t^n = 0$ for $n \geqslant N$. But $u_n t - 1$ is clearly a unit, as $t \in \mathfrak{m}$. Therefore $a_n = 0$ for $n \geqslant N$, and hence $a = 0$.

From this it follows that any $f \in A$ can be written as $f = ut^n$, where $n \geqslant 0$ and $u \in A^\times$ is a unit. Explicitly, $n$ is the unique integer $n$ such that $f \in \mathfrak{m}^n - \mathfrak{m}^{n+1}$. (Note that $\mathfrak{m}^n - \mathfrak{m}^{n+1}$ nonempty for every $n$ by (A.10).) If we extend this via $\nu(f/g) = \nu(f) - \nu(g)$ for $f/g \in K^\times$ and $\nu(0) = \infty$, then $A$ is exactly the discrete valuation associated to $\nu$.

(ii) $\Rightarrow$ (iii). Let $t$ be a generator for the maximal ideal $\mathfrak{m}$ and let $\mathfrak{a} \subset A$ be a non-zero ideal. Let $n$ be the largest integer such that $\mathfrak{a} \subset \mathfrak{m}^n$. (Such an $n$ exists beccause $\mathfrak{a}$ cannot be contained in all powers of $\mathfrak{m}$ by (A.10).) Since $\mathfrak{a} \nsubseteq \mathfrak{m}^{n+1}$, there is an $a \in \mathfrak{a}$ such that $a = bt^n$ with $b \notin \mathfrak{m}$; that is, $b$ is a unit since the ring is local. It follows that $(t^n) \subset \mathfrak{a}$, and hence $\mathfrak{a} = \mathfrak{m}^n$.

(iv) $\Rightarrow$ (ii). Assume that $A$ is integrally closed and let $x \in \mathfrak{m}$ be any non-zero element. Since $A$ is Noetherian and of dimension 1, $\mathfrak{m}$ is the only non-zero prime ideal in $A$, and so $\mathfrak{m}$ is associated to $(x)$.

There is an integer $n$ such that $\mathfrak{m}^n \subset (x)$. If we choose $n$ to be minimal among these, so that $\mathfrak{m}^{n-1} \nsubseteq (x)$, and $\mathfrak{m}^n - (x)$ is non-empty. Pick $y \in \mathfrak{m}^{n-1} - (x)$. Set $z = x/y \in K$. Then we claim that $z \in A$ and $\mathfrak{m} = (t)$.

$$z^{-1}\mathfrak{m} \subset x^{-1}\mathfrak{m} \subset x^{-1}\mathfrak{m}^n \subset x^{-1}(x) \subset A$$

Therefore $z^{-1}A$ is an ideal of $A$. There are two cases:

Case 1: $z^{-1} \in \mathfrak{m}$. Then consider the endomorphism $\phi\colon A \to A$ given by $\phi(a) = z^{-1}a$, takes $A$ into $\mathfrak{m}$. By the Cayley–Hamilton theorem, this would mean that $z^{-1}$ is integral over

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$A$. By assumption, $A$ is normal, so $z^{-1} \in A$. Hence $z^{-1} = yx^{-1} \in A$ and hence $y \in A$. But this contradicts the definition of $y$.

Case 2: $z^{-1} \notin \mathfrak{m}$. Then $z^{-1}A = A$ is the unit ideal. But then $\mathfrak{m} = (z)$, and $\mathfrak{m}$ is principal.

(ii) $\Leftrightarrow$ (v). If $\mathfrak{m} = (x)$, then $\mathfrak{m}/\mathfrak{m}^2$ is generated by the class of $x$ modulo $\mathfrak{m}^2$. We also have $\mathfrak{m} \neq \mathfrak{m}^2$ (since $A$ has dimension 1), so $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$. The converse implication follows by Nakayama's lemma. □

## A.19 Normal domains

**Theorem A.50 (Serre).** Let $A$ be Noetherian local ring. Then $A$ is normal if and only if
(i) $A_\mathfrak{p}$ is regular for every prime ideal of height at most 1.
(ii) $A$ is reduced, and for every nonzerodivisor $f \in A$, the associated primes of $A/(f)$ are of height 1 in $A$.

**Theorem A.51 (Krull).** Let $A$ be an integral domain, then $A$ is normal if and only if
(i) $A_\mathfrak{p}$ is a discrete valuation ring for every non-zero minimal prime ideal $\mathfrak{p}$
(ii) $A = \bigcap_\mathfrak{p} A_\mathfrak{p}$, where the intersection is taken over all minimal prime ideals of $A$.

Let us recall that an ideal quotient $(b : a) = \{\, x \mid xa \in (b) \,\}$ equals the annihilator of the class of $a$ in $A/(b)A$. A basic result from the theory of primary decomposition in Noetherian rings asserts that each proper annihilator is contained in a maximal annihilator. In this case, these are precisely the prime ideals associated with $(b)$.

**Lemma A.52.** A Noetherian domain $A$ equals the intersection $\bigcap_\mathfrak{p} A_\mathfrak{p}$, where $\mathfrak{p}$ runs through the prime ideals associated with principal ideals.

*Proof* Assume for contradiction that there is an element $ab^{-1}$ in the fraction field of $A$ that lies in $\bigcap_\mathfrak{p} A_\mathfrak{p}$ but not in $A$. The ideal quotient $(b : a) = \{\, x \in A \mid xa \in (b) \,\}$ is a proper ideal because $ab^{-1} \notin A$, and so $(b : a)$ is contained in a maximal ideal, that is, a prime $\mathfrak{p}$ associated with $(b)$. Since $ab^{-1} \in A_\mathfrak{p}$, we can write $ab^{-1} = cd^{-1}$ with $c, d \in A$ and $d \notin \mathfrak{p}$. Hence $ad = bc$, implying that $d \in (b : a) \subset \mathfrak{p}$, which is a contradiction. □

**Theorem A.53.** Let $A$ be a Noetherian domain. Then $A$ is normal if and only if the following conditions are met:
(i) The local ring $A_\mathfrak{p}$ at each height 1 prime ideal $\mathfrak{p}$ is a DVR.
(ii) Each principal ideal has no embedded components, that is, all associated primes have height 1.

*Proof* Assume first that the two conditions are fulfilled. By Lemma A.52, $A$ is the intersection of the local rings $A_\mathfrak{p}$ where $\mathfrak{p}$ runs over the prime ideals associated to a principal ideal. Each such $\mathfrak{p}$ has height 1 by Krull's Principal Ideal Theorem, and by assumption (i) the

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localization $A_{\mathfrak{p}}$ is a DVR, hence normal. Intersections of normal rings are normal, so $A$ is normal.

Conversely, assume that $A$ is normal. Then the localizations $A_{\mathfrak{p}}$ at height 1 primes are also normal, and being 1-dimensional, they are discrete valuation rings XXX. For item (ii): let $\mathfrak{p}$ be a prime in $A$ associated with a principal ideal $(b)$. In the local ring $A_{\mathfrak{p}}$, its maximal ideal $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}}$ remains associated with the principal ideal $(b)A_{\mathfrak{p}}$. By Proposition A.49, $\mathfrak{m}$ is principal. Hence, $A_{\mathfrak{p}}$ is a DVR, so $\mathfrak{p}$ is of height 1 and cannot be an embedded prime. □

When the first condition of the criterion is met, the $\mathfrak{p}$-primary ideals of height 1 are well understood. They are all symbolic powers $\mathfrak{p}^{(\nu)} = A \cap \mathfrak{p}^{\nu}A_{\mathfrak{p}}$. Indeed, if $\mathfrak{q}$ is $\mathfrak{p}$-primary, then $\mathfrak{q} = A \cap \mathfrak{q}A_{\mathfrak{p}}$, and since $A_{\mathfrak{p}}$ is a DVR, every ideal in $A_{\mathfrak{p}}$, including $\mathfrak{q}A_{\mathfrak{p}}$ is a power of the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

If $A$ is normal and $f \in A$ is a nonzero element, then the primary decomposition of $(f)$ has the form

$$(f) = \mathfrak{p}_1^{(\nu_1)} \cap \cdots \cap \mathfrak{p}_r^{(\nu_r)}.$$

where the $\mathfrak{p}_i$ are height 1 primes. The primes $\mathfrak{p}_i$ and the exponents $\nu_i$ completely determine $(f)$, that is, they determine $f$ up to an invertible factor.

**Theorem A.54 (Algebraic Hartogs' theorem).** Let $A$ be a Noetherian normal integral domain. Then

$$A = \bigcap_{\mathrm{ht}\mathfrak{p}=1} A_{\mathfrak{p}}. \tag{A.11}$$

*Proof* If $g \in A$ is a nonzero element, then each prime ideal $\mathfrak{p}$ associated to $(g)$ is of height 1. This is because $\mathfrak{p}A_{\mathfrak{p}}$ is associated to $(g)A_{\mathfrak{p}}$, and $A_{\mathfrak{p}}$ is a discrete valuation ring, hence $\mathrm{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}} = 1$.

Now assume for a contradiction that there is an element $f/g \in K$ which lies in every $A_{\mathfrak{p}}$ for $\mathfrak{p}$ of height 1, but not in $A$. In other words, $f \in (g)A_{\mathfrak{p}}$ for every height 1 prime $\mathfrak{p}$.

Let $(g) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be a primary decomposition of $(g)$, and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ be the assocated prime ideals. By the above paragraph, the $\mathfrak{p}_i$ have height 1, we get $f \in \mathfrak{p}_i$ for every $i$. By Proposition A.26, we get

$$f \in \bigcap_{i=1}^{r} ((g)A_{\mathfrak{p}_i} \cap A) = \bigcap_{i=1}^{r} \mathfrak{q}_i = (g)$$

In other words, there is an element $a \in A$ such that $f = a \cdot g$, or in other words $f/g = a \in A$. □

**Theorem A.55 (Generic freeness).** Let $A$ be a Noetherian integral domain and let $B$ be a finitely generated $A$-algebra.

*Proof* See (Matsumura, 1970, p.185) or (**?**, Tag 051S). □



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## A.20 Unique factorization domains

In a Noetherian ring $A$, any nonzero nonunit $x \in A$ admits a factorization

$$x = x_1 \cdots x_n \tag{A.12}$$

where each $x_i$ is irreducible (that is, a non-zero element that is not a unit and is not the product of two non-units). To see this, suppose for a contradiction that no such factorization exists. Set $a_1 = x$. We must be able to write $x$ as $x = yz$ where neither $y$ nor $z$ is irreducible, nor a unit. Without loss of generality, we may assume that $y$ cannot be written as a product of irreducibles. Set $a_2 = y$. Writing $y$ in the same way, and so on, we arrive at a sequence $a_1, a_2, a_3, \ldots$ of elements of $A$ such that $a_{i+1}$ divides $a_i$ for each $i \geqslant 1$ and $a_{i+1}/a_i$ is not a unit. This means that the chain $(a_1) \subset (a_2) \subset (a_3) \subset \ldots$ is an infinite strictly increasing sequence of ideals, contradicting the fact that $A$ is Noetherian.

We say that $A$ is a unique factorization domain (UFD) if the factorization in (A.12) is unique up to order and multiplication by units.

**Example A.56.** Examples of UFDs include $\mathbb{Z}$, $\mathbb{Z}[i]$, fields, and principal ideal domains. If $A$ is a UFD, then so is $A[x]$. △

**Proposition A.57 (Gauss' Lemma).** Let $A$ be a UFD with fraction field $K$. If $f \in A[x]$ is a polynomial which factors into a product of non-constant polynomials in $K[x]$, then it does so also in $A[x]$.

**Proposition A.58.** Let $A$ be an integral domain and let $f \in A$ be a nonzero element which is not a unit. Then if $(f)$ is prime, then $f$ is irredcucible. If $A$ is a UFD, then the converse also holds.

*Proof* If $(f)$ is prime and $f = gh$ is a factorization, then $g \in (f)$ or $h \in (f)$. If e.g., $g = a \cdot f$, then $f = (af)h$, implying that $h$ is a unit. Hence $f$ is irreducible.

Suppose that $f$ is irreducible and that $xy \in (f)$. Then we can write $xy = af$ for some $a \in A$. Write $x = x_1 \cdot x_s$, $y = y_1 \cdots y_s$ where the $x_i$ and $y_j$ are irreducible. As $A$ is a UFD and $f$ is irreducible, we see that $f$ has to be one of the $x_i$ or the $x_j$, that is, $f$ divides either $x$ or $y$. □

**Proposition A.59.** Let $A$ be a Noetherian integral domain. Then $A$ is a UFD if and only if every height 1 prime ideal is principal.

*Proof* Suppose first that $A$ is a UFD and let $\mathfrak{p}$ be a height 1 prime ideal. Take $x \in \mathfrak{p}$ non-zero and let $x = x_1 \cdots x_n$ be a factorization into irreducible elements. Since $\mathfrak{p}$ is prime, we must have, say, $x_1 \in \mathfrak{p}$. Then $\mathfrak{p} \supset (x_1) \supset (0)$. $A$ is UFD, $(x_1)$ is prime, and hence $\mathfrak{p} = (x_1)$ because $\mathfrak{p}$ has height 1.

Conversely, suppose that every height 1 prime is principal. By (A.12), any element can be written as a product of irreducibles, so we need to show that the factorization is unique up to order and multiplication. For this, it suffices to prove that an irreducible element is prime. But this follows by Proposition A.58. □

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## A.21 Modules over a PID

A very useful theorem for modules over a principal ideal domain $R$ is the fact that any matrix with entries in $R$ can be put into so-called *Smith normal form* (see below). This is a generalization of the diagonalization process from Linear Algebra; finding the normal form involves only basic elementary row and column operations, and multiplying by units.

**Theorem A.60 (Smith Normal form).** Let $R$ be a PID and let $A$ be an $m \times n$ matrix with entries from $R$. Then there exist invertible matrices $P$ and $Q$, with dimensions $m \times m$ and $n \times n$ respectively, such that

$$PAQ = \text{diag}(s_1, \ldots, s_v, 0, \ldots, 0)$$

is a diagonal matrix where the non-zero entries satisfy $s_i | s_{i+1}$ for $i = 1, \ldots, v-1$. The elements $s_i$ are unique up to multiplication by a unit of $R$ and are called the *invariant factors* of $A$.

If $A$ is a matrix $A$ and $s_i \in R$ are the elements in the Smith Normal Form of $A$, then we have

$$\text{Coker } A \simeq R/(s_1) \oplus \cdots \oplus R/(s_v) \oplus R^{m-v}.$$

Over a PID, any finitely generated module $M$ has a presentation $0 \to E \to F \to M \to 0$, where $E$ and $F$ are free modules of finite rank.

**Corollary A.61 (Modules over a PID).** Let $R$ be a principal ideal domain and let $M$ be a finitely generated $R$-module. Then there are elements $d_1, \ldots, d_m \in R$ with $d_i | d_{i+1}$ for each $i$, such that

$$M \simeq R^r \oplus R/(d_1) \oplus \cdots \oplus R/(d_m).$$

In particular, if $M$ is a free $R$-module of rank $m$ (so $M \simeq R^m$), then any submodule $N \subset M$ is free of rank at most $m$.

## A.22 Symmetric powers and Exterior powers

For an $A$-module $M$, the *tensor algebra* $T(M)$ is defined as

$$T(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n}$$

where $M^{\otimes n}$ is the $n$-fold tensor product $M \otimes_A \cdots \otimes_A M$. $T(M)$ has a multiplication given by the tensor product, but this is not commutative except in a few special cases. However, we can define $\text{Sym}(M)$ as the quotient of $T(M)$ by the two-sided ideal generated by all elements of type $x \otimes y - y \otimes x$ for $x, y \in M$. Then $\text{Sym}(M)$ is a graded, commutative, $A$-algebra. For $n \in \mathbb{N}$, we define $\text{Sym}^n(M)$ to be the $n$-graded piece of $\text{Sym}(M)$. As the multiplication is commutative, we write $x_1 x_2 \cdots x_n$ for the image of $x_1 \otimes \cdots \otimes x_n$ in $\text{Sym}(M)$. So, explicitly, $\text{Sym}^n(M)$ is generated as an $A$-module by all symbols $x_1 x_2 \cdots x_n$ where $x_1, \ldots, x_n \in M$, modulo the relations $x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ where $\sigma \in S_n$ is a permutation.

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**Example A.62.** If $M = A^r$, then $\mathrm{Sym}(M) = A[x_1, \ldots, x_r]$ and $\mathrm{Sym}^n(A^r)$ is the degree-$n$ piece. In particular, if $M$ is a free $A$-module of rank $r$, then $\mathrm{Sym}^n(M)$ is a free module of rank $\binom{n+r-1}{r-1}$, generated by elements of the form $x_1^{e_1} \cdots x_r^{e_r}$, where $e_1 + \cdots + e_r = n$. $\triangle$

Next, we define the *exterior algebra* $\bigwedge M$ as the quotient of $T(M)$ by the two-sided ideal generated by products $x \otimes x$, where $x \in M$. We let $x_1 \wedge \cdots \wedge x_n$ denote the image of $x_1 \otimes \cdots \otimes x_n$ in $\wedge(M)$. In this notation, we have $x \wedge y = (-1)^{kl} y \wedge x$ for $x \in \bigwedge^k M$, $y \in \bigwedge^l M$, and more generally $x_{\sigma(1)} \wedge x_{\sigma(2)} \cdots \wedge x_{\sigma(n)} = (\mathrm{sign}(\sigma)) x_1 \wedge x_2 \wedge \cdots \wedge x_n$ for a permutation $\sigma \in S_n$. In particular, the wedge product in $\bigwedge M$ is skew-commutative.

**Example A.63.** If $M$ is free of rank $r$, then $\bigwedge^n M$ is free of rank $\binom{n}{r}$, generated by the elements $x_{i_1} \wedge \cdots \wedge x_{i_n}$ where $i_1 < i_2 < \cdots < i_n$. $\triangle$

**Example A.64.** An important special case is when $M = A^r$ is a free module of rank $r$. Then the $A$-module $\bigwedge^r M$ is called the *determinant* of $M$. The name comes from the following: $x_1, \ldots, x_r$ and $y_1, \ldots, y_r$ are two bases of $M$, related by $(y_1, \ldots, y_r) = (x_1, \ldots, x_r) \cdot S$ for some matrix $S$, then $y_1 \wedge \cdots y_r = (\det S) x_1 \wedge \cdots \wedge x_r$. $\triangle$

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# Appendix B

# Further constructions and examples

## B.1 Toric varieties

In this section, we explore a large class of examples, known as *toric varieties*. Like projective spaces, toric varieties are defined in terms of algebras generated by monomials. The correspondence between monomials an lattice points in $\mathbb{R}^n$ sets up an interesting interplay between algebra and combinatorics, which allows us to do computations with toric varieties using combinatorial methods (and in some cases, solve combinatorial problems using tools from algebraic geometry). We will confine ourself to discussing affine and projective toric varieties. We refer to the standard texts Fulton (1993) and Cox et al. (2024) for more information about this rich subject.

**B.1.0 Affine toric varieties.** Let $k$ be a field and consider the ring of Laurent polynomials $k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We will use multiindex notation for monomials, that is, if $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, then $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$.

Given a subset $S \subset \mathbb{R}^n$, we define the *cone* of $S$ to be the non-negative span of $S$, that is,

$$\operatorname{Cone}(S) = \left\{ \lambda_1 \mathbf{v}_1 + \cdots + \lambda_s \mathbf{v}_s \;\mid\; \mathbf{v}_i \in S, \text{ and } \lambda_i \geqslant 0, \forall i = 1, \ldots, s \right\}.$$

For a cone $C$, we define the following subalgebra generated by the monomials corresponding to the lattice points in $C$:

$$k[C] = \bigoplus_{\mathbf{m} \in C \cap \mathbb{Z}^n} k\, \mathbf{x}^{\mathbf{m}}. \tag{B.1}$$

The *toric variety* associated to $C$ is the affine scheme $X = \operatorname{Spec} k[C]$.

We will usually assume that $C$ is *rational polyhedral*, meaning that it is generated by a finite set of integer points $S \subset \mathbb{Z}^n$. This implies that there are finitely many lattice points $\mathbf{m}_1, \ldots, \mathbf{m}_r \in C$ so that every point in $C \cap \mathbb{Z}^n$ can be expressed as a non-negative $\mathbb{Z}$-linear combination of the $\mathbf{m}_1, \ldots, \mathbf{m}_r$. On the level of rings, this means that the following map of $k$-algebras

$$\phi \colon k[y_1, \ldots, y_r] \longrightarrow k[C]; \qquad y_i \mapsto \mathbf{x}^{\mathbf{m}_i}. \tag{B.2}$$

is surjective, and hence induces a closed embedding

$$\iota \colon X \longrightarrow \mathbb{A}_k^r.$$

To find the ideal defining $X$ inside $\mathbb{A}_k^r$, we need to find the kernel of $\phi$. Write $\mathcal{A}$ for the $n \times r$ matrix with the vectors $\mathbf{m}_1, \ldots, \mathbf{m}_r$ as column vectors.



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**Proposition B.1.** The kernel of $\phi$ is given by the ideal

$$I_{\mathcal{A}} = (\mathbf{y}^u - \mathbf{y}^v \mid u - v \in \operatorname{Ker} \mathcal{A}) \tag{B.3}$$

*Proof* It is clear that $I_{\mathcal{A}} \subset \operatorname{Ker} \phi$. For the reverse inclusion, we use a monomoial order argument. More precisely, we will consider the lexicographic ordering $\prec$ on monomials in $k[y_1, \ldots, y_n]$, so that for instance

$$y_1^2 > y_1 y_2 > y_1 y_3 > y_2^2 > y_2 y_3 > y_3^2.$$

If $g \in \operatorname{Ker} \phi$, we can write it as

$$g = c_u \mathbf{y}^u + \sum_{v \prec u} c_v \mathbf{y}^v.$$

where $c_u \mathbf{y}^u \neq 0$ is the leading term with respect to $\prec$. Applying $\phi$, we get

$$0 = \phi(g) = c_u \mathbf{x}^{\mathcal{A}u} + \sum_{v \prec u} c_v \mathbf{x}^{\mathcal{A}v}.$$

This is an identity of polynomials in $k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, so there must be cancellations between the monomials. In particular, the term $c_u \mathbf{x}^{\mathcal{A}u}$ must cancel with some other terms in the sum. This means that there exists some $v$ with $v \prec u$ such that $\mathcal{A}u = \mathcal{A}v$, or in other words, $u - v \in \operatorname{Ker} \mathcal{A}$. Replacing $g$ with $g - c_u(\mathbf{x}^u - \mathbf{x}^v)$, we obtain a polynomial which has a leading term which is strictly smaller than that of $g$ with respect to $\prec$. Note that $\mathbf{x}^u - \mathbf{x}^v$ belongs to the ideal $I_{\mathcal{A}}$. Continuing in this manner, we eventually obtain the zero polynomial, which means that $g$ is an element of $I_{\mathcal{A}}$. □

Hence $I_{\mathcal{A}}$ is a prime ideal defined by binomials. To find a *finite* generating set, a few more computations are usually needed. A basic algorithm is to keep adding binomials $\mathbf{y}^u - \mathbf{y}^v$ until one obtains a prime ideal with the right codimension.

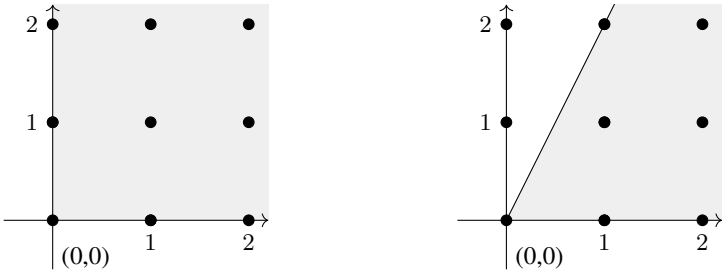


**Figure B.1** Two cones in $\mathbb{R}^2$. On the left, the cone generated by $(1, 0)$ and $(0, 1)$. On the right, the cone generated by $(1, 0)$ and $(1, 2)$.

**Example B.2.** If $C$ is the first quadrant, generated by $(1, 0)$, $(0, 1)$, then $k[C] = k[x, y]$. The corresponding toric variety is $\mathbb{A}^2_k$. △

**Example B.3.** If $C$ is the cone generated by $(1, 0)$, $(1, 2)$, then $k[C] = k[x, xy, xy^2]$. In

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this case,

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

which has kernel generated by the vector $(1, -2, 1)$. This gives $\mathcal{I}_{\mathcal{A}} = (v^2 - uw)$, and $k[x, xy, xy^2] \simeq k[u, v, w]/(v^2 - uw)$. Therefore, the corresponding toric variety is the quadratic surface $v^2 = uw$ in $\mathbb{A}^3_k$. △

**Example B.4.** Note that $C = \mathbb{R}^n$ itself is a cone. The corresponding toric variety is the *algebraic torus*

$$T = \mathbf{G}_m^n = \operatorname{Spec} k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

The reason behind the name is the following. When $k = \mathbb{C}$, the set of $\mathbb{C}$-points $T(\mathbb{C})$ is identified with $(\mathbb{C}^\times)^n$, which in the Euclidean topology is homotopy equivalent to a product $(S^1)^n$. Thus, for $n = 2$, this is the classical torus $S^1 \times S^1$.

If $X$ is any toric variety, then $T$ embeds as a distinguished open subscheme in $X$. Indeed, if $\mathbf{v} \in C$ is any vector in the interior of $C$, then the localization $k[C]_{\mathbf{x}^{\mathbf{v}}}$ is isomorphic to $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Thus toric varieties can be viewed as 'compactifications' of the torus $T$. △

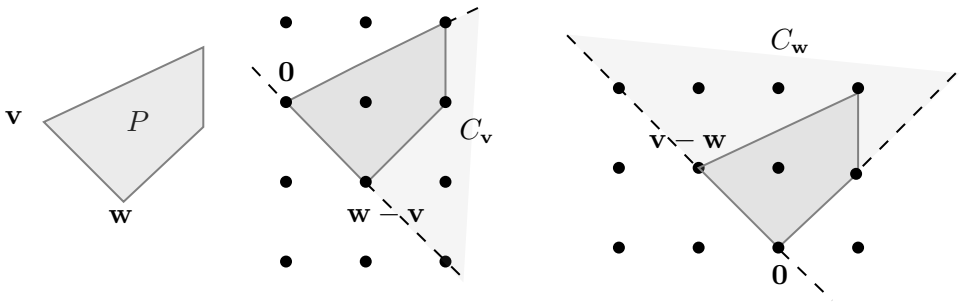
**B.1.0 Projective toric varieties.** A *lattice polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{Z}^n$. In other words, there are $\mathbf{m}_1, \dots, \mathbf{m}_r \in \mathbb{Z}^n$ such that

$$P = \left\{ \lambda_1 \mathbf{m}_1 + \cdots + \lambda_r \mathbf{m}_r \;\mid\; \sum_{i=1}^{r} \lambda_i = 1 \text{ and } \lambda_1, \dots, \lambda_r \geqslant 0 \right\}.$$

Starting with a lattice polytope $P$, we can define a collection of cones as follows. For each $\mathbf{v} \in P \cap \mathbb{Z}^n$, let $P - \mathbf{v} = \{ x \in \mathbb{R}^n \mid x + \mathbf{v} \in P \}$ denote the translate of $P$ by $-\mathbf{v}$, so that the vertex $\mathbf{v}$ ends up at the origin. Then the cone

$$C_{\mathbf{v}} = \operatorname{Cone}(P - \mathbf{v})$$

is a cone spanned by the elements of $P$ as viewed from the vertex $\mathbf{v}$. This is pictured in the figure below.




Let $k$ be a field and consider the ring of Laurent polynomials $S = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and use multiindex notation for monomials in $S$, that is, if $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$, then $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$.

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For a cone $C$ we define the following subalgebra of $S$, generated by the monomials corresponding to the lattice points in $C$:

$$k[C] = \bigoplus_{\mathbf{m} \in C \cap \mathbb{Z}^n} k\, \mathbf{x}^{\mathbf{m}}. \tag{B.4}$$

In particular, to the polytope $P$, we can attach $k$-algebras $k[C_{\mathbf{v}}]$, one for each $\mathbf{v} \in P \cap \mathbb{Z}^n$.

As $P$ has only finitely many vertices, the cones $C_{\mathbf{v}}$ are each spanned by finitely many points with integer coordinates. Morever, there are finitely many lattice points $\mathbf{m}_1, \ldots, \mathbf{m}_r \in C_{\mathbf{v}}$ so that every point in $C_v \cap \mathbb{Z}^n$ can be expressed as a non-negative $\mathbb{Z}$-linear combination of the $\mathbf{m}_1, \ldots, \mathbf{m}_r$. This implies that each $k[C_{\mathbf{v}}]$ is a finitely generated $k$-algebra.

**Lemma B.5.** If $\mathbf{v}$ and $\mathbf{w}$ are two vertices of $P$, then

$$k\left[C_{\mathbf{v}}\right]_{\mathbf{x}^{\mathbf{w}-\mathbf{v}}} = k\left[C_{\mathbf{w}}\right]_{\mathbf{x}^{\mathbf{v}-\mathbf{w}}}. \tag{B.5}$$

*Proof* Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be the vertices of $P$, ordered so that $\mathbf{v} = \mathbf{v}_1$ and $\mathbf{w} = \mathbf{v}_2$. Then the cone $C_{\mathbf{v}}$ is the non-negative span of the vectors $\mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_r - \mathbf{v}_1$, and $C_{\mathbf{w}}$ is the non-negative span of the vectors $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_3 - \mathbf{v}_2, \ldots, \mathbf{v}_r - \mathbf{v}_2$. Clearly, these spans are the same up to translation by $\mathbf{v}_2 - \mathbf{v}_1$, so we get

$$C_{\mathbf{v}} + \mathbb{R}(\mathbf{w} - \mathbf{v}) = C_{\mathbf{w}} + \mathbb{R}(\mathbf{w} - \mathbf{v}). \tag{B.6}$$

The equality (B.5) follows, because both sides are generated by the same monomials. □

Consider the affine schemes

$$X_{\mathbf{v}} = \operatorname{Spec}(k[C_{\mathbf{v}}])$$

where $v \in P$ is a vertex. If $\mathbf{w} \in P$ is another vertex, the affine scheme

$$X_{\mathbf{v},\mathbf{w}} = \operatorname{Spec}(k[C_{\mathbf{v}}]_{\mathbf{x}^{\mathbf{w}-\mathbf{v}}})$$

is identified with a distinguished open set in $X_{\mathbf{v}}$ (defined by the monomial $\mathbf{x}^{\mathbf{w}-\mathbf{v}}$). The gluing conditions for gluing the $X_{\mathbf{v}}$ are then satisfied. The resulting scheme, $X_P$, is called the *toric variety* associate to the polytope $P$.

**Example B.6** (The projective line)**.** Let $P = [0, 1]$ be the unit interval in $\mathbb{R}$. Then $P$ has two vertices 0 and 1. For the vertex 0, which is already at the origin, the cone $S_0$ is the non-negative span of $P$, that is, $S_0 = [0, \infty)$. Likewise $S_1 = (-\infty, 0]$. This gives the following two $k$-algebras:: $A_0 = k[x]$ and $A_1 = k[x^{-1}]$. The corresponding toric variety is $\mathbb{P}^1_k$. △

**Example B.7** (Projective space)**.** Let $P \subset \mathbb{R}^2$ be the triangle with vertices at $0, e_1$ and $e_2$. There are three cones: $S_1 = \{x \geqslant 0, y \geqslant 0\}$, $S_2 = \{x \leqslant 0, x + y \leqslant 0\}$ and $S_2 = \{y \leqslant 0, x + y \leqslant 0\}$. These are depicted in Figure **??**.
The cones $S_1, S_2, S_3$ give the following three $k$-algebras

$$A_1 = k[x, y], \quad A_2 = k[x^{-1}, x^{-1}y], \quad A_3 = k[y^{-1}, xy^{-1}].$$

In light of Example **??**, we recoginze the corresponding toric variety as the projective plane $\mathbb{P}^2_k$.

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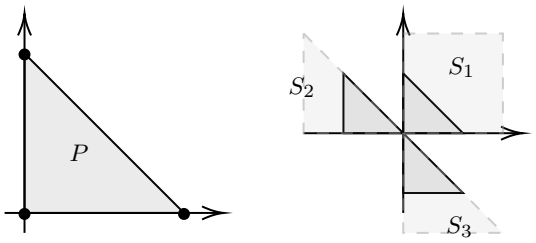
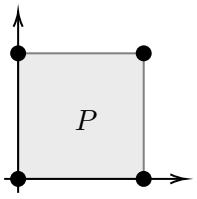



**Figure B.2** Constructing the projective plane

More generally, if $P \subset \mathbb{R}^n$ is the *standard $n$-simplex*, that is, the polytope with vertices at the origin and the $n$ standard basis vectors $0, e_1, \ldots, e_n$, then the corresponding toric variety is the projective space $\mathbb{P}^n_k$.

△

**Example B.8.** Consider the polytope $P$ given by the unit square, that is, the convex hull of the four points $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$.

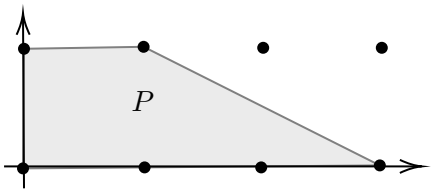



Then the four cones $S_1, \ldots, S_4$ are the four quadrants in $\mathbb{R}^2$. Hence

$$A_1 = k[x,y] \qquad A_2 = k[x^{-1},y]$$
$$A_3 = k[x^{-1},y^{-1}] \qquad A_4 = k[x,y^{-1}]$$

The corresponding toric variety is $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. We will define products of general schemes in Chapter 7. Note that the polytope $P$ is itself a product: $P = [0,1] \times [0,1]$, and $[0,1]$ is the polytope defining $\mathbb{P}^1_k$. This is a general fact about products of toric varieties (see Exercise ???). △

**Example B.9** (Hirzebruch surfaces)**.** Let $r$ be an integer and consider the polytope $P$ defined by the four points $(0,0)$, $(0,1)$, $(1,1)$ and $(r+1,1)$. This is pictured below for $r = 1$. The case $r = 0$ was considered in the previous example.




This gives the following $k$-algebras:

$$A_1 = k[x,y] \qquad A_2 = k[x^{-1},x^{-r}y]$$
$$A_3 = k[x^{-1},x^{r}y^{-1}] \qquad A_4 = k[x,y^{-1}]$$



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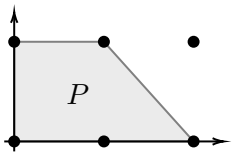
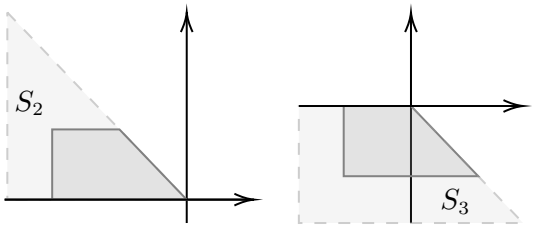
The corresponding toric variety $\mathbb{F}_r$ is called a *Hirzebruch surface*.




There are four vertices, so four cones. Two of these are shown below:




We compute that

$$A_1 = k[x, y] \qquad A_2 = k[x^{-1}, x^{-1}y]$$
$$A_3 = k[x^{-1}, xy^{-1}] \qquad A_4 = k[x, y^{-1}]$$

△

Each of the rings $k[C_{\mathbf{v}}]$ and their localizations are subrings of $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and the morphisms $T \to \operatorname{Spec} k[C_{\mathbf{v}}]$ glue to an open embedding $T \to X$.

The toric variety $X_P$ admits a closed embedding into projective space. More precisely, let $\mathbf{m}_1, \dots, \mathbf{m}_r$ be the lattice points of $P$, and assume that $P$ satisfies the following property: for each $i$, the semigroup $C_{\mathbf{m}_i} \cap \mathbb{Z}^n$ is generated by the elements $\mathbf{m}_1 - \mathbf{m}_i, \mathbf{m}_2 - \mathbf{m}_i, \dots, \mathbf{m}_r - \mathbf{m}_i$. This condition is not too restrictive, as it is satisfied for sufficently large scalings of $P$, and $X_P$ remains unchanged under scaling $P$. Moreover, it is automatically satsfied if $P$ has dimension at most 2.

In this setting, $k[C_{\mathbf{m}_i}]$ is generated by the monomials $\mathbf{x}^{\mathbf{m}_1 - \mathbf{m}_i}, \dots, \mathbf{x}^{\mathbf{m}_r - \mathbf{m}_i}$, and we can define a surjective map of $k$-algebras

$$\phi_i \colon k\left[\frac{y_1}{y_i}, \dots, \frac{y_r}{y_i}\right] \longrightarrow k[C_{\mathbf{m}_i}]; \qquad \frac{y_j}{y_i} \mapsto \mathbf{x}^{\mathbf{m}_j - \mathbf{m}_i} \tag{B.7}$$

and hence a closed embedding $\operatorname{Spec} k[C_{\mathbf{m}_i}] \to U_i = \operatorname{Spec} k[\frac{y_1}{y_i}, \dots, \frac{y_r}{y_i}]$. It is straightforward to check that these embeddings glue to an embedding

$$f \colon X \longrightarrow \mathbb{P}^{r-1}_k.$$

Alternatively, one can consider the morphism

$$f_0 \colon \operatorname{Spec} k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \longrightarrow \mathbb{P}^{r-1}$$

defined by the monomials $\mathbf{x}^{\mathbf{m}_1}, \dots, \mathbf{x}^{\mathbf{m}_r}$. Then $X$ is isomorphic to the Zariski closure of image of $f_0$ in $\mathbb{P}^{r-1}_k$.

Another nice property of the toric varieties $X_P$ is that they provide natural ways to

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'compactify' affine hypersurfaces. If $f \in k[x_1, \ldots, x_n]$ is a nonzero polynomial, we can consider the associated *Newton polytope* $P_f \subset \mathbb{R}^n$ by taking the convex hull of the points in $\mathbb{Z}^n$ corresponding to monomials appearing in $f$. This is best explained through an example

**Example B.10.** Consider the polynomial

$$f = 1 + x^2y + xy^2$$

Then the Newton polytope $P = P_f$ is the convex hull of the vertices $(0,0)$, $(2,1)$, $(1,2)$. The toric variety associated to $P$ is covered by the three affine charts

$$\begin{aligned} R_1 &= k[x^2y, xy, xy^2] = k[u_1, u_2, u_3]/(u_2^2 - u_1u_3) \\ R_2 &= k[x^{-2}y^{-1}, x^{-1}, x^{-1}y] = k[v_1, v_2, v_3]/(v_2^3 - v_1v_3) \\ R_3 &= k[x^{-1}y^{-2}, y^{-1}, xy^{-1}] = k[w_1, w_2, w_3]/(w_2^3 - w_1w_3) \end{aligned}$$

The hypersurface given by $V(f) \subset T = \operatorname{Spec} k[x^{\pm 1}, y^{\pm 1}]$ is an open set in a hypersurface $Y \subset X_P$. Over the open set $U_1 = \operatorname{Spec} R_1$ this is simply given by the linear equation $1 + u_1 + u_3 = 0$. Over $U_2 = \operatorname{Spec} R_2$ it is given by the 'dehomogenization' $x^{-2}y^{-1} + 1 + x^{-1}y = v_1 + 1 + v_3$ and so on. △

While any affine hypersurface can be embedded into some projective space by homogenizing, the toric varieties $X_P$ tend to be 'more economical', as they take into account only the monomials appearing in $f$.

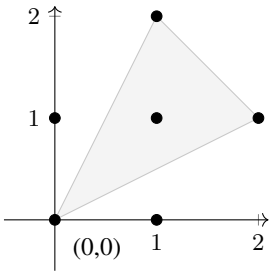


**Figure B.3** The polytope $P$

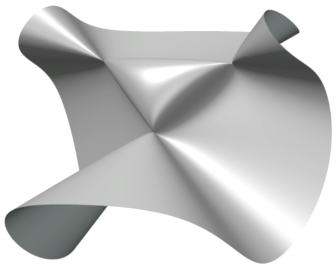

**Figure B.4** A cubic surface.

**Exercise B.1.1.** Let $P \subset \mathbb{R}^3$ be a the convex hull of the lattice points $0, e_1, e_2, e_1 + e_2 + 3e_3$. Show that $P$ has only 4 lattice points and that $P$ does not satisfy the condition mentioned on page 447.

**Exercise B.1.2.** In the notation of Proposition B.1, let $T = \{u_1, \ldots, u_r\}$ be a $\mathbb{Z}$-basis for $\operatorname{Ker} \mathcal{A}$ and let

$$I'_{\mathcal{A}} = (\mathbf{y}^{u_+} - \mathbf{y}^{u_-} \mid u \in S)$$

where we decompose $u = u_+ - u_- \in \mathbb{Z}^n$ in terms of its non-negative and non-positive entries. Show that $I_{\mathcal{A}}$ is the saturation of $I'_{\mathcal{A}}$ with respect to the maximal ideal at the origin, i.e., $I_{\mathcal{A}} = I'_{\mathcal{A}} : (y_1, \ldots, y_m)^{\infty}$.

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## B.2 The blow-up of the affine plane

In this section we will construct the *blow-up of* $\mathbb{A}^2_k$ *at the origin*, by gluing together two affine planes.

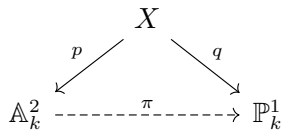
Consider the morphism $\pi\colon \mathbb{A}^2_k - \{(0,0)\} \to \mathbb{P}^1_k$ from (5.5). On $k$-points, the morphism is given by $(a,b) \mapsto (a:b)$, where $(a:b)$ are the homogeneous coordinates on $\mathbb{P}^1_k$. It will be helpful to think geometrically, so that $\mathbb{P}^1(k)$ parameterizes lines through the origin in $\mathbb{A}^2(k)$. Then $\pi$ sends a point $(a,b) \neq (0,0)$ to the unique line passing through $(0,0)$ and $(a,b)$ In particular, all the points on the line $bx - ay = 0$ get mapped to the point $(a:b) \in \mathbb{P}^1_k$.

From this perspective, it is not surprising that $\pi$ cannot be extended to a morphism $\mathbb{A}^2_k \to \mathbb{P}^1_k$. In fact, there is not even an extension as a continuous map, because $\pi$ takes different values along lines $bx - ay = 0$ and $(0,0)$ is contained in all of them.

While $\pi$ cannot be extended to $\mathbb{A}^2_k$, it is possible to embed $\mathbb{A}^2_k - \{(0,0)\}$ as an open set in a scheme $X$ that does admit a morphism to $\mathbb{P}^1_k$ extending $\pi$. This scheme is called the *blow-up* of $\mathbb{A}^2_k$ at the origin. The blow-up amits two morphisms, $p\colon X \to \mathbb{A}^2_k$ and $q\colon X \to \mathbb{P}^1_k$ such that

(i) The scheme theoretic fiber $E = p^{-1}(0,0)$ over the origin $(0,0)$ is isomorphic to $\mathbb{P}^1_k$.
(ii) $p$ defines an isomorphism betwen $X - E$ and $\mathbb{A}^2_k - \{(0,0)\}$.
(iii) The morphism $q \circ p^{-1}$, defined over $\mathbb{A}^2_k - \{(0,0)\}$, coincides with $\pi$.

This is pictured in the diagram below (the dashed arrow indicates that $\pi$ is only defined on an open set):




Before we construct the scheme $X$, let us describe what the set of $k$-points of $X$ looks like. The $k$-points of $X$ can be thought of as pairs $(p, [L])$ where $p \in \mathbb{A}^2(k)$ is a $k$-point and $L \subset \mathbb{A}^2(k)$ is a line containing $p$. In terms of coordinates, we can define this as

$$X(k) = \left\{ ((x,y),(u_0:u_1)) \;\middle|\; u_1 x - u_0 y = 0 \right\} \subset \mathbb{A}^2(k) \times \mathbb{P}^1(k). \tag{B.8}$$

In other words, $X(k)$ is the *graph* of the mapping $(x,y) \mapsto (x:y)$.

We can verify that the properties (i), (ii), (iii) hold as follows. Firstly, note that $p^{-1}(0,0) = (0,0) \times \mathbb{P}^1(k)$ is a copy of the projective line $\mathbb{P}^1(k)$, as the equation $u_1 x - u_0 y = 0$ puts no conditions on $u_0$ and $u_1$. On the other hand, if $(x,y)$ is a $k$-point in the open set $\mathbb{A}^2_k - \{(0,0)\}$, then $(x:y)$ is a well-defined point in $\mathbb{P}^1(k)$, and $p^{-1}(x,y) = ((x,y),(x:y))$. Therefore, $p$ defines a bijection over the set $\mathbb{A}^2(k) - \{(0,0)\}$. Now composing the inverse of $p^{-1}$ with the second projection sends $(x,y)$ to $(x:y)$, so (iii) follows as well.

Note that (B.8) is just as *set* of $k$-points. We haven't yet defined a scheme structure on the blow-up. However, this set provides a clue for constructing $X$ as a scheme.

We consider the usual construction of $\mathbb{P}^1_k$ via gluing $V_0 = \operatorname{Spec} k[u]$ to $V_1 = \operatorname{Spec} k[v]$ using the identification $v = u^{-1}$ over the overlaps. In light of (B.8), we then define

$$X_0 = \operatorname{Spec} B_0; \qquad B_0 = k[x,y,u]/(y - ux). \tag{B.9}$$

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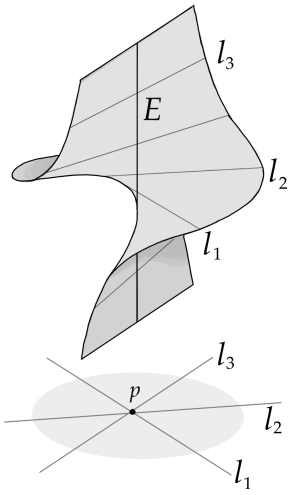


**Figure B.5** *The blow-up of the plane at a point*

and over the open set $V_1$, we define

$$X_1 = \operatorname{Spec} B_1; \qquad B_1 = k[x, y, v]/(x - vy).$$

Note that $B_0 \simeq k[x, u]$ and $B_1 \simeq k[y, v]$, so $X_0$ and $X_1$ are both isomorphic to affine planes.

Furthermore, the map $u \mapsto v^{-1}$ defines an isomorphism of rings

$$(B_0)_u = k[x, y, u, u^{-1}]/(y - ux) \simeq k[x, y, v, v^{-1}]/(vy - x) = (B_1)_v.$$

This means that we may identify the distinguished open $D(u) \subset X_0$ with the distinguished open $D(v) \subset X_1$, and hence glue $X_0$ and $X_1$ together to form a new scheme $X$.

We next define the morphisms $p$ and $q$. Note that there is a morphism $X_0 \to V_0$ induced by the ring map $k[u] \to B_0$. Likewise, there is a morphism $X_1 \to V_1$ induced by $k[v] \to B_1$. These morphisms are compatible with the gluing isomorphism $u \mapsto v^{-1}$, so they glue to a morphism $X \to \mathbb{P}^1_k$. Similarly, the two ring maps $k[x, y] \to B_0$ and $k[x, y] \to B_1$ induce a morphism $p\colon X \to \mathbb{A}^2_k$.

The $k$-points of the scheme $X$ are precisely given by the set (B.8). To see this, consider a morphism $\iota\colon \operatorname{Spec} k \to X$. Such a morphism must have image contained in either $X_0$ and $X_1$. Assuming, say, $\iota$ maps to $(a, b, c) \in X_1(k)$, then $b = ac$, which corresponds to the point $((a, b), (1 : c))$ in the set (B.8). Conversely, any pair $(a, b), (c_0 : c_1)$ satisfying $c_1 a - c_0 b = 0$, we can assume without loss of generality that $c_0 = 1$. Then $b = ac_1$, and the pair defines the $k$-point $(a, b, c_1)$ in $X_0$.

Let us consider the fiber of the morphism $q\colon X \to \mathbb{P}^1$. On the level of $k$-points, if $(u_0 : u_1) \in \mathbb{P}^1(k)$ is fixed, the $k$-points of the fiber $q^{-1}(u_0 : u_1)$ consists of the points $(x, y), (u_0 : u_1)$ such that $u_1 x - u_0 y = 0$, that is, the points on the line $u_1 x - u_0 y = 0$ in $\mathbb{A}^2(k)$. In fact, the scheme-theoretic fibers are all isomorphic to affine lines. To see this, take a point $p \in \mathbb{P}^1_k$, which we may assume lies in the open set $V_0 = \operatorname{Spec} k[u]$ and corresponds

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to a prime ideal $\mathfrak{p} \subset k[u]$. Then the scheme-theoretic fiber is given by

$$\operatorname{Spec} k[x,y,u]/\left((y-ux)+\mathfrak{p}\right) = \operatorname{Spec} k[x,u]/\mathfrak{p} \simeq \mathbb{A}^1_{k(p)}$$

The fibers of the morphism $p\colon X \to \mathbb{A}^2_k$ are also interesting. First off all, the scheme theoretic fiber $E$ over the origin $(0,0)$, which corresponds to $\mathfrak{m} = (x,y) \subset k[x,y]$, is glued together by the two affine schemes

$$E_0 = \operatorname{Spec} k[x,y,u]/(y-ux,x,y) = \operatorname{Spec} k[u]$$

and

$$E_1 = \operatorname{Spec} k[x,y,v]/(vx-y,x,y) = \operatorname{Spec} k[v].$$

Hence, the scheme theoretic fiber $E$ is isomorphic to $\mathbb{P}^1_k$.

We claim that $p$ restricts to an isomophism of schemes

$$X - E \longrightarrow \mathbb{A}^2_k - \{(0,0)\}. \tag{B.10}$$

Indeed, note that $E \cap X_0 \subset X_0$ is the closed subscheme $V(x)$ inside $X_0$. Likewise, $E \cap X_1$ is given by $V(y)$ in $X_1$. This means that $X - E$ is covered by the two affine subsets $D(x) \subset X_0$ and $D(y) \subset X_1$. Note that $p$ maps $D(x) \subset X_0$ isomorphically to $D(x) \subset \mathbb{A}^2_k$. This follows because it is induced by ring isomorphism

$$k[x,x^{-1},y] \to (B_0)_x = k[x,x^{-1},y,u]/(y-ux)$$

Likewise $p$ restricts to an isomorphism over $D(y)$, proving the claim.

While the discussion on $k$-points required working over a field, the scheme $X$ constructed via gluing two copies of the plane works over any ring. So for instance, there is a blow-up of $\mathbb{A}^2_{\mathbb{Z}}$ at the origin $(x,y)$.

One can also define the blow-up of $\mathbb{A}^n_k$ at the origin, or in fact, along any linear subspace. While the details can be worked out, they are more naturally understood in the context of the Proj-construction, as we will see in Example **??**.

## B.3 Line bundles on $\mathbb{P}^1$

The sheaf $\mathcal{O}_{\mathbb{P}^1_k}(n)$ on the projective line $\mathbb{P}^1_k$, which we constructed in Example 5.1, has a geometric counterpart, known as a line bundle over $\mathbb{P}^1_k$. This is a scheme equipped with a morphism

$$\pi\colon X \longrightarrow \mathbb{P}^1_k. \tag{B.11}$$

whose fibers are affine lines (hence the name 'line bundle').

In this section, we will consider a family of schemes, constructed by gluing two affine planes $\mathbb{A}^2_k$ along distinguished opens. Let $n \in \mathbb{Z}$, and consider the affine schemes

$$X_0 = \operatorname{Spec} k[x,t], \quad X_1 = \operatorname{Spec} k[x^{-1},x^n t], \tag{B.12}$$

These are glued along the open set $X_{01} = \operatorname{Spec} k[x,x^{-1},t]$. Here, $X_{01}$ is a distinguished open set in both $X_0$ and $X_1$: it arises by inverting $x$ in $k[x,t]$ and $x^{-1}$ in $k[x^{-1},x^n t]$. The gluing conditions are in this case automatically fulfilled, and hence we obtain a scheme, which we denote by $X$.

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The scheme $X$ admits a morphism $\pi\colon X \to \mathbb{P}^1_k$. This is defined by gluing together the two morphisms $\operatorname{Spec} k[x,t] \to \mathbb{A}^1_k = \operatorname{Spec} k[x]$ and $\operatorname{Spec}[x^{-1}, x^n t] \to \operatorname{Spec} k[x^{-1}]$. These glue to a morphism $X \to \mathbb{P}^1_k$ by applying Spec to the following commutative diagram:

$$\begin{array}{ccccc} k[x,t] & \hookrightarrow & k[x,x^{-1},t] & \hookleftarrow & k[x^{-1},x^n t] \\ \uparrow & & \uparrow & & \uparrow \\ k[x] & \hookrightarrow & k[x,x^{-1}] & \hookleftarrow & k[x^{-1}] \end{array} \tag{B.13}$$

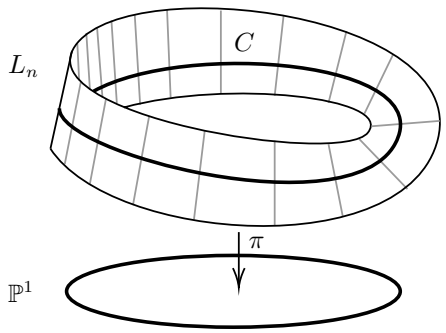
In summary, the scheme $X$ and the morphism $X \to \mathbb{P}^1_k$ is obtained by gluing together two copies of $\mathbb{A}^2_k$ and the projection morphisms $\mathbb{A}^2_k \to \mathbb{A}^1_k$ onto the first factor, but there is a twist – the two morphisms $X_{01} \to \operatorname{Spec} k[x,x^{-1}]$ are related by a multiplication of $x^m$.

The diagram (B.13) also shows that if $P \in \operatorname{Spec} k[x]$ is a point, then $\pi^{-1}(P)$ is contained in $X_0$. One can see directly that the fiber is an affine line as follows: if $P$ corresponds to a maximal ideal $\mathfrak{m} \subset k[x]$, then

$$\pi^{-1}(P) = \operatorname{Spec}(k[x,y]/\mathfrak{m}) = \operatorname{Spec}(k[x][y]/\mathfrak{m}) = \operatorname{Spec}(\kappa(P)[y]) \simeq \mathbb{A}^1_{\kappa(P)}.$$

As noted above, this is the reason for the term 'line bundle'. Intuitively, $X$ is a family of affine lines parameterized by the base space $\mathbb{P}^1_k$.

The scheme $X$ admits a closed subscheme isomorphic to $\mathbb{P}^1_k$. In $X_0$, the closed subscheme $V(t)$, corresponding to '$x$-axis', is isomorphic to $\mathbb{A}^1_k$ and similarly for $V(x^n t) \subset X_1$. Since $x$ is invertible in $k[x,x^{-1},t]$, these two subschemes coincide on $X_{01}$ (with the subscheme defined by the ideal $(t)$), and hence glue to a closed subscheme $C \to X$. By construction, $C$ is construced by gluing $\operatorname{Spec} k[x,t]/(t) \simeq \operatorname{Spec} k[x]$ and $\operatorname{Spec} k[x^{-1}, x^n t]/(x^n t) \simeq \operatorname{Spec} k[x^{-1}]$, so it is isomorphic to $\mathbb{P}^1_k$. We also see that the composition $C \to X \to \mathbb{P}^1_k$ is the identity, as it restricts to the identity maps over $X_0$ and $X_1$. In other words, $C$ is a *section* of the morphism $\pi$.




The ring of globaly sections $\mathcal{O}_X(X)$ can be obtained from the exact sequence

$$0 \longrightarrow \mathcal{O}_X(X) \longrightarrow k[x,t] \times k[x^{-1}, x^n t] \longrightarrow k[x,x^{-1},t] \tag{B.14}$$

which shows that

$$\mathcal{O}_X(X) = k[x,t] \cap k[x^{-1}, x^m t]. \tag{B.15}$$

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This intersection depends on $n$. If $n \geqslant 0$, then

$$k[x,t] \cap k[x^{-1}, x^n t] = k[x^n t, x^{n-1}t, \ldots, t] \tag{B.16}$$

If $n < 0$, then

$$k[x,t] \cap k[x^{-1}, x^n t] = k.$$

Let us study a few special cases.

**Example B.11** ($n = 0$)**.** The scheme $X$ is glued together of two copies of $\mathbb{A}^2_k$ via the inclusions

$$k[x,t] \hookrightarrow k[x, x^{-1}, t] \hookleftarrow k[x^{-1}, t].$$

where the variable $t$ is unchanged in the process. In this case, we notice that $X$ is isomorphic to $\mathbb{P}^1_{k[t]}$. In addition to the morphism $\sigma\colon X \to \mathbb{P}^1_k$, there is the projection $X \to \operatorname{Spec} \mathbb{A}^1_k = \operatorname{Spec} k[t]$. The scheme $X$ is in fact equal to the 'fiber product' $\mathbb{P}^1 \times_k \mathbb{A}^1_k$ (fiber products will be studied in Chapter 7). The $k$-points of $X$ are given by $\mathbb{P}^1(k) \times \mathbb{A}^1(k)$. △

**Example B.12** ($n = -1$)**.** When $n = -1$, the scheme $X$ is isomorphic to the complement of a closed point $P$ in the projective plane. Indeed, with the standard affine covering of $\mathbb{P}^2_k$, the open set $U = \mathbb{P}^2_k - \{(0:0:1)\}$ is covered by the two affine subschemes $V_0 = \operatorname{Spec} k[x_1/x_0, x_2/x_0]$ and $V_1 = \operatorname{Spec} k[x_0/x_1, x_2/x_1]$. By setting $x = x_1/x_0$ and $t = x_2/x_0$ in $V_0$ and observing that $x_2/x_1 = x_0/x_1 \cdot x_2/x_0 = x^{-1}t$, we recover exactly the gluing data for $L_{-1}$.

Geometrically, the morphism $\mathbb{P}^2_k - \{P\} \to \mathbb{P}^1_k$ is given by 'projection from the point $P$'. The fibers are the lines in $\mathbb{P}^2_k$ through $P$ with the point $P$ removed, and the zero section is identified with $V_+(x_2)$ (the 'line at infinity'). △

**Example B.13** ($n = 1$)**.** The line bundle $L_1$ can be identified with the blow-up of $\mathbb{A}^2_k$ at the origin. Recall that the blow-up $X$ comes equipped with a map $q\colon X \to \mathbb{P}^1_k$, which is described in Section B.2. It is not hard

One checks without much difficulties that the gluing maps used for forming $X$ are the same as for making $L_{-1}$. The zero-section $C$ corresponds to the exceptional divisor $E$ in the blow-up.

In this example, the canonical map $X \to \operatorname{Spec}(\mathcal{O}_X(X))$ is exactly the blow-up morphism $X \to \mathbb{A}^2_k$. Again, it makes sense that the exceptional divisor $E$ is contracted by this map, as any map from $\mathbb{P}^1_k$ to an affine scheme must factor via a $k$-point. △

**Example B.14** ($n = 2$)**.** The scheme $L_2$ is quite interesting. It is a so-called *desingularization of a quadratic cone*.

The ring of global sections $R = \mathcal{O}_X(X)$ has the following description:

$$R = k[x^2 t, xt, t] \simeq k[u,v,w]/(v^2 - uw)$$

via the identification $u = x^2 t$, $v = xt$ and $w = t$.

The inclsuions $R \subset k[x,t]$ and $R \subset k[x^{-1}, x^2 t]$ induce morphisms $\sigma_0\colon X_0 \to \operatorname{Spec} R$ and $\sigma_1\colon X_1 \to \operatorname{Spec} R$ which glue to a morphism $\sigma\colon X \to \operatorname{Spec} R$ (over the intersection they both correspond to the inclusion $R \subset k[x, x^{-1}, t]$).

We claim that $X \to \operatorname{Spec} R$ is surjective and induces an isomorphism between $X - C$

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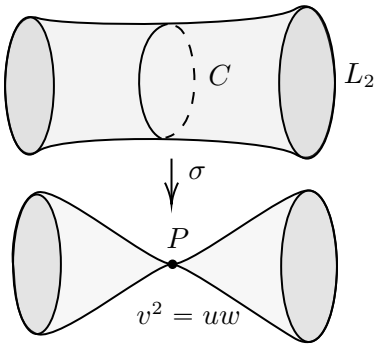
and $\operatorname{Spec} R - \{P\}$, where $P = (0,0,0)$ is the origin corresponding to the maximal ideal $(u,v,w) \subset R$.




To understand the fibers, consider the restriction $\sigma_0 \colon \operatorname{Spec} k[x,t] \longrightarrow \operatorname{Spec} R$ to $X_0 \subset X$. The fiber over $\mathfrak{m} = (u,v,w)$ is given by

$$\operatorname{Spec}(k[x,t]/\mathfrak{m}k[x,t]) = \operatorname{Spec}(k[x,t]/(x^2t, xt, t)) = \operatorname{Spec}(k[x,t]/(t))$$

Hence the fiber equals the closed set $\sigma_0^{-1}(V(\mathfrak{m})) = V(t)$. This means that the '$x$-axis' $V(t)$ in $\mathbb{A}^2 = \operatorname{Spec} k[x,t]$ is collapsed to the origin in $\operatorname{Spec} R$. Likewise, the '$x^{-1}$-axis' in $\mathbb{A}^2_k = \operatorname{Spec} k[x^{-1}, x^m t]$ is collapsed to the origin, and hence the entire zero-section $C \subset X$ is mapped to the origin. This is consistent with the fact that any morphism from $\mathbb{P}^1_k$ to an affine scheme is constant.

To see the isomorphism $X - C \simeq \operatorname{Spec} R - \{P\}$, note that $\operatorname{Spec} R - \{P\}$ is covered by the two distinguished open sets $D(u)$ and $D(w)$ (note that $D(v) = D(v^2) = D(uv)$ by the quadratic relation defining $R$). Similarly, the complement $X - C$ is covered by the distinguished open subsets $D(t) \subset X_0 = \operatorname{Spec} k[x,t]$ and $D(t) \subset V_1 = \operatorname{Spec} k[u^{-1}, t]$.

Note that the inclusion $R \subset k[x,t]$ becomes an isomorphism after inverting $t$, because $x = (xt)t^{-1}$. Likewise, $R_{x^2t} \subset k[x^{-1}, x^2t, x^{-2}t^{-1}]$ is an isomorphism because $x^{-1} = (xt)(x^{-2}t^{-1})$. Therefore, the two morphisms $\phi_0$ and $\phi_1$ restrict to isomorphisms over the respective open sets $D(t) \subset X_0$ and $D(x^2t) \subset X_1$. All together, $\sigma$ is an isomorphism outside $C$.

The morphism $\sigma$ is helpful for understanding the quadratic cone. In the terminology of Chapter 10, $\operatorname{Spec} R$ has a 'singularity' at the origin, and the morphism '$X \to \operatorname{Spec} R$' resolves this singularity. △

**Exercise B.3.1.** Show that when $n = -3$, the ring $R = \mathcal{O}_X(X)$ is isomorphic to the homogeneous coordinate ring of the projective twisted cubic curve. Describe the morphism $L_{-3} \to \operatorname{Spec} R$.

## B.4 Double covers

**B.4.0 Hyperelliptic curves.** Let $k$ be a field and let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree $n$. We write $n = 2g+1$ if $n$ is odd and $n = 2g+2$ if $n$ is even.

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Consider the two affine schemes $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$, where

$$A = k[x, y]/(y^2 - p(x)) \text{ and } B = k[u, v]/(v^2 - u^{2g+2}p(u^{-1})).$$

Here

$$u^{2g+2}p(u^{-1}) = a_n + a_{n-1}u \cdots + a_1 u^{n-1} + a_0 u^n \tag{B.17}$$

if $n$ is even, and

$$u^{2g+2}p(u^{-1}) = a_n u + a_{n-1}u^2 \cdots + a_1 u^{n-1} + a_0 u^{n+1} \tag{B.18}$$

if $n$ is odd.

The gluing of $X_1$ and $X_2$ is achieved via the isomorphism $\phi\colon B_u \to A_x$ defined by $\phi(u) = x^{-1}$ and $\phi(v) = x^{-g-1}y$. This is well-defined, since

$$\phi(v^2 - u^{2g+2}p(u^{-1})) = x^{-2(g+1)}y^2 - x^{-2g-2}p(x) = x^{-(2g+2)}(y^2 - p(x))$$

and its inverse is defined by $x \mapsto u^{-1}$ and $y \mapsto u^{-g-1}v$.

The resulting scheme $X$ is called a *hyperelliptic curve*. In the case $g = 1$, the curve $X$ is an example of an *elliptic curve*. As a set, the scheme $X$ consists of the points of $\operatorname{Spec} A$, along

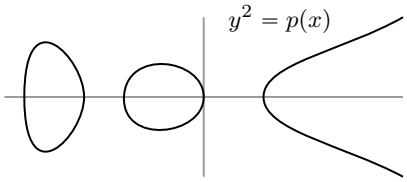


**Figure B.6** A hyperelliptic curve

with the points of $\operatorname{Spec} B$ outside $D(u) = D(x)$, that is, the points of $V(u) \subset \operatorname{Spec} B$. The number of points in $V(u)$ depends on whether $n$ is even or odd. If $n$ is odd, then the equation (B.18) implies that $V(u)$ consists of a single point, corresponding to the maximal ideal $(u, v)$. If $n$ is even, then

$$V(u) = V(u, v^2 - a_n)$$

which consists of two points if $a_n$ has a square root in $k$ and a single point otherwise.

The scheme $X$ admits a morphism $\pi\colon X \to \mathbb{P}^1_k$ to the projective line. Consider the two inclusions $k[x] \subset A$ and $k[u] \subset B$. Under the isomorphism $\phi\colon B_u \to A_x$ above, $k[u]$ is mapped into $k[x]$ and $u$ maps to $x^{-1}$, so there is a commutative diagram:

$$\begin{array}{ccc} k[u] & \xrightarrow{u \mapsto x^{-1}} & k[x] \\ \downarrow & & \downarrow \\ B_u & \xrightarrow{\phi} & A_x. \end{array}$$

The two inclusions give maps $X_1 \to U_0 = \operatorname{Spec} k[x] \subset \mathbb{P}^1_k$ and $X_2 \to U_1 = \operatorname{Spec} k[u] \subset \mathbb{P}^1_k$, where $U_0$ and $U_1$ are joined together to a $\mathbb{P}^1_k$ according to the rule $x \leftrightarrow u^{-1}$. By the

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observation above, this is compatible with the way $X_1$ and $X_2$ are joined together, and the two maps glue to the morphism $\pi$.

The morphism $\pi$ is also called a *double cover of* $\mathbb{P}^1_k$. Over each affine, $\pi$ restricts to a double cover of $\mathbb{A}^1_A$. Therefore, if $A$ is an algebraically closed field, the fibers remain either two points, or one point with 'multiplicity 2' as in the previous example.

Notice that the gluing map defining $X$ is very similar to the one involved in the construction of the schemes $L_m$ from Section B.3. In fact, $X$ is a closed subscheme of $L_{-g-1}$. Indeed, $L_{-g-1}$ is obtained by gluing $U_1 = \operatorname{Spec} k[x,y]$ and $U_2 = \operatorname{Spec} k[u,v]$ and $X_1$ and $X_2$ are naturally closed subschemes of $U_1$ and $U_2$ respectively. As the isomorphism defining $L_{-g-1}$ is exactly by the same formula as $\phi$, we find that $X_1$ and $X_2$ glue together to a closed subscheme of $L_{-g-1}$. The gluing isomorphisms are moreover compatible with the two morphisms to $\mathbb{P}^1_k$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} X & \hookrightarrow & L_{-g-1} \\ & \searrow & \downarrow \\ & & \mathbb{P}^1_k \end{array}$$

**B.4.0 Higher-dimensional double coverings.** The above construction generalizes in a straightforward manner to higher-dimensional projective spaces. We will even consider projective spaces over any ring $A$.

Let $A$ be a ring and let $R = A[x_0, \ldots, x_n]$ with the usual grading. Let $f \in R$ be a homogeneous polynomial of degree $2d$, and for each $0 \leqslant i \leqslant n$ let

$$S_i = A\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{y}{x_i^d}\right] \Big/ \left(\left(\frac{y}{x_i^d}\right)^2 - f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)\right) \tag{B.19}$$

For each pair $i, j$, we define $S_{ij} = S_i[x_i/x_j]$. Then we have equalities $S_{ij} = S_{ji}$, by the following identity

$$\left(\frac{x_i}{x_j}\right)^{2d}\left(\left(\frac{y}{x_i^d}\right)^2 - f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right)\right) = \left(\frac{y}{x_j^d}\right)^2 - f\left(\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

As in the example of projective space, the $\operatorname{Spec} S_i$'s glue together along the open subschemes $\operatorname{Spec} S_{ij}$'s to a scheme $X$. Moreover, keeping the notation $R_i$ from the previous section, the morphisms $\operatorname{Spec} S_i \to \operatorname{Spec} R_i$, induced by the inclusions $R_i \to S_i$, glue together to a morphism $\pi\colon X \to \mathbb{P}^n_A$.

## B.5 Exercises

**Exercise B.5.1.** Consider $\mathbb{P}^n(k)$ as defined on page 93. Show that every closed subset of $\mathbb{P}^n(k)$ is of the form $Z_+(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a}$.

**Exercise B.5.2.** Let $\mathbb{F}_{p^r}$ denote the field with $p^r$ elements.

a) Show that $\#\mathbb{A}^n(\mathbb{F}_{p^r}) = p^{rn}$.

b) Show that $\#\mathbb{P}^n(\mathbb{F}_{p^r}) = p^{rn} + p^{r(n-1)} + \cdots + p^n + 1$.

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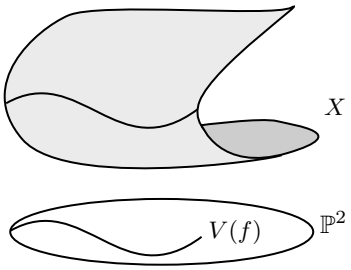


**Figure B.7** A double cover of the projective plane.

**Exercise B.5.3.** Let $X = \operatorname{Spec} A$ be an affine scheme over a field $k$. Show that every morphism $\mathbb{P}^1_k \to X$ is constant, i.e. it factors through some $k$-valued point of $X$.

**Exercise B.5.4.** Show that $\mathbb{P}^1_A$ is not affine for any ring $A$. HINT: The canonical map $\mathbb{P}^1_A \to \operatorname{Spec} A$ is never an isomorphism.

**Exercise B.5.5.** Describe the closed subsets of $\mathbb{P}^1_{\mathbb{C}}, \mathbb{P}^1_{\mathbb{R}}$ and $\mathbb{P}^1_{\mathbb{Q}}$.

**Exercise B.5.6.** Prove that $\mathbb{P}^n_k$ is irreducible for any $n \geqslant 0$.

**Exercise B.5.7.** Show that the $\mathbb{Z}$-valued points of $\mathbb{P}^1_{\mathbb{Z}}$ is given by $\{\, (a : b) \mid a, b \in \mathbb{Z} \,\}$.

**Exercise B.5.8.** Let $A$ be a ring and let $a_0, \ldots, a_n$ be elements generating the unit ideal in $A$. Show that the closed subscheme defined by (5.11) corresponds to the $A$-valued point $\iota_a \colon \operatorname{Spec} A \to \mathbb{P}^n_A$ defined in (5.7). HINT: If $\pi \colon \mathbb{P}^n_A \to \operatorname{Spec} A$ is the structure map, show that $\pi^{-1} D(a_i) = \mathbb{P}^n_{A_{a_i}}$ and reduce to the case where $a_0$ is invertible.

**Exercise B.5.9.** Let $k$ be a field and let $F \in k[x_0, x_1]$ be a nonzero homogeneous polynomial of degree $d$. Show that the subscheme $Z = V_+(F)$ consists of $d$ points counted with multiplicity. More precisely, show that $d = \sum_{x \in Z} \dim_k \mathcal{O}_{Z,x}$.

**Exercise B.5.10.** Let $X$ be the affine line with two origins, as defined in Section **??**.

a) Imitate the construction of the sheaves $\mathcal{O}_{\mathbb{P}^1_k}(n)$ on $\mathbb{P}^1_k$ to form a collection of sheaves $\mathcal{O}_X(m)$ on $X$, one for each integer $m$.
b) Show that $\mathcal{O}_X(m)$ and $\mathcal{O}_X(n)$ are not isomorphic unless $m = n$. HINT: Consider the behaviour of sections at the two origins.

**Exercise B.5.11.** Verify the claims in Examples 4.37 and 4.38 above that $X$ is isomorphic respectively to $\operatorname{Spec} \mathbb{Z}_2 \cap \mathbb{Z}_3$ and to $\operatorname{Spec} A_P$. HINT: Use the uniqueness statement in Proposition 4.35 on page 81.

**Exercise B.5.12.** Glue $\operatorname{Spec} \mathbb{Z}_{(2)}$ to itself along the generic point to obtain a scheme $X$. Show that $X$ is not affine. HINT: Show that $\mathcal{O}_X(X) = \mathbb{Z}_{(2)}$.

**Exercise B.5.13.** Show that for any ring $A$, $\Gamma(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}) = A$.

**Exercise B.5.14.** Compute the space $\mathcal{O}_X(X)$ of global sections of the blow-up $X$ and describe the canonical map $X \to \operatorname{Spec} \mathcal{O}_X(X)$.

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**Exercise B.5.15.** Imitate the construction above to define the blow-up of $\mathbb{A}^n$ along a codimension 2 linear space $V(x, y)$.

**Exercise B.5.16.** Let $k$ be a field and consider the scheme $\mathrm{GL}_2 = \operatorname{Spec} k[x_{11}, x_{12}, x_{21}, x_{22}, \frac{1}{\det}]$ where $\det = x_{11}x_{22} - x_{12}x_{21}$.

a) Find a non-constant morphism $f\colon \mathrm{GL}_2 \to \mathbb{P}^1_k$.
b) Show that $f$ is surjective. Deduce that the image of an affine scheme may not be contained in an affine scheme in general.
c) Describe the fibers of $f$.

**Exercise B.5.17.** Check that $L_{-1}$ is indeed the blow-up constructed in Section B.2.

**Exercise B.5.18.** Show that for $m \geqslant 0$, the scheme $L_{-m}$ admits a morphism $\sigma\colon L_{-m} \to Y$ contracting the zero-section $C$ to a point.

**Exercise B.5.19.** For the canonical morphism $\pi\colon L_m \to \mathbb{P}^1_k$, show that

$$\pi_* \mathcal{O}_{L_m} = \bigoplus_{i \geqslant 0} \mathcal{O}_{\mathbb{P}^1_k}(-im).$$

**Exercise B.5.20.** When $k$ is an algebraically closed field the $k$-points of $L_m$ are described by expressions resembling homogeneous coordinates.

a) Show that the $k$-points of $L_m$ are precisely the equivalence classes of triples

$$(x_0 : x_1 \mid t),$$

where $x_0, x_1, t \in k$, with $(x_0, x_1) \neq (0, 0)$ under the relation

$$(x_0 : x_1 \mid t) = (\alpha x_0 : \alpha x_1 \mid \alpha^m t),$$

for $\alpha \in k$ a non-zero scalar.
b) Show that the zero section is the set of points of the form $(x_0 : x_1 \mid 0)$, and that if $m \geqslant 0$ and $p(x_0, x_1)$ is a homogeneous polynomial of degree $m$, then the map $\mathbb{P}^1(k) \to L_m(k)$ given by the assignment

$$(x_0 : x_1) \mapsto (x_0 : x_1 \mid q(x_0, x_1)t)$$

is a well-defined section of $L_m(k) \to \mathbb{P}^1(k)$ (at least in a set-theoretic sense).

**Exercise B.5.21.** Define $f\colon L_{-m}(k) \to \mathbb{A}^{m+1}(k)$ by

$$(x_0 : x_1 \mid t) \mapsto (tx_0^m, tx_0^{m-1}x_1, \ldots, tx_0x_1^{m-1}t, x_1^m)$$

Show that this map is well-defined and collapses the zero-section to the origin. Define and describe a scheme version of this map.

**Exercise B.5.22.** Assume that $k$ is algebraically closed. Let $a_{2g+1} = 1$ and $a_1 = -1$ and $a_i = 0$ for the other indices. Determine the image of $D(x)$ and $D(u)$ in $\mathbb{P}^1_k$. Find all points in $\mathbb{P}^1_k$ where the fiber of the double covering $f$ does not consist of exactly two points. How many are there?

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**Exercise B.5.23.** Let $k$ be an algebraically closed field. Show that the $k$-points of $\mathbb{F}_m(k)$ are in a one-to-one correspondence with the equivalence classes of quadruples

$$(x_0 : x_1 \mid y_0 : y_1)$$

under the equivalence relation

$$(x_0 : x_1 \mid y_0 : y_1) \sim (\alpha x_0 : \alpha x_1 \mid \alpha^m \beta y_0 : \beta y_1)$$

for non-zero scalars $\alpha$ and $\beta$.

**Exercise B.5.24.** The different Hirzebruch surfaces are closely related, as this exercise shows.

a) Show that for some point $P$ there is a map $\mathbb{F}_m - P \to \mathbb{F}_{m-1}$ that induces an isomorphism on the complement of two fibers.
b) Show that for some point $P$ there is a map $\mathbb{F}_{m-1} - P \to \mathbb{F}_m$ that induces an isomorphism on the complement of two fibers.

HINT: On $k$-points, these are $(x_0 : x_1 \mid y_0 : y_1) \mapsto (x_0 : x_1 \mid y_0 : x_1 y_1)$ with $P = (1 : 0 \mid 0 : 1)$ and $(x_0 : x_1 \mid y_0 : y_1) \mapsto (x_0 : x_1 \mid x_1 y_0 : y_1)$ with $P = (1 : 0 \mid 1 : 0)$.

**Exercise B.5.25.** Show that the open subschemes $L_m - C_{-m}$ and $L_m - C_m$ of respectively $L_m$ and $L_{-m}$ are isomorphic over $\mathbb{P}^1_k$. Show that gluing them together gives $\mathbb{F}_m$.

**Exercise B.5.26.** Let $X = \mathbb{P}^1_k$. Show that any element $\mathcal{O}_X(X)$ corresponding to a map $X \to \mathbb{A}^1$ factors via a "constant map" $\operatorname{Spec} k \to \mathbb{A}^1$.

**Exercise B.5.27.** Let $R$ be a local ring. Show that $\mathbb{P}^n(R) = \mathbb{P}^n(R) = (R^n - 0) / \sim$ .
HINT: The maximal ideal must land in some $U_i$; show that the other points must be contained there as well.

**Exercise B.5.28.** Show that the $\mathbb{Z}$-points of $\mathbb{P}^1$ are in bijection with the set of pairs $(a, b) \in \mathbb{Z}^2$ with $a, b$ coprime up to multiplication by $-1$.

**Exercise B.5.29.** A *cyclic cover of* $\mathbb{A}^n_A$ is a scheme described by the equation $y^r = f(x_1, \ldots, x_n)$ in $\mathbb{A}^{n+1}_A$. Generalize (B.19) to define cyclic covers of $\mathbb{P}^n_A$.

**Exercise B.5.30.** Let $P$ be a lattice polytope and let $Q \subset P$ be a *face* of $P$, that is, there is a linear form $\ell$ on $\mathbb{R}^n$ so that $P \subset \{\ell \geqslant 0\}$ and $P \cap \{\ell = 0\} = Q$. Show that there is a closed embedding $X_Q \to X_P$. Describe the corresponding embeddings for the examples $\mathbb{P}^1_k$ and $\mathbb{P}^2_k$.

**Example B.15.** Let $K$ be a field and let $\{A_i\}_{i \in I}$ be a collection of subrings of $K$. Suppose that for each pair $i, j \in I$, there is an element $g_{ij} \in A_i$ such that for all $i, j, k \in I$:

(i) $(A_i)_{g_{ij}} = (A_j)_{g_{ji}}$
(ii) $(A_i)_{g_{ij}g_{ik}} = (A_j)_{g_{ji}g_{jk}} = (A_k)_{g_{ki}g_{kj}}$

where the equalities are as subrings of $K$. Show that the affine schemes $U_i = \operatorname{Spec} A_i$ can be glued together to a scheme $X$. △

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## B.6 Grassmannians

Grassmannian varieties are important examples in algebraic geometry. Just like projective space parameterize lines through the origin of a vector space, Grassmannians are designed to parameterize linear subspaces of a fixed vector space. In fact, for natural numbers $d$ and $n$, there is a scheme $\mathrm{Gr}(d,n)$ of finite type over $\mathbb{Z}$ such that for any field $k$, the $k$-points are given by the set

$$\mathrm{Gr}(d,n)(k) = \Big\{ [L] \mid L \subset k^n \text{ is a linear space of dimension } d \Big\}. \tag{B.20}$$

Projective space $\mathbb{P}^{n-1}_{\mathbb{Z}}$ is the special case when $r = 1$.

In the present discussion, we will find it notationally simpler to work with linear spaces of *codimension* $r$. The basic idea is to parameterize the $r = n - d$ linear equations defining $L$.

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= 0 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= 0 \\ &\vdots \\ a_{r,1}x_1 + a_{q,2}x_2 + \cdots + a_{r,n}x_n &= 0 \end{aligned} \tag{B.21}$$

or equivalently, we write

$$L = \mathrm{Ker}(A) \subset k^n \quad \text{where} \quad A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{r,1} & \cdots & a_{r,n} \end{pmatrix}$$

Of course, the matrix $A$ is not unique in determining $L$. For instance, the two matrices

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} \tag{B.22}$$

give rise to the same 2-dimensional subspace in $k^4$, as the second matrix is the row reduction of the first. In fact, two matrices $A$ and $A'$ give rise to the same subspace $L$ precisely they are related by an element of $\mathrm{GL}_r(k)$, that is, $A' = gA$ for some $g \in \mathrm{GL}_r(k)$. Moreover, the matrix $A$ must have rank $r$ for $L$ to have dimension $n - r$.

Let $\mathbb{A}^{rn}(k)$ denote the $k$-points of the affine space of dimension $rn$ with affine cordinates $x_{ij}$, $i = 1, \ldots, r$, $j = 1, \ldots, n$. We will think of the points of $\mathbb{A}^{rn}(k)$ as $r \times n$-matrices with entries in $k$. Let $Z \subset \mathbb{A}^{rn}(k)$ denote the closed subset defined by the vanishing set of all $r \times r$-minors of the generic matrix $(x_{ij})$. Then $Z$ corresponds to the matrices of rank at most $r - 1$.

Based on the above paragraph, we want to consider the quotient space

$$(\mathbb{A}^{rn}(k) - Z) \,/\, \mathrm{GL}_r(k) \tag{B.23}$$

This is of course possible in the category of topological spaces, as we can simply give the set on the right-hand side the quotient topology. What is not yet clear, is that this works on the level of schemes. While there is a vast theory on constructing quotients of schemes, it turns

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out in the present setting to be simpler to construct the scheme $\mathrm{Gr}(r,k)$ by gluing together affine spaces, like we did for projective space.

To explain how the gluing works we consider again the situation above. If a subspace $L \subset k^n$ is represented by $A$, we can by Gaussian elimination find a more canonical representative, by putting the matrix $A$ in reduced echelon form. In other words, we may represent each $L$ by a matrix $A$ with some $r \times r$ identity matrix as a submatrix. Conversely, any such matrix $A$ determines a subspace $L$ of $k^n$ and now two matrices $A, B$ give the same $L$ if and only if $A = B$. Note that matrices $A$ with a fixed $r \times r$ identity submatrix are parameterized by an affine space of dimension $nr - r^2 = r(n-r)$. It therefore makes sense to try to construct the variety $\mathrm{Gr}(r,n)$ by gluing together these affine spaces.

Write

$$\mathbb{A}^{rn} = \operatorname{Spec} \mathbb{Z}[x_{ij} : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant n]$$

We think of $\mathbb{A}^{rn}$ as an affine space parameterizing $r \times n$-matrices.

We write $M = (x_{ij})$ for the $r \times n$-matrix of indeterminates. If $I \subset \{1, \dots, n\}$ is a subset of size $r$ and $A$ is an $r \times n$-matrix, we let $A_I$ denote the $r \times r$ submatrix of $A$ given by the columns in $I$.

For each subset $I \subset \{1, \dots, n\}$ of size $r$, consider the closed subscheme of $\mathbb{A}^{rn}$ defined by

$$U_I = V\left(x_{ij} - \delta_{ij} : i, j \in I\right) \simeq \mathbb{A}^{r(n-r)}$$

Note that there are $\binom{n}{r}$ choices for the $r$ columns, so $\binom{n}{r}$ affine spaces in total.

If $J$ is another $r$-subset, let $U_{I,J} \subset U_I$ denote the distinguished open set $D(\det M_J)$ in $U_I$.

**Example B.16.** For $n = 4, r = 2$, there are 6 affine spaces, each isomorphic to $\mathbb{A}^4$. In suggestive notation:

$$U_{12} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} 1 & 0 & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \end{bmatrix} \quad U_{13} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} 1 & x_{12} & 0 & x_{14} \\ 0 & x_{22} & 1 & x_{24} \end{bmatrix}$$
$$U_{14} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} 1 & x_{12} & x_{13} & 0 \\ 0 & x_{22} & x_{23} & 1 \end{bmatrix} \quad U_{23} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} x_{11} & 1 & 0 & x_{14} \\ x_{21} & 0 & 1 & x_{24} \end{bmatrix}$$
$$U_{24} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} x_{11} & 1 & x_{13} & 0 \\ x_{21} & 0 & x_{23} & 1 \end{bmatrix} \quad U_{34} = \operatorname{Spec} \mathbb{Z}\begin{bmatrix} x_{11} & x_{12} & 1 & 0 \\ x_{21} & x_{22} & 0 & 1 \end{bmatrix}.$$

△

If $T$ is a scheme, the $T$-valued points $U_I(T)$ is the set of $r \times n$-matrices with entries in $\mathcal{O}_T(T)$ and the identity matrix in the columns indexed by $I$. Furthermore, $U_{I,J}(T)$ is the set of $r \times n$-matrices $A \in U_I(T)$ such that the determinant of $M_J$ is invertible, i.e., belongs to $\mathcal{O}_T(T)^\times$.

Note that if $A$ is an $r \times n$ matrix and $\det A_I$ is invertible, then $A_I^{-1}A$ has the identity matrix in columns $I$. Hence, $A \mapsto (A_I)^{-1}A$ defines a bijection of sets $U_{J,I}(T) \to U_{I,J}(T)$. Note that if $f\colon S \to T$ is a morphism of schemes, the pullback induces a map $U_I(T) \to U_I(S)$. In this way, we obtain natural transformations betwen the functors of points $h_{U_{I,J}}$ and $h_{U_{J,I}}$. By Yoneda's Lemma, we obtain isomorphisms of affine schemes

$$\tau_{JI} : U_{I,J} \longrightarrow U_{J,I}$$

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These isomorphisms moreover satisfy the gluing condition

$$\tau_{KI} = \tau_{KJ} \circ \tau_{JI}. \tag{B.24}$$

Indeed, both sides of the equation, when applied to a matrix $A$ with the identity matrix in the $I$-columns, row-reduce $A$ to a matrix with the identity matrix in the $K$-columns. The left-hand side does this directly, while the right-hand side does so in two steps: first to the identity in columns $J$, then to the identity in columns $K$.

It follows that the affine spaces $U_I$ glue to a scheme, which we call the *Grassmannian* $\mathrm{Gr}(d, n)$. The next proposition tells us that the $k$-points of $\mathrm{Gr}(d, n)$ are in bijection with the set of $d$-dimensional linear subspaces of $k^n$, as we alluded to in (B.20).

**Proposition B.17.** Let $k$ be a field. Then the $k$-points of $\mathrm{Gr}(r, n)$ are in bijection with the set of equivalence classes of $r \times n$-matrices of rank $r$ modulo the action of $\mathrm{GL}_r(k)$.

*Proof* If $A$ is an $r \times n$-matrix with entries in $k$, and the rank of $A$ is equal to $r$, then some submatrix $A_I$ must be invertible. The bijection is defined by sending $A$ to $A_I^{-1}A$, which defines a $k$-point in $U_I$, and hence in $\mathrm{Gr}(d, n)$. If $A'$ is another matrix equivalent to $A$, then also $A'_I$ is invertible, and $A'^{-1}_I A = A_I^{-1} A$. It follows that the assignment is injective. It is also surjective, as any $k$-point $\mathrm{Gr}(d, n)$ lies in some $U_I$, and hence comes from a matrix $A$ with identity matrix in the $I$-columns. □

The Grassmannians resemble projective spaces in several ways:

**Generalized homogeneous coordinates.** In light of Proposition B.17, we may also talk about *generalized homogeneous coordinates* on $\mathrm{Gr}(r, n)$: for a field $k$, the $k$-points are equivalence classes $[M]$ of $r \times n$-matrices, where we say $[M] = [M']$ if $M' = gM$ for some $g \in \mathrm{GL}_r(k)$. For instance, for the two matrices in (B.22) we have

$$\left[\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix}\right].$$

**Local coordinates.** Just as the ratios $x_0/x_i, \ldots, x_n/x_i$ form affine coordinates on $D_+(x_i) \subset \mathbb{P}^n$, the affine coordinates of $U_I$ are given by the entries of the matrix $M_I^{-1}M$. More precisely, if $\mathbb{Z}[M_I^{-1}M]$ denotes the $k$-algebra generated by the entries in the matrix $M_I^{-1}M$, then $\mathrm{Gr}(r, n)$ is obtained by gluing together the affine spaces

$$U_I = \operatorname{Spec} \mathbb{Z}[M_I^{-1}M] \simeq \mathbb{A}^{rn} \tag{B.25}$$

using the isomorphisms given by the following equality of subrings of $\mathbb{Q}(x_{11}, \ldots, x_{nn})$:

$$\mathbb{Z}[M_I^{-1}M]_{\det(M_I^{-1}M_J)} = \mathbb{Z}[M_J^{-1}M]_{\det(M_J^{-1}M_I)}. \tag{B.26}$$

**Quotient morphism.** If $Z \subset \mathbb{A}^{rn}$ denotes the closed subset defined by all the $r \times r$-minors of the matrix of variables $M$, then there is a a quotient morphism

$$\pi \colon \mathbb{A}^{rn} - Z \longrightarrow \mathrm{Gr}(r, n). \tag{B.27}$$

This morphism is obtained by gluing together morphisms between affine schemes

$$\pi_I \colon D(\det M_I) \longrightarrow U_I \tag{B.28}$$

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To define (B.28), we can use Yoneda's Lemma. Note that a $T$-point of the open subscheme $D(\det M_I) \subset \mathbb{A}^{rn}$ is given by an $r \times n$-matrix $A$ with entries in $\mathcal{O}_T(T)$ so that $\det A_I$ is invertible in $\mathcal{O}_T(T)$. On the level of $T$-points, the morphism (B.28) then sends the matrix $A$ to $A_I^{-1}A$ which is a well defined point in $U_I(T)$. It is clear that this defines a natural transformation between the functors of points, and that the morphism is surjective. Moreover, two matrices $A, B \in D(\det M_I)$ map to the same point in $U_I$ if and only if $A_I^{-1}A = B_I^{-1}B$, which happens if and only if $B = gA$ where $g = A_I B_I^{-1}$ is an element in $\mathrm{GL}_r(\mathcal{O}_T(T))$.

In fact, the multiplication map defines a bijection between $\mathrm{GL}_r(\mathcal{O}_T(T)) \times U_I(T)$ and $D(\det M_I)(T)$, and consequently an isomorphism of schemes

$$\mathrm{GL}_r \times U_I \xrightarrow{\simeq} D(\det M_I) \tag{B.29}$$

It is straightforward to check that the morphisms (B.28) are compatible with the gluing of $\mathrm{Gr}(d, n)$, so we obtain the morphism (B.27). In light of (B.29), $\pi$ is a so-called 'fiber bundle' over $U_I$ with fiber $\mathrm{GL}_r$'.

**Universal sheaves.** On projective space, $\mathbb{P}^n$ there is the invertible sheaf $\mathcal{O}(1)$ which is glued together by copies of $\mathcal{O}_{U_i}$ using the transition functions $g_{ij} = \frac{x_j}{x_i}$ over the distinguished opens $D_+(x_i) \simeq \mathbb{A}^n$. Moreover, the homogeneous coordinates $x_0, \ldots, x_n$ determine a surjection

$$\mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

Likewise, the Grassmannian carries a *universal quotient sheaf* $\mathcal{Q}$, which satisfies $\mathcal{Q}|_{U_I} \simeq \mathcal{O}_{U_I}^r$, and it is glued together using the transition functions

$$g_{IJ} = M_I^{-1} M_J \in \mathrm{GL}_r(\mathcal{O}_{U_{I,J}}). \tag{B.30}$$

and there is a surjection

$$\mathcal{O}_G^n \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{B.31}$$

Note that on $\mathbb{A}^{rn}$, there is such a map, namely, the map given by the $r \times n$ matrix $M$

$$\mathcal{O}_{\mathbb{A}^{rn}}^n \xrightarrow{M} \mathcal{O}_{\mathbb{A}^{rn}}^r \tag{B.32}$$

This is surjective on all stalks contained in the open set $\bigcup_I D(\det M_I)$. Moreover, restricting this to the $U_I$ we obtain

$$\mathcal{O}_{U_I}^n \xrightarrow{M} \mathcal{O}_{U_I}^r \tag{B.33}$$

In terms of the affine charts in (B.25), the map (B.33) is induced by

$$\mathbb{Z}[M_I^{-1}M]^n \xrightarrow{M_I^{-1}M} \mathbb{Z}[M_I^{-1}M]^r$$

Passing to the localization, these maps glue to the sheaf map (B.31), because the following diagram commutes.

$$\begin{array}{ccc} (\mathbb{Z}[M_J^{-1}M]_{\det(M_J^{-1}M_I)})^r & \xrightarrow{\quad M_I^{-1}M_J \quad} & (\mathbb{Z}[M_I^{-1}M]_{\det(M_I^{-1}M_J)})^r \\ & {\scriptstyle M_J^{-1}M} \searrow \quad \nearrow {\scriptstyle (M_I)^{-1}M} & \\ & (\mathbb{Z}[M_I^{-1}M]_{\det(M_I^{-1}M_J)})^n & \end{array}$$

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**The Plucker embedding** Consider the morphism

$$p\colon \mathbb{A}^{rn} \longrightarrow \mathbb{A}^{\binom{n}{r}} = \operatorname{Spec} \mathbb{Z}[y_I]$$

which is induced by all the $r \times r$ determinants of $M$, that is, $p^\sharp(y_I) = \det M_I$. This induces a morphism

$$\rho\colon \mathbb{A}^{rn} - Z \longrightarrow \mathbb{P}^{\binom{n}{r}-1}$$

where $Z = \bigcap_I V(\det M_I)$ is the closed subset defined by all the $r \times r$-minors of $M$.

We claim that $\rho$ is a closed embedding. To do this, it suffices to show that $\rho$ restricts to an isomorphism over each $U_I$, which maps into the distinguished open $D_+(y_I) \subset \mathbb{P}^{\binom{n}{r}-1}$. We write $\rho_I\colon U_I \to D_+(y_I)$ for this morphism.

**Lemma B.18.** For each $I$, the morphism $\rho_I$ is a closed embedding.

*Proof* After re-indexing, it suffices to consider the case $I = \{1, 2, \dots, r\}$. Then, if we expand the minors of the $r \times n$ matrix (corresponding to a matrix $U_I$)

$$\begin{pmatrix} 1 & & & x_{1,r+1} & \dots & x_{1,n} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & x_{r,r+1} & \dots & x_{r,n} \end{pmatrix}$$

it is clear that the set of minors contains (up to sign) all the variables $x_{ij}$ with $j \notin I$. Hence $\rho_I : \mathbb{A}^{r(n-r)} \to \mathbb{A}^{\binom{n}{r}-1}$ is given by the graph of a morphism $\mathbb{A}^{r(n-r)} \to \mathbb{A}^{\binom{n}{r}-1-r(n-r)}$, and hence is a closed immersion (Exercise 4.11.56). □

This means that $\mathrm{Gr}(r, n)$ embeds as a closed subscheme of $\mathbb{P}^{\binom{n}{r}-1}$. The image of $\rho$ is defined by all the polynomial relations between the $r \times r$-minors of the matrix $M$. These relations are known to be generated by the so called *Plucker quadrics*. These quadrics take the form

$$\sum_{l=1}^{r+1} (-1)^l y_{i_1,\dots,i_{r-1},j_l} y_{j_1,\dots,\hat{j_l},\dots j_{r+1}} = 0. \tag{B.34}$$

There is one relation for each pair of increasing sequences $I = (i_1 < \dots < i_{r-1})$, $J = (j_1 < \dots < j_{r+1})$ of length $r-1$ and $r+1$ respectively, and $j_1, \dots, \hat{j_l} \dots j_{r+1}$ denotes the sequence $j_1, \dots, \dots j_{r+1}$ with the term $j_l$ omitted. Moreover, in the sum (B.34), the indexes in the variables may not be increasing: here we adopt the convention that $y_I$ is equal to 0 if $I$ contains repeated elements, and $y_{i_1,\dots,i_r} = -y_{i_1,\dots,i_{s-1},i_{s+1},\dots,i_r}$.

The proof that these quadrics generate the entire ideal of the Grassmannian is not particularly difficult, but it requires algebraic manipulations that make it a little bit lengthy (see (Dolgachev, 2003, Corollary 2.5)). Besides we do not need this fact in what follows. In fact, $\mathrm{Gr}(k, n)$ is probably the best example of a projective variety that is not best studied by its equations in some projective space.

**Example B.19.** Let us continue the example of $n = 4$, $r = 2$. In this case, the image of $\phi$ is the quadric

$$Q = V(y_{12}y_{34} - y_{13}y_{24} + y_{14}y_{23}) \subset \mathbb{P}^5$$

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This is a consequence of the following relation between the minors of a $2 \times 4$ matrix

$$\begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} - \begin{vmatrix} x_{11} & x_{13} \\ x_{21} & x_{23} \end{vmatrix} \begin{vmatrix} x_{12} & x_{14} \\ x_{22} & x_{24} \end{vmatrix} + \begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix} = 0$$

and the fact that both $\mathrm{Gr}(2,4)$ and $Q$ are integral and of the same dimension (over a field, they are 4-dimensional). △

If $R$ is a ring, we define $\mathrm{Gr}_R(d,n) \simeq \mathrm{Gr}_{\mathbb{Z}}(d,n) \times_{\mathbb{Z}} \operatorname{Spec} R$. The main case of interest is when $R = k$ is a field.

**Proposition B.20.** For a field $k$, the Grassmannian $\mathrm{Gr}_k(r,n)$ is a non-singular projective variety of dimension

$$\dim \mathrm{Gr}_k(r,n) = r(n-r).$$

**B.6.0 The Grassmannian functor.** Let $r$ and $n$ be non-negative integers and consider the functor

$$\mathbf{G}_{r,n}(T) = \left\{ \begin{array}{l} \text{short exact sequences } 0 \to \mathcal{U} \to \mathcal{O}_T^n \to \mathcal{Q} \to 0 \\ \text{where } \mathcal{U}, \mathcal{Q} \text{ are locally free of ranks } n-r \text{ and } r \end{array} \right\} / \sim$$

where two exact sequences are said to be equivalent if there are isomorphisms making the following diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ & & \downarrow \simeq & & \| & & \downarrow \simeq & & \\ 0 & \longrightarrow & \mathcal{U}' & \longrightarrow & \mathcal{O}_T^n & \longrightarrow & \mathcal{Q}' & \longrightarrow & 0 \end{array}$$

**Theorem B.21.** The functor $\mathbf{G}_{r,n}$ is represented by the Grassmannian $\mathrm{Gr}(r,n)$.

*Proof* Suppose we have a scheme $T$ and a short exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{U}' \longrightarrow \mathcal{O}_T^n \longrightarrow \mathcal{Q}' \to 0. \tag{B.35}$$

Choose an affine open covering of $T$ given by $T_i = \operatorname{Spec}(A_i)$ so that $\mathcal{Q}'|_{T_i} \simeq \mathcal{O}_{T_i}^r$. The surjection $\mathcal{O}_{T_i}^n \to \mathcal{Q}'|_{T_i} \simeq \mathcal{O}_T^r$ is given by some $n \times r$ matrix $M_i$ with entries in $A_i$, which naturally determines a morphism $g_i : \operatorname{Spec} A_i \to \mathbb{A}^{rn}$.

As $\mathcal{O}_T^n \to \mathcal{Q}$ is surjective, all of these morphisms factor through $\mathbb{A}^{rn} - Z$, so they induce morphisms $\varphi \circ g_i : \operatorname{Spec}(A_i) \to \mathrm{Gr}(r,n)$. The morphisms $h_i = \varphi \circ g_i$ glue to give a morphism $h\colon T \to \mathrm{Gr}(r,n)$. Indeed, the $h_i$ are defined from the morphism of locally free sheaves, which implies that they agree on overlaps. Moreover, it means that the construction of $h$ does not depend on the trivializations used, and also is independent of choice of the sequence (B.35) within its equivalence class (as two equivalent sequences may be simultaneously trivialized by the same data).

If $\phi\colon T \to \mathrm{Gr}(r,n)$ is any morphism, covering $\mathrm{Gr}(r,n)$ by the open affine pieces $U_I$ and then covering $h^{-1}(U_I)$ by open affine pieces $\operatorname{Spec}(A_i)$ over which $\mathcal{Q}'$ is trivial, we see that $\phi$ pulls back the universal family to $S$ if the following diagram commutes:

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But this happens if and only if $h|_{\mathrm{Spec}(A_i)} = g_i$. Therefore $g$ is the unique morphism pulling back the universal family to $T$, as required. □

*Remark* B.22. The Plucker embedding $P\colon \mathrm{Gr}(k,n) \to \mathbb{P}_{\mathbb{Z}}^{\binom{n+r}{r}-1}$ has an elegant interpretation in terms of the functor of points. The morphism $P$ comes from the natural transformation which takes a $T$-valued point of $\mathrm{Gr}(k,n)$, that is, a quotient $\mathcal{O}_T^n \to \mathcal{Q} \to 0$, and sends it to $\wedge^r\mathcal{O}^n \to \wedge^r Q \to 0$, which since $\wedge^r Q$ has rank 1, defines a point in $\mathbb{P}_{\mathbb{Z}}^{\binom{n+r}{r}-1}(T)$. It is possible to use this to produce a completely coordinate-free proof of the fact that $P$ is an embedding in this functorial language.

**Example B.23.** The functor $F\colon \mathsf{Sch}^{op} \to \mathsf{Sets}$ given by

$$F(T) = \{\text{locally free sheaves of rank } r \text{ on } T\}/\text{isomorphism}$$

is not representable. To see why, assume that there is a scheme $X$ such that $F \simeq h_X$. For a scheme $T$, let $o_T \in F(T)$ denote the element corresponding to the trivial sheaf $\mathcal{O}_T^r$.

Now, let $T$ be any scheme, and let $\mathcal{E}$ be a locally free sheaf $\mathcal{E}$ or rank $r$ on it. We may choose a covering $U_i$ of $T$ such that $\mathcal{E}|_{U_i} \simeq \mathcal{O}_{U_i}^r$. This means that the corresponding element $e \in F(T)$ must map to $o_{U_i} \in F(U_i)$ via the maps $F(T) \to F(U_i)$. The same is of course true for the element $o_T \in F(T)$. So by the above remark, we must have that $e = o_T \in F(T)$. However, this would mean that any locally free sheaf on $T$ is trivial, which is false in general. For instance, for $T = \mathbb{P}^n$ and $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^r$, then $\mathcal{E}$ is non-trivial. △

**B.6.0 Enumerative problems.** The Grassmannian appears in classical algebraic geometry involving enumerative questions involving linear spaces in projective space. One example is the following theorem, which is one of the most celebrated results in classical algebraic geometry.

**Theorem B.24 (Cayley–Salmon).** Over an algebraically closed field, every smooth cubic surface $S \subset \mathbb{P}^3$ contains exactly 27 lines.

**Example B.25.** For illustration, let us consider the lines on the Fermat cubic surface defined by

$$F = x_0^3 + \cdots + x_3^3 = 0$$

We identify lines in $\mathbb{P}_k^3$ with planes in $k^4$. Let us find equations for the set of points in $\mathrm{Gr}(2,4)$ corresponding to lines in $S = V(F)$. Let us consider lines $L$ of the form

$$x_0 + a_2x_2 + b_2x_3 = 0,$$
$$x_1 + a_3x_2 + b_3x_3 = 0.$$

These correspond to points contained in the affine chart $U_{\{1,2\}} \subset \mathrm{Gr}(2,4)$. Then $L$ is contained in $S$ if and only if the cubic $F$ restricts to the zero polynomial in $L$. In other words, $F(-a_2s - b_2t, -a_3s - b_3t, s, t)$ is the zero polynomial in $s$ and $t$. Expanding and collecting

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coefficients, we obtain the following system of equations for $a_2, a_3, b_2, b_3$:

$$\begin{aligned} a_2^3 + a_3^3 &= 1 \\ a_2^2 b_2 + a_3^2 b_3 &= 0 \\ a_2 b_2^2 + a_3 b_3^2 &= 0 \\ b_2^3 + b_3^3 &= 1. \end{aligned}$$

Let $\mathfrak{a}$ denote the ideal generated by these equations. Then $\mathfrak{a}$ can be simplied to:

$$\mathfrak{a} = (a_2a_3, a_2b_2, a_3b_3, b_2b_3, b_2^3 + b_3^3 - 1, a_3^3 + b_3^3 - 1, a_2^3 - b_3^3).$$

From this we see that either $a_2 = 0$ or $a_3 = 0$. If $a_2 = 0$, then the last equation implies that $b_3 = 0$ as well, and the two remaining equations reduce to $b_2^3 = 1$ and $a_3^3 = 1$. Here $a_3$ and $b_3$ can be any cubic root of unity, leading to $3 \times 3 = 9$ solutions. Doing the same for $a_3 = 0$ leads to a number of $9 + 9 = 18$ lines in $U_I$. By permuting the coordinates, we find the lines contained in the other $U_I$ as well, leading to a total of 27 lines contained in $V(F)$:

$$\begin{aligned} x_0 + \omega^k x_1 = x_2 + \omega^j x_3 = 0, \quad 0 \leqslant j, k \leqslant 2 \\ x_0 + \omega^k x_2 = x_1 + \omega^j x_3 = 0, \quad 0 \leqslant j, k \leqslant 2 \\ x_0 + \omega^k x_3 = x_1 + \omega^j x_2 = 0, \quad 0 \leqslant j, k \leqslant 2 \end{aligned}$$

where $\omega$ is a primitive cube root of 1. △

## B.7 The Picard group of a Grassmannian

**Proposition B.26.** Let $k$ be a field. Then the Picard group of the Grassmannian $\mathrm{Gr}(r, n)$ is isomorphic to $\mathbb{Z}$. It is generated by the restriction of $\mathcal{O}(1)$ from the Plücker embedding.

*Proof* Consider $\mathrm{Gr}(r, n) \subset \mathbb{P}^N$ embedded via the Plücker embedding. Fix an $r$-tuple $I \subset \{1, \ldots, n\}$. Then the divisor $H = \mathrm{div}(x_I)|_X$ is a well-defined divisor on $\mathrm{Gr}(r, n)$. The complement $\mathrm{Gr}(r, n)$ is the open set $U_I \subset \mathrm{Gr}(r, n)$ (corresponding to $n \times r$-matrices whose $I$-th minor is nonzero). As $U_I \simeq \mathbb{A}^{r(n-r)}$ has trivial class group, the exact sequence (16.5) shows that the class group of $\mathrm{Gr}(r, n)$ is generated by $H$. Clearly, no multiple of $H$ is zero in $\mathrm{Cl}(\mathrm{Gr}(r, n))$. That would mean that some $\mathcal{O}_X(m) \simeq \mathcal{O}_X$. This is not possible, as $\mathcal{O}(1)$ has $N + 1$ linearly independent global sections. □

**Exercise B.7.1.** Show that the set of lines on the quadric surface $S = V(x_0x_3 - x_1x_2) \subset \mathbb{P}^3_k$ is a union of two conics in $\mathrm{Gr}(2, 4) \subset \mathbb{P}^5_k$.

**Exercise B.7.2** (Gluing relative schemes)**.** Let $X$ be a scheme and suppose that we are given the following data:

(i) For each affine open $U \subset X$, a scheme $Y_U$ and a morphism $\pi_U \colon Y_U \to U$.
(ii) For any pair of affine open subschemes $V \subset U \subset X$, a morphism $\rho_{UV} \colon Y_V \to$

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$Y_U$ such that the following diagram commutes

$$\begin{array}{ccc} Y_V & \xrightarrow{\rho_{U,V}} & Y_U \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi_U} \\ V & \hookrightarrow & U \end{array} \tag{B.36}$$

and induces an isomorphism $Y_V \simeq \pi_U^{-1}(V)$.

(iii) For any affine open subschemes $W \subset V \subset U \subset X$, we have $\rho_{U,W} = \rho_{V,U} \circ \rho_{W,V}$.

Show that there exists a scheme $Y$ together with a morphism $\pi\colon Y \to X$, and isomorphisms $\iota_U\colon \pi^{-1}(U) \to Y_U$, such that for each $V \subset U$ affine, the following diagram commutes:

$$\begin{array}{ccccc} \pi^{-1}(U) & \xrightarrow{\iota_U} & Y_U & \xrightarrow{\pi_U} & U \\ \uparrow & & {\scriptstyle \rho_{U,V}}\uparrow & & \uparrow \\ \pi^{-1}(V) & \xrightarrow{\iota_V} & Y_V & \xrightarrow{\pi_V} & V \end{array}$$

and $\pi_U \circ \iota_U = \pi|_U$ for each $U$. Show that as an $X$-scheme, $Y$ is unique up to isomorphism. HINT: See (**?**, Tag 01LG).

## B.8 Relative Spec

Let $X$ be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-*algebras*. This means that $\mathcal{A}$ is a quasi-coherent sheaf and for each open set $U \subset X$, the group $\mathcal{A}(U)$ is an algebra over the ring $\mathcal{O}_X(U)$.

Let us apply Proposition **??** to the case where

$$Y_U = \operatorname{Spec} \mathcal{A}(U)$$

and $\pi_U\colon Y_U \to U$ is the morphism induced by the ring map $\mathcal{O}_X(U) \to \mathcal{A}(U)$. Let us check the that the second condition in the proposition is satisfied. If $V \subset U$ is another affine subset, we have there is a ring map $\mathcal{A}(U) \to \mathcal{A}(V)$ which induces a morphism $Y_V \to Y_U$ inducing a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec}(\mathcal{A}(V)) & \xrightarrow{\rho_{U,V}} & \operatorname{Spec}(\mathcal{A}(V)) \\ \downarrow{\scriptstyle \pi_V} & & \downarrow{\scriptstyle \pi_U} \\ V & \hookrightarrow & U \end{array} \tag{B.37}$$

The diagram is actually Cartesian, because $\mathcal{A}(V) = \mathcal{A}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V)$, as $\mathcal{A}$ is quasi-coherent. The third condition is also satisfied, because the ring map $\mathcal{A}(U) \to \mathcal{A}(W)$ factors via restriction to $V$.

It follows that the schemes $Y_U$ glue together to a scheme, which we denote $\mathbf{Spec}(\mathcal{A})$ which we call the 'relative spectrum of $\mathcal{A}$'. There is a morphism $\pi\colon \mathbf{Spec}(\mathcal{A}) \to X$ which satisfies

$$\pi_* \mathcal{O}_{\mathbf{Spec}(\mathcal{A})} = \mathcal{A}.$$

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The scheme $\mathbf{Spec}(\mathcal{A})$ satisfies the following universal property: For each morphism $h : Z \to X$ with a map of $\mathcal{O}_X$-algebras $\mathcal{A} \to h_*\mathcal{O}_Z$, there is a unique morphism $f : Z \to \mathbf{Spec}(\mathcal{A})$ such that $h = \pi \circ f$.

**Example B.27.** For $\mathcal{A} = \mathcal{O}_X[t_1, \ldots, t_n]$, the relative Spec coincides with the relative affine space $\mathbb{A}^n_X = X \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{A}^n$. △

**Example B.28.** Let $X = \mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$ and let $f \in k[x_1, \ldots, x_n]$ be a polynomial. Then

$$\mathcal{A} = \mathcal{O}_X[t]/(t^m - f(x_1, \ldots, x_n))$$

is an $\mathcal{O}_X$-algebra. The relative spec $Y = \operatorname{Spec} \mathcal{A}$ is exactly the degree $m$ cover of $\mathbb{A}^n$ branched over the zero set of $f$. △

**Example B.29.** More generally, let $X$ be a normal integral scheme, $D \subset X$ an effective divisor, and let $L$ be an invertible sheaf on $X$ such that $L^{\otimes m} \simeq \mathcal{O}_X(D)$. Let $s \in \mathcal{O}_X(D)$ be the section that defines $D$; we will view it as a map $s : \mathcal{O}_X \to L^{\otimes m}$. Define the $\mathcal{O}_X$-module

$$\mathcal{A} = \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-m+1}$$

This becomes an $\mathcal{O}_X$-algebra via the multiplication

$$L^{-a} \otimes L^{-b} \simeq L^{-a-b} \otimes \mathcal{O}_X \xrightarrow{id \otimes s} L^{-a-b} \otimes L^m \simeq L^{-a-b+m}.$$

Let $Y = \mathbf{Spec}\mathcal{A}$ with the projection $\pi \colon Y \to X$. We call $Y$ the ramified cyclic cover of $s$.

Over an open set $U$ where $L \simeq \mathcal{O}_U$, pick a local generator $s$. The image $s^m \in \Gamma(U, L^m)$. On such an open, we have $\mathcal{A}|_U \simeq \mathcal{O}_U^m$, which is generated by 1 and $f$ subject to the relation $z^m = f$.

It is not hard to show that $Z$ is nonsingular if and only if $X$ and $D$ are. △

**Exercise B.8.1.** Check that the scheme $\mathbf{Spec}(\mathcal{A})$ and the morphism $\pi$ satisfies the above universal property.

**Example B.30** (Closed subschemes)**.** An important special case is when $\mathcal{A} = \mathcal{O}_X/\mathcal{I}$ for some quasi-coherent ideal $\mathcal{I}$. In this case there is a morphism

$$i : \operatorname{Spec}(\mathcal{O}_X/\mathcal{I}) \longrightarrow \operatorname{Spec}(\mathcal{O}_X) = X$$

and $Y = \operatorname{Spec}(\mathcal{O}_X/\mathcal{I})$ is exactly the closed subscheme associated to $\mathcal{I}$. △

**Example B.31** (Vector bundles)**.** Let $\mathcal{E}$ denote a locally free sheaf of rank $r$. The symmetric algebra

$$\operatorname{Sym}^*(\mathcal{E}^\vee) = \mathcal{O}_X \oplus \mathcal{E}^\vee \oplus S^2(\mathcal{E}^\vee) \oplus \cdots$$

is naturally an algebra over $\mathcal{O}_X$. The corresponding relative Spec is denoted by $\mathbb{V}(\mathcal{E})$. The projection $\pi \colon \mathbb{V}(\mathcal{E}) \to X$ is what is known as a *vector bundle*; all the scheme-theoretic fibers are affine spaces of dimension $r$. More precisely, if $x \in X$, the fiber $\mathcal{E}(x) = \mathcal{E} \otimes_{\mathcal{O}_X} k(x)$ is isomorphic to $k(x)^r$, and so the scheme-theoretic fiber of $\pi$ over $x$ is isomorphic to the spectrum of

$$\operatorname{Sym}^*(k(x)^r) \simeq k(x)[t_1, \ldots, t_r]$$

△

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**Example B.32** (Normalization)**.** Now suppose $X$ is a general integral scheme. For an affine subset $U = \operatorname{Spec} A \subset X$, we set $Y_U = \operatorname{Spec} \overline{A}$, where $\overline{A}$ is the integral closure of $A$ in $K(X)$. We check that the collection of morphisms $\pi_U \colon Y_U \to U$ satisfy the conditions of Proposition **??**, so that they glue to a morphism $\pi \colon \overline{X} \to X$.

If $U, V$ are two affines with $V \subset U$, we can consider the open subscheme $W = \pi_U^{-1}(V) \subset Y_U$. By assumption, this scheme is affine, because $\pi_U$ is an affine morphism, and integral and normal, being an open set in $Y_U$. Furthermore, inside $K(X)$, we have

$$\mathcal{O}_W(W) = \bigcap_{y \in W} \mathcal{O}_{Y_U, y}.$$

As the local rings $\mathcal{O}_{Y_U,y}$ are integrally closed, we see that $\mathcal{O}_W(W)$ is normal. As $W \to V$ is finite, $\mathcal{O}_X(V) \to \mathcal{O}_W(W)$ a finite ring extension, and we see that $\mathcal{O}_W(W)$ is the integral closure of $\mathcal{O}_X(V)$ in $K(X)$. This allows us to canonically identify $Y_V$ with $\pi_U^{-1}(V)$, and consequently the fiber product $Y_U \times_U V$. Finally, if $W \subset V \subset U$ are three affines, the map $Y_W \to Y_U$ clearly factors via $Y_V$. Therefore the conditions of Proposition **??** are satisfied, and the schemes $Y_V$ glue to a scheme which we will denote by $\overline{X}$.

Finally, we prove that the scheme $\overline{X}$ and $\pi_X \colon \overline{X} \to X$ satisfy the universal property. So let $h \colon Z \to X$ be a dominant morphism from a normal integral scheme $Z$. Over each $U_i$, we have an induced dominant morphism $h^{-1}(U_i) \to U_i$, which by the universal property over the $U_i$ must factor uniquely via $\overline{U_i}$ via a morphism $h_i \colon h^{-1}(U_i) \to \overline{U_i}$. Again the uniqueness in the universal property tells us that these maps must agree over the overlaps $h^{-1}(U_{ij})$. Since the $h^{-1}(U_i)$ form an open cover of $Z$, these maps glue to a map $\overline{h} \colon Z \to \overline{X}$ factoring $h$. △

## B.9 Relative Proj

Let $X$ be a scheme and let $\mathcal{R}$ be a quasi-coherent sheaf of *graded $\mathcal{O}_X$-algebras*. This means that for each open set $U \subset X$, the group $\mathcal{R}(U)$ is a graded ring with degree 0 isomorphic to $\mathcal{O}_X(U)$.

For an open affine $U \subset X$, set $Y_U = \operatorname{Proj} \mathcal{R}(U)$, with projection $\pi \colon Y_U \to U$ induced by the natural map $\operatorname{Proj} \mathcal{R}(U) \to \operatorname{Spec} \mathcal{O}_X(U) = U$. If $V \subset U$ is another affine, the map $\mathcal{R}(U) \to \mathcal{R}(V)$ is a map of graded rings, this induces a map $Y_V \to Y_U$. Checking that the conditions of Proposition **??** are satisfied is similar to the Relative Spec-case. We call the resulting scheme $\mathbf{Proj}(\mathcal{R}) \to X$ the 'relative Proj of $\mathcal{R}$'.

**Example B.33** (Projective bundles)**.** Let $\mathcal{E}$ be a locally free sheaf on a scheme $X$ and consider the symmetric algebra $\operatorname{Sym}^*(\mathcal{E})$, which we for notational convenience write as

$$\operatorname{Sym}^*(\mathcal{E}) = \mathcal{O}_X \oplus \mathcal{E}t \oplus S^2(\mathcal{E})t^2 \oplus \cdots$$

The scheme $\mathbf{Proj}(\operatorname{Sym}^*(\mathcal{E}))$ is usually[1] denoted by $\mathbb{P}(\mathcal{E})$. △

**Example B.34** (Hirzebruch surfaces)**.** Consider the sheaf $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-d)$, where

[1] Warning: The notation in the literature here is not always consistent; in many texts $\mathbb{P}(\mathcal{E})$ refers to $\mathbf{Proj}(\operatorname{Sym}^*(\mathcal{E}^\vee))$. The differences in notation is rooted in the contrasting viewpoints of denoting $\mathbb{P}^n_k$ by lines or hyperplanes in $k^{n+1}$. The notation here is the so-called 'Grothendieck notation' for projective bundles.

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$d \in \mathbb{Z}$. The corresponding projective bundle $\mathbb{P}(\mathcal{E})$ is exactly the Hirzebruch surface $\mathbb{F}_m$ from Example XX. △

## B.10 Blow-ups

Let $A$ be a $k$-algebra. We are interested in morphisms

$$\operatorname{Spec}(A) \to \mathbb{P}^n_k = \operatorname{Proj}(k[x_0, \ldots, x_n]).$$

Interpreting $A$ as $\mathbb{P}^0_A = \operatorname{Proj} A[t]$, the map of graded $k$-algebras

$$\phi\colon k[x_0, \ldots, x_n] \to A[t]; \quad x_i \mapsto f_i t$$

defines an $A$-valued point $(f_0 : \cdots : f_n)$ of $\mathbb{P}^n_k$.

The base locus $\operatorname{Bs}(\phi)$ is empty if and only if $(f_0, \ldots, f_n) = (1)$, which happens if and only if $(f_0 t, \ldots, f_n t) = (t)$ in $A[t]$, that is, $f_0 t, \ldots, f_n t$ generate the irrelevant ideal.

If $(f_0, \ldots, f_n) = (g)$ is a principal ideal, then we have $f_i = g c_i$ for some elements $c_i \in A$. This means that we can extend the morphism over $\operatorname{Bs}(\phi)$ by defining the $A$-valued point $(c_0, \ldots, c_n)$.

In general, let $\mathfrak{a} = (f_0, \ldots, f_n) \subset A$, and consider the graded ring

$$R(\mathfrak{a}) = \bigoplus_{i \geqslant 0} \mathfrak{a}^i t^i,$$

where $t$ is a variable, i.e., $R$ is the subring $A[t]$ of polynomials $\sum_{i \geqslant 0} a_i t^i$ with $a_i \in \mathfrak{a}^i$. In the ring $R$, $t$ has degree 1, while the elements of $A$ have degree 0. Since $R_0 = A$, $\operatorname{Proj} R$ is a scheme over $\operatorname{Spec} A$ with structure morphism

$$\pi\colon \operatorname{Proj} R \longrightarrow \operatorname{Spec} A,$$

(this was introduced just after Definition **??** on page ??). We claim that $\pi$ is an isomorphism outside the closed subset $\pi^{-1}V(\mathfrak{a})$, and so $\pi$ merits being called the *blow-up* of $V(\mathfrak{a})$. Indeed, if $f \in \mathfrak{a}$, it holds that $\mathfrak{a}A_f = A_f$ and consequently $\mathfrak{a}^i A_f = A_f$ for all $i$. Therefore, we have the equality $R_f = A_f[t]$. By Exercise 5.4.13 and Example **??**, we then find that $\pi$ induces an isomorphism

$$\pi^{-1}D(f) = \operatorname{Proj} R_f = \operatorname{Proj} A_f[t] \simeq \operatorname{Spec} A_f = D(f).$$

Note that the elements $f_0 t, \ldots, f_n t$ by definition generate the irrelevant ideal of $R$. This means that they induce a morphism

$$\widetilde{F}\colon \operatorname{Proj} R \to \mathbb{P}^n_k,$$

which fits into the diagram

$$\begin{array}{ccc} & \operatorname{Proj} R & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\widetilde{F}} \\ \operatorname{Spec} A & \overset{F}{\dashrightarrow} & \mathbb{P}^n_k \end{array}$$

The scheme $\operatorname{Proj} R$ is called the *blow-up* of $\operatorname{Spec} A$ along $\mathfrak{a}$.

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The blow-ups can be computed as follows. Let $\mathfrak{a} \subset A$ be an ideal, generated by $f_0, \ldots, f_r$. Let $J$ be the kernel of the map of graded rings

$$A[u_0, \ldots, u_r] \longrightarrow R(\mathfrak{a}) = \bigoplus_{n \geqslant 0} \mathfrak{a}^n t^n, \tag{B.38}$$

sending $u_i$ to $f_i t$. Then the blow-up of $\mathfrak{a}$ is the closed subscheme

$$\mathrm{Bl}_{\mathfrak{a}}(X) = V_+(J) \subset \mathbb{P}^r_A,$$

where $X = \operatorname{Spec} A$.

**Proposition B.35.** Let $\mathfrak{a} \subset A$ be an ideal and let $X = \operatorname{Spec} A$. The blow-up $\mathrm{Bl}_{\mathfrak{a}}(X)$ and the morphism $\pi\colon \mathrm{Bl}_{\mathfrak{a}}(X) \to X$ have the following properties:

(i) $E = \pi^{-1}(V(\mathfrak{a}))$ is a Cartier divisor on $\mathrm{Bl}_{\mathfrak{a}}(X)$.

(ii) $\pi$ is surjective, and if $U = X - V(\mathfrak{a})$, the restriction $\pi|_U\colon \pi^{-1}(U) \to U$ is an isomorphism.

Let $\mathfrak{a} = (f_0, \ldots, f_n) \subset A$. We define the morphism

$$\widetilde{F}\colon \mathrm{Bl}_{\mathfrak{a}}(X) \to \mathbb{P}^n_k$$

by the composition

$$\mathrm{Bl}_{\mathfrak{a}}(X) \to \mathbb{P}^n_A \to \mathbb{P}^n_k.$$

Then $\pi$ is proper, because it is a composition of a closed embedding and $\mathbb{P}^n_A \to \mathbb{P}^n_k$, which is proper.

Let $U = X - V(\mathfrak{a})$. Then $U = \bigcup_{f \in \mathfrak{a}} D(f)$. For any $f \in \mathfrak{a}$, we have

$$\pi^{-1}(D(f)) = \operatorname{Proj}(R(\mathfrak{a}) \otimes_A A_f) = \operatorname{Proj}(A_f[t]) \simeq \operatorname{Spec} A_f.$$

These isomorphisms glue to an isomorphism

$$\pi\colon \pi^{-1}(U) \simeq U.$$

Therefore, $\pi$ is surjective, being closed and the image containing a dense open $U$.

Now consider $D_i = D_+(f_i t) = \operatorname{Spec} A\left[\frac{f_0 t}{f_i t}, \ldots, \frac{f_n t}{f_i t}\right]$. Then $\pi^{-1}(V(\mathfrak{a})) \cap D_i = V(f_i)$, as $f_j = f_i \cdot \frac{f_j t}{f_i t}$.

Proposition **??** gives a description of the scheme-theoretic inverse image of $V(\mathfrak{a})$, as

$$\Big(\bigoplus_{i \geqslant 0} \mathfrak{a}^i t^i\Big) \otimes_A A/\mathfrak{a} = \bigoplus_{i \geqslant 0} \mathfrak{a}^i/\mathfrak{a}^{i+1},$$

and hence $\pi^{-1}V(\mathfrak{a}) = \operatorname{Proj} \bigoplus_{i \geqslant 0} \mathfrak{a}^i/\mathfrak{a}^{i+1}$.

**Example B.36** (The blow-up of the plane)**.** Consider the polynomial ring $A = k[x, y]$ and the ideal $\mathfrak{a} = (x, y)$. Let $R$ be the graded ring

$$R = \bigoplus_{i \geqslant 0} \mathfrak{a}^i t^i,$$

where the graded piece of degree $i$ equals $\mathfrak{a}^i t^i$. The irrelevant ideal $R_+$ is generated by $xt$

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and $yt$, and consequently $\operatorname{Proj} R$ is the union of the two open affine subschemes $\operatorname{Spec}(R_{xt})_0$ and $\operatorname{Spec}(R_{yt})_0$.

There is a map of graded rings $\phi\colon A[u,v] \to R$, where $u$ and $v$ are of degree 1, given by $u \mapsto xt$ and $v \mapsto yt$. This map is surjective because $\mathfrak{a}$ is generated by $x$ and $y$. The kernel of $\phi$ contains the element $xv - yu$, and by Exercise 5.4.27, we have

$$R \simeq A[u,v]/(xv - yu). \tag{B.39}$$

From this description, we see that $\operatorname{Proj} R$ is covered by the two distinguished open sets

$$D_+(u) = \operatorname{Spec}(R_u)_0 \quad \text{and} \quad D_+(v) = \operatorname{Spec}(R_v)_0,$$

where

$$(R_u)_0 \simeq k[x, vu^{-1}] \quad \text{and} \quad (R_v)_0 \simeq k[y, uv^{-1}].$$

These affine schemes are glued together along their intersection, which is

$$\operatorname{Spec}(R_{uv})_0 \simeq (A[u,v]_{uv}/(xv - yu))_0.$$

One finds that

$$(A[u,v]_{uv}/(xv - yu))_0 \simeq k[x, y, uv^{-1}, vu^{-1}]/(xvu^{-1} - y) \simeq k[x, uv^{-1}, vu^{-1}].$$

Therefore, $\operatorname{Proj} R$ coincides with the blow-up construction described in Section B.2 on page 449. △

**Example B.37.** Let $A = \mathbb{Z}[t]$ and $R = A[u_0, u_1]/(tu_1 - 7u_0)$. The projective scheme $\operatorname{Proj} R$ with its structure morphism

$$\pi\colon \operatorname{Proj} R \longrightarrow \mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} A,$$

can be viewed as the blow-up of $\mathbb{A}^1_{\mathbb{Z}}$ in the closed point $\mathfrak{m} = (7, t) \in \mathbb{A}^1_{\mathbb{Z}}$. Then

$$\pi^{-1}(D(t)) = \operatorname{Proj}\left(R \otimes_A A[\tfrac{1}{t}]\right) = \operatorname{Proj}(A[u_0, \tfrac{1}{t}]) \simeq D(t)$$
$$\pi^{-1}(D(7)) = \operatorname{Proj}\left(R \otimes_A A[\tfrac{1}{7}]\right) = \operatorname{Proj}(A[u_1, \tfrac{1}{7}]) \simeq D(7)$$

This shows that $\pi$ restricts to an isomorphism over the open subset $\operatorname{Proj} R - \pi^{-1}V(\mathfrak{m})$. On the other hand, the schreme-theoretic fiber of $\mathfrak{m}$ is given by

$$\operatorname{Proj}(A[u_0, u_1] \otimes_A A/\mathfrak{m}) = \operatorname{Proj}(A[u_0, u_1]/(7, t)) = \mathbb{P}^1_{\mathbb{F}_7}.$$

△

**Proposition B.38.** Let $X = \operatorname{Spec} A$ and $Y = V(\mathfrak{a})$, and let $\widetilde{X} = \operatorname{Bl}_{\mathfrak{a}}(X)$.

(i) If $X$ is reduced/irreducible/integral, then so is $\widetilde{X}$.

(ii) We have $\widetilde{X} \subset Z = V_+(f_i u_j - f_j u_i) \subset \mathbb{P}^n_A$. If $f_0, \dots, f_n$ are a minimal set of generators for $\mathfrak{a}$ and $Z$ is integral, then $\widetilde{X} = Z$.

*Proof* (i): Let $\widetilde{X} = \operatorname{Bl}_{\mathfrak{a}}(X) = \operatorname{Proj} R(\mathfrak{a})$, where $R(\mathfrak{a}) = \bigoplus_{d \geqslant 0} \mathfrak{a}^d t^d$ is the Rees algebra of $\mathfrak{a}$. If $X = \operatorname{Spec} A$ is reduced, then $A$ has no nilpotent elements. Since $R(\mathfrak{a})$ is a graded subring of $A[t]$, it inherits the reduced property from $A$. Thus, $\widetilde{X} = \operatorname{Proj} R(\mathfrak{a})$ is reduced. The same argument shows that $\widetilde{X}$ is also irreducible, integral.

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(ii): $\widetilde{X} \subset Z = V_+(f_iu_j - f_ju_i) \subset \mathbb{P}^n_A$, and if $f_0, \ldots, f_n$ are a minimal set of generators for $\mathfrak{a}$ and $Z$ is integral, then $\widetilde{X} = Z$.

The blow-up $\widetilde{X}$ is defined by the kernel $J$ of the graded surjection (B.38). This kernel contains the relations $f_iu_j - f_ju_i$ for all $i, j$, since $f_i(f_jt) = f_j(f_it)$. Thus, $\widetilde{X} = \operatorname{Proj} R(\mathfrak{a})$ is a closed subscheme of $Z = V_+(f_iu_j - f_ju_i) \subset \mathbb{P}^n_k$.

Suppose $f_0, \ldots, f_n$ are a minimal set of generators for $\mathfrak{a}$, and $Z$ is integral. Since $\widetilde{X} \subset Z$ and both are integral schemes, it suffices to show that the map $\widetilde{X} \to Z$ is an isomorphism on an open dense subset.

Let $U = X - V(\mathfrak{a})$. On $U$, the blow-up $\widetilde{X}$ restricts to an isomorphism $\pi^{-1}(U) \simeq U$, as shown earlier. Similarly, $Z$ restricts to an open subset isomorphic to $U$ because the relations $f_iu_j = f_ju_i$ define the same structure as the blow-up on $U$. Since $Z$ is integral and $\widetilde{X} \subset Z$ is a closed subscheme that agrees with $Z$ on a dense open subset, we conclude that $\widetilde{X} = Z$. □

**Example B.39.** Let $A = k[x, y]/(y^2 - x^3)$ and $\mathfrak{a} = (x, y)$. The blow-up $\widetilde{X} = \operatorname{Bl}_{\mathfrak{a}}(\operatorname{Spec} A)$ embeds into $\mathbb{P}^1_A = \operatorname{Proj} A[U, V]$. We compute $\widetilde{X}$ on the affine charts $D_+(U)$ and $D_+(V)$.

On $D_+(U)$: Let $u = V/U$. The blow-up is defined by the relation $y = ux$. Substituting into $y^2 = x^3$, we get:

$$(ux)^2 = x^3 \implies u^2x^2 = x^3 \implies u^2 = x.$$

Thus, $\widetilde{X} \cap D_+(U) = \operatorname{Spec} A[u]/(u^2 - x)$, with $y = u^3$. The morphism $\operatorname{Spec} k[u] \to \operatorname{Spec} A$ is given by:

$$x \mapsto u^2, \quad y \mapsto u^3.$$

On $D_+(V)$: Let $v = U/V$. The blow-up is defined by the relation $x = vy$. Substituting into $y^2 = x^3$, we get:

$$y^2 = (vy)^3 \implies y^2 = v^3y^3 \implies 1 = v^3y.$$

Thus, $\widetilde{X} \cap D_+(V) = \operatorname{Spec} A[v]/(1 - v^3y)$, with $x = vy$. The morphism $\operatorname{Spec} k[v] \to \operatorname{Spec} A$ is given by:

$$x \mapsto v^3, \quad y \mapsto v^2.$$

The blow-up morphism $\pi\colon \widetilde{X} \to \operatorname{Spec} A$ is obtained by gluing these two charts. On $D_+(U)$, it is induced by $x \mapsto u^2$, $y \mapsto u^3$, and on $D_+(V)$, by $x \mapsto v^3$, $y \mapsto v^2$. △

## B.11 Pushouts of affine schemes

Gluing schemes along open subschemes have been a central theme in this book. In some cases, we can also glue two schemes along a common *closed* subsheme. In this section, we explain how this can be done for affine schemes.

Let $A, B, C$ be rings and let $f : A \to C$, $g : B \to C$ be surjections. From this data, we can form the pullback ring $A \times_C B$ arising in the pullback diagram

$$\begin{array}{ccc} A \times_C B & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle g} \\ A & \xrightarrow{\ f\ } & C \end{array}$$

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Explicitly, the ring $A \times_C B$ is defined by

$$A \times_C B = \{\, (a,b) \in A \times B \mid f(a) = g(b) \,\}.$$

The diagram above induces a *pushout diagram* of schemes

$$\begin{array}{ccc} \operatorname{Spec} C & \hookrightarrow & \operatorname{Spec} B \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec}(A \times_C B) \end{array}$$

This means that $\operatorname{Spec} A \times_C B$ satisfies a universal property dual to that of the fiber product: it is universal among diagrams of the form (B.11) with $\operatorname{Spec}(A \times_C B)$ replaced by some other scheme.

**Proposition B.40.** As a topological space, $\operatorname{Spec}(A \times_C B)$ is homeomorphic to

$$(\operatorname{Spec} A) \cup_{\operatorname{Spec} C} (\operatorname{Spec} B) \tag{B.40}$$

**Example B.41.** The nodal cubic curve can be obtained from this construction; it is obtained by identifying two points of $\mathbb{A}^1_k$. [ADD MORE DETAILS.] △

**Example B.42.** Here is an example of a non-normal surface with an isolated singularity. We let $X$ be the scheme obtained by identifying two points in $\mathbb{A}^2_k$; $X$ is the affine variety given by the $k$-algebra

$$A = \{f \in k[x,y] \mid f(0,0) = f(0,1)\}.$$

Then the normalization $\overline{X}$ is the affine plane. The algebra $A$ is generated by the 4 polynomials

$$x, xy, y^2 - y, y^3 - y \tag{B.41}$$

To see this, note that if $f(x,y)$ is any polynomial satisfying $f(0,0) = f(0,1)$, we may subtract products of the form $b(x)(y^2-y)^k$ or $b(x)(y^2-y)^k(y^3-y)$ until the $y$-degree of $f$ is at most 1; the remaining polynomials can be written as polynomials in $x$ and $xy$.

The polynomials (B.41) define a closed embedding of the surface $X$ in $\mathbb{A}^4$. △

**Theorem B.43.** Let $X$ and $Y$ be schemes and let $i\colon Z \to X$ and $j\colon Z \to Y$ be closed embeddings. Then the pushout $X \cup_Z Y$ exists as a scheme.

**Example B.44** ($\operatorname{Spec} \mathbb{Z}\sqrt{-3}$)**.** The spectrum of the ring $R = \mathbb{Z}[\sqrt{-3}]$ is rather interesting. It can be viewed as a sort of singular curve over $\operatorname{Spec} \mathbb{Z}$. As such it shares many properties with the nodal cubic curve of Example 10.37.

Note first that $R$ is not a unique factorization domain. For example,

$$2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3}).$$

It is also not normal, because the element $\omega = (1+\sqrt{-3})/2 \in \mathbb{Q}(\sqrt{-3})$ satisfies the monic equation $x^2 + x + 1 = 0$, but $\omega \notin R$. In fact, the integral closure of $R$ is given by the ring the ring of Eisenstein integers $\mathbb{Z}[\omega]$, and $\mathbb{Z}[\omega]$ *is* a unique factorization domain. In particular,

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$\mathbb{Z}[\omega]$ is normal, and equals the integral closure of $R$ inside the fraction field $\mathbb{Q}(\sqrt{-3})$. It follows that the morphism

$$\operatorname{Spec} \mathbb{Z}[\omega] \to \operatorname{Spec} \mathbb{Z}[\sqrt{-3}],$$

induced by the inclusion $\mathbb{Z}[2\omega] \to \mathbb{Z}[\omega]$, is the normalization map.

Note that $R$ is not a Dedekind domain: an integral domain is Dedekind if and only if each of its localizations is a discrete valuation ring. However, the localization $R_{\mathfrak{p}}$ at the prime ideal $\mathfrak{p} = (2, 1 + \sqrt{-3})$ is not a discrete valuation ring; the maximal ideal requires two generators. However the square of $\mathfrak{p}$ is principal; it satisfies

$$\mathfrak{p}^2 = (2) \subset \mathbb{Z}[\sqrt{-3}].$$

There are two ring maps $\phi, \psi \colon \mathbb{Z}[\omega] \to \mathbb{F}_2[x]/(x^2 + x + 1) = \mathbb{F}_4$, one sending $\omega$ to $x$, and the other sending $\omega$ to $x + 1$. The subring of $\mathbb{Z}[\omega]$ where these coincide is exactly $\mathbb{Z}[2\omega] = R$. We get a pushout diagram

$$\begin{array}{ccc} \mathbb{F}_4 & \xleftarrow[\psi]{} & \mathbb{Z}[\omega] \\ {\scriptstyle\phi}\uparrow & & \uparrow \\ \mathbb{Z}[\omega] & \longleftarrow & \mathbb{Z}[\sqrt{-3}] \end{array}$$

This induces a homeomorphism between $\operatorname{Spec} \mathbb{Z}[\sqrt{-3}]$ and $\operatorname{Spec} \mathbb{Z}[\omega]$ with two points identified. Hence $\operatorname{Spec} R$ is obtained by identifying two points in the spectrum of the Eisenstein integers. △

## B.12 Multigraded rings

The Proj-construction has the following multigraded analogue.

Let $R = k[x_1, \ldots, x_n]$ be a ring graded by the group $\mathbb{Z}^n$. This means that each variable $x_i$ is assigned a degree $\mathbf{d}_i \in \mathbb{Z}^n$. Let $f \in R$ denote a homogeneous element with respect to the $\mathbb{Z}^n$-grading. Let $(R_f)_{\mathbf{0}}$ denote all the elements in the localization of degree $\mathbf{0} \in \mathbb{Z}^n$.

**Definition B.45.** For $\mathbf{w} \in \mathbb{Z}^n$, we define the subring

$$R(\mathbf{w}) = \bigoplus_{n \geqslant 0} R_{n\mathbf{w}} \subset R$$

and the $\mathbf{w}$-*irrelevant ideal* as the graded ideal

$$I_{\mathbf{w}} = \bigoplus_{n > 0} R_{n\mathbf{w}} \subset R$$

Note that $R(\mathbf{w})$ and $I_{\mathbf{w}}$ are graded $R_{\mathbf{0}}$-modules, in the usual sense. In fact, $R(\mathbf{w})$ is a graded algebra over $R_{\mathbf{0}}$.

**Definition B.46.** For a given $\mathbf{w} \in \mathbb{Z}^n$, we define the multigraded projective spectrum $\mathbf{w}$-Proj$(R)$ as the set of homogeneous prime ideals $\mathfrak{p}$ that do not contain the irrelevant ideal $I_{\mathbf{w}}$.

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As in the usual Proj-construction, the set $\mathbf{w}$-Proj$(R)$ inherits a Zariski-topology, by declaring that the closed sets are exactly the sets $V(\mathfrak{a})$ of homogeneous prime ideals $\mathfrak{p} \supset \mathfrak{a}$ (not containing the irrelevant ideal $I_{\mathbf{w}}$). There is also the set of distinguished opens $D_+(f) = \mathbf{w}\text{-Proj}(R) - V(f)$, defined for $\mathbb{Z}^n$-homogeneous $f$. As before, these give a basis for the topology on $\mathbf{w}$-Proj$(R)$ and

$$D_+(f) \simeq \operatorname{Spec}(R_f)_{\mathbf{0}}.$$

Next, we define the structure sheaf on the topological space $X = \mathbf{w}\text{-Proj}(R)$. We define it on the basis consisting on distinguished opens by

$$\mathcal{O}_X(D_+(f)) = (R_f)_{\mathbf{0}}$$

and the restriction maps are as usual given by the localization maps $R_f \to R_g$ for $D_+(f) \supset D_+(g)$. This defines a sheaf of rings $\mathcal{O}_X$ on $X$ whose stalks at are the local rings $(R_{\mathfrak{p}})_{\mathbf{0}}$. Therefore, the locally ringed space $(X, \mathcal{O}_X)$ is a scheme.

## B.13 Examples

**Example B.47.** Let $m, n \geqslant 1$ be integers and consider the polynomial ring $R = k[x_0, \ldots, x_m, y_0, \ldots, y_n]$ with the $\mathbb{Z}^2$-grading given by $\deg x_i = (1, 0)$ and $\deg y_j = (0, 1)$.

For the vector $\mathbf{w} = (1, 1)$, we find

$$I_{\mathbf{w}} = (x_0y_0, \ldots, x_my_n) = (x_0, \ldots, x_m) \cap (y_0, \ldots, y_n).$$

Note that $\mathbf{w}$-Proj$(R)$ is covered by the affine schemes

$$U_{ij} = \operatorname{Spec}(R_{x_iy_j})_{\mathbf{0}} \simeq \operatorname{Spec} k\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, \frac{y_0}{y_j} \ldots, \frac{y_0}{y_j}\right] \simeq \mathbb{A}_k^{m+n}.$$

This is the usual affine cover of $X_{\mathbf{w}} = \mathbb{P}_k^m \times \mathbb{P}_k^n$.

△

**Example B.48.** Consider the polynomial ring $R = k[x_0, x_1, y_0, y_1]$ with the grading given by the columns of the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Let us choose the vector $w = (1, 1)$ as the degree vector. Then the irrelevant ideal is generated by all monomials of bidegree $\mathbf{w}$, i.e., $I_{\mathbf{w}} = (x_0^2y_1, x_0x_1y_1, x_0y_0, x_1^2y_1, x_1y_0)$. This has radical $\sqrt{I_{\mathbf{w}}} = (x_0y_0, x_1y_0, x_0y_1, x_1y_1) = (x_0, x_1) \cap (y_0, y_1)$. The localizations are given by

$$\begin{aligned} (R_{(x_0y_0)})_{\mathbf{0}} &= k\left[\tfrac{x_1}{x_0}, \tfrac{x_1y_1}{y_0}\right] & (R_{(x_0y_1)})_{\mathbf{0}} &= k\left[\tfrac{x_1}{x_0}, \tfrac{x_0y_0}{y_1}\right] \\ (R_{(x_1y_0)})_{\mathbf{0}} &= k\left[\tfrac{x_0}{x_1}, \tfrac{x_0y_1}{y_0}\right] & (R_{(x_1y_1)})_{\mathbf{0}} &= k\left[\tfrac{x_0}{x_1}, \tfrac{x_0y_0}{y_1}\right]. \end{aligned}$$

This is the same affine cover as the blow-up of $\mathbb{P}_k^2$ at a point. △

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