STAT 110A HWI

Problem 1. About Example 1. Suppose we flip a fair coin independently n times.

(1) Specify the sample space Ω. What is the size of Ω?

(2) Let A be the event that there are k heads, where k is an integer between 0 and n. What is the size of A?

(3) Calculate $\Pr(A)$.

Problem 2. Continue from Problem 1. For each sequence $\omega \in \Omega$, let $X(\omega)$ be the number of heads in ω. For $i = 1, 2, ..., n$, let $Z_i(\omega) = 1$ if the i-th flip of ω is head, and let $Z_i(\omega) = 0$ if the i-th flip of ω is tail.

(1) Prove that $X = Z_1 + Z_2 + ... + Z_n$. Here for notational simplicity, we ignore ω in $X(\omega)$ and $Z_i(\omega)$.

(2) Calculate the distribution of X, i.e., for each possible value k that X can take, calculate $\Pr(X = k)$.

Problem 3. Continue from Problem 2.

(1) Explain that the coin flipping experiment can be mapped to the quincunx experiment.

(2) Explain that the distribution of X can be calculated by the Pascal triangle.

Problem 4. Continue from Problem 3. Let X/n be the frequency of heads.

(1) Calculate $\Pr(X/n \in [.4, .6])$ for $n = 5, 6, 7, ...$, using the Pascal triangle. You may continue to increase n as much as you can.

(2) How does $\Pr(X/n \in [.4, .6])$ changes with n? For a small $\epsilon > 0$, what would be the behavior of $\Pr(X/n \in [1/2 - \epsilon, 1/2 + \epsilon])$ as $n \to \infty$? You do not have to prove your conclusion.

(3) Explain how we can reconcile the following two statements about flipping a fair coin n times independently. (a) All the sequences are equally likely. (b) The frequency of heads is about 1/2 if n is large.

Problem 5. Continue from Problem 3. Suppose $Z_i = 1$ if the i-th flip is a head, and $Z_i = -1$ if the i-th flip is a tail. Let $X_t = \sum_{i=1}^{t} Z_i$.

(1) Explain that X_t is a random walk on integers. Find the distribution of X_t.

(2) Suppose 1 million people start from 0, and each person walks on the integers according to the random walk in (1). Explain that the distribution in (1) can be viewed as the distribution of the population of these 1 million people.

Problem 6. Consider an alphabet with four letters α, β, γ, δ. Consider a probability distribution on these four letters with $\Pr(\alpha) = 1/2$, $\Pr(\beta) = 1/4$, $\Pr(\gamma) = 1/8$ and $\Pr(\delta) = 1/8$. Such a distribution reflects the frequencies of these four letters.

(1) Explain how to generate a random letter according to the above distribution using coin flipping.

(2) Explain that the scheme in (1) can be mapped to a binary code for the four letters. Calculate the expected coding length per letter.

(3) If we have a sequence $\alpha\alpha\beta\gamma\beta\delta$, then what is the corresponding binary sequence? Explain that we can decode the original sequence from this binary sequence.

Problem 7. Example 2. Suppose we throw a fair die twice independently. Let X and Y be the two numbers we get.

(1) Calculate $\Pr(X + Y > 8)$.

(2) Calculate $\Pr(X > 4 | X + Y > 8)$.

(3) Calculate the distributions of $X + Y$ and $|X - Y|$.

Problem 8. Suppose we roll a fair die 4 times independently.

(1) What is the probability that we get at least one 6?

(2) What is the probability that we get four different numbers?

Problem 9. Example 3. Suppose we generate $X, Y \sim \text{Uniform}[0,1]$ independently.

(1) Calculate $\Pr(X + Y > 1)$.

(2) Calculate $\Pr(|X - Y| > 1/2)$.

(3) Calculate $\Pr(X^2 + Y^2 < 1)$.

(4) Calculate $\Pr(X > 1/3 | X + Y > 1)$.

Problem 10.

(1) Using Venn diagrams, explain the four axioms of probability calculus.

(2) Using geometric diagrams, explain the difference between mutual exclusiveness and independence.

(3) Using Venn diagrams, show that $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$. Generalized the above result to $\Pr(A \cup B \cup C)$.