STAT 110A HW2 Solution

1. (a) See class note.
   (b) See class note.

2. (a) $X \sim B(n=4, p=0.2)$
   $Pr(\text{two or more people support Mr. A}) = Pr(X \geq 2)$
   $= \binom{4}{2}0.2^2 0.8^2 + \binom{4}{3}0.2^3 0.8 + \binom{4}{4}0.2^4$
   (b) $X \sim B(n=100, p=0.2)$
   $Pr(\text{at least 24 people support Mr. A}) = Pr(X \geq 24) = \sum_{i=24}^{100}\binom{100}{i}0.2^i 0.8^{100-i}$

3. (a) $Pr(A) = Pr(A \cap B_1) + Pr(A \cap B_2) + \cap \cdots + Pr(A \cap B_n)$
   $= Pr(B_1)Pr(A|B_1) + Pr(B_2)Pr(A|B_2) + \cdots Pr(B_n)Pr(A|B_n) = \sum_{i=1}^{n} Pr(B_i)Pr(A|B_i)$
   (b) $Pr(B_j|A) = Pr(B_j \cap A)/Pr(A) = Pr(A \cap B_j)/Pr(A) = P(B_j)Pr(A|B_j)/Pr(A)$
   $= P(B_j)Pr(A|B_j)/\sum_{i=1}^{n} Pr(B_i)Pr(A|B_i)$
   (c) venn diagram

4. (a) $Pr(\text{the second ball is red})$
   $= Pr(\text{1st=red})Pr(\text{2nd=red}|\text{1st=red}) + Pr(\text{1st=black})Pr(\text{2nd=red}|\text{1st=black})$
   $= \frac{r}{r+b}\frac{r+1}{r+b+1} + \frac{b}{r+b}\frac{r}{r+b+1} = \frac{r}{r+b}$
   (b) $Pr(\text{the third ball is red})$
   $= Pr(\text{2nd=red})Pr(\text{3rd=red}|\text{2nd=red}) + Pr(\text{2nd=black})Pr(\text{3rd=red}|\text{2nd=black})$
   $= Pr(\text{1st=red})Pr(\text{2nd=red}|\text{1st=red})Pr(\text{3rd=red}|\text{2nd=red}) +$
   $Pr(\text{1st=black})Pr(\text{2nd=red}|\text{1st=black})Pr(\text{3rd=red}|\text{2nd=red}) +$
   $Pr(\text{1st=red})Pr(\text{2nd=black}|\text{1st=red})Pr(\text{3rd=red}|\text{2nd=black}) +$
   $Pr(\text{1st=black})Pr(\text{2nd=black}|\text{1st=black})Pr(\text{3rd=red}|\text{2nd=black})$
   $= \frac{r}{r+b}\frac{r+1}{r+b+1}\frac{r+2}{r+b+2} + \frac{b}{r+b}\frac{r}{r+b+1}\frac{r+1}{r+b+2} + \frac{r}{r+b}\frac{b}{r+b+1}\frac{r+1}{r+b+2} + \frac{b}{r+b}\frac{b+1}{r+b+1}\frac{r}{r+b+2}$

5. $\frac{1\% \times 95\%}{1\% \times 95\% + 99\% \times 10\%} = 8.76\%$

6. (a) $T_{ii} = 0; T_{ij} = 1/2, i \neq j$

$$T = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

(b) $M_{ij} = Pr(X_{t+2} = j|X_t = i)$
$= \sum_{k=1}^{3} Pr(X_{t+1} = k|X_t = i)Pr(X_{t+2} = j|X_{t+1} = k) = \sum_{k=1}^{3} T_{ik}T_{kj}$
$\implies M = T^2$

(c) $M_{ij} = Pr(X_{t+s} = j|X_t = i)$
$= \sum_{a=1}^{3} \sum_{b=1}^{3} \cdots \sum_{r=1}^{3} Pr(X_{t+1} = a|X_t = i) \times Pr(X_{t+2} = b|X_{t+1} = a)$
$\times\, Pr(X_{t+3} = c|X_{t+2} = b) \times \cdots \times Pr(X_{t+s} = j|X_{t+s-1} = r)$
$= \sum_{a=1}^{3} \sum_{b=1}^{3} \cdots \sum_{r=1}^{3} T_{ia}T_{ab}T_{bc} \cdots T_{rj}$
$\implies M = T^s$

7. (a) $p_0 = (1, 0, 0)$
$p_1(i) = Pr(X_1 = i) = \sum_{j=1}^{3} Pr(X_0 = j)Pr(X_1 = i|X_0 = j)$ for i =1,2,3
$p_1(1) = 1 \times 0 + 0 \times 1/2 + 0 \times 1/2 = 0,$
$p_1(2) = 1 \times 1/2 + 0 \times 0 + 0 \times 1/2 = 1/2,$
$p_1(3) = 1 \times 1/2 + 0 \times 1/2 + 0 \times 0 = 1/2$
$\implies p_1 = (0, 1/2, 1/2)$
$p_2(i) = Pr(X_2 = i) = \sum_{j=1}^{3} Pr(X_1 = j)Pr(X_2 = i|X_1 = j)$ for i =1,2,3
$p_2(1) = 0 \times 0 + 1/2 \times 1/2 + 1/2 \times 1/2 = 1/2,$
$p_2(2) = 0 \times 1/2 + 1/2 \times 0 + 1/2 \times 1/2 = 1/4,$
$p_2(3) = 0 \times 1/2 + 1/2 \times 1/2 + 1/2 \times 0 = 1/4$
$\implies p_2 = (1/2, 1/4, 1/4)$
$p_3(i) = Pr(X_3 = i) = \sum_{j=1}^{3} Pr(X_2 = j)Pr(X_3 = i|X_2 = j)$ for i =1,2,3
$p_3(1) = 1/2 \times 0 + 1/4 \times 1/2 + 1/4 \times 1/2 = 1/4,$
$p_3(2) = 1/2 \times 1/2 + 1/4 \times 0 + 1/4 \times 1/2 = 3/8,$
$p_3(3) = 1/2 \times 1/2 + 1/4 \times 1/2 + 1/4 \times 0 = 3/8$
$\implies p_3 = (1/4, 3/8, 3/8)$

(b) $p_{t+1}(i) = Pr(X_{t+1} = i) = \sum_{j=1}^{3} Pr(X_t = j)Pr(X_{t+1} = i|X_t = j)$
$= \sum_{j=1}^{3} p_t(j)T_{ji}$ for $i = 1, 2, 3$
$\implies p_{t+1} = p_tT$

(c) It is a model for an individual walking on the three states who at each point of time takes one step to the other two states with probability 1/2 respectively. The probability of the person in the state of time t just depends on the state of time t-1. When it runs enough time, the probability of the person in each state will be 1/3.