STAT 110A HW4 Solution

1. (a) $\int xf(x)dx$: [the summation of $x\times$ the number of a_i in $(x, x+\triangle x)$ over all x]/N=μ
   $\int(x-\mu)^2 f(x)dx$: [the summation of $(x-\mu)^2\times$ the number of a_i in $(x, x+\triangle x)$ over all x]/N=σ^2

   (b) $\int h(x)f(x)dx$: [the summation of $h(x)\times$ the number of a_i in $(x, x+\triangle x)$ over all x]/N $= \mu_h$
   $\int(h(x)-\mu_h)^2 f(x)dx$: [the summation of $(h(x)-\mu_h)^2\times$ the number of a_i in $(x, x+\triangle x)$ over all x]/N $= \sigma_h^2$

   (c) $\mu_h = (\beta a_1 + \gamma + \cdots + \beta a_N + \gamma)/N = \beta(a_1 + \cdots + a_N)/N + N\gamma/N = \beta\mu + r$
   or
   $\mu_h = \int h(x)f(x)dx = \int(\beta x+\gamma)f(x)dx = \beta\int xf(x)dx + \gamma\int f(x)dx = \beta\mu+\gamma$
   $\sigma_h^2 = \sum_{i=1}^N[(\beta a_i+\gamma)-(\beta\mu-\gamma)]^2/N = \beta^2\sum_{i=1}^N(a_i-\mu)^2/N = \beta^2\sigma^2$
   or
   $\sigma_h^2 = \int(h(x)-\mu_h)^2 f(x)dx = \int(\beta x+\gamma-\beta\mu-\gamma)^2 f(x)dx$
   $= \beta^2\int(x-\mu)^2 f(x)dx = \beta^2\sigma^2$

2. (a) $\mu = (a_1+\cdots+a_N)/N = \sum_x x\times$[number of $a_i's$ that are equal to $x]/N = \sum_x xp(x)$
   $\sigma^2 = \sum_{i=1}^N(a_i-\mu)^2/N = \sum_x(x-\mu)^2\times$ [number of $a_i's$ that are equal to $x]/N = \sum_x(x-\mu)^2 p(x)$

   (b) $\mu_h = \sum_x h(x)p(x), \sigma_h^2 = \sum_x(h(x)-\mu_h)^2 p(x)$

   (c) $\mu_h = \sum_x(\beta x+\gamma)p(x) = \beta\sum_x xp(x) + \gamma\sum_x p(x) = \mu+\gamma$
   $\sigma_h^2 = \sum_x(\beta x+\gamma-\beta\mu-\gamma)^2 p(x) = \beta^2\sum_x(x-\mu)^2 p(x) = \beta^2\sigma^2$

3. (a) $f_X(x)$: $\frac{\text{The number of people with height in } (x,x+\triangle x)/N}{\triangle x}$
   $f_Y(y)$: $\frac{\text{The number of people with weight in } (y,y+\triangle y)/N}{\triangle y}$

   (b) $f(x,y) = \frac{\text{The number of people with height in } (x,x+\triangle x) \text{ and weight in } (y,y+\triangle y)/N}{\triangle x\triangle y}$

   (c) $f_X(x) = \sum_y f(x,y)\triangle y, f_Y(y) = \sum_x f(x,y)\triangle x$

   (d) $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f(x,y)}{\sum_y f(x,y)\triangle y}$

4. (a) $p_X(x)$ =number of people having height=x/N
   $p_Y(y)$ =number of people having weight=y/N

   (b) $p(x,y)$ =number of people having geight=x and weight=y/N

   (c) $p_X(x) = Pr(X=x) = \sum_y Pr(X=x, Y=y) = \sum_y p(x,y)$
   $p_Y(y) = Pr(Y=y) = \sum_x Pr(X=x, Y=y) = \sum_x p(x,y)$

   (d) $p_{Y|X}(y|x) = Pr(Y=y|X=x) = \frac{Pr(X=x,Y=y)}{Pr(X=x)} = \frac{p(x,y)}{p_x(x)} = \frac{p(x,y)}{\sum_y p(x,y)}$

5. (a) $f_X(x)$: How often a person's height falls in $(x, x + \triangle x)/\triangle x$
$f_Y(y)$: How often a person's weight falls in $(y, y + \triangle y)/\triangle y$
$f(x, y)$: How often a person's height and weight falls in $(x, x + \triangle x)$ and $(y, y + \triangle y)/\triangle x\triangle y$
$f_{Y|X}(y|x)$: When a person's height falls in $(x, x + \triangle x)$, how often his/her weight falls in $(y, y + \triangle y)/\triangle y$

(b) $p_X(x)$: How often a person's height is x
$p_Y(y)$: How often a person's weight is y
$p(x, y)$: How often a person's height and weight is x and y
$p_{Y|X}(y|x)$: When a person's height is x, how often his/her weight is y

(c) μ : the average height of the population
μ_h : the average value of h(x) of the population

6. (a) $\int\int h(x, y)f(x, y)dxdy$: the summation of $h(x, y)\times$ the probability of X and Y in $(x, x + \triangle x)$ and $(y, y + \triangle y)$ over all x and y $= \mu_h$
$\int\int(h(x, y) - \mu_h)^2 f(x, y)dxdy$: the summation of $(h(x) - \mu_h)^2\times$ the probability of X and Y in $(x, x + \triangle x)$ and $(y, y + \triangle y)$ over all x and y $= \sigma_h^2$
$\sum\sum h(x, y)p(x, y)$: the summation of $h(x, y)\times$ the probability of X and Y in x and y over all x and y $= \mu_h$
$\sum\sum(h(x, y) - \mu_h)^2 p(x, y)$: the summation of $(h(x) - \mu_h)^2\times$ the probability of X and Y in x and y over all x and y $= \sigma_h^2$

(b) $Var(X) = E(X - \mu_X)^2 = \sum_i(a_i - \mu_X)^2 p(X = a_i) = \sum_i(a_i - \mu_X)^2 \times \frac{1}{N} = \sum_i c_i^2/N = \|c\|^2/N$
$Var(Y) = E(Y - \mu_Y)^2 = \sum_i(b_i - \mu_Y)^2 p(X = b_i) = \sum_i(b_i - \mu_Y)^2 \times \frac{1}{N} = \sum_i d_i^2/N = \|d\|^2/N$

(c) $Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \sum_i(a_i - \mu_X)(b_i - \mu_Y)p(X = a_i, Y = b_i)$
$= \sum_i c_i d_i/N = (c_1, c_2, \cdots, c_N)(d_1, d_2, \cdots, d_N)'/N = c \cdot d/N$
$\rho = Cov(X, Y)/\sigma_X\sigma_Y = \frac{c\cdot d/N}{\|c\|\|d\|/N}$ = cos(angel between c nd d)

(d) Covariance is the measure of how much two variables change together. If two variables tend to vary together (that is, when one of them is above its expected value, then the other variable tends to be above its expected value too), then the covariance between the two variables will be positive. On the other hand, if one of them is above its expected value and the other variable tends to be below its expected value, then the covariance between the two variables will be negative.

7. (a) $Var(X + Y) = E[(X + Y) - E(X + Y)]^2 = E[(X - E(X)) + (Y - E(Y))]^2$
$= \sum_{i=1}^N(a_i - \mu_X + b_i - \mu_Y)^2/N = \sum_i[(a_i - \mu_X)^2 + (b_i - \mu_Y)^2 + 2(a_i - \mu_X)(b_i - \mu_Y)]/N$
$= Var(X) + Var(Y) + 2Cov(X, Y)$, or,
$Var(X + Y) = E[(X - E(X)) + (Y - E(Y))]^2 = E[(X - E(X))^2 + (Y - E(Y))^2 + 2(X - E(X))(Y - E(Y))]$
$= Var(X) + Var(Y) + 2Cov(X, Y)$

(b) $Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = \sum_i(a_i - \mu_X)(b_i - \mu_Y)/N$
$= \sum_i(a_i b_i - \mu_X b_i - \mu_Y a_i + \mu_X\mu_Y)/N = \sum_i a_i b_i/N - \mu_X\sum_i b_i/N - \mu_Y\sum_i a_i/N + N\mu_x\mu_Y/N = E(XY) - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y = E(XY) - \mu_X\mu_Y$,or
$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY - X\mu_y - \mu_X Y + \mu_X\mu_Y] = E(XY) - \mu_x\mu_Y = 0$

(c) $Cov(X, Y) = 0 \Longrightarrow corr(X, Y) =$ cos(angel between c nd d) $= 0$
$\Longrightarrow$ X and Y are perpendicular