STAT 110A Midterm Solution

1. (a) $Pr(X+Y<z) = \frac{\text{area of } \{(x,y):x\in[0,1],y\in[0,1],x+y<z\}}{\text{area of unit square } [0,1]\times[0,1]}$

   $$F(z) = Pr(X+Y<z) = \begin{cases} 0 & z<0 \\ \frac{z^2}{2} & 0\le z\le 1 \\ 1-\frac{(1-z)^2}{2} & 1<z\le 2 \\ 1 & z>2 \end{cases}$$

   (b) $Pr(X>\frac{1}{2}|X+Y<\frac{3}{2}) = \frac{Pr(X>\frac{1}{2}\cap X+Y<\frac{3}{2})}{Pr(X+Y<\frac{3}{2})} = \frac{\frac{1}{2}-\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}{1-\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}} = \frac{\frac{2}{8}}{\frac{7}{8}} = \frac{3}{8}$

2. $Pr(\text{fire}|\text{alarm})$
   $= \frac{Pr(\text{fire}\cap\text{alarm})}{Pr(\text{alarm})} = \frac{Pr(\text{fire})Pr(\text{alarm}|\text{fire})}{Pr(\text{fire})Pr(\text{alarm}|\text{fire})+Pr(\text{no fire})Pr(\text{alarm}|\text{no fire})} = \frac{\alpha\beta}{\alpha\beta+(1-\alpha)\gamma}$

3. (a) $Pr(X_0=1)=1$
   $Pr(X_1=1)=1/2, Pr(X_1=2)=1/4, Pr(X_1=3)=1/4$
   $Pr(X_2=i)=\sum_{k=1}^{3} Pr(X_1=k\cap X_2=i) = \sum_{k=1}^{3} Pr(X_1=k)Pr(X_2=i|X_1=k)$
   $\Longrightarrow$
   $Pr(X_2=1) = Pr(X_1=1)Pr(X_2=1|X_1=1)$
   $+ Pr(X_1=2)Pr(X_2=1|X_1=2) + Pr(X_1=3)Pr(X_2=1|X_1=3)$
   $= 1/2\times1/2 + 1/4\times1/4 + 1/4\times1/4 = 3/8$
   $Pr(X_2=2) = 1/2\times1/4 + 1/4\times1/2 + 1/4\times1/4 = 5/16$
   $Pr(X_2=3) = 1/2\times1/4 + 1/4\times1/4 + 1/4\times1/2 = 5/16$

   (b) $Pr(X_1=i|X_2=3) = \frac{Pr(X_1=i\cap X_2=3)}{Pr(X_2=3)} = \frac{Pr(X_1=i)Pr(X_2=3|X_1=i)}{5/16}$
   $Pr(X_1=1|X_2=3) = \frac{1/2\times1/4}{5/16} = 2/5$
   $Pr(X_1=2|X_2=3) = \frac{1/4\times1/4}{5/16} = 1/5$
   $Pr(X_1=3|X_2=3) = \frac{1/4\times1/2}{5/16} = 2/5$

4. (a) $Pr(X=i\cap Y=j) = Pr(X=i\cap Y=j\cap Z=0) + Pr(X=i\cap Y=j\cap Z=1)$
   $= Pr(X=i\cap Y=j|Z=0)Pr(Z=0) + Pr(X=i\cap Y=j|Z=1)Pr(Z=1)$
   $= Pr(X=i\|Z=0)Pr(Y=j|Z=0)Pr(Z=0)$
   $+ Pr(X=i\|Z=1)Pr(Y=j|Z=1)Pr(Z=1)$
   $Pr(X=1\cap Y=1) = \alpha\beta_1\gamma_1 + (1-\alpha)\beta_0\gamma_0$
   $Pr(X=1\cap Y=0) = \alpha\beta_1(1-\gamma_1) + (1-\alpha)\beta_0(1-\gamma_0)$
   $Pr(X=0\cap Y=1) = \alpha(1-\beta_1)\gamma_1 + (1-\alpha)(1-\beta_0)\gamma_0$
   $Pr(X=0\cap Y=0) = \alpha(1-\beta_1)(1-\gamma_1) + (1-\alpha)(1-\beta_0)(1-\gamma_0)$

   (b) $Pr(Y=1|X=i) = \frac{Pr(Y=1\cap X=i)}{Pr(X=i)}$
   $Pr(X=i) = Pr(X=i\cap Z=0) + Pr(X=i\cap Z=1)$
   $= Pr(Z=0)Pr(X=i|Z=0) + Pr(Z=1)Pr(X=i|Z=1)$
   $Pr(X=1) = (1-\alpha)\beta_0 + \alpha\beta_1$
   $Pr(X=0) = (1-\alpha)(1-\beta_0) + \alpha(1-\beta_1)$

$$Pr(Y=1|X=1) = \frac{\alpha\beta_1\gamma_1+(1-\alpha)\beta_0\gamma_0}{(1-\alpha)\beta_0+\alpha\beta_1}$$

$$Pr(Y=1|X=0) = \frac{\alpha(1-\beta_1)\gamma_1+(1-\alpha)(1-\beta_0)\gamma_0}{(1-\alpha)(1-\beta_0)+\alpha(1-\beta_1)}$$