UCLA STAT 110A
Applied Probability & Statistics for Engineers

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Chapters 3 – Discrete Variables, Probabilities, CLT

- Random Variables (RV’s)
- Probability Density Functions (PDF’s) for discrete RV’s
- Binomial, Negative Binomial, Geometric,
- Hypergeometric, Poisson distributions

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**Frequency Distributions - damaged boxes**

<table>
<thead>
<tr>
<th>Type</th>
<th>Total</th>
<th>Relative</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>A - Flap out</td>
<td>16</td>
<td>0.0096</td>
<td>1</td>
</tr>
<tr>
<td>B - Flap torn</td>
<td>17</td>
<td>0.0102</td>
<td>1</td>
</tr>
<tr>
<td>C - End smashed</td>
<td>132</td>
<td>0.0793</td>
<td>8</td>
</tr>
<tr>
<td>D - Puncture</td>
<td>95</td>
<td>0.0571</td>
<td>6</td>
</tr>
<tr>
<td>E - Glue problem</td>
<td>87</td>
<td>0.0523</td>
<td>5</td>
</tr>
<tr>
<td>F - Corner gouge</td>
<td>984</td>
<td>0.5913</td>
<td>59</td>
</tr>
<tr>
<td>G - Compr. wrinkle</td>
<td>15</td>
<td>0.0090</td>
<td>1</td>
</tr>
<tr>
<td>H - Tip crushed</td>
<td>303</td>
<td>0.1821</td>
<td>18</td>
</tr>
<tr>
<td>I - Tot. destruction</td>
<td>15</td>
<td>0.0090</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>1664</td>
<td>1.0000</td>
<td><strong>100</strong></td>
</tr>
</tbody>
</table>

(* the relative frequencies do not add to 1.0000 due to rounding)

The frequency distribution of a variable is often presented graphically as a bar-chart/bar-plot. For example, the data in the frequency table above can be shown as:

The vertical axis can be frequencies or relative frequencies or percentages. On the horizontal axis all boxes should have the same width leave gaps between the boxes (because there is no connection between them) the boxes can be in any order.

Relative frequency for type A is: \( \frac{16}{1664} = 0.0096 \)

Percentage for type A is: \( \frac{16}{1664} \times 100 = 0.96 = 1 \%

The usefulness of relative frequencies and percentages is clear: for example, it is easily seen that corner gouge accounts for 59% of the total number of damages.
Experiments, Models, RV’s

- **Experiment** is a naturally occurring phenomenon, a scientific study, a sampling trial or a test, in which an object (unit/subject) is selected at random (and/or treated at random) to observe/measure different outcome characteristics of the process the experiment studies.
- **Model** – generalized hypothetical description used to analyze or describe a phenomenon.
- A **random variable** is a type of measurement taken on the outcome of a random experiment.

Definitions

- The **probability function** for a discrete random variable \( X \) gives the chance that the observed value for the process equals a specific outcome, \( x \).
  - \( P(X = x) \) [denoted \( \text{pr}(x) \) or \( P(x) \)] for every value \( x \) that the R.V. \( X \) can take
- E.g., number of heads when a coin is tossed twice

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{pr}(x) )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{4} )</td>
</tr>
</tbody>
</table>

Stopping at one of each or 3 children

**Sample Space** – complete/unique description of the possible outcomes from this experiment.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>GGG</th>
<th>GGB</th>
<th>GB</th>
<th>BG</th>
<th>BBG</th>
<th>BBB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{4} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
</tr>
</tbody>
</table>

- For R.V. \( X \) = number of girls, we have

| \( x \) | \( 0 \) | \( 1 \) | \( 2 \) |
|---|---|---|
| \( \text{pr}(x) \) | \( \frac{1}{8} \) | \( \frac{5}{8} \) | \( \frac{1}{8} \) |

Tossing a biased coin twice

- For each toss, \( P(\text{Head}) = p \) \( \rightarrow \) \( P(\text{Tail}) = P(\text{comp}(H)) = 1-p \)
- Outcomes: HH, HT, TH, TT
- Probabilities: \( p.p, \ (1-p)\text{p}, \ (1-p)(1-p) \)
- Count \( X \), the number of heads in 2 tosses

| \( x \) | \( 0 \) | \( 1 \) | \( 2 \) |
|---|---|---|
| \( \text{pr}(x) \) | \( (1-p)^2 \) | \( 2p(1-p) \) | \( p^2 \) |

Calculating Interval probabilities from cumulative probabilities

- A Bernoulli trial is an experiment where only two possible outcomes are possible (0 / 1).
- Examples:
  - Coin tosses
  - Computer chip (0 / 1) signal.
  - Poll supporters/opponents; yes/no; for/against.
The two-color urn model

\[ N \text{ balls in an urn, of which there are} \]
\[ M \text{ black balls} \]
\[ N - M \text{ white balls} \]

Sample \( n \) balls and count \( X = \# \text{ black balls in sample} \)

We will compute the probability distribution of the R.V. \( X \).

The biased-coin tossing model

\[ \begin{align*}
\text{toss 1} & : \quad \Pr(H) = p \\
\text{toss 2} & : \quad \Pr(H) = p \\
\text{toss } n \quad \quad & : \quad \Pr(H) = p
\end{align*} \]

Perform \( n \) tosses and count \( X = \# \text{ heads} \)

We also want to compute the probability distribution of this R.V. \( X \).
Are the two-color urn and the biased-coin models related? How do we present the models in mathematical terms?

The answer is: Binomial distribution

- The distribution of the number of heads in \( n \) tosses of a biased coin is called the Binomial distribution.

Binomial \((N, p)\) – the probability distribution of the number of Heads in an \( N \)-toss coin experiment, where the probability for Head occurring in each trial is \( p \).

\( \text{E.g., Binomial}(6, 0.7) \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pr(X=x) )</td>
<td>0.001</td>
<td>0.010</td>
<td>0.060</td>
<td>0.185</td>
<td>0.324</td>
<td>0.580</td>
<td>0.882</td>
</tr>
<tr>
<td>( \Pr(X \leq x) )</td>
<td>0.001</td>
<td>0.011</td>
<td>0.070</td>
<td>0.256</td>
<td>0.580</td>
<td>0.882</td>
<td>1.000</td>
</tr>
</tbody>
</table>

For example \( \Pr(X=0) = \Pr(\text{all 6 tosses are Tails}) = (1 - 0.7)^6 = 0.3^6 = 0.001 \)

Binary random process

The biased-coin tossing model is a physical model for situations which can be characterized as a series of trials where:
- each trial has only two outcomes: success or failure;
- \( p = \Pr(\text{success}) \) is the same for every trial; and trials are independent.
- The distribution of \( X = \text{ number of successes (heads) in} \ N \text{ such trials is} \)

\( \text{Binomial}(N, p) \)

Sampling from a finite population – Binomial Approximation

If we take a sample of size \( n \)

- from a much larger population (of size \( N \))
- in which a proportion \( p \) have a characteristic of interest, then the distribution of \( X \), the number in the sample with that characteristic,
- is approximately Binomial \((n, p)\).
  - (Operating Rule: Approximation is adequate if \( n/N < 0.1 \).)
- Example, polling the US population to see what proportion is/has-been married.
### Binomial Probabilities – the moment we all have been waiting for!

- Suppose \( X \sim \text{Binomial}(n, p) \), then the probability
  \[
P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad 0 \leq x \leq n
  \]
- Where the binomial coefficients are defined by
  \[
  \binom{n}{x} = \frac{n!}{(n-x)!x!}, \quad n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n
  \]

### Examples – Birthday Paradox

- **The Birthday Paradox**: In a random group of \( N \) people, what is the chance that at least two people have the same birthday?
  - E.g., if \( N = 23 \), \( P \approx 0.5 \). Main confusion arises from the fact that in real life we rarely meet people having the same birthday as us, and we meet more than 23 people.
  - The reason for such high probability is that any of the 23 people can compare their birthday with any other one, not just you comparing your birthday to anybody else’s.
  - There are \( N \)-Choose-2 = \( \binom{N}{2} = \frac{N(N-1)}{2} \) ways to select a pair of people.
  - \( P(\text{one-particular-pair-failure}) = 1 - \frac{1}{365} \approx 0.99726 \).
  - Assume there are 365 days in a year, \( P(\text{one-particular-pair-same-b-day}) = \frac{1}{365} \), and
  - \( P(\text{one-particular-pair-failure}) = 1 - \frac{1}{365} \approx 0.99726 \).
  - For \( N = 20 \), \( 20 \)-Choose-2 = 190. \( E = \{\text{No 2 people have the same birthday}\} \).
  - \( P(\text{one-particular-pair-failure}) = 1 - \frac{1}{365} \approx 0.99726 \).
  - Hence, \( P(\text{at-least-one-success}) = 1 - 0.59 = 0.41 \), quite high.
  - Note: for \( N = 42 \), \( P > 0.5 \). Main confusion arises from the fact that in real life we rarely meet people having the same birthday as us, and we meet more than 23 people.
  - The Birthday Paradox: In a random group of \( N \) people, what is the change that at least two people have the same birthday?

### Binomial Formula with examples

- Does the Binomial probability satisfy the requirements?
  \[
  \sum_x P(X = x) = \sum_x \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = 1
  \]
- Explicit examples for \( n = 2 \), do the case \( n = 3 \) at home:
  \[
  \sum_{x=0}^{2} \binom{2}{x} p^x (1-p)^{2-x} = \left\{ \text{Three terms in the sum} \right\}
  \]
  \[
  \frac{2}{2} p(p^2) + \frac{3}{2} p^2(1-p) + \frac{1}{2} p^3(1-p)^2 = \frac{1}{1} x(1-x)^2 + \frac{2}{1} x^2(1-x) + \frac{1}{1} x^3
  \]
  \[
  (p + (1-p))^2 = 1
  \]

### Expected values

- The game of chance: cost to play: $1.50; Prices $1, $2, $3, probabilities of winning each price are \( \{0.6, 0.3, 0.1\} \), respectively.
- **Should we play the game?** What are our chances of winning/loosing?

<table>
<thead>
<tr>
<th>Prize ($)</th>
<th>x \times \text{Frequency}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>100</td>
</tr>
</tbody>
</table>

- **What we would “expect” from 100 games**

### Definition of the expected value, in general.

- The expected value:
  \[
  E(X) = \sum_x x \cdot P(x) = \int_{X} P(x) \cdot dx
  \]
  - Sum of (value times probability of value)
Example

In the at least one of each or at most 3 children example, where \( X = \{ \text{number of Girls} \} \) we have:

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(X) )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{5}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
</tr>
</tbody>
</table>

\[
E(X) = \sum xP(X)
\]

\[
= 0 \times \frac{1}{8} + 1 \times \frac{5}{8} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8}
\]

\[
= .25
\]

The expected value and population mean

\( \mu = E(X) \) is called the mean of the distribution of \( X \).

\( \mu_N = E(X) \) is usually called the population mean.

\( \mu_x \) is the point where the bar graph of \( P(X=x) \) balances.

Population standard deviation

The population standard deviation is

\[
\sigma(X) = \sqrt{E[(X-\mu)^2]}
\]

Note that if \( X \) is a RV, then \( (X-\mu) \) is also a RV, and so is \( (X-\mu)^2 \). Hence, the expectation, \( E[(X-\mu)^2] \), makes sense.

For the Binomial distribution . . . mean

\[
X \sim \text{Binomial}(n, p) \Rightarrow
\]

\[
E(X) = np
\]

\[
\sigma(X) = \sqrt{np(1-p)}
\]

\[
X = Y_1 + Y_2 + Y_3 + \ldots + Y_n
\]

where \( Y_k \sim \text{Bernoulli}(p) \),

\[
E(Y_k) = p \Rightarrow
\]

\[
E(X) = E(Y_1 + Y_2 + Y_3 + \ldots + Y_n) = np
\]

Binomial and Multinomial Distributions

- **Multinomial Distribution**
  - \( k \) possible outcomes \( (E_i, \ldots, E_k) \)
  - Each outcome has probability \( p_i \) \( (p_1 + \ldots + p_k = 1) \)
  - In \( n \) independent trials,
    \[
    X_1 + X_2 + \ldots + X_k = n
    \]

\[
f(x_1, \ldots, x_k; p_1, \ldots, p_k, n) = \binom{n}{x_1, \ldots, x_k} p_1^{x_1} \ldots p_k^{x_k}
\]

with \( \sum x_i = n, \sum p_i = 1 \)

Marginal distribution of \( X_j \): \( \text{Bin}(n, p) \)

Binomial and Multinomial Distributions

Ex. Suppose we have 9 people arriving at a meeting.

- \( P(\text{by Air}) = 0.4 \), \( P(\text{by Bus}) = 0.2 \)
- \( P(\text{by Automobile}) = 0.3 \), \( P(\text{by Train}) = 0.1 \)

- \( P(3 \text{ by Air, 3 by Bus, 1 by Auto, 2 by Train}) = ? \)
- \( P(2 \text{ by air}) = ? \)
For any constants \(a\) and \(b\), the expectation of the RV \(aX + b\) is equal to the sum of the product of \(a\) and the expectation of the RV \(X\) and the constant \(b\).

\[
E(aX + b) = aE(X) + b
\]

And similarly for the standard deviation (\(b\), an additive factor, does not affect the SD).

\[
SD(aX + b) = |a| SD(X)
\]

### Example:
1. \(X = \{-1, 2, 0, 3, 4, 0, -2, 1\}; P(X=x) = 1/8\), for each \(x\)
2. \(Y = 2X - 5 = \{-7, -1, -5, 1, 3, -5, -9, -3\}\)
3. \(E(X) =\)
4. \(E(Y) =\)
5. Does \(E(Y) = 2E(X) - 5\) ?
6. Compute \(SD(X), SD(Y)\). Does \(SD(Y) = 2SD(X)\)?
For the Binomial distribution . . . SD

\[ X \sim \text{Binomial}(n, p) \Rightarrow \\
X = Y_1 + Y_2 + Y_3 + \ldots + Y_n, \]

where \( Y_k \sim \text{Bernoulli}(p), \)

\[ \text{Var}(Y_k) = (1-p)^2p + (0-p)^2(1-p) \Rightarrow \\
\text{Var}(Y_k) = (1-p)(p-p^2+p^2) = (1-p)p \Rightarrow \\
\text{Var}(X) = \text{Var}(Y_1) + \ldots + \text{Var}(Y_n) = n(1-p)p \\
\text{SD}(X) = \sqrt{\text{Var}(X)} = \sqrt{n(1-p)p} \]

Sample vs. theoretical mean & variance

- The Expected value:
  \[ E(X) = \sum x \cdot P(x) = \left[ x \cdot P(x) \right]_{x} \]

- Sample mean \[ \bar{X} = \frac{1}{N} \sum x_i \]

- (Theoretical) Variance
  \[ \text{Var}(X) = \sum (x - \mu)^2 P(x) = \left[ (x - \mu)^2 P(x) \right]_{x} \]

- (Sample) variance
  \[ \text{Var}(X) = \frac{1}{N-1} \sum (x_i - \bar{X})^2 = \sum (x_i - \bar{X})^2 P(x) \]

Poisson Distribution – Definition

- Used to model counts – number of arrivals \( k \) on a given interval …
- The Poisson distribution is also sometimes referred to as the distribution of rare events. Examples of Poisson distributed variables are number of accidents per person, number of sweeps taken per person, or the number of catastrophic defects found in a production process.

Functional Brain Imaging – Positron Emission Tomography (PET)

Isotope Energy (MeV)   Range(mm)  1/2-life  Appl.
---   ---   ---   ---
C   0.96   1.1   20 min   receptors
O   1.7   1.5   2 min   stroke/activation
F   0.6   1.0   110 min   neurology
I   ~2.0   1.6   4.5 days   oncology
Statistical Brain Imaging –
Positron Emission Tomography (PET)

Poisson Distribution – Mean

- Used to model counts – number of arrivals (k) on a given interval ...
- Y~Poisson(λ), then P(Y=k) = \( \frac{\lambda^k e^{-\lambda}}{k!} \), k = 0, 1, 2, ...
- Mean of Y, \( \mu_Y = \lambda \)

\[
E(Y) = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda \]

Poisson Distribution – Variance

- Y~Poisson(λ), then P(Y=k) = \( \frac{\lambda^k e^{-\lambda}}{k!} \), k = 0, 1, 2, ...
- Variance of Y, \( \sigma_Y^2 = \lambda \), since

\[
\sigma_Y^2 = Var(Y) = \sum_{k=0}^{\infty} (k-\lambda)^2 \frac{\lambda^k e^{-\lambda}}{k!} = \lambda
\]

Poisson Distribution - Example

- For example, suppose that Y denotes the number of blocked shots in a randomly sampled game for the UCLA Bruins men's basketball team. Poisson distribution with mean=4 may be used to model Y.

Poisson as an approximation to Binomial

- Suppose we have a sequence of Binomial(n, p_r) models, with \( \lim(n \rightarrow \infty) \rightarrow \lambda \), as \( n \rightarrow \infty \).
- For each 0<=y<=n, if Y~ Binomial(n, p_r), then

\[
P(Y=y) = \binom{n}{y} p_r^y (1-p_r)^{n-y}
\]
- But this converges to:

\[
\binom{n}{y} p_r^y (1-p_r)^{n-y} \xrightarrow{\lambda \rightarrow \infty} \frac{\lambda^y e^{-\lambda}}{y!}
\]
- Thus, Binomial(n, p_r) \rightarrow Poisson(\lambda)

Poisson as an approximation to Binomial

- Rule of thumb is that approximation is good if:
  - \( n \geq 100 \)
  - \( p \leq 0.01 \)
  - \( \lambda = np \leq 20 \)
- Then, Binomial(n, p) \rightarrow Poisson(\lambda)
Example using Poisson approx to Binomial

- Suppose $P(\text{defective chip}) = 0.0001=10^{-4}$. Find the probability that a lot of 25,000 chips has \( \geq 2 \) defective!
- $Y \sim \text{Binomial}(25,000, 0.0001)$, find $P(Y>2)$. Note that $Z \sim \text{Poisson}(\lambda = np = 25,000 \times 0.0001 = 2.5)$

\[
P(Z > 2) = 1 - P(Z \leq 2) = 1 - \sum_{z=0}^{2} \frac{2.5^z}{z!} e^{-2.5} = 1 - \left( \frac{2.5^0}{0!} e^{-2.5} + \frac{2.5^1}{1!} e^{-2.5} + \frac{2.5^2}{2!} e^{-2.5} \right) = 0.456
\]

Normal approximation to Binomial – Example

- Roulette wheel investigation:
  - Compute $P(Y>58)$, where $Y \sim \text{Binomial}(100, 0.47)$
  - The proportion of the Binomial(100, 0.47) population having more than 58 reds (successes) out of 100 roulette spins (trials).
  - Since $np=47 > 10$ & $n(1-p)=53 > 10$ Normal approx is justified.
  - $Z=(Y-np)/\sqrt{np(1-p)} = (58 - 100*0.47)/\sqrt{100*0.47*0.53} = 2.2$
  - $P(Y>58) \iff P(Z>2.2) = 0.0139$
  - True $P(Y>58) = 0.177$, using SOCR (demo!)
  - Binomial approx useful when no access to SOCR avail.

Normal approximation to Poisson – example

- Let $X_i \sim \text{Poisson}(\lambda)$ & $X_2 \sim \text{Poisson}(\mu) \Rightarrow X_i + X_2 \sim \text{Poisson}(\lambda+\mu)$
- Let $X_1, X_2, X_p, \ldots, X_{200} \sim \text{Poisson}(2)$, and independent,
- $Y_k = X_1 + X_2 + \ldots + X_k \sim \text{Poisson}(400)$, $E(Y_k)=\text{Var}(Y_k)=400$.
- By CLT the distribution of the standardized variable $(Y_k - 400) / \sqrt{400} \Rightarrow N(0, 1)$, as $k$ increases to infinity.
- $Z_k = (Y_k - 400) / 20 \sim N(0, 1) \Rightarrow Y_k \sim N(400, 400)$.
- $P(2 < Y_k < 400) = (\text{std} \leq 2 & 400) = P(2 < Y_k < 400) = 0.5$

Poisson or Normal approximation to Binomial?

- Poisson Approximation ($\text{Binomial}(n, p) \Rightarrow \text{Poisson}(\lambda)$): $\left( \frac{n}{y} \right) p^y (1-p)^{n-y}$
- $n=100$ & $p=0.01$ & $\lambda = np = 1$
- Normal Approximation ($\text{Binomial}(n, p) \Rightarrow N(\mu, (np(1-p)))^{1/2}$)
- $np = 10$ & $n(1-p) = 10$
Geometric, Hypergeometric, Negative Binomial

- **Geometric** (p), then the probability mass function is:
  \[ P(X = x) = (1 - p)^x p \]
  Probability of first failure at \( x \)th trial.
  \[ E(X) = \frac{1 - p}{p} \]
  \[ Var(X) = \frac{1 - p}{p^2} \]

- **Ex**: Stat dept purchases 40 light bulbs; 5 are defective.
  Select 5 components at random.
  Find: \( P(3rd \text{ bulb used is the first that does not work}) = ? \)

Geometric, Hypergeometric, Negative Binomial

- **Hypergeometric** – \( X \sim \text{HyperGeom}(n, M, N) \)
  Total objects: \( N \). Successes: \( M \).
  Sample-size: \( n \) (without replacement).
  \( X = \) number of successes in sample.

- **Ex**: 40 components in a lot; 3 components are defective.
  Select 5 components at random.
  \( P(\text{obtain one defective}) = P(X=1) = ? \)

Geometric, Hypergeometric, Negative Binomial

- **Negative binomial pmf** \( [X \sim \text{NegBin}(r, p), \text{if } r=1 \Rightarrow \text{Geometric } (p)] \)
  Number of failures until the \( r \)th success (negative), since
  number of successes (\( r \)) is fixed & number of trials (\( X \)) is random.

- **Ex**: 4,000 out of 10,000 residents are against a new tax.
  15 residents are selected at random.
  \( P(\text{at most 7 favor the new tax}) = ? \)

Geometric, Hypergeometric, Negative Binomial

- **Relation among Distributions**
  
  - Normal (\( X \))
  - \( \mu, \sigma^2 \)
  - \( Z = \frac{X - \mu}{\sigma} \)
  - \( Z \sim \text{N}(0,1) \)
  - Chi-square (\( \chi^2 \))
  - \( \nu \)
  - Exponential (\( X \))
  - \( \beta = 1 \)
  - Weibull (\( X \))
  - \( \gamma = 1 \)
  - Beta (\( X \))
  - \( \alpha = 1 \)
  - Uniform (\( U \))
  - \( \alpha, \beta \)
  - Lognormal (\( X \))
  - \( \mu, \sigma^2 \)
  - Weibull (\( X \))
  - \( \gamma = 1 \)
  - Gamma (\( X \))
  - \( \alpha = 1 \)
  - Uniform (\( U \))
  - \( \alpha, \beta \)
  - Exponential (\( X \))
  - \( \beta = 1 \)
  - Weibull (\( X \))
  - \( \gamma = 1 \)
  - Beta (\( X \))
  - \( \alpha = 1 \)
  - Uniform (\( U \))
  - \( \alpha, \beta \)
  - Exponential (\( X \))
  - \( \beta = 1 \)
  - Weibull (\( X \))
  - \( \gamma = 1 \)
  - Beta (\( X \))
  - \( \alpha = 1 \)