Definitions

- An **experiment** is a naturally occurring phenomenon, a scientific study, a sampling trial or a test, in which an object (unit/subject) is selected at random (and/or treated at random) to observe/measure different outcome characteristics of the process the experiment studies.

- A **random variable** is a type of measurement taken on the outcome of a random experiment.

The **probability function** for a discrete random variable \( X \) gives \( P(X = x) \) [denoted \( pr(x) \) or \( P(x) \)] for every value \( x \) that the R.V. \( X \) can take.

For R.V. \( X = \) number of girls, we have

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( pr(x) )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
<td>( \frac{1}{8} )</td>
</tr>
</tbody>
</table>

Plotting the probability function
Tossing a biased coin twice

- For each toss, \( P(\text{Head}) = p \Rightarrow P(\text{Tail}) = 1 - p \)
- Outcomes: HH, HT, TH, TT
- Probabilities: \( pp, p(1-p), (1-p)p, (1-p)(1-p) \)
- Count \( X \), the number of heads in 2 tosses

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( pr(X) )</td>
<td>((1-p)^2)</td>
<td>(2p(1-p))</td>
<td>(p^2)</td>
</tr>
</tbody>
</table>

Hospital stays

<table>
<thead>
<tr>
<th>Days stayed</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>30</td>
<td>113</td>
<td>79</td>
<td>21</td>
<td>8</td>
<td>2</td>
<td>263</td>
</tr>
<tr>
<td>Proportion</td>
<td>0.038</td>
<td>0.114</td>
<td>0.430</td>
<td>0.300</td>
<td>0.080</td>
<td>0.030</td>
<td>0.008</td>
<td>1.000</td>
</tr>
<tr>
<td>Cumulative</td>
<td>0.038</td>
<td>0.152</td>
<td>0.582</td>
<td>0.882</td>
<td>0.962</td>
<td>0.992</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>Proportion</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Calibrating Interval probabilities from cumulative probabilities

- \( x \)-values: 1 2 3 4 5 6 7 8 9 10 11 12
- To get 4 to 8, \( P(3 < X < 9) \)
- \( P(X < 9) \)
- \( P(X = 3) \)
- \( P(X = 8) \)

\[ \frac{[4, 8]}{[1, 12]} \]

Figure 5.2.2 Interval probabilities from cumulative probabilities. 

Review

- What is a random variable? What is a discrete random variable? (type of measurement taken on the outcome of random experiment)
- What general principle is used for finding \( P(X=x) \)? (Adding the probabilities of all outcomes of the experiment where we have measured the RV, \( X=x \))
- What two general properties must be satisfied by the probabilities making up a probability function? (\( P(X)=0; \sum P(x)=1 \))
- What are the two names given to probabilities of the form \( P(X \leq x) \)? (cumulative & lower/left-tail)

Review

- How do we find an upper/right-tail probability from a cumulative probability? \( P(X=x) = P(X \geq x) \)
- When we use \( P(X \leq 12) - P(X \leq 5) \) to calculate the probability that \( X \) falls within an interval of values, what numbers are included in the interval? \( [6:12] \)

The two-color urn model

- \( N \) balls in an urn, of which there are \( M \) black balls, \( N-M \) white balls
- Sample \( n \) balls and count \( X = \# \) black balls in sample

We will compute the probability distribution of the R.V. \( X \)
Perform \( n \) tosses and count \( X = \# \) heads.

We also want to compute the probability distribution of this R.V. \( X \).
Are the two-color urn and the biased-coin models related? How do we present the models in mathematical terms?

The biased-coin tossing model

Each trial has only two outcomes:
success or failure;

\( p = P(\text{success}) \) is the same for every trial; and

trials are independent.

The distribution of \( X = \) number of successes (heads) in \( N \) such trials is Binomial(\( N, p \)).

Sampling from a finite population – Binomial Approximation

If we take a sample of size \( n \)

- from a much larger population (of size \( N \))

- in which a proportion \( p \) have a characteristic of interest, then the distribution of \( X \), the number in the sample with that characteristic,

is approximately Binomial(\( n, p \)).

(Operating Rule: Approximation is adequate if \( n / N < 0.1 \).)

Example, polling the US population to see what proportion is has-been married.

The answer is: Binomial distribution

- The distribution of the number of heads in \( n \) tosses of a biased coin is called the Binomial distribution.

Binomial(\( N, p \)) – the probability distribution of the number of Heads in an \( N \)-toss coin experiment, where the probability for Head occurring in each trial is \( p \).

E.g., Binomial(6, 0.7)

For example \( P(X=0) = P(\text{all 6 tosses are Tails}) = (1 - 0.7)^6 = 0.3^6 = 0.001 \)

Odds and ends …

- For what types of situation is the urn-sampling model useful? For modeling binary random processes. When sampling with replacement, Binomial distribution is exact, where as, in sampling without replacement Binomial distribution is an approximation.

- For what types of situation is the biased-coin sampling model useful? Defective parts. Approval poll of cloning for medicinal purposes. Number of Boys in 151 presidential children (90).

- Give the three essential conditions for its applicability. (two outcomes; same \( p \) for every trial; independence)
Odds and ends …

- What is the distribution of the number of heads in \( n \) tosses of a biased coin?
- Under what conditions does the Binomial distribution apply to samples taken without replacement from a finite population? When interested in assessing the distribution of a R.V., \( X \), the number of observations in the sample (of \( n \)) with one specific characteristic, where \( \frac{n}{N} < 0.1 \) and a proportion \( p \) have the characteristic of interest in the beginning of the experiment.

Binomial Probabilities – the moment we all have been waiting for!

- Suppose \( X \sim \text{Binomial}(n, p) \), then the probability
  
  \[
  P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad 0 \leq x \leq n
  \]

  Where the binomial coefficients are defined by
  
  \[
  \binom{n}{x} = \frac{n!}{x!(n-x)!}, \quad n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times n \text{ n-factorial}
  \]

Binomial Formula with examples

- Does the Binomial probability satisfy the requirements?
  
  \[
  \sum_x P(X = x) = \sum_x \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = 1
  \]

- Explicit examples for \( n=2 \), do the case \( n=3 \) at home!
  
  \[
  \sum_{x=0}^{2} \binom{2}{x} p^x (1-p)^{2-x} = \text{Three terms in the sum}
  \]

  \[
  \left( \begin{array}{c}
  2
  
  \end{array} \right) p^2 (1-p)^0 + \left( \begin{array}{c}
  2
  
  \end{array} \right) p^1 (1-p)^1 + \left( \begin{array}{c}
  2
  
  \end{array} \right) p^0 (1-p)^2 = 1
  \]

  \[
  \begin{align*}
  &1 \times (1-p)^2 + 2 \times p \times (1-p) + 1 \times p^2 \\
  \Rightarrow & (p + (1-p))^2 = 1
  \end{align*}
  \]

Expected values

- The game of chance: cost to play: $1.50; Prices {$1, $2, $3}, probabilities of winning each price are {0.6, 0.3, 0.1}, respectively.
- Should we play the game? What are our chances of winning/losing?

**Definition of the expected value, in general.**

- The expected value:
  
  \[
  E(X) = \sum_{x} x P(x) = \int_{all X} x P(x)dx
  \]

  = Sum of (value times probability of value)
Example

In the at least one of each or at most 3 children example, where \( X = \{\text{number of Girls}\} \) we have:

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\[
E(X) = \sum x P(x)
\]

\[
= 0 \times \frac{1}{8} + 1 \times \frac{5}{8} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8}
\]

\[
= 1.25
\]

The expected value and population mean

\( \mu_x = E(X) \) is called the mean of the distribution of \( X \).

\( \mu_x = E(X) \) is usually called the population mean.

\( \mu_x \) is the point where the bar graph of \( P(X = x) \) balances.

The expected value as the point of balance

The mean \( \mu_x \) is the balance point.

Population standard deviation

The population standard deviation is

\[
sd(X) = \sqrt{E((X - \mu)^2)}
\]

Note that if \( X \) is a RV, then \( (X - \mu)^2 \) is also a RV, and so is \( (X - \mu)^2 \). Hence, the expectation, \( E[(X - \mu)^2] \), makes sense.

Population mean & standard deviation

Expected value:

\[
E(X) = \sum \sum x P(X = x)
\]

Variance

\[
Var(X) = \sum (x - E(x))^2 P(X = x)
\]

Standard Deviation

\[
SD(X) = \sqrt{Var(X)} = \sqrt{\sum (x - E(x))^2 P(X = x)}
\]

For the Binomial distribution . . . mean

\[
E(X) = np
\]

\[
sd(X) = \sqrt{np(1-p)}
\]

\[
E(X) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}
\]

\[
= \sum_{x=1}^{n} \binom{n}{x} p^x (1-p)^{n-x}
\]

\[
= \sum_{x=0}^{n-1} \binom{n}{x} p^x (1-p)^{n-x} + \binom{n}{n} p^n (1-p)^0
\]

If \( x = 0 \), the entire term is zero.

Change variables: \( x \to (x+1) \)
For any constants $a$ and $b$, the expectation of the RV $aX + b$ is equal to the sum of the product of $a$ and the expectation of the RV $X$ and the constant $b$.

\[
E(aX + b) = aE(X) + b
\]

And similarly for the standard deviation ($b$, an additive factor, does not affect the SD).

\[
SD(aX + b) = |a| SD(X)
\]

**Linear Scaling (affine transformations)** $aX + b$

And why do we care?

\[
E(aX + b) = aE(X) + b; \quad SD(aX + b) = |a| SD(X)
\]

- E.g., say the rules for the game of chance we saw before change and the new pay-off is as follows: \{$0, $1.50, $3\}, with probabilities of \{0.6, 0.3, 0.1\}, as before. What is the newly expected return of the game? Remember the old expectation was equal to the entrance fee of $1.50, and the game was fair!

\[
Y = 3(X - 1)/2
\]

\{$1, $2, $3\} \rightarrow \{$0, $1.50, $3\},

\[
E(Y) = 3/2 \cdot E(X) - 3/2 - 3 / 4 = 0.75
\]

And the game became clearly biased. Note how easy it is to compute $E(Y)$.