Stat 100a: Introduction to Probability.
Outline for the day:
1. Ly vs. Negreanu flush draw example.
2. A or K on the board example.
3. Addiction and poker legality.
4. Kohlberg and Murphy example.
5. A♠ vs 2♣ after first ace.
6. Conditional probability, independence, and multiplication rule.
7. Independence and dependence situations.
8. Negreanu and Elezra example.
On problem 2.4, use the convention that a royal flush is an example of a straight flush.

Note that the answer to 2.9 is in the back of the book.

1. Ly vs. Negreanu, p66.

Ex. Suppose you have 2 ♣️s, and there are exactly 2 ♣️s on the flop. Given this info, what is P(at least one more ♣️ on turn or river)?

Answer: 52-5 = 47 cards left (9 ♣️s, 38 others).

So n = choose(47,2) = 1081 combinations for next 2 cards. Each equally likely (and obviously mutually exclusive).

Two-♣️ combos: choose(9,2) = 36. One-♣️ combos: 9 × 38 = 342.

Total = 378. So answer is 378/1081 = 35.0%.

Answer #2: Use the addition rule…
ADDITION RULE, revisited…..

Axioms (initial assumptions/rules) of probability:

1) \( P(A) \geq 0 \).
2) \( P(A) + P(A^c) = 1 \).
3) Addition rule:
   If \( A_1, A_2, A_3, \ldots \) are mutually exclusive, then
   \( P(A_1 \text{ or } A_2 \text{ or } A_3 \text{ or } \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots \)

As a result, even if \( A \) and \( B \) might not be mutually exclusive,
\( P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \).  
(p6 of book)
Ex. You have 2 ♣s, and there are exactly 2 ♣s on the flop. Given this info, what is P(at least one more ♣ on turn or river)?

Answer #1: 52-5 = 47 cards left (9 ♣s, 38 others).
So n = \text{choose}(47,2) = 1081 combinations for next 2 cards.
Each equally likely (and obviously mutually exclusive).
Two-♣ combos: \text{choose}(9,2) = 36. One-♣ combos: 9 \times 38 = 342.
Total = 378. So answer is \frac{378}{1081} = 35.0\%.

Answer #2: Use the addition rule.
\begin{align*}
P(\geq 1 \text{ more ♣}) &= P(♣ \text{ on turn OR river}) \\ &= P(♣ \text{ on turn}) + P(♣ \text{ on river}) - P(\text{both}) \\ &= \frac{9}{47} + \frac{9}{47} - \frac{\text{choose}(9,2)\text{/choose}(47,2)} \\ &= 19.15\% + 19.15\% - 3.3\% = 35.0\%.
\end{align*}
2. A or K on the board.

**Ex.** You have AK. Given this, what is
P(at least one A or K comes on board of 5 cards)?

**Wrong Answer:**
P(A or K on 1st card) + P(A or K on 2nd card) + …
= 6/50 x 5 = 60.0%.
No: these events are NOT Mutually Exclusive!!!

**Right Answer:**
choose(50,5) = 2,118,760 boards possible.
How many have exactly one A or K? 6 x choose(44,4) = 814,506
# with exactly 2 aces or kings? choose(6,2) x choose(44,3) = 198,660
# with exactly 3 aces or kings? choose(6,3) x choose(44,2) = 18,920 …
… altogether, 1,032,752 boards have at least one A or K,
So it’s 1,032,752 / 2,118,760 = 48.7%.

**Easier way:** P(no A and no K) = choose(44,5)/choose(50,5)
= 1086008 / 2118760 = 51.3%, so answer = 100% - 51.3% = 48.7%

3. Addiction and legality of poker.

Linda Johnson $543,000  
Kathy Kolberg $300,000  
Phil Laak $475,000  
Sue Murphy $155,000  
Tammy Pescatelli $377,000  
Mark Curry $0.

No small blind. Johnson in big blind for $8000.
Murphy (8♥ 8♠). Calls $8,000.
Kolberg. (9♣ 9◆). Raises to $38,000.
Pescatelli (K♥ 3♠) folds, Laak (9♥ 3♥) folds, Johnson (J♥ 6◆) folds.
Murphy calls.
TV Screen:  
Kolberg. (9♣ 9◆) 81%  
Murphy (8♥ 8♠) 19%
Flop: 8♣ 10♦ 10♠.

Murphy quickly goes all in. Kolberg thinks for 2 min, then calls.
Laak (to Murphy): “You’re 92% to take it down.”

TV Screen:  
Kolberg. (9♣ 9◆) 17%  
Murphy (8♥ 8♠) 83%

Who’s right?
(Turn 9♠ river A♦), so Murphy is eliminated. Laak went on to win.
Murphy quickly goes all in. Kolberg thinks for 2 min, then calls. Laak (to Murphy): “You’re 92% to take it down.”

Laak (about Kolberg): “She has two outs twice.”

P(9 on the turn or river, given just their 2 hands and the flop)?

\[
P(9 \text{ on the turn or river}) = P(9 \text{ on turn}) + P(9 \text{ on river}) - P(9 \text{ on both})
\]

\[
= \frac{2}{45} + \frac{2}{45} - \frac{1}{\binom{45}{2}} = 8.8\%
\]
Given just their 2 hands and the flop, what is

\[ P(9 \text{ or } 10 \text{ on the turn or river, but not } 98 \text{ or } 108) \]?

\[ P(9 \text{ or } 10 \text{ on the turn}) + P(9 \text{ or } 10 \text{ on river}) - P(910) - [P(98) + P(108)] \]

\[ = \frac{4}{45} + \frac{4}{45} - \left[ \binom{4}{2} + 2 + 2 \right]/\binom{45}{2} = 16.77\% \]
5. Deal til first ace appears. Let $X =$ the *next* card after the ace. 
$P(X = A♠)? \; P(X = 2♣)?$ 

(a) How many permutations of the 52 cards are there? 
\[52!\] 

(b) How many of these perms. have $A♠$ right after the 1st ace? 
(i) How many perms of the *other* 51 cards are there? 
\[51!\] 
(ii) For *each* of these, imagine putting the $A♠$ right after the 1st ace. 

1:1 correspondence between permutations of the other 51 cards 
& permutations of 52 cards such that $A♠$ is right after 1st ace. 
So, the answer to question (b) is 51!. 
Answer to the overall question is \(51! / 52! = 1/52\). 
Obviously, same goes for $2♣$. 

3. Negreanu and Elezra.
P(A & B) is often written “P(AB)”.
“P(A U B)” means P(A or B [or both]).

Conditional Probability:
P(A given B) [written “P(A|B)”] = P(AB) / P(B).

Independent: A and B are “independent” if P(A|B) = P(A).

Fact (multiplication rule for independent events):
If A and B are independent, then P(AB) = P(A) x P(B)

Fact (general multiplication rule):
P(AB) = P(A) P(B|A)
P(ABC…) = P(A) x P(B|A) x P(C|A&B) …
7. Independence and Dependence Examples

Independence: \( P(A \mid B) = P(A) \) [and \( P(B \mid A) = P(B) \)].

So, when independent, \( P(A \& B) = P(A)P(B \mid A) = P(A)P(B) \).

Reasonable to assume the following are independent:

a) Outcomes on different rolls of a die.

b) Outcomes on different flips of a coin.

c) Outcomes on different spins of a spinner.

d) Outcomes on different poker hands.

e) Outcomes when sampling from a large population.

Ex: \( P(\text{you get AA on 1st hand and I get AA on 2nd hand}) \)

\[ = P(\text{you get AA on 1st}) \times P(\text{I get AA on 2nd}) \]
\[ = 1/221 \times 1/221 = 1/48841. \]

\( P(\text{you get AA on 1st hand and I get AA on 1st hand}) \)

\[ = P(\text{you get AA}) \times P(\text{I get AA} \mid \text{you have AA}) \]
\[ = 1/221 \times 1/(50 \text{ choose } 2) = 1/221 \times 1/1225 = 1/270725. \]
Greenstein folds, Todd Brunson folds, Harman folds. Elezra calls $600, Farha raises to $2600, Sheikhan folds. Negreanu calls, Elezra calls. Pot is $8,800.

Flop: 6♠ 10♠ 8♥.

Negreanu bets $5000. Elezra raises to $15000. Farha folds. Negreanu thinks for 2 minutes….. then goes all-in for another $96,000.

Elezra: 8♣ 6♣. (Elezra calls. Pot is $214,800.)

Negreanu: A◆ 10♥.

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At this point, the odds on tv show 73% for Elezra and 25% for Negreanu. They “run it twice”. First: 2♠ 4♥. Second time? A♥ 8♦!

P(Negreanu hits an A or 10 on turn & still loses)?
Given both their hands, and the flop, and the first “run”, what is $P(\text{Negreanu hits an A or 10 on the turn & loses})$?

Since he can’t lose if he hits a 10 on the turn, it’s:

$P(\text{A on turn & Negreanu loses})$

$= P(\text{A on turn}) \times P(\text{Negreanu loses | A on the turn})$

$= 3/43 \times 4/42$

$= 0.66\% \ (1 \text{ in } 150.5)$

Note: this is very different from:

$P(\text{A or 10 on turn}) \times P(\text{Negreanu loses}),$

which would be about $5/43 \times 73\% = 8.49\% \ (1 \text{ in } 12)$