Some nonregular designs from the Nordstrom and Robinson code and their statistical properties

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Summary

The Nordstrom and Robinson code is a well-known nonlinear code in coding theory. This paper explores the statistical properties of this nonlinear code. Many nonregular designs with 32, 64, 128 and 256 runs and 7–16 factors are derived from it. It is shown that these nonregular designs are better than regular designs of the same size in terms of resolution, aberration and projectivity. Furthermore, many of these nonregular designs are shown to have generalised minimum aberration among all possible designs. Seven orthogonal arrays are shown to have unique wordlength pattern and four of them are shown to be unique up to isomorphism.

Some key words: Generalised minimum aberration; Generalised resolution; Generalised wordlength pattern; Linear programming; MacWilliams identity; Orthogonal array; Projectivity.

1. Introduction

Fractional factorial designs with factors at two levels are among the most widely used experimental designs. Regular fractional factorial designs are specified by some defining relations among the factors. They are typically chosen by the minimum aberration criterion (Fries & Hunter, 1980), which includes the maximum resolution criterion (Box & Hunter, 1961) as a special case. There are many recent results on the construction and properties of minimum aberration designs; see Wu & Hamada (2000, Ch. 4) for details and references.
There has been increasing interest in the study of nonregular designs because they enjoy some good projection properties; see Lin & Draper (1992), Wang & Wu (1995), Cheng (1995, 1998) and Box & Tyssedal (1996). The concepts of resolution and aberration have been extended to nonregular designs; see Deng & Tang (1999), Tang & Deng (1999), Ma & Fang (2001), Xu & Wu (2001) and Xu (2003). Nonregular designs from Hadamard matrices of order 16, 20 and 24 have been catalogued by Deng & Tang (2002) with a computer search. The construction of good nonregular designs remains challenging especially when the size is large.

This paper studies some nonregular designs derived from a well-known code in coding theory. A regular design is known as a linear code and a nonregular design is simply a nonlinear code. The connection between codes in coding theory and designs in statistics was first observed by Bose (1961). The Nordstrom & Robinson code, a well-known nonlinear code, was originally constructed by Nordstrom & Robinson (1967) and has been studied extensively in coding theory; see MacWilliams & Sloane (1977, Ch. 2 & 15). On the statistical side, the Nordstrom & Robinson code is a nonlinear orthogonal array with 256 runs, 16 factors, two levels and strength 5 while a linear orthogonal array of the same size has strength at most 4 (Hedayat et al., 1999, Ch. 5 § 10). However, the statistical properties of the Nordstrom & Robinson code have not been fully explored.

The Nordstrom & Robinson code, as well as some background information on notation and definitions, is described in § 2. Many nonregular designs with 32, 64, 128 and 256 runs and 7–16 factors are derived from it and their statistical properties are studied in § 3. It is shown that these nonregular designs are better than regular designs of the same size in terms of resolution, aberration and projectivity. Some associated theoretic questions are
addressed in § 4. With MacWilliams identities and linear programming, many of these nonregular designs are shown to have generalised minimum aberration among all possible designs. Furthermore, seven orthogonal arrays are shown to have unique wordlength pattern and four of them are shown to be unique up to isomorphism. Concluding remarks are given in § 5.

2. Background

2.1. Notation and definitions

A design $D$ of $N$ runs and $n$ factors is represented by an $N \times n$ matrix where each row corresponds to a run and each column a factor. A two-level design takes on only two symbols, say $-1$ or $+1$. For $s = \{c_1, \ldots, c_k\}$, a subset of $k$ columns of $D$, define

$$J_k(s) = \left| \sum_{i=1}^{N} c_{i1} \cdots c_{ik} \right|, \quad (1)$$

where $c_{ij}$ is the $i$th component of column $c_j$. When $D$ is a regular design, $J_k(s)$ takes on only two values: 0 or $N$. In general, $0 \leq J_k(s) \leq N$. If $J_k(s) = N$, these $k$ columns in $s$ form a word of length $k$.

Suppose that $r$ is the smallest integer such that $\max_{|s|=r} J_r(s) > 0$, where the maximisation is over all subsets of $r$ columns of $D$. The generalised resolution (Deng & Tang, 1999) of $D$ is defined as $R(D) = r + \lfloor 1 - \max_{|s|=r} J_r(s)/N \rfloor$. Let

$$A_k(D) = N^{-2} \sum_{|s|=k} [J_k(s)]^2. \quad (2)$$

The vector $(A_1(D), \ldots, A_n(D))$ is the generalised wordlength pattern. The generalised minimum aberration criterion, called minimum $G_2$-aberration by Tang & Deng (1999), is to sequentially minimise $A_1(D), A_2(D), \ldots, A_n(D)$. When restricted to regular designs, generalised resolution, generalised wordlength pattern and generalised minimum aberration
reduce to the traditional resolution, wordlength pattern and minimum aberration, respectively. In the rest of the paper, we simply use resolution, wordlength pattern and minimum aberration for both regular and nonregular designs.

A two-level design \( D \) of \( N \) runs and \( n \) factors is an orthogonal array of strength \( t \), denoted by \( OA(N, n, 2, t) \), if all possible \( 2^t \) level combinations for any \( t \) factors appear equally often. Deng & Tang (1999) showed that a design has resolution \( r \leq R < r + 1 \) if and only if it is an orthogonal array of strength \( t = r - 1 \).

A two-level design is said to have projectivity \( p \) (Box & Tyssedal, 1996) if any \( p \)-factor projection contains a complete \( 2^p \) factorial design, possibly with some points replicated. A regular design with resolution \( R = r \) is an orthogonal array of strength \( r - 1 \) and hence has projectivity \( r - 1 \). Deng & Tang (1999) showed that a design with resolution \( R > r \) has projectivity \( p \geq r \).

A two-level design is also called a binary code in coding theory. For two row vectors \( a \) and \( b \), the Hamming distance \( d_H(a, b) \) is the number of places where they differ. Let

\[
B_j(D) = N^{-1} |\{(a, b) : a, b \text{ are row vectors of } D, \text{ and } d_H(a, b) = j\}|.
\]

The vector \( (B_0(D), B_1(D), \ldots, B_n(D)) \) is called the distance distribution of \( D \). The minimum distance \( d \) is the smallest integer \( k \geq 0 \) such that \( B_k(D) > 0 \). A design \( D \) of \( N \) runs, \( n \) factors and minimum distance \( d \) is called an \((n, N, d)\) code in coding theory. See Hedayat et al. (1999, Ch. 4) for an introduction to coding theory and applications to orthogonal arrays.

Xu & Wu (2001) showed that the wordlength pattern is the MacWilliams transform of the distance distribution, i.e.,

\[
A_j(D) = N^{-1} \sum_{i=0}^{n} P_j(i; n) B_1(D) \quad \text{for } j = 0, \ldots, n, \tag{3}
\]
where \( P_j(x; n) = \sum_{i=0}^{j} (-1)^i \binom{i}{j-i} \binom{n-x}{j-i} \) are the Krawtchouk polynomials and \( A_0(D) = 1 \). By the orthogonality of the Krawtchouk polynomials, it is easy to show that

\[
B_j(D) = N \sum_{i=0}^{n} P_j(i; n) A_i(D) \quad \text{for} \; j = 0, \ldots, n.
\]

The equations (3) and (4) are known as the generalised MacWilliams identities.

### 2.2. Nordstrom and Robinson code

The original Nordstrom & Robinson code (Nordstrom & Robinson, 1967) has 15 columns, labelled as

\[
X_0, X_1, \ldots, X_7, Y_0, Y_1, \ldots, Y_6,
\]

where \( X_0, \ldots, X_7 \) are information bits (or independent columns) and \( Y_0, \ldots, Y_6 \) are redundant bits (or dependent columns). Each \( Y \) is a Boolean function of the \( X \)'s. \( Y_0 \) is defined as follows:

\[
Y_0 = X_7 \oplus X_6 \oplus X_0 \oplus X_1 \oplus X_3 \oplus (X_0 \oplus X_4)(X_1 \oplus X_2 \oplus X_3 \oplus X_5) \oplus (X_1 \oplus X_2)(X_3 \oplus X_5),
\]

where \( \oplus \) denotes modulo 2 addition. Note that the \( X \)'s and \( Y \)'s take on value 0 or 1 here. The remaining \( Y \)'s are found by cyclically shifting \( X_0 \) through \( X_6 \); i.e., for \( Y_j \) substitute \( X_{i+j \pmod{7}} \) for \( X_i \) in (5) where \( i = 0, 1, \ldots, 6 \) for each \( j = 0, 1, \ldots, 6 \).

The extended Nordstrom & Robinson code has an additional column, labelled as \( Y_7 \), where

\[
Y_7 = X_0 \oplus X_1 \oplus \cdots \oplus X_7 \oplus Y_0 \oplus Y_1 \oplus \cdots \oplus Y_6.
\]

The extended Nordstrom & Robinson code is a design of 256 runs and 16 factors when \( X_0, \ldots, X_7 \) are evaluated at \( 2^8 \) possible level combinations. It is well known in coding theory that the distance distribution coincides with the MacWilliams transform of the distance distribution, and they are:
The extended Nordstrom & Robinson code is a (16, 256, 6) code and an OA(256, 16, 2, 5).

There are several different constructions for the Nordstrom & Robinson code; see MacWilliams & Sloane (1977, Ch. 2) and Hedayat et al. (1999, Ch. 5 § 10). Nevertheless, it is known that a (16, 256, 6) code is unique up to isomorphism (MacWilliams & Sloane, 1977, p. 74–75). Two designs or codes are said to be isomorphic if one can be obtained from the other by permuting the rows, the columns and the symbols of each column.

3. Nonregular designs from the Nordstrom and Robinson code

3.1. Designs of 256 runs

First we study the projection property of the extended Nordstrom & Robinson code in term of the $J_k(s)$ values defined in (1). The $J_k$’s are zero except for $J_6$, $J_8$, $J_{10}$ and $J_{16}$ because the $A_k$’s are zero except for $A_6$, $A_8$, $A_{10}$ and $A_{16}$. With a computer, it is straightforward to verify that there are 448 six-factor projections with $J_6 = 128$, 30 words of length 8 (i.e., $J_8 = 256$), 448 ten-factor projections with $J_{10} = 128$, one word of length 16 (i.e., $J_{16} = 256$), and all other $J_k(s) = 0$. Therefore, the frequencies of the nonzero $J_k$ values are

<table>
<thead>
<tr>
<th>$J_6$</th>
<th>$J_8$</th>
<th>$J_{10}$</th>
<th>$J_{16}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>128</td>
<td>256</td>
<td>128</td>
<td>256</td>
</tr>
<tr>
<td>448</td>
<td>30</td>
<td>448</td>
<td>1</td>
</tr>
</tbody>
</table>

The extended Nordstrom & Robinson code has resolution 6.5 and hence has projectivity at least 6. Indeed, it can be verified that it has projectivity 7. For comparison, a regular minimum aberration design of the same size has resolution 5 and projectivity 4.

Note that a regular minimum aberration design of 256 runs has resolution 5 for 13–16 factors and resolution 6 for 10–12 factors (Draper & Lin, 1990). An immediate conclusion is
that any projection design with 10–16 columns from the extended Nordstrom & Robinson code has higher resolution and better projectivity than any regular design of the same size.

Next we study the projection designs (or subdesigns). It is evident that any subdesign of 15 factors is an $OA(256, 15, 2, 5)$ and a $(15, 256, 5)$ code. The MacWilliams identities uniquely determine the wordlength pattern because the only nonzero $A_i$ values are $A_0 = 1, A_6, A_8$ and $A_{10}$, and $B_0 = 1, B_i = 0$ for $i = 1, 2, 3, 4$. Indeed, the first three identities of (4) are

\[
2^{-7}(1 + A_6 + A_8 + A_{10}) = B_0 = 1,
\]
\[
2^{-7}(15 + 3A_6 - A_8 - 5A_{10}) = B_1 = 0,
\]
\[
2^{-7}(105 - 3A_6 - 7A_8 + 5A_{10}) = B_2 = 0.
\]

There is a unique solution: $A_6 = 70, A_8 = 15, A_{10} = 42$ and all other $A_i = 0$.

Similarly, any subdesign of 14 factors is an $OA(256, 14, 2, 5)$ and a $(14, 256, 4)$ code. The MacWilliams identities completely determine the wordlength pattern: $A_6 = 42, A_8 = 7, A_{10} = 14$ and all other $A_i = 0$. Any subdesign of 13 factors is an $OA(256, 13, 2, 5)$ and a $(13, 256, 3)$ code. The wordlength pattern, determined by the MacWilliams identities, is $A_6 = 24, A_8 = 3, A_{10} = 4$ and all other $A_i = 0$. The wordlength patterns are not unique for subdesigns of 6–12 factors.

Table 1 shows minimum aberration subdesigns for 9–16 factors from the extended Nordstrom & Robinson code, their resolutions and wordlength patterns where $A_1 = \cdots = A_5 = 0$ are omitted. Corresponding regular minimum aberration designs can be found in Chen & Wu (1991) for 9–12 factors, Chen (1992) for 13 factors, and Franklin (1984) for 14–16 factors. Whether Franklin’s designs have minimum aberration needs to be verified, though. Compared with the regular minimum aberration design, the minimum aberration design
given in Table 1 has more aberration for 9–10 factors, the same aberration for 11–12 factors and less aberration for 13–16 factors. The minimum aberration design for 9 factors is a regular design.

3.2. Designs of 128 runs

From the extended Nordstrom & Robinson code, one gets a design of 128 runs and 15 factors by taking the runs that begin with $X_0 = 0$ and omitting the column $X_0$. This technique is known as shortening in coding theory and the resulting design is known as the shortened Nordstrom & Robinson code. The shortened Nordstrom & Robinson code is an $OA(128, 15, 2, 4)$ and a $(15, 128, 6)$ code. It is known that a $(15, 128, 6)$ code is unique (MacWilliams & Sloane, 1977, p. 75).

The wordlength pattern of the shortened Nordstrom & Robinson code is again uniquely determined by the MacWilliams identities. The only nonzero $B_i$ values are $B_0 = 1$, $B_6$, $B_8$ and $B_{10}$; and $A_0 = 1$, $A_i = 0$ for $i = 1, 2, 3, 4$. Indeed, the first three identities of (3) are

$$2^{-7}(1 + B_6 + B_8 + B_{10}) = A_0 = 1,$$
$$2^{-7}(15 + 3B_6 - B_8 - 5B_{10}) = A_1 = 0,$$
$$2^{-7}(105 - 3B_6 - 7B_8 + 5B_{10}) = A_2 = 0.$$

There is a unique solution: $B_6 = 70, B_8 = 15, B_{10} = 42$. By the MacWilliams identities (3), one gets $A_0 = 1, A_5 = 42, A_6 = 70, A_7 = 15, A_8 = 15, A_9 = 70, A_{10} = 42, A_{15} = 1$ and all other $A_i = 0$. It is interesting to note that the wordlength pattern is the MacWilliams transform of the wordlength pattern of the $OA(256, 15, 2, 5)$ described in § 3.1.

The frequencies of the nonzero $J_k$ values are

<table>
<thead>
<tr>
<th>$J_5$ : 64</th>
<th>$J_6$ : 64</th>
<th>$J_7$ : 128</th>
<th>$J_8$ : 128</th>
<th>$J_9$ : 64</th>
<th>$J_{10}$ : 64</th>
<th>$J_{15}$ : 128</th>
</tr>
</thead>
<tbody>
<tr>
<td>168</td>
<td>280</td>
<td>15</td>
<td>15</td>
<td>280</td>
<td>168</td>
<td>1</td>
</tr>
</tbody>
</table>
The shortened Nordstrom & Robinson code has resolution 5.5 and projectivity 6 (not 5). For comparison, a regular minimum aberration design of the same size has resolution 4 and projectivity 3.

Note that a regular minimum aberration design of 128 runs has resolution 4 for 12–15 factors and resolution 5 for 10–11 factors (Draper & Lin, 1990). Therefore, any projection design with 10–15 columns from the shortened Nordstrom & Robinson code has higher resolution and better projectivity than any regular design of the same size.

Table 2 shows minimum aberration subdesigns for 8–15 factors from the shortened Nordstrom & Robinson code, their resolutions and wordlength patterns. Corresponding regular minimum aberration designs can be found in Chen & Wu (1991) for 8–11 factors, Chen (1992) for 12 factors, Chen (1998) for 13–14 factors, and Franklin (1984) for 15 factors. Again, whether the Franklin’s design has minimum aberration needs to be verified. Compared with the regular minimum aberration design, the minimum aberration design given in Table 2 has more aberration for 9 factors, the same aberration for 8, 10–11 factors and less aberration for 12–15 factors. The minimum aberration design for 9 factors is of interest because it has projectivity 6 while a regular minimum aberration design of the same size has projectivity 5. The minimum aberration design for 8 factors is a regular design.

3.3. Designs of 64 runs

From the shortened Nordstrom & Robinson code, one gets a design of 64 runs and 14 factors by taking the runs that begin with $X_1 = 0$ and omitting the column $X_1$. The resulting design is an $OA(64,14,2,3)$ and a $(14,64,6)$ code. It is again known that a $(14,64,6)$ code is unique (MacWilliams & Sloane, 1977, p. 75).

The wordlength pattern of the $OA(64,14,2,3)$, determined by the MacWilliams iden-
tities, is the MacWilliams transform of the wordlength pattern of the $OA(256, 14, 2, 5)$ described in § 3-1. The frequencies of the nonzero $J_k$ values are

<table>
<thead>
<tr>
<th>$J_4$ : 32</th>
<th>$J_5$ : 32</th>
<th>$J_6 : (64, 32)$</th>
<th>$J_7 : 64$</th>
<th>$J_8 : (64, 32)$</th>
<th>$J_9 : 32$</th>
<th>$J_{10} : 32$</th>
<th>$J_{14} : 64$</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>224</td>
<td>(7, 168)</td>
<td>16</td>
<td>(7, 168)</td>
<td>224</td>
<td>56</td>
<td>1</td>
</tr>
</tbody>
</table>

The $OA(64, 14, 2, 3)$ has resolution 4.5 and projectivity 5 (not 4). For comparison, a regular minimum aberration design of the same size has resolution 4 and projectivity 3.

Note that a regular minimum aberration design of 64 runs and 9–14 factors has resolution 4 (Draper & Lin, 1990). Therefore, any projection design with 9–14 columns from the $OA(64, 14, 2, 3)$ has higher resolution and better projectivity than any regular design of the same size.

Cheng (1998) showed that as long as an orthogonal array of strength three has no defining word of length four, its projection onto any five factors allows the estimation of all the main effects and two-factor interactions when the higher-order interactions are negligible. It can be verified that the $OA(64, 14, 2, 3)$ has the stronger property that its projection onto any seven factors allows the estimation of all the main effects and two-factor interactions.

Table 3 shows minimum aberration subdesigns for 7–14 factors, their resolutions and wordlength patterns. Compared with the corresponding regular minimum aberration design (Chen et al., 1993), the minimum aberration design given in Table 3 has the same aberration for 7–12 factors and less aberration for 13–14 factors. The minimum aberration design for 8 factors has resolution 5.5 while the regular minimum aberration $2^8-2$ design has resolution 5 although they have the same wordlength pattern. The minimum aberration design for 7 factors is a regular design.

3-4. Designs of 32 runs
From the \((14,64,6)\) code, one gets a design of 32 runs and 13 factors by taking the runs that begin with \(X_2 = 0\) and omitting the column \(X_2\). The resulting design is an \(OA(32,13,2,2)\) and a \((13,32,6)\) code. A \((13,32,6)\) code is unique (MacWilliams & Sloane, 1977, p. 75).

The wordlength pattern of the \(OA(32,13,2,2)\) is the MacWilliams transform of the wordlength pattern of the \(OA(256,13,2,5)\) described in §3.1. The frequencies of the nonzero \(J_k\) values are

\[
\begin{array}{cccccccc}
16 & 120 & (3,216) & (12,96) & (12,96) & (3,216) & 120 & 16 & 1
\end{array}
\]

The \(OA(32,13,2,2)\) has resolution 3.5 and projectivity 4 (not 3). Note that a regular minimum aberration design of 32 runs and 7–13 factors has resolution 4 and projectivity 3. A conclusion is that any projection design with 7–13 columns from the \(OA(32,13,2,2)\) has better projectivity than any regular design of the same size.

Cheng (1995) showed that as long as an orthogonal array of strength two has no defining word of length three or four, its projection onto any four factors allows the estimation of all the main effects and two-factor interactions when the higher-order interactions are negligible. It can be verified that the \(OA(32,13,2,2)\) has the stronger property that its projection onto any five factors allows the estimation of all the main effects and two-factor interactions.

Table 4 shows minimum aberration subdesigns for 6–13 factors, their resolutions and wordlength patterns. Compared with the corresponding regular minimum aberration design (Chen et al., 1993), the minimum aberration design given in Table 4 has the same aberration for 6–9 factors and more aberration for 10–13 factors. The minimum aberration design for
7-9 factors has resolution 4.5 while the corresponding regular minimum aberration design has resolution 4 although they have the same wordlength pattern. The minimum aberration design for 6 factors is a regular design.

4. SOME THEORETICAL RESULTS

The MacWilliams identities provide a powerful tool in the study of coding theory and factorial design. Based on the MacWilliams identities and the fact that the wordlength pattern \( A_i \) and the distance distribution \( B_i \) are always nonnegative, linear programming technique can be used to establish bounds on the maximum size of a code for given length and distance and bounds on the minimum size of an orthogonal array for given number of constraints and strength; see MacWilliams and Sloane (1977, Ch. 17 § 4) and Hedayat et al. (1999, Ch. 4 § 5). Here we use MacWilliams identities and linear programming to show that many nonregular designs derived from the Nordstrom & Robinson code indeed have minimum aberration among all possible designs.

From the definition (2), it is easy to see that \( 0 \leq A_k \leq \binom{n}{k} < 2^n \) for all \( k \). Because \( N^2 A_k \) is an integer, sequentially minimising \( A_1, A_2, \ldots, A_n \) is equivalent to minimising \( \sum_{j=1}^{n} \lambda^{n-j} A_j \), where \( \lambda \) is any number that is larger than or equal to \( N^2 2^n \).

Suppose an \( OA(N, n, 2, t) \) exists. Then a minimum aberration design of \( N \) runs and \( n \) factors must satisfy \( A_1 = \ldots = A_t = 0 \) and sequentially minimises \( A_{t+1}, A_{t+2}, \ldots, A_n \). So we consider the following linear programming problem:

\[
\begin{align*}
\text{minimise} & \quad \sum_{j=t+1}^{n} (N^2 2^n)^{n-j} A_j \\
\text{subject to} & \quad \sum_{j=t+1}^{n} A_j \geq N^{-1} 2^n - 1,
\end{align*}
\]

subject to

\[
\sum_{j=t+1}^{n} A_j \geq N^{-1} 2^n - 1,
\]
\[
\sum_{j=t+1}^{n} P_i(j; n) A_j \geq -P_i(0; n) \text{ for } i = 1, \ldots, n, \tag{8}
\]
\[A_j \geq 0 \text{ for } j = t + 1, \ldots, n. \tag{9}\]

where inequality (7) corresponds to \(B_0 \geq 1\) and (8) corresponds to \(B_i \geq 0\).

An optimal solution to (6) gives a feasible minimum aberration wordlength pattern. If the wordlength pattern of a design coincides with the optimal solution, then the design must have minimum aberration among all possible designs.

As an example, consider the case of \(N = 32\) and \(n = 7\) where an \(OA(32, 7, 2, 3)\) exists. The constraints (7)–(9) are

\[
A_4 + A_5 + A_6 + A_7 \geq 3, \tag{10}
\]
\[-A_4 - 3A_5 - 5A_6 - 7A_7 \geq -7, \tag{11}\]
\[-3A_4 + A_5 + 9A_6 + 21A_7 \geq -21, \]
\[3A_4 + 5A_5 - 5A_6 - 35A_7 \geq -35, \]
\[3A_4 - 5A_5 - 5A_6 + 35A_7 \geq -35, \]
\[-3A_4 + A_5 + 9A_6 - 21A_7 \geq -21, \]
\[-A_4 + 3A_5 - 5A_6 + 7A_7 \geq -7, \]
\[A_4 - A_5 + A_6 - A_7 \geq -1, \]
\[A_4 \geq 0, A_5 \geq 0, A_6 \geq 0, A_7 \geq 0. \tag{12}\]

Adding \(3 \times (10)\) to (11) yields \(2A_4 - 2A_6 - 4A_7 \geq 2\) or \(A_4 \geq A_6 + 2A_7 + 1 \geq 1\) due to (12).

When \(A_4 = 1, A_6\) and \(A_7\) must be zero, and hence \(A_5 \geq 2\) from (10). Therefore, an optimal solution is \(A_4 = 1, A_5 = 2, A_6 = 0, A_7 = 0\), which satisfies all constraints. Recall that the minimum aberration design for 7 factors given in Table 4 has this wordlength pattern. Therefore, it has minimum aberration among all possible designs.
The coefficients of constraints are large and complicated in general, so a computer software is used to solve the linear programming problem. We use Mathematica (a software of Wolfram Research, Inc.) in this task because it gives exact solutions.

Table 5 lists optimal solutions to the linear programming problem (6) for various parameters ($A_1 = A_2 = A_3 = 0$ are omitted). It is obvious that a solution cannot be the wordlength pattern of a design whenever $N^2A_i$ is not an integer for some $i$. Nevertheless, there are 20 cases where the wordlength pattern of a design coincides with the optimal solution. The last column of Table 5 indicates such a design where R refers to a regular minimum aberration design and NR refers to a nonregular minimum aberration design derived from the Nordstrom & Robinson code. In summary, we have the following result.

**Theorem 1.** The nonregular minimum aberration designs given in Tables 1–4 have minimum aberration among all possible designs for the following 13 cases: 256 runs and 14–16 factors, 128 runs and 13–15 factors, 64 runs and 8–9, 12–14 factors, and 32 runs and 7–8 factors.

From Table 5, we observe that a regular minimum aberration $2^{n-1}$ design has minimum aberration for $n=6$–9. This is true in general because an $OA(2^{n-1}, n, 2, n-1)$ is unique up to isomorphism.

We also observe that a regular minimum aberration $2^{n-2}$ design has minimum aberration for $n=7$–10 (although there also exist nonregular minimum aberration designs). The following theorem shows that this is also true in general.

**Theorem 2.** For $n$ factors and $N = 2^{n-2}$ runs, a regular minimum aberration $2^{n-2}$ design has minimum aberration among all possible designs.

**Proof.** Chen & Wu (1991) showed that a regular minimum aberration $2^{n-2}$ design has
resolution \( R = \lfloor 2n/3 \rfloor \), where \( \lfloor x \rfloor \) is the largest integer that is less than or equal to \( x \).

They also showed that the minimum aberration wordlength pattern is \( A_R = 3R - 2n + 3 \), \( A_{R+1} = 2n - 3R \) and other \( A_i = 0 \). We show that this is the optimal solution to the linear programming problem (6).

The first two inequalities of (7) and (8) are

\[
\sum_{j=R}^{n} A_j \geq N^{-1}2^n - 1 = 3, \quad (13)
\]

\[
\sum_{j=R}^{n} (n-2j)A_j \geq -n. \quad (14)
\]

Multiplying (13) by \( 2R + 2 - n \) and adding it to (14), one gets

\[
2A_R + \sum_{j=R+1}^{n} (2R - 2j + 2)A_j \geq 3(2R + 2 - n) - n.
\]

Since \( 2R - 2j + 2 \leq 0 \) for \( j \geq R + 1 \) and \( A_j \geq 0 \), one gets \( A_R \geq 3R - 2n + 3 \). Therefore, the smallest \( A_R \) value is \( 3R - 2n + 3 \). Next multiplying (13) by \( 2R + 4 - n \) and adding it to (14), one gets

\[
4A_R + 2A_{R+1} + \sum_{j=R+2}^{n} (2R - 2j + 4)A_j \geq 3(2R + 4 - n) - n.
\]

Since \( 2R - 2j + 4 \leq 0 \) for \( j \geq R + 2 \) and \( A_j \geq 0 \), one gets \( A_{R+1} \geq 3R - 2n + 6 - 2A_R \). When \( A_R = 3R - 2n + 3 \), the smallest \( A_{R+1} \) value is \( 2n - 3R \). Therefore, the optimal solution is \( A_R = 3R - 2n + 3 \), \( A_{R+1} = 2n - 3R \) and other \( A_i = 0 \). \( \square \)

We do not know whether Theorem 2 can be extended to \( N = 2^{n-3} \). From Table 5, a regular minimum aberration \( 2^{n-3} \) design has minimum aberration for \( n = 8, 9 \). However, the optimal solutions are different from the minimum aberration wordlength patterns for \( n = 10, 11 \). For example, Table 5 shows that the optimal solution is \((0, 0, 0, 0, 2, 5, 0, 0, 0, 0)\) for \( n = 10 \), whereas a regular minimum aberration \( 2^{10-3} \) design has wordlength pattern
(0, 0, 0, 0, 3, 3, 1, 0, 0, 0). Note that the nonregular minimum aberration design from the Nordstrom & Robinson code has the same wordlength pattern as the regular minimum aberration design. It is of interest to investigate whether there exists a nonregular design of 128 runs and 10 factors having wordlength pattern (0, 0, 0, 0, 2, 5, 0, 0, 0, 0).

The linear programming technique can also be used to determine whether an orthogonal array has a unique wordlength pattern in some cases. Consider the following linear programming problem:

\[
\text{maximise } \sum_{j=t+1}^{n} (N^22^n)^{n-j} A_j \\
\text{subject to the constraints (7)–(9). The optimal solution to (15) sequentially maximises } A_{t+1}, A_{t+2}, \ldots, A_{n} \text{ while the optimal solution to (6) sequentially minimises them. When the two solutions are the same, the wordlength pattern of all } OA(N, n, 2, t) \text{'s must be unique.}
\]

Using Mathematica, we have proved the following result.

**Theorem 3.** The following seven orthogonal arrays have unique wordlength pattern: \(OA(256, n, 2, 5)\) for \(n = 14, 15, 16\); \(OA(128, n, 2, 4)\) for \(n = 13, 14, 15\); and \(OA(64, 8, 2, 4)\).

From Theorem 3, an \(OA(256, 16, 2, 5)\) must have the same wordlength pattern and distance distribution as the extended Nordstrom & Robinson code. Therefore, it has minimum distance 6. In other words, an \(OA(256, 16, 2, 5)\) is a \((16, 256, 6)\) code. The same argument shows that an \(OA(256, 15, 2, 5)\) is a \((15, 256, 5)\) code, an \(OA(128, 15, 2, 4)\) is a \((15, 128, 6)\) code and an \(OA(128, 14, 2, 4)\) is a \((14, 128, 5)\) code. It is known that the \((16, 256, 6)\), \((15, 256, 5)\), \((15, 128, 6)\) and \((14, 128, 5)\) codes are unique up to isomorphism (MacWilliams & Sloane, 1977, p. 74–75). Therefore, we have the following result.

**Theorem 4.** The \(OA(256, 16, 2, 5)\), \(OA(256, 15, 2, 5)\), \(OA(128, 15, 2, 4)\) and \(OA(128, 14, 2, 4)\) are unique up to isomorphism.
Hedayat et al. (1999, p. 109) wrote that the $OA(256, 16, 2, 5)$ is unique up to isomorphism known from coding theory; however, they did not provide the detail. It is interesting to note that the $(14, 64, 6)$ and $(13, 32, 6)$ codes are unique but the $OA(64, 14, 2, 3)$ and $OA(32, 13, 2, 2)$ are not. The $OA(64, 8, 2, 4)$ is not unique although it has a unique wordlength pattern.

5. Concluding remarks

This paper explores the statistical properties of the Nordstrom & Robinson code and derives many nonregular designs with 32, 64, 128 and 256 runs and 7–16 factors from it. These nonregular designs are better than regular designs of the same size in terms of resolution, aberration and projectivity.

Regular designs have a simple aliasing structure: Any two factorial effects are either orthogonal or fully aliased. In contrast, nonregular designs have a complex aliasing structure: Some factorial effects are partially aliased. The partial aliasing of nonregular designs can be an advantage and a disadvantage. An advantage is that partially aliased effects can be estimated simultaneously under some circumstance (Hamada & Wu, 1992). A disadvantage is that the design and analysis can be complicated if there is some form of blocking or split-plot structure. There are many researches on blocked or split-plot regular designs; see Sun et al. (1997), Sitter et al. (1997) and Chen and Cheng (1999) for blocked regular designs and Bingham & Sitter (1999a, 1999b) for split-plot regular designs. The study on blocked or split-plot nonregular designs is still primitive.

The structure of the Nordstrom & Robinson code allows us to derive some good blocked or split-plot nonregular designs easily. For example, if the first column is used as a blocking variable, the extended Nordstrom & Robinson code is a blocked nonregular design with 256
runs and 15 factors in 2 blocks. The extended Nordstrom & Robinson code can also be
used to study 14 factors in 4 blocks or 13 factors in 8 blocks. The shorten Nordstrom &
Robinson code can be used to study 14 factors in 2 blocks, 13 factors in 4 blocks, or 12
factors in 8 blocks. These blocked nonregular designs have higher resolution than blocked
regular designs of the same size. These blocked nonregular designs can also be used as
split-plot designs, which have higher resolution than split-plot regular designs of the same
size.

In summary, the nonregular designs given in this paper are recommended with the
exception of 32 runs, and 64 runs with 12 or less factors. Of course, the choice may depend
on how important the practitioner values orthogonality in the design.

Generalisations of the Nordstrom & Robinson code include the Kerdock and Preparata
codes (MacWilliams & Sloane, 1977, Ch. 15). For even $m \geq 4$, the Kerdock code $K(m)$ is
a nonlinear $(2^m, 2^{2m}, 2^{m-1} - 2^{(m-2)/2})$ code and the Preparata code $P(m)$ is a nonlinear
$(n = 2^m, 2^{n-2m}, 6)$ code. Both the first Kerdock code $K(4)$ and the first Preparata code
$P(4)$ are equivalent to the extended Nordstrom & Robinson code. The second Kerdock
code $K(6)$ is an $OA(4096, 64, 2, 5)$ and good nonregular designs can be derived from it. For
example, shortening $K(6)$ yields an $OA(2048, 63, 2, 4)$. For comparison, a regular design of
the same size has strength at most 3. Other Kerdock and Preparata codes are too large to
be useful as fractional factorial designs.

ACKNOWLEDGEMENT

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the editor and the referees for helpful comments.
References


Table 1: Minimum aberration designs of 256 runs from the Nordstrom & Robinson code

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<thead>
<tr>
<th>$n$</th>
<th>Columns</th>
<th>$R$</th>
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<td>1–8, 16</td>
<td>8</td>
<td>(0,0,1,0)</td>
</tr>
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<td>1–9, 16</td>
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<td>(2,0,1,0,0)</td>
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<tr>
<td>11</td>
<td>1–10, 16</td>
<td>6.5</td>
<td>(6,0,1,0,0,0)</td>
</tr>
<tr>
<td>12</td>
<td>1–10, 14, 16</td>
<td>6.5</td>
<td>(12,0,3,0,0,0,0)</td>
</tr>
<tr>
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<td>6.5</td>
<td>(24,0,3,0,4,0,0,0)</td>
</tr>
<tr>
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<td>any 14 columns</td>
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<tr>
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Table 2: Minimum aberration designs of 128 runs from the Nordstrom & Robinson code

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<td>5.5</td>
<td>(3,3,1,0,0,0)</td>
</tr>
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<td>1–9, 13, 15</td>
<td>5.5</td>
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<tr>
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<td>1–12</td>
<td>5.5</td>
<td>(11,13,2,1,3,1,0,0)</td>
</tr>
<tr>
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Table 3: Minimum aberration designs of 64 runs from the Nordstrom & Robinson code

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<td>4.5</td>
<td>(2, 8, 4, 1, 0, 0)</td>
</tr>
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<td>1–9, 11, 13</td>
<td>4.5</td>
<td>(4, 14, 8, 0, 3, 2, 0)</td>
</tr>
<tr>
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<td>(6, 24, 16, 0, 9, 8, 0, 0)</td>
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<tr>
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Table 4: Minimum aberration designs of 32 runs from the Nordstrom & Robinson code

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<td>(0, 3, 4, 0, 0, 0)</td>
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<td>(0, 6, 8, 0, 0, 1, 0)</td>
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