13-1 DESIGNING ENGINEERING EXPERIMENTS

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  13-2.2 Analysis of Variance
  13-2.3 Multiple Comparisons Following the ANOVA
  13-2.4 Residual Analysis and Model Checking
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  13-4.1 Design and Statistical Analysis
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CHAPTER OUTLINE
13-1 Designing Engineering Experiments

Every experiment involves a sequence of activities:

1. **Conjecture** – the original hypothesis that motivates the experiment.

2. **Experiment** – the test performed to investigate the conjecture.

3. **Analysis** – the statistical analysis of the data from the experiment.

4. **Conclusion** – what has been learned about the original conjecture from the experiment. Often the experiment will lead to a revised conjecture, and a new experiment, and so forth.
13-2 The Completely Randomized Single-Factor Experiment

13-2.1 An Example

A manufacturer of paper used for making grocery bags is interested in improving the tensile strength of the product. Product engineering thinks that tensile strength is a function of the hardwood concentration in the pulp and that the range of hardwood concentrations of practical interest is between 5 and 20%. A team of engineers responsible for the study decides to investigate four levels of hardwood concentration: 5%, 10%, 15%, and 20%. They decide to make up six test specimens at each concentration level, using a pilot plant. All 24 specimens are tested on a laboratory tensile tester, in random order. The data from this experiment are shown in Table 13-1.
13-2 The Completely Randomized Single-Factor Experiment

13-2.1 An Example

Table 13-1 Tensile Strength of Paper (psi)

<table>
<thead>
<tr>
<th>Hardwood Concentration (%)</th>
<th>Observations</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>Totals</th>
<th>Averages</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td></td>
<td>7</td>
<td>8</td>
<td>15</td>
<td>11</td>
<td>9</td>
<td>10</td>
<td>60</td>
<td>10.00</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>12</td>
<td>17</td>
<td>13</td>
<td>18</td>
<td>19</td>
<td>15</td>
<td>94</td>
<td>15.67</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>14</td>
<td>18</td>
<td>19</td>
<td>17</td>
<td>16</td>
<td>18</td>
<td>102</td>
<td>17.00</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>19</td>
<td>25</td>
<td>22</td>
<td>23</td>
<td>18</td>
<td>20</td>
<td>127</td>
<td>21.17</td>
</tr>
</tbody>
</table>

383 15.96
13-2 The Completely Randomized Single-Factor Experiment

13-2.1 An Example

- The levels of the factor are sometimes called treatments.
- Each treatment has six observations or replicates.
- The runs are run in random order.
13-2 The Completely Randomized Single-Factor Experiment

13-2.1 An Example

Figure 13-1 (a) Box plots of hardwood concentration data. (b) Display of the model in Equation 13-1 for the completely randomized single-factor experiment
13-2 The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

Suppose there are \( a \) different levels of a single factor that we wish to compare. The levels are sometimes called treatments.

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Observations</th>
<th>Totals</th>
<th>Averages</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( y_{11} )</td>
<td>( y_{12} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>2</td>
<td>( y_{21} )</td>
<td>( y_{22} )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( a )</td>
<td>( y_{a1} )</td>
<td>( y_{a2} )</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

\( y_{..} \) | \( \bar{y}_{..} \) |
13-2 The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

We may describe the observations in Table 13-2 by the linear statistical model:

\[ Y_{ij} = \mu + \tau_i + \epsilon_{ij} \]

where:
- \( Y_{ij} \) is the observation
- \( \mu \) is the overall mean
- \( \tau_i \) is the effect of the ith treatment
- \( \epsilon_{ij} \) is the random error

The model could be written as

\[ Y_{ij} = \mu_i + \epsilon_{ij} \]

where:
- \( \mu_i \) is the mean of the ith treatment
13-2 The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

**Fixed-effects Model**

The treatment effects are usually defined as deviations from the overall mean so that:

$$\sum_{i=1}^{a} \tau_i = 0$$

Also,

$$y_{i.} = \sum_{j=1}^{n} y_{ij} \quad \bar{y}_{i.} = y_{i.}/n \quad i = 1, 2, \ldots, a$$

$$y_{..} = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij} \quad \bar{y}_{..} = y_{..}/N$$
13-2 The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

We wish to test the hypotheses:

\[ H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0 \]

\[ H_1: \tau_i \neq 0 \quad \text{for at least one } i \]

The analysis of variance partitions the total variability into two parts.
13-2 The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

The sum of squares identity is

\[ \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{..})^2 = n \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2 + \sum_{i=1}^{a} \sum_{j=1}^{n} (y_{ij} - \bar{y}_{i.})^2 \]  \hspace{1cm} (13-5)

or symbolically

\[ SS_T = SS_{\text{Treatments}} + SS_E \]  \hspace{1cm} (13-6)
13-2 The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

The expected value of the treatment sum of squares is

\[ E(SS_{\text{Treatments}}) = (a - 1)\sigma^2 + n \sum_{i=1}^{a} \tau_i^2 \]

and the expected value of the error sum of squares is

\[ E(SS_E) = a(n - 1)\sigma^2 \]

The ratio \( MS_{\text{Treatments}} = \frac{SS_{\text{Treatments}}}{(a - 1)} \) is called the mean square for treatments.
13-2 The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

The appropriate test statistic is

\[ F_0 = \frac{SS_{\text{Treatments}}/(a - 1)}{SS_{E}/[a(n - 1)]} = \frac{MS_{\text{Treatments}}}{MS_{E}} \]  (13-7)

We would reject \( H_0 \) if \( f_0 > f_{\alpha,a-1,a(n-1)} \)
13-2 The Completely Randomized Single-Factor Experiment

13-2.2 The Analysis of Variance

The sums of squares computing formulas for the ANOVA with equal sample sizes in each treatment are

\[ SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N} \]  \hspace{1cm} (13-8)

and

\[ SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y_{i..}^2}{n} - \frac{y_{..}^2}{N} \]  \hspace{1cm} (13-9)

The error sum of squares is obtained by subtraction as

\[ SSE = SS_T - SS_{\text{Treatments}} \]  \hspace{1cm} (13-10)

where \( N=na \) is the total number of observations.
### 13-2 The Completely Randomized Single-Factor Experiment

#### 13-2.2 The Analysis of Variance

**Analysis of Variance Table**

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>$F_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>$SS_{Treatments}$</td>
<td>$a - 1$</td>
<td>$MS_{Treatments}$</td>
<td>$\frac{MS_{Treatments}}{MS_E}$</td>
</tr>
<tr>
<td>Error</td>
<td>$SS_E$</td>
<td>$a(n - 1)$</td>
<td>$MS_E$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$SS_T$</td>
<td>$an - 1$</td>
<td>$MS_E$</td>
<td></td>
</tr>
</tbody>
</table>

Table 13-3  The Analysis of Variance for a Single-Factor Experiment, Fixed-Effects Model
Example 13-1

Consider the paper tensile strength experiment described in Section 13-2.1. We can use the analysis of variance to test the hypothesis that different hardwood concentrations do not affect the mean tensile strength of the paper.

The hypotheses are

\[ H_0: \tau_1 = \tau_2 = \tau_3 = \tau_4 = 0 \]

\[ H_1: \tau_i \neq 0 \text{ for at least one } i \]
Example 13-1

We will use $\alpha = 0.01$. The sums of squares for the analysis of variance are computed from Equations 13-8, 13-9, and 13-10 as follows:

$$SS_T = \sum_{i=1}^{4} \sum_{j=1}^{6} y_{ij}^2 - \frac{y_{..}^2}{N}$$

$$= (7)^2 + (8)^2 + \cdots + (20)^2 - \frac{(383)^2}{24} = 512.96$$

$$SS_{\text{Treatments}} = \sum_{i=1}^{4} \frac{y_{..}^2}{n} - \frac{y_{..}^2}{N}$$

$$= \frac{(60)^2 + (94)^2 + (102)^2 + (127)^2}{6} - \frac{(383)^2}{24} = 382.79$$

$$SS_E = SS_T - SS_{\text{Treatments}}$$

$$= 512.96 - 382.79 = 130.17$$
13-2 The Completely Randomized Single-Factor Experiment

Example 13-1

The ANOVA is summarized in Table 13-4. Since $f_{0.01,3,20} = 4.94$, we reject $H_0$ and conclude that hardwood concentration in the pulp significantly affects the mean strength of the paper. We can also find a $P$-value for this test statistic as follows:

$$P = P(F_{3,20} > 19.60) \approx 3.59 \times 10^{-6}$$

Since $P \approx 3.59 \times 10^{-6}$ is considerably smaller than $\alpha = 0.01$, we have strong evidence to conclude that $H_0$ is not true.

Table 13-4  ANOVA for the Tensile Strength Data

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>$f_0$</th>
<th>$P$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hardwood concentration</td>
<td>382.79</td>
<td>3</td>
<td>127.60</td>
<td>19.60</td>
<td>3.59 E-6</td>
</tr>
<tr>
<td>Error</td>
<td>130.17</td>
<td>20</td>
<td>6.51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>512.96</td>
<td>23</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 13-5  Minitab Analysis of Variance Output for Example 13-1

One-Way ANOVA: Strength versus CONC

Analysis of Variance for Strength

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conc</td>
<td>3</td>
<td>382.79</td>
<td>127.60</td>
<td>19.61</td>
<td>0.000</td>
</tr>
<tr>
<td>Error</td>
<td>20</td>
<td>130.17</td>
<td>6.51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>23</td>
<td>512.96</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Individual 95% CIs For Mean Based on Pooled StDev

<table>
<thead>
<tr>
<th>Level</th>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>6</td>
<td>10.000</td>
<td>2.828</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>15.667</td>
<td>2.805</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>6</td>
<td>17.000</td>
<td>1.789</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>6</td>
<td>21.167</td>
<td>2.639</td>
<td></td>
</tr>
</tbody>
</table>

Pooled StDev = 2.551

10.0  15.0  20.0  25.0

Fisher's pairwise comparisons

Family error rate = 0.192
Individual error rate = 0.0500

Critical value = 2.086

Intervals for (column level mean) − (row level mean)

<table>
<thead>
<tr>
<th></th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8.739</td>
<td>5.94</td>
<td></td>
</tr>
<tr>
<td></td>
<td>−2.594</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>10.072</td>
<td>−4.406</td>
<td></td>
</tr>
<tr>
<td></td>
<td>−3.928</td>
<td>1.739</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>14.239</td>
<td>−8.572</td>
<td>−7.239</td>
</tr>
<tr>
<td></td>
<td>−8.094</td>
<td>−2.428</td>
<td>−1.094</td>
</tr>
</tbody>
</table>
A 100(1 − α) percent confidence interval on the mean of the $i$th treatment $\mu_i$ is

$$\bar{y}_i - t_{\alpha/2, a(n-1)} \sqrt{\frac{MSE}{n}} \leq \mu_i \leq \bar{y}_i + t_{\alpha/2, a(n-1)} \sqrt{\frac{MSE}{n}}$$

(13-11)

The 95% CI on the mean of the 20% hardwood is

$$\left[ \bar{y}_{4.} \pm t_{0.025, 20} \sqrt{\frac{MSE}{n}} \right]$$

$$\left[ 21.167 \pm (2.086) \sqrt{6.51 / 6} \right]$$

$$19.00 \text{ psi} \leq \mu_4 \leq 23.34 \text{ psi}$$
A 100(1 − α) percent confidence interval on the difference in two treatment means \( \mu_i - \mu_j \) is

\[
\overline{y}_i - \overline{y}_j - t_{\alpha/2, a(n-1)} \sqrt{\frac{2MS_E}{n}} \leq \mu_i - \mu_j \leq \overline{y}_i - \overline{y}_j + t_{\alpha/2, a(n-1)} \sqrt{\frac{2MS_E}{n}}
\]  
(13-12)

For the hardwood concentration example,

A 95% CI on the difference in means \( \mu_3 - \mu_2 \) is

\[
\left[ \overline{y}_3 - \overline{y}_2 \pm t_{0.025, 20} \sqrt{\frac{2MS_E}{n}} \right]
\]

\[
\left[ 17.00 - 15.67 \pm (2.086) \sqrt{2(6.51) / 6} \right]
\]

\[-1.74 \leq \mu_3 - \mu_2 \leq 4.40\]
13-2 The Completely Randomized Single-Factor Experiment

An Unbalanced Experiment

The sums of squares computing formulas for the ANOVA with unequal sample sizes \( n_i \) in each treatment are

\[
SS_T = \sum_{i=1}^{a} \sum_{j=1}^{n_i} y_{ij}^2 - \frac{y^2.}{N} \quad (13-13)
\]

\[
SS_{\text{Treatments}} = \sum_{i=1}^{a} \frac{y^2_i}{n_i} - \frac{y^2.}{N} \quad (13-14)
\]

and

\[
SS_E = SS_T - SS_{\text{Treatments}} \quad (13-15)
\]
13-2 The Completely Randomized Single-Factor Experiment

13-2.3 Multiple Comparisons Following the ANOVA

The least significant difference (LSD) is

\[
LSD = t_{\alpha/2, n-1} \sqrt{\frac{2MSE}{n}} \tag{13-16}
\]

If the sample sizes are different in each treatment:

\[
LSD = t_{\alpha/2, N-a} \sqrt{MSE \left( \frac{1}{n_i} + \frac{1}{n_j} \right)}
\]
We will apply the Fisher LSD method to the hardwood concentration experiment. There are \( a = 4 \) means, \( n = 6 \), \( MS_E = 6.51 \), and \( t_{0.025,20} = 2.086 \). The treatment means are

\[
\bar{y}_1. = 10.00 \text{ psi} \\
\bar{y}_2. = 15.67 \text{ psi} \\
\bar{y}_3. = 17.00 \text{ psi} \\
\bar{y}_4. = 21.17 \text{ psi}
\]

The value of LSD is

\[
LSD = t_{0.025,20} \sqrt{\frac{2MS_E}{n}} = 2.086 \sqrt{2(6.51)/6} = 3.07
\]

Therefore, any pair of treatment averages that differs by more than 3.07 implies that the corresponding pair of treatment means are different.
The comparisons among the observed treatment averages are as follows:

4 vs. 1 = 21.17 − 10.00 = 11.17 > 3.07
4 vs. 2 = 21.17 − 15.67 = 5.50 > 3.07
4 vs. 3 = 21.17 − 17.00 = 4.17 > 3.07
3 vs. 1 = 17.00 − 10.00 = 7.00 > 3.07
3 vs. 2 = 17.00 − 15.67 = 1.33 < 3.07
2 vs. 1 = 15.67 − 10.00 = 5.67 > 3.07

From this analysis, we see that there are significant differences between all pairs of means except 2 and 3. This implies that 10 and 15% hardwood concentration produce approximately the same tensile strength and that all other concentration levels tested produce different tensile strengths. It is often helpful to draw a graph of the treatment means, such as in Fig. 13-2, with the means that are not different underlined. This graph clearly reveals the results of the experiment and shows that 20% hardwood produces the maximum tensile strength.
13-2 The Completely Randomized Single-Factor Experiment

Example 13-2

Figure 13-2 Results of Fisher’s LSD method in Example 13-2.
13-2 The Completely Randomized Single-Factor Experiment

13-2.4 Residual Analysis and Model Checking

Table 13-6  Residuals for the Tensile Strength Experiment

<table>
<thead>
<tr>
<th>Hardwood Concentration (%)</th>
<th>Residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-3.00</td>
</tr>
<tr>
<td>10</td>
<td>-3.67</td>
</tr>
<tr>
<td>15</td>
<td>-3.00</td>
</tr>
<tr>
<td>20</td>
<td>-2.17</td>
</tr>
</tbody>
</table>
13-2 The Completely Randomized Single-Factor Experiment

13-2.4 Residual Analysis and Model Checking

Figure 13-4 Normal probability plot of residuals from the hardwood concentration experiment.
13-2 The Completely Randomized Single-Factor Experiment

13-2.4 Residual Analysis and Model Checking

Figure 13-5 Plot of residuals versus factor levels (hardwood concentration).
13-2 The Completely Randomized Single-Factor Experiment

13-2.4 Residual Analysis and Model Checking

Figure 13-6 Plot of residuals versus $\bar{y}_i$
13-4 Randomized Complete Block Designs

13-4.1 Design and Statistical Analyses

The *randomized block design* is an extension of the paired t-test to situations where the factor of interest has more than two levels.

![Randomized Block Design Diagram](image)

**Figure 13-9** A randomized complete block design.
13-4 Randomized Complete Block Designs

13-4.1 Design and Statistical Analyses

For example, consider the situation of Example 10-9, where two different methods were used to predict the shear strength of steel plate girders. Say we use four girders as the experimental units.

<table>
<thead>
<tr>
<th>Treatments (Method)</th>
<th>Block (Girder)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$y_{11}$</td>
</tr>
<tr>
<td>2</td>
<td>$y_{21}$</td>
</tr>
<tr>
<td>3</td>
<td>$y_{31}$</td>
</tr>
</tbody>
</table>
13-4 Randomized Complete Block Designs

13-4.1 Design and Statistical Analyses

General procedure for a randomized complete block design:

<table>
<thead>
<tr>
<th>Treatments</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>b</td>
<td>Totals</td>
<td>Averages</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>b</td>
<td>Totals</td>
<td>Averages</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Totals</td>
<td>Averages</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>1</td>
<td>2</td>
<td>...</td>
<td>b</td>
<td>Totals</td>
<td>Averages</td>
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<td>Totals</td>
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<td></td>
<td>Totals</td>
<td>Averages</td>
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<td></td>
</tr>
<tr>
<td>Averages</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Totals</td>
<td>Averages</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 13-10   A Randomized Complete Block Design with a Treatments and b Blocks
13-4 Randomized Complete Block Designs

13-4.1 Design and Statistical Analyses

The appropriate linear statistical model:

\[ Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \]

\[ i = 1, 2, \ldots, a \]

\[ j = 1, 2, \ldots, b \]

We assume

- treatments and blocks are initially fixed effects
- blocks do not interact
- \( \sum_{i=1}^{a} \tau_i = 0 \) and \( \sum_{j=1}^{b} \beta_j = 0 \)
13-4 Randomized Complete Block Designs

13-4.1 Design and Statistical Analyses

We are interested in testing:

\[ H_0: \tau_1 = \tau_2 = \cdots = \tau_a = 0 \]
\[ H_1: \tau_i \neq 0 \text{ at least one } i \]

The sum of squares identity for the randomized complete block design is

\[
\sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar{y}_{..})^2 = b \sum_{i=1}^{a} (\bar{y}_{i.} - \bar{y}_{..})^2 + a \sum_{j=1}^{b} (\bar{y}_{.j} - \bar{y}_{..})^2 \\
+ \sum_{i=1}^{a} \sum_{j=1}^{b} (y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..})^2 \quad (13-27)
\]

or symbolically

\[ SS_T = SS_{\text{Treatments}} + SS_{\text{Blocks}} + SS_E \]
13-4 Randomized Complete Block Designs

13-4.1 Design and Statistical Analyses

The mean squares are:

\[
MS_{\text{Treatments}} = \frac{SS_{\text{Treatments}}}{a - 1}
\]

\[
MS_{\text{Blocks}} = \frac{SS_{\text{Blocks}}}{b - 1}
\]

\[
MS_E = \frac{SS_E}{(a - 1)(b - 1)}
\]
13-4 Randomized Complete Block Designs

13-4.1 Design and Statistical Analyses

The expected values of these mean squares are:

\[
E(MS_{\text{Treatments}}) = \sigma^2 + \frac{b \sum_{i=1}^{a} \tau_i^2}{a - 1}
\]

\[
E(MS_{\text{Blocks}}) = \sigma^2 + \frac{a \sum_{j=1}^{b} \beta_j^2}{b - 1}
\]

\[
E(MS_E) = \sigma^2
\]
### 13-4 Randomized Complete Block Designs

#### 13-4.1 Design and Statistical Analyses

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>$F_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatments</td>
<td>$SS_{\text{Treatments}}$</td>
<td>$a - 1$</td>
<td>$\frac{SS_{\text{Treatments}}}{a - 1}$</td>
<td>$\frac{MS_{\text{Treatments}}}{MS_E}$</td>
</tr>
<tr>
<td>Blocks</td>
<td>$SS_{\text{Blocks}}$</td>
<td>$b - 1$</td>
<td>$\frac{SS_{\text{Blocks}}}{b - 1}$</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>$SS_E$ (by subtraction)</td>
<td>$(a - 1)(b - 1)$</td>
<td>$\frac{SS_E}{(a - 1)(b - 1)}$</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>$SS_T$</td>
<td>$ab - 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 13-5

An experiment was performed to determine the effect of four different chemicals on the strength of a fabric. These chemicals are used as part of the permanent press finishing process. Five fabric samples were selected, and a randomized complete block design was run by testing each chemical type once in random order on each fabric sample. The data are shown in Table 13-12. We will test for differences in means using an ANOVA with $\alpha = 0.01$.

<table>
<thead>
<tr>
<th>Chemical Type</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Treatment Totals $y_p$</th>
<th>Treatment Averages $\bar{y}_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.3</td>
<td>1.6</td>
<td>0.5</td>
<td>1.2</td>
<td>1.1</td>
<td>5.7</td>
<td>1.14</td>
</tr>
<tr>
<td>2</td>
<td>2.2</td>
<td>2.4</td>
<td>0.4</td>
<td>2.0</td>
<td>1.8</td>
<td>8.8</td>
<td>1.76</td>
</tr>
<tr>
<td>3</td>
<td>1.8</td>
<td>1.7</td>
<td>0.6</td>
<td>1.5</td>
<td>1.3</td>
<td>6.9</td>
<td>1.38</td>
</tr>
<tr>
<td>4</td>
<td>3.9</td>
<td>4.4</td>
<td>2.0</td>
<td>4.1</td>
<td>3.4</td>
<td>17.8</td>
<td>3.56</td>
</tr>
<tr>
<td>Block totals</td>
<td>9.2</td>
<td>10.1</td>
<td>3.5</td>
<td>8.8</td>
<td>7.6</td>
<td>39.2($\bar{y}_\cdot$)</td>
<td></td>
</tr>
<tr>
<td>Block averages</td>
<td>2.30</td>
<td>2.53</td>
<td>0.88</td>
<td>2.20</td>
<td>1.90</td>
<td>1.96($\bar{y}_{..}$)</td>
<td></td>
</tr>
</tbody>
</table>
Example 13-5

Table 13-13  Analysis of Variance for the Randomized Complete Block Experiment

<table>
<thead>
<tr>
<th>Source of Variation</th>
<th>Sum of Squares</th>
<th>Degrees of Freedom</th>
<th>Mean Square</th>
<th>$f_0$</th>
<th>$P$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical types</td>
<td>18.04</td>
<td>3</td>
<td>6.01</td>
<td>75.13</td>
<td>4.79 E-8</td>
</tr>
<tr>
<td>(treatments)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fabric samples</td>
<td>6.69</td>
<td>4</td>
<td>1.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(blocks)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>0.96</td>
<td>12</td>
<td>0.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>25.69</td>
<td>19</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The ANOVA is summarized in Table 13-13. Since $f_0 = 75.13 > f_{0.01,3,12} = 5.95$ (the $P$-value is $4.79 \times 10^{-8}$), we conclude that there is a significant difference in the chemical types so far as their effect on strength is concerned.
### 13-4 Randomized Complete Block Designs

#### Minitab Output for Example 13-5

<table>
<thead>
<tr>
<th>Table 13-14</th>
<th>Minitab Analysis of Variance for the Randomized Complete Block Design in Example 13-5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Analysis of Variance (Balanced Designs)</strong></td>
<td></td>
</tr>
<tr>
<td>Factor</td>
<td>Type</td>
</tr>
<tr>
<td>Chemical</td>
<td>fixed</td>
</tr>
<tr>
<td>Fabric S</td>
<td>fixed</td>
</tr>
</tbody>
</table>

#### Analysis of Variance for strength

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical</td>
<td>3</td>
<td>18.0440</td>
<td>6.0147</td>
<td>75.89</td>
<td>0.000</td>
</tr>
<tr>
<td>Fabric S</td>
<td>4</td>
<td>6.6930</td>
<td>1.6733</td>
<td>21.11</td>
<td>0.000</td>
</tr>
<tr>
<td>Error</td>
<td>12</td>
<td>0.9510</td>
<td>0.0792</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>19</td>
<td>25.6880</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

F-test with denominator: Error

Denominator MS = 0.079250 with 12 degrees of freedom

<table>
<thead>
<tr>
<th>Numerator</th>
<th>DF</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemical</td>
<td>3</td>
<td>6.015</td>
<td>75.89</td>
<td>0.000</td>
</tr>
<tr>
<td>Fabric S</td>
<td>4</td>
<td>1.673</td>
<td>21.11</td>
<td>0.000</td>
</tr>
</tbody>
</table>
Fisher’s Least Significant Difference for Example 13-5

\[
\bar{y}_1 = 1.14 \quad \bar{y}_2 = 1.76 \quad \bar{y}_3 = 1.38 \quad \bar{y}_4 = 3.56
\]

\[
\text{LSD} = t_{0.025, 12} \sqrt{\frac{2MSE}{b}} = 2.179 \sqrt{\frac{2(0.08)}{5}} = 0.39
\]

Figure 13-10 Results of Fisher’s LSD method.
13-4 Randomized Complete Block Designs

13-4.3 Residual Analysis and Model Checking

(a) Normal prob. plot of residuals

(b) Residuals versus $\hat{y}_{ij}$
13-4 Randomized Complete Block Designs

(a) Residuals by block.  
(b) Residuals by treatment