Problem 1
Based on a sample of 50 observations having mean 35.36 and standard deviation 4.26, find a 95% confidence interval for the population mean. Note: Use R to find $t_{0.025;49} = 2.009575$.

Answer
We don’t know the population standard deviation, therefore we will use the $t$ distribution:

$$\bar{x} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$$

Substitute to get

$$35.36 \pm 2.009575 \frac{4.26}{\sqrt{50}} \quad \text{or} \quad 35.36 \pm 1.21$$

We conclude that a 95% confidence interval for $\mu$ is $(34.15, 36.57)$.

Problem 2
Consider the confidence interval for $\mu$ based on a sample of size $n$ from a population which is assumed normal. Assume $\sigma$ is unknown.

a. If $n = 10$, how much longer (percentagewise) is a 95% confidence interval than a 90% confidence interval?

b. If $n = 20$, how much longer (percentagewise) is a 95% confidence interval than a 90% confidence interval?

c. Finding the limiting answer, as $n$ gets very large, to the solution to the questions above.

Answer

a. The confidence interval is $\bar{x} \pm t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$. Therefore the length of this confidence interval is $2 \times t_{\alpha/2, n-1} \frac{s}{\sqrt{n}}$. The ratio of the length of a 95% confidence interval to the length of a 90% confidence interval is

$$\frac{2 \times t_{0.05;9} \frac{s}{\sqrt{9}}}{2 \times t_{0.025;9} \frac{s}{\sqrt{9}}} = \frac{t_{0.025;9}}{t_{0.05;9}}$$

We see that the ratio depends only on the values from the $t$ table and it is equal to $\frac{2.262}{1.833} = 1.234$. It follows that the 95% confidence intervals is 23.4% longer.

b. Exactly as in part (a), except that the degrees of freedom are 19, and the ratio is $\frac{2.093}{1.729} = 1.211$. Therefore, the 95% confidence interval is 21.1% longer.

c. As $n$ gets larger the $t$ distribution approaches the $Z$ distribution, therefore the ratio is $\frac{1.96}{1.645} = 1.191$. Therefore, the 95% confidence interval is 19.1% longer.
Problem 3
A sample of 612 voters were asked whether they would vote for candidate A in the upcoming election. Of these, 344 said that they intended to vote for candidate A. Find a 95% confidence interval for the population parameter \( p \) (corresponding to the proportion who will vote for candidate A).

Answer
\[
\hat{p} = \frac{344}{612} \\
\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\
0.562 \pm 1.96 \sqrt{\frac{0.562(1-0.562)}{612}} \\
0.562 \pm 0.039, \text{ or } 0.523 \leq p \leq 0.601.
\]

Problem 4
The distributor of a certain variety of tomato seed has promised that 80% of the seeds will germinate under standard greenhouse conditions. You test this claim with 500 seeds, and you find that 362 germinate successfully. At the 5% level of significance, test the claim as \( H_0 : p = 0.80 \) against the alternative \( H_a : p \neq 0.80 \). Also find the p-value for this test.

Answer
a. Formulate the hypothesis:
\[
H_0 : p = 0.80 \\
H_a : p \neq 0.80
\]

b. Test statistic:
\[
z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.724 - 0.80}{\sqrt{\frac{0.80(1-0.80)}{500}}} = -4.25.
\]

c. Rejection region
Since this is a two-sided test with \( \alpha = 0.05 \) the two reject regions are \( z > 1.96 \) or \( z < -1.96 \). Since \( z = -42.5 < -1.96 \) we reject the null hypothesis.

The p-value is \( P(\hat{p} < 0.724) = P(Z < -4.25) \approx 0. \)

Problem 5
if the lifetimes of 15 of batteries (which constitutes a random sample from a normal population) have:
\[
\sum_{i=1}^{15} x_i = 250, \quad \sum_{i=1}^{15} x_i^2 = 8000.
\]

where \( x \) denotes the lifetime of a battery, construct a 95% confidence interval for \( \sigma^2 \).

Answer
This is a confidence interval for the population variance \( \sigma^2 \) given by
\[
\left[ \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}, n-1}}, \quad \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}, n-1}} \right]
\]

In this problem, \( n = 15, \ 1 - \alpha = 0.95 \), and \( s^2 = \frac{1}{n-1} \left[ \sum_{i=1}^{n} x_i^2 - \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^2 \right] = \frac{1}{15-1}(8000 - \frac{2560^2}{15}) = 273.81 \). The confidence interval is
\[
\left[ \frac{(15 - 1)273.81}{26.12}, \quad \frac{(15 - 1)273.81}{5.63} \right] \text{ or } (146.76, 680.88).
\]
Problem 6
Answer the following questions:

a. A sample of size 16 gave a variance of 5.76. Find $c$ such that $P(|\bar{X} - \mu| < c) = 0.95$, where $\bar{X}$ is the sample mean and $\mu$ is the population mean. Assume that the sample comes from a normal distribution.

The statement $P(|\bar{X} - \mu| < c) = 0.95$ is equivalent to

$P(-c < \bar{X} - \mu < c) = 0.95$ and if we divide all sides of the inequality with $\frac{s}{\sqrt{n}}$

$P\left(-\frac{c}{\sqrt{5.76}} < t_{15} < \frac{c}{\sqrt{5.76}}\right) = 0.95$

Therefore using the $t$ table with 15 degrees of freedom we find $\frac{4c}{\sqrt{5.76}} = 2.131 \Rightarrow c = 1.2786$.

b. A sample from a normal population produced a sample variance of 4.0. Find the sample size if the sample mean deviates from the population mean by no more than 2.0 with probability at least 0.95. This statement is written as $P(|\bar{X} - \mu| < 2) \geq 0.95$ or $P(-2 < \bar{X} - \mu < 2) \geq 0.95$ or $P\left(-\frac{2}{s} < \frac{\bar{X} - \mu}{s} < \frac{2}{s}\right) \geq 0.95$ or $P\left(-\sqrt{n} < t_{n-1} < \sqrt{n}\right) \geq 0.95$

To find $n$ let’s look at some entries in the $t$ table under $t_{0.975}$:

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<th>$df$</th>
<th>$t_{0.975}$</th>
<th>$n$</th>
<th>$\sqrt{n}$</th>
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<td>4</td>
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<td>3.0</td>
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</table>

Therefore, $n = 7$.

c. A bottling machine is supposed to discharge an average of $\mu$ ounces per bottle with variance $\sigma^2 = 0.04$ ounce. It is also assumed that the amount dispensed by the machine is normally distributed. A sample of $n = 9$ filled bottles is randomly selected and it is found that the sample variance is $s^2 = 0.13$ ounce. Does the result suggest that the machine does not meet the standard deviation specification?

Hint: Find the probability that the sample variance will exceed 0.13 when $\sigma^2 = 0.04$.

Answer:

In class we learned that $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$. In order to produce this we must multiply by $n - 1$ and divide by $\sigma^2$ both sides of the inequality below:

$P(S^2 > 0.13) = P\left(\frac{(9-1)s^2}{\sigma^2} > \frac{8(0.13)}{0.04}\right) = P(\chi^2 > 26) < 0.005$. We found this probability by examine the $\chi^2$ table with 8 degrees of freedom. Therefore, the machine does not meet the standard deviation specification because to obtain a value $s^2 = 0.13$ or more extreme seems to be a very unusual!