Problem 1  (25 points)
Answer the following questions:

a. Let $X \sim \Gamma(\alpha, \beta)$. Show that $Y = cX$ follows $\Gamma(\alpha, c\beta)$.

$$\begin{align*}
F_Y(y) &= P(Y \leq y) = P(cX \leq y) \\
&= P(X \leq \frac{y}{c}) = F_X\left(\frac{y}{c}\right) \\
\Rightarrow f_Y(y) &= \frac{1}{c} f_X\left(\frac{y}{c}\right) = \frac{1}{c} \left[ \frac{y^{a-1} e^{-\frac{y}{\beta}}}{\Gamma(a)} \right] \\
&= \frac{\alpha - 1}{c \Gamma(a)} \frac{y^{a-1} e^{-\frac{y}{c\beta}}}{\Gamma(a)(c\beta)^a} \Rightarrow Y \sim \Gamma(\alpha, c\beta)
\end{align*}$$

b. Let $X \sim N(\mu, \sigma)$. Find the distribution of $Y = e^X$.

$$
f_Y(y) = P(Y \leq y) = P(e^X \leq y) \\
= P(X \leq \ln y) = F_X(\ln y)
$$

$$
f_Y(y) = \int_{0}^{y} f_X(\ln y) d\ln y = \frac{1}{\sqrt{2\pi} \sigma} \int_{0}^{y} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}} d\ln y
$$

C. Suppose the radius of a circle $X$ is a random variable that follows the exponential distribution with parameter $\lambda$. Find the distribution of the area of the circle: $Y = \pi X^2$.

$$f_Y(y) = P(Y \leq y) = P(\pi X^2 \leq y) = P\left(X^2 \leq \frac{y}{\pi}\right)$$

$$= P\left(-\sqrt{\frac{y}{\pi}} \leq X \leq \sqrt{\frac{y}{\pi}}\right) = P\left(X \leq \sqrt{\frac{y}{\pi}}\right) - P\left(X \leq -\sqrt{\frac{y}{\pi}}\right)$$

$$= F_X(\sqrt{\frac{y}{\pi}}) - f_X(\sqrt{\frac{y}{\pi}}) = \int_{0}^{\sqrt{\frac{y}{\pi}}} f_X(x) dx =$$

$$f_Y(y) = \frac{1}{2\sqrt{\pi} \lambda} e^{-\lambda \sqrt{\frac{y}{\pi}}}
$$
Problem 2 (25 points)
In California earthquakes of magnitude 1-2 in the Richter scale are recorded at the rate of 8 per hour according to a Poisson distribution. Answer the following questions:

a. What is the probability that more than 12 earthquakes (of magnitude 1-2 in the Richter scale) will be recorded in the next hour. Please write the expression that computes the exact probability (no computations).

\[ P(X > 12) = \sum_{x=13}^{\infty} \frac{8^x e^{-8}}{x!} = 1 - \sum_{x=0}^{12} \frac{8^x e^{-8}}{x!} \]

b. Approximate the probability of part (a) using the normal distribution.

\[ P(X > 12) = P\left( Z > \frac{12.5 - 8}{\sqrt{8}} \right) = P\left( Z > 1.59 \right) \]

\[ = 1 - 0.9441 = 0.0559 \]

c. What is the distribution of the time that you have to wait until the 30th earthquake (of magnitude 1-2 in the Richter scale) is recorded. Please write the complete density of the distribution.

\[ T \sim \Gamma(30, \frac{1}{8}) \]

\[ f(t) = \frac{30^{30} t^{29} e^{-\frac{30t}{8}}}{\Gamma(30)} = \frac{30^{30} t^{29} e^{-\frac{30t}{8}}}{\Gamma(30)} \]

d. Using the distribution of part (c) write the expression that computes the probability that the 30th earthquake (of magnitude 1-2 in the Richter scale) will be recorded in less than 4 hours from now.

\[ P(T < 4) = \int_{0}^{30} \frac{30^{30} t^{29} e^{-\frac{30t}{8}}}{\Gamma(30)} dt \]

e. Approximate the probability of part (d).

\[ E(T) = \alpha T = 30 \frac{1}{8} = 3.75 \]

\[ \text{Var}(T) = \alpha \beta^2 = 30 \frac{1}{64} = 0.3 \Rightarrow SD(T) = 0.55 \]

\[ P(T < 4) = P\left( Z \leq \frac{4 - 3.75}{0.55} \right) = P\left( Z \leq 0.54 \right) = 0.6766 \]

f. Write the expression that computes the same probability of part (d) using the Poisson distribution (no computations).

\[ P(T < 4) = P\left( X > 30 \right) = \sum_{x=30}^{\infty} \frac{32 e^{-32}}{x!} \]
Problem 3  (25 points)
Answer the following questions:

a. Using your class notes and the one-page handout on the beta distribution show that the variance of beta distribution is

\[
\text{var}(X) = \frac{\alpha \beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}
\]

Karl's note: \[E X^k = \frac{\beta (\alpha + k, \beta)}{\beta (\alpha, \beta)}\]

\[
E X = \frac{\alpha}{\alpha + \beta}
\]

\[
E X^2 = \frac{B(\alpha + 2, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha + 2) \Gamma(\beta)}{\Gamma(\alpha + 2 + \beta)} - \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + 2 + \beta)}
\]

\[
= \frac{(\alpha + 1)\Gamma(\alpha + 1) \Gamma(\beta)}{(\alpha + \beta + 1)\Gamma(\alpha + 2 + \beta)} - \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + 2 + \beta)} = \frac{\alpha (\alpha + 1)}{(\alpha + \beta)(\alpha + 2 + \beta)} - \frac{\alpha^2}{(\alpha + 2 + \beta)}
\]

\[
\text{Var}(X) = \frac{\alpha (\alpha + 1)}{(\alpha + \beta)(\alpha + 2 + \beta)} - \frac{\alpha^2}{(\alpha + 2 + \beta)}
\]

b. Scores on a certain standardized test, IQ scores, follow the normal distribution with mean \( \mu = 100 \) and standard deviation \( \sigma = 13 \). An individual is selected at random. What is the probability that his score satisfies \( 120 < X < 130 \)?

\[
P\left(\frac{120 - 100}{13} < Z < \frac{130 - 100}{13}\right)
\]

\[
= P\left(1.54 < Z < 2.31\right) = P\left(Z < 2.31\right) - P\left(Z < 1.54\right)
\]

\[
= 0.9896 - 0.9382 = 0.0514.
\]

c. Refer to part (b). Find the 40th percentile of the distribution of these scores.

\[
0.755 = \frac{c - 100}{13}
\]

\[
c = 100 - (0.755)13
\]

\[
c = 96.685
\]
Problem 4 (25 points)
Answer the following questions:

a. It is said that a random variable $X$ follows the Pareto distribution with parameters $x_0$ and $\alpha$ with $x_0 > 0, \alpha > 0$, if $X$ has the following probability density function:

$$f(x) = \frac{\alpha x_0^\alpha}{x_0^{\alpha+1}}, \text{ for } x \geq x_0.$$

Show that $Y = \ln\left(\frac{x}{x_0}\right)$ follows the exponential distribution with parameter $\alpha$. Note: $x_0$ is a constant.

$$F_Y(y) = P(\ln\left(\frac{x}{x_0}\right) \leq y) = P\left(\frac{x}{x_0} \leq e^y\right) = P(X \leq x_0 e^y)$$

$$F_Y(y) = F_X(x_0 e^y) \Rightarrow f_Y(y) = x_0 \frac{d}{dx} F_X(x_0 e^y)$$

$$= x_0 e^y \frac{\alpha x_0}{\alpha+1} \frac{x_0}{e} = \alpha e^{-\alpha y}$$

$$\Rightarrow Y \sim \text{exp}(\alpha).$$

b. Let $X \sim \Gamma(\alpha, \beta)$, with $\alpha > 2, \beta > 0$. Show that the variance of $\frac{1}{X}$ is $\frac{1}{\beta^2(\alpha-1)^2(\alpha-2)}$. 

$$EX^k = \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)} \Rightarrow EX^{-1} = \frac{\theta^{-1} \Gamma(\alpha-1)}{\Gamma(\alpha)} = \frac{1}{\beta(\alpha-1)}$$

$$EX^{-2} = \frac{\theta^{-2} \Gamma(\alpha-2)}{\Gamma(\alpha)} = \frac{1}{\beta^2(\alpha-1)(\alpha-2)}$$

$$\text{Var} \left(\frac{1}{X}\right) = \left(EX^{-1}\right)^2 - \left(EX^{-1}\right)^2 = \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)} - \frac{1}{\beta^2(\alpha-1)}^2$$

$$= \frac{(\alpha-2) - (\alpha-1)}{\beta^2(\alpha-1)^2(\alpha-2)} = \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)}$$

c. Let $U$ be a uniform random variable on $[0,1]$, and let $V = \frac{1}{U}$. Find the probability density function of $V$. For what values of $V$ is this density valid? On the previous page please draw the density of $U$ and the density of $V$ on two separate graphs.

$$F_U(v) = P(U \leq v) = P\left(\frac{1}{U} \leq v\right) = P\left(U > \frac{1}{v}\right)$$

$$= 1 - P\left(U \leq \frac{1}{v}\right) \Rightarrow F_V(v) = 1 - F_U\left(\frac{1}{v}\right)$$

$$= 1 - \frac{1}{v}$$

$$f_U(v) = \frac{1}{v^2} f_U\left(\frac{1}{v}\right) \rightarrow 1 \quad \text{for } v > 0$$

$$f_V(v) = \frac{1}{v^2}$$

$$f_V(v) = \frac{1}{v^2}$$