EXERCISE 1
A coin is tossed 3 times independently. One of the variables of interest is the number of tails $X$. Let $Y$ denote the amount of money won on a side bet in the following manner:
If the first tail occurs on the first toss, you win $1$.
If the first tail occurs on the second toss, you win $2$.
If the first tail occurs on the third toss, you win $3$.
If no tails appear you lose $1$.
Construct the joint probability distribution of $X$ and $Y$. In other words complete the following table where the entries are the probabilities for each pair of values of the variables $X$ and $Y$.

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>-1</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>1</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>2</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>3</td>
<td>?</td>
<td>?</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

EXERCISE 2
A die is rolled and the number observed $X$ is recorded. Then a coin is tossed number of times equal to the value of $X$. For example if $X = 2$ then the coin is tossed twice, etc. Let $Y$ be the number of heads observed. Note: Assume that the die and the coin are fair.

a. Construct the joint probability distribution of $X$ and $Y$.
b. Find the conditional expected value of $Y$ given $X = 5$.
c. Find the conditional variance of $Y$ given $X = 5$.

EXERCISE 3
There are three checkout counters at a local supermarket. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random and independently of the other. Let $X$ denote the number of customers who choose counter 1 and $Y$ the number of customers who select counter 2. Find the joint probability distribution of $X$ and $Y$.

EXERCISE 4
Let $X$ and $Y$ denote the proportion of time, out of the workday, that employees $I$ and $II$, respectively, actually spend performing their assigned tasks. The joint probability density function of $X$ and $Y$ is as follows:

$$ f_{XY}(x, y) = \begin{cases} 
  x + y & 0 \leq x \leq 1; 0 \leq y \leq 1 \\
  0 & \text{elsewhere}
\end{cases} $$

a. Find $P(X < \frac{1}{2}, Y > \frac{1}{4})$. [Ans. $\frac{3}{16}$]
b. Find $P(X + Y \leq 1)$. [Ans. $\frac{1}{2}$]

EXERCISE 5
A particular fast-food outlet is interested in the joint behavior of the random variables $X$, defined as the total time between a customer’s arrival at the store and leaving the service window, and $Y$, the time that a customer waits in line before reaching the service window. Because $X$ contains the time a customer waits in line, we must have $X \geq Y$. Suppose the joint probability density function of $X$ and $Y$ is as follows:

$$ f_{XY}(x, y) = \begin{cases} 
  e^{-x} & 0 \leq y \leq x < \infty \\
  0 & \text{elsewhere}
\end{cases} $$

with time measured in minutes.

a. Find $P(X < 2, Y > 1)$. [Ans. $e^{-1} - 2e^{-2}$]
b. Find $P(X \geq 2Y)$. [Ans. $\frac{1}{2}$]
c. Find $P(X - Y \geq 1)$. Note that $X - Y$ denotes the time spent at the service window. [Ans. $e^{-1}$]
EXERCISE 6
Let $X$ and $Y$ have the joint probability density function given by

$$f_{XY}(x, y) = \begin{cases} kxy & 0 \leq x \leq 1; 0 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the constant $k$ that makes this a probability density function. [Ans. 4]
b. Find $P(X \leq \frac{1}{2}, Y \leq \frac{3}{4})$. [Ans. $\frac{9}{64}$]

EXERCISE 7
Refer to exercise 4.

a. Find the marginal density functions for $X$ and $Y$. [Ans. $f_X(x) = x + \frac{1}{2}, f_Y(y) = y + \frac{1}{2}$]
b. Find $P(X \geq \frac{1}{2}|Y \geq \frac{1}{2})$. [Ans. $\frac{2}{3}$]
c. If employee II spends exactly 50% of the day on assigned duties, find the probability that employee I spends more than 75% of the day on similar duties. In other words find $P(X > 0.75|Y = 0.5)$. [Ans. $\frac{11}{12}$]

EXERCISE 8
Refer to exercise 6.

a. Find the marginal density functions of $X$ and $Y$. [Ans. $f_X(x) = 2x, f_Y(y) = 2y$]
b. Find $P(X \leq \frac{1}{2}|Y \geq \frac{3}{4})$. [Ans. $\frac{1}{4}$]
c. Find the conditional density function of $X$ given $Y = y$. [Ans. $2x$]
d. Find the conditional density function of $Y$ given $X = x$. [Ans. $2y$]
e. Find $P(X \leq \frac{3}{4}|Y = \frac{1}{2})$. [Ans. $\frac{9}{16}$]