Problem 1  (25 points)
Answer the following questions:

a. The pdf of a random variable $X$ is $f(x) = \frac{1}{x}e^{-\frac{x}{2}}$. What is the distribution of $X$.

$$
X \sim \Gamma\left(\frac{4}{2}, \frac{4}{2}\right) = \frac{\Gamma\left(\frac{4}{2}ight)}{\Gamma\left(\frac{2}{2}\right)} = \frac{\Gamma\left(2\right)}{\Gamma\left(1\right)} = \frac{1}{2} = \frac{\Gamma\left(2\right)}{\Gamma\left(1\right)}
$$

$$
\frac{X}{2} \sim \Gamma\left(1, \frac{2}{2}\right) = \frac{\Gamma\left(1\right)}{\Gamma\left(1\right)} = \frac{1}{1} = \frac{\Gamma\left(1\right)}{\Gamma\left(1\right)}
$$

b. Let $X \sim \Gamma\left(\frac{3}{2}, \beta\right)$. Find the distribution of $Y = \frac{2X}{\beta}$.

$$
M_Y(t) = M_{\frac{2X}{\beta}}(t) = M_X\left(\frac{2t}{\beta}\right) = \left(1 - \frac{2t}{\beta}\right)^{-\frac{1}{2}}
$$

$$
\Rightarrow Y \sim \chi^2_n
$$

c. Let $X \sim \Gamma(\alpha, \beta)$. A random sample $X_1, X_2, \ldots, X_n$ is selected from this distribution. Suppose that $\alpha$ is known and $\hat{\beta} = \frac{X}{n}$ is an estimator of $\beta$. Is $\hat{\beta}$ MVUE of $\beta$?

$$
\text{Var} \left(\frac{X}{\alpha}\right) = \frac{\sigma^2}{n\alpha^2} = \frac{\alpha \beta^2}{n\alpha^2} = \text{Var} (\hat{\beta}) = \frac{\beta^2}{n\alpha^2}
$$
Problem 2 (25 points)
Let $X_1, X_2, X_3, X_4$ be iid random variables each one with probability $N(10, 4\sigma)$. Let $Y_1, Y_2, Y_3$ be iid random variables each one with probability $N(5, 3\sigma)$. The two samples are independent. Also, let $\bar{X}, \bar{Y}, S^2_\bar{X}, S^2_\bar{Y}$ be the corresponding sample means and sample variances of the two samples. Answer the following questions:

a. What is the distribution of $\bar{X} - \bar{Y}$?

$$\bar{X} - \bar{Y} \sim N\left[5, \sigma\sqrt{\frac{1}{4} + \frac{3}{9}}\right]$$

b. What is the distribution of $\frac{(n-1)S^2_X}{\sigma^2} + \frac{(m-1)S^2_Y}{\sigma^2}$?

$$\frac{(n-1)S^2_X}{\sigma^2} + \frac{(m-1)S^2_Y}{\sigma^2} \sim \chi^2(n-1) + \chi^2(m-1)$$

(c) Use parts (a) and (b) to form a ratio that follows the t distribution. What are the degrees of freedom?

$$\frac{\bar{X} - \bar{Y} - 5}{\sqrt{\frac{3S^2_X}{16\sigma^2} + \frac{2S^2_Y}{9\sigma^2}}} \sim t_{n+m-2}$$

(d) Using the ratio of part (c) find $k$ such that:

$$P\left(\frac{\bar{X} - \bar{Y} - 5}{\sqrt{\frac{3S^2_X}{16\sigma^2} + \frac{2S^2_Y}{9\sigma^2}}} > k\right) = 0.05$$

$$P\left[\frac{\bar{X} - \bar{Y} - 5}{\sigma\sqrt{\frac{3S^2_X}{16\sigma^2} + \frac{2S^2_Y}{9\sigma^2}}} > \frac{\sigma\sqrt{\frac{3S^2_X}{16\sigma^2} + \frac{2S^2_Y}{9\sigma^2}}}{\sigma\sqrt{\frac{3S^2_X}{16\sigma^2} + \frac{2S^2_Y}{9\sigma^2}}}\right] = 0.05$$

$$k\sqrt{\frac{3S^2_X}{16\sigma^2} + \frac{2S^2_Y}{9\sigma^2}} = 2.015 \Rightarrow k = 2.384$$
Problem 3  (25 points)
Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution with mean zero and unknown variance $\sigma^2$.

a. Let $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} X_i^2}{n}$ be an estimator of $\sigma^2$. Is it unbiased?

$$E \hat{\sigma}^2 = E \frac{\sum_{i=1}^{n} X_i^2}{n} = \frac{\sum_{i=1}^{n} EX_i^2}{n} = \frac{n \sigma^2}{n} = \sigma^2$$

YES

b. Find the variance of $\hat{\sigma}^2$. Is it consistent?

$$\sum_{i=1}^{n} \frac{(X_i - 0)^2}{\sigma^2} \sim \chi_n^2 \quad \text{or} \quad \frac{\sum_{i=1}^{n} X_i^2}{\sigma^2} \sim \chi_n^2$$

$$\text{VAR} \left( \frac{\sum_{i=1}^{n} X_i^2}{n \sigma^2} \right) = \frac{\sigma^4}{n} \quad \text{VAR} \left( \frac{\sum_{i=1}^{n} X_i^2}{\sigma^2} \right) = \frac{\sigma^4}{n} \quad \text{as} \ n \to \infty \quad \text{VAR} \to \sigma^4$$

c. Show that the variance of the estimate of part (a) is equal to the Cramér-Rao lower bound.

$$f(x) = \frac{1}{\sigma} e^{-\frac{1}{2 \sigma^2} x^2} \quad \mu(f(x)) = \frac{1}{\sigma} \left( \frac{1}{\sigma^2} \right)^{\frac{1}{2}} = \frac{1}{\lambda^2} \quad \text{if} \quad \lambda = \frac{1}{\sigma^2}$$

$$\frac{\partial^2 \ln f(x)}{\partial \sigma^2} = \frac{1}{2 \sigma^4} - \frac{x^2}{2 \sigma^6}$$

$$\text{MSE}(\hat{\sigma}^2) = \text{VAR} \left( \frac{\sum_{i=1}^{n} X_i^2}{n \sigma^2} \right) = \frac{\sigma^4}{n} \quad \text{as} \ n \to \infty \quad \text{MSE} \to \frac{\sigma^4}{n}$$

YES

d. Another estimator for $\sigma^2$ is $S^2 = \frac{\sum_{i=1}^{n} X_i^2}{n-1}$. Find the MSE of $S^2$ and compare it to the MSE of $\hat{\sigma}^2$.

$$S^2 = \frac{n \sum_{i=1}^{n} X_i^2}{n-1} = \frac{n}{n-1} \hat{\sigma}^2$$

$$\text{MSE}(S^2) = \text{MSE}(\hat{\sigma}^2) + \beta = \frac{\sigma^4}{n} + \beta = \frac{2 \sigma^4}{n} + \frac{\sigma^4}{n-1} = \sigma^4 \left( \frac{2n+1}{n(n-1)} \right)$$

$$\beta = \text{ES}^2 - \sigma^2 = \frac{n}{n-1} \sigma^2 - \sigma^2 = \sigma^2 \left( \frac{n}{n-1} - 1 \right) = \frac{\sigma^2}{n-1}$$

$$\text{MSE}(\hat{\sigma}^2) = \frac{\sigma^4}{n}$$

$$\text{MSE}(S^2) = \frac{\sigma^4 (2n+1)}{(n-1)^2} \quad \text{as} \ n \to \infty \quad \text{MSE} \to \frac{\sigma^4}{n-1}$$

$$2 \hat{\sigma}^2 + \hat{\sigma}^2 + \frac{\sigma^4}{n-1} - \frac{\sigma^4}{n-1} > 0$$
Problem 4  (25 points)
Part A:
A population has mean $\mu = 1.25$ and standard deviation $\sigma = 2$. Repeated samples each one of size $n = 40$ are selected from this population. It is claimed that the histogram below represents the distribution of the total ($T$) of these 40 observations. Clearly explain if there is anything wrong with this histogram.

\[
T \sim N \left( 40 \times 1.25, 2 \sqrt{40} \right) \\
T \sim N \left( 50, 12.65 \right) \\
50 \pm 3 \times 12.65 \\
12 \text{ and } 88 \\
\text{too NARROW!}
\]

Part B:
Let $X_1, X_2, \ldots, X_{19}$ be a random sample from $N(0, \sigma)$, and let $\bar{X}, S^2$ represent the sample mean and sample variance of this sample. Answer the following questions:

a. For what value of $c$ the expression $c \frac{S^2}{\sigma^2}$ follows the $F$ distribution? What are the degrees of freedom?

\[
19 \frac{\bar{X}^2}{S^2} \sim F_{19, \infty} \\
\frac{\left( \frac{\bar{X} - 0}{\sigma / \sqrt{19}} \right)^2}{\frac{18 \cdot \frac{S^2}{\sigma^2}}{18}} = \frac{2^{11, 1}}{18} = F_{11, 18}
\]

b. Using only your statistical tables (please no calculator!) find the 80th percentile of the distribution of $c \frac{S^2}{\sigma^2}$. Show all your work.

\[
F_{0.80, 11, 18} = (1.33)^2 = 1.7689
\]