Problem 1  (25 points)
Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ independent with $X_i \sim \text{exp}\left(\frac{1}{\theta}\right)$, i.e. $\lambda = \frac{1}{\theta}$. Let $R = \sum_{i=1}^{n} \frac{X_i}{n}$. Find the distribution of $R$.

b. Let $X_1, X_2, \ldots, X_n$ independent and identically distributed random variables with $X_i \sim N(\mu, \sigma)$. Consider an estimator of $\sigma^2$ of the form $\theta = cS^2$, where $S^2$ is the sample variance and $c$ is a constant. Find the MSE of $\theta$.

c. Suppose we observe $X_1$ from an exponential distribution with parameter $\frac{1}{\lambda}$. Consider an estimator of $\lambda$ of the form $\hat{\lambda} = cX_1$. Find the MSE of $\hat{\lambda}$.

d. Show that the choice of $c$ that minimizes the MSE in part (c) is $c = 0.5$.

e. Which estimator of $\lambda$ will choose, $cX_1$ or $X_1$. Please explain using the MSE criterion with $c = 0.5$. 

Problem 2  (25 points)
Answer the following questions:

a. Let $X_1, X_2, \ldots, X_n$ independent and identically distributed random variables with $X_i \sim N(\mu, \sigma)$. Consider $S^2$, the estimator of $\sigma^2$. Find the moment generating function of $S^2$.

b. Let $Y_1, Y_2, \ldots, Y_n$ independent bernoulli random variables, with probability of success $p$ and therefore probability of failure $1 - p$. Find the Fisher information $I_1(\theta)$ for this problem.

c. Refer to part (b). Show that $\hat{p} = \frac{X}{n}$ achieves the Cramér-Rao lower bound and is therefore MVUE. Note: $X = \sum_{i=1}^n Y_i$. The goal here is to estimate the success probability $p$ using $n$ independent Bernoulli trials.

d. Suppose $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with $X_i \sim N(\mu, \sigma)$. Two unbiased estimators of $\sigma^2$ are $S^2 = \frac{\sum_{i=1}^n (X_i - \overline{X})^2}{n-1}$ and $\hat{\sigma}^2_1 = \frac{1}{2} (X_1 - X_2)^2$. Find the relative efficiency of $S^2$ relative to $\hat{\sigma}^2_1$.2
Problem 3  (25 points)
Answer the following questions:

a. Let \( X \sim N_n(\mu 1, \Sigma) \), where \( 1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \), and \( \Sigma \) is the variance covariance matrix of \( X \). Let \( \Sigma = (1 - \rho)I + \rho J \), with \( \rho > -\frac{1}{n-1} \), \( I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix} \) and \( J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix} \). Therefore, when \( \rho = 0 \) we have \( X \sim N_n(\mu 1, I) \), and in this case we showed in class that \( \bar{X} \) and \( \sum_{i=1}^{n}(X_i - \bar{X})^2 \) are independent. Are they independent when \( \rho \neq 0 \)?

b. Suppose \( \epsilon \sim N_3(0, \sigma^2 I_3) \) and that \( Y_0 \sim N(0, \sigma^2) \), independently of the \( \epsilon_i \)'s. Therefore the vector \( \begin{pmatrix} Y_0 \\ \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} \), is multivariate normal. Define \( Y_i = \rho Y_{i-1} + \epsilon_i \) for \( i = 1, 2, 3 \). Express \( Y_1, Y_2, Y_3 \) in terms of \( \rho, Y_0, \) and the \( \epsilon_i \)'s.

c. Refer to part (b). Find the covariance matrix of \( Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \),

d. What is the distribution of \( Y \)?
Problem 4  (25 points)
Let $X_1, X_2, X_3, X_4, X_5$ be independent and identically distributed random variables with $X_i \sim N(0, \sigma)$. Answer the following questions:

a. Find $c$ so that the distribution of $cX_1^2$ is $\chi^2$. What are the degrees of freedom?

b. Find $c$ so that $c(X_1 + X_2 + X_3)$ follows that standard normal distribution.

c. What is the distribution of $\frac{X_1^2}{X_2^2}$?

d. Does the ratio $\frac{X_5}{\sqrt{X_3^2 + X_4^2}}$ follow the $t$ distribution? If not, can you multiply the ratio by a constant $c$ so that it follows the $t$ distribution? What are the degrees of freedom?

e. Is $\frac{1}{4}(X_1 + X_2 + X_3)^2$ unbiased estimator of $\sigma^2$?