Problem 1  (25 points)

Answer the following questions:

a. The pdf of a random variable $X$ is $f(x) = \frac{1}{4}xe^{-\frac{x}{2}}$. What is the distribution of $X$.

b. Let $X \sim \Gamma\left(\frac{n}{2}, \beta\right)$. Find the distribution of $Y = \frac{2X}{\beta}$.

c. Let $X \sim \Gamma(\alpha, \beta)$. A random sample $X_1, X_2, \ldots, X_n$ is selected from this distribution. Suppose that $\alpha$ is known and $\hat{\beta} = \frac{\bar{X}}{\alpha}$ is an estimator of $\beta$. Is $\hat{\beta}$ MVUE of $\beta$?
Problem 2  (25 points)
Let $X_1, X_2, X_3, X_4$ be iid random variables each one with probability $N(10, 4\sigma)$.
Let $Y_1, Y_2, Y_3$ be iid random variables each one with probability $N(5, 3\sigma)$.
The two samples are independent. Also, let $\bar{X}, \bar{Y}, S_x^2, S_y^2$ be the corresponding sample means and sample variances of the two samples. Answer the following questions:

a. What is the distribution of $\bar{X} - \bar{Y}$?

b. What is the distribution of $\frac{3S_x^2}{16\sigma^2} + \frac{2S_y^2}{9\sigma^2}$?

c. Use parts (a) and (b) to form a ratio that follows the $t$ distribution. What are the degrees of freedom?

d. Using the ratio of part (c) find $k$ such that:

$$P \left( \frac{\bar{X} - \bar{Y} - 5}{\sqrt{\frac{3S_x^2}{16\sigma^2} + \frac{2S_y^2}{9\sigma^2}}} > k \right) = 0.05.$$
Problem 3  (25 points)
Let $X_1, X_2, \ldots, X_n$ denote a random sample from a normal distribution with mean zero and unknown variance $\sigma^2$.

a. Let $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2$ be an estimator of $\sigma^2$. Is it unbiased?

b. Find the variance of $\hat{\sigma}^2$. Is it consistent?

c. Show that the variance of the estimate of part (a) is equal to the Cramér-Rao lower bound.

d. Another estimator for $\sigma^2$ is $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} x_i^2$. Find the MSE of $S^2$ and compare it to the MSE of $\hat{\sigma}^2$. 
Problem 4  (25 points)

Part A:
A population has mean $\mu = 1.25$ and standard deviation $\sigma = 2$. Repeated samples each one of size $n = 40$ are selected from this population. It is claimed that the histogram below represents the distribution of the total ($T$) of these 40 observations. Clearly explain if there is anything wrong with this histogram.

![Histogram](image)

Part B:
Let $X_1, X_2, \ldots, X_{19}$ be a random sample from $N(0, \sigma)$, and let $\bar{X}, S^2$ represent the sample mean and sample variance of this sample. Answer the following questions:

a. For what value of $c$ the expression $c \frac{\bar{X}^2}{S^2}$ follows the $F$ distribution? What are the degrees of freedom?

b. Using only your statistical tables (please no calculator!) find the 80th percentile of the distribution of $c \frac{\bar{X}^2}{S^2}$. Show all your work.